EXTRACTA MATHEMATICAE Vol. **25**, Núm. 1, 1 – 61 (2010)

Kleisli and Eilenberg-Moore Constructions as Parts of Biadjoint Situations

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Presented by Antonio M. Cegarra Received February 8, 2010

Abstract: We consider monads over varying categories, and by defining the morphisms of Kleisli and of Eilenberg-Moore from a monad to another and the appropriate transformations (2-cells) between morphisms of Kleisli and between morphisms of Eilenberg-Moore, we obtain two 2-categories $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$. Then we prove that $\mathbf{Mnd}_{\mathrm{Kl}}$ and $\mathbf{Mnd}_{\mathrm{EM}}$ are, respectively, 2-isomorphic to the conjugate of **Kl** and to the transpose of **EM**, for two suitably defined 2-categories **Kl** and **EM**, related, respectively, to the constructions of Kleisli and of Eilenberg-Moore. Next, by considering those morphisms and transformations of monads that are simultaneously of Kleisli and of Eilenberg-Moore, we obtain a 2-category $\mathbf{Mnd}_{\mathrm{alg}}$, of monads, algebraic morphisms, and algebraic transformations, and, to confirm its naturalness, we, on the one hand, prove that its underlying category can be obtained by applying the Ehresmann-Grothendieck construction to a suitable contravariant functor, and, on the other, we provide an explicit 2-embedding of a certain 2-category, $\mathbf{Sig}_{\mathfrak{pd}}$, of many-sorted signatures (hence also of another 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications), arising from the field of many-sorted universal algebra, into a 2-category of the type $\mathbf{Mnd}_{\mathrm{alg}}$. Moreover, we investigate for the adjunctions between varying categories the counterparts of the concepts previously defined for the monads, obtaining several 2-categories of adjunctions, as well as several 2-functors from them to the corresponding 2-categories of monads, and all in such a way that the classical Kleisli and Eilenberg-Moore constructions are left and right biadjoints, respectively, for these 2-functors. Finally, we define a 2-category $\mathbf{Ad}_{\mathrm{alg}}$, of adjunctions, algebraic squares, and algebraic transformations, and prove that there exists a canonical 2-functor $\mathrm{Md}_{\mathrm{alg}}$ from $\mathbf{Ad}_{\mathrm{alg}}$ to $\mathbf{Mnd}_{\mathrm{alg}}$.

Key words: Morphism of Kleisli, morphism of Eilenberg-Moore, transformation of Kleisli, transformation of Eilenberg-Moore, adjoint square of Kleisli, adjoint square of Eilenberg-Moore, algebraic square of adjunctions, transformation of algebraic squares, algebraic morphism of monads, algebraic transformation.

AMS *Subject Class.* (2000): 18A40, 18C15, 18C20, 18D05, 03B05, 03B45.

"In memory of our dear friend Fuensanta Andreu Vaillo (1955–2008)"

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