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The Hitchhiker Guide to Categorical Banach Space Theory. Part I.

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Abstract: What has category theory to offer to Banach spacers? In this survey-like paper we will focus on some of the five basic elements of category theory –namely, i) The definition of category, functor and natural transformation; ii) Limits and colimits; iii) Adjoint functors; plus a naive presentation of Kan extensions– to support the simplest answer "tools that work and a point of view that helps to understand problems, even if one does not care at all about categories". Homology will be treated in a second part.

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1. Prologue

Functional analysis, in general, and Banach space theory, in particular, are unthinkable outside the ambient of algebraic structures. In fact, functional analysis can be defined as the blend of algebra and topology. One of the major achievements of mathematics in the XXth century is category theory, whose paramount importance comes from the fact of establishing a common language for all mathematics. Indeed, it becomes easier for mathematicians working in different areas to communicate if they are able to identify their basic constructions, knowing that certain objects of their disciplines are particular instances of the same concept. And the only part of mathematics able today to provide such common language is categorical algebra. The organization of this paper is as follows:

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Contents

Now, focusing in Banach space theory:

What is categorical Banach space theory?

It is the blend of categorical algebra and Banach space theory. The first steps should be the understanding and interpretation of the basic categorical concepts in the domain of Banach space theory. In this regard, probably the five basic elements of category theory are

1. The definition of category, functor and natural transformation.
2. Limits and colimits.
3. Adjoint functors.
4. Derived functors.
5. Kan extensions.

In this survey we will focus on the first three of those elements plus a naive presentation of Kan extensions. The study of derived functors conforms what is called homology theory, and will be treated in part II. Impatient readers will find a few homological elements in [82]; and most of them in [80].

What has category theory to offer to Banach spacers?

The answer is simple: Tools that work and a point of view that helps to understand problems, even if one does not care at all about categories.

A second type of answer could be: almost the same as Banach space theory has to offer to category theory. Indeed, there are good arguments to sustain the idea that Banach spaces is a very interesting category, even if one just cares about category theory.

The purpose of this survey is therefore to establish a few travel signals along the road towards "Categorical Banach space theory".

$\downarrow$ (*downwards direction*) Making the basic ideas, elements and techniques of categorical algebra accessible to Banach spacers. This implies, but is not restricted to, bringing the categorical concepts to life in a Banach space ambient.

$\uparrow$ (*upwards direction*) To encourage research in the complementary aspect of the problem: identifying the categorical elements that correspond to Banach space constructions which are useful inside the theory; and, if necessary, to invent and develop them.

The categorical approach to Banach space theory has a far richer and longer history than one can expect; a brief account of how categorical concepts have appeared and have been studied in Banach space theory will also be presented.

2. The Eilenberg-MacLane program

The Eilenberg-MacLane program, from now on EM, lays the foundations for any theory having expectations to be "well-done". As stated in [3], it reads:

> "The theory emphasizes that, whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects as an integral part of their definition. The pursuit of this program entails a simultaneous consideration of objects and their mappings (in modern terminology, the category). This emphasis on the specification of the type of mappings employed gives more insight into the degree of invariance of the various concepts involved [...] For instance, the concept of the commutator subgroup of a group is in a sense a more invariant one than that of the center, which in turn is more invariant than the concept of the automorphism group of a group, even though in the classical sense all three concepts are invariant.
>
> The invariant character of a mathematical discipline can be formulated in these terms. Thus, in group theory all the basic constructions can be regarded as the definitions of co- or contravariant functors, so we may formulate the dictum: The subject of group theory is essentially the study of those constructions of groups which behave in a covariant or contravariant manner under induced homomorphisms."

It is a pity that The Eilenberg-MacLane paper did not contemplate Banach space theory, although this had been formalized almost fifteen years ago with Banach's book [1]. Otherwise, it should not sound so strange that "The subject of *Banach space theory* is essentially the study of those constructions of *Banach spaces* which behave in a covariant or contravariant manner under induced homomorphisms".

3. The principles of categorical Banach space theory

A functor $\mathcal{F} : \mathbf{A} \to \mathbf{C}$ between two categories is a correspondence assigning objects to objects and arrows to arrows which respects composition and identities. the functor is called covariant if whenever $f : A \to C$ then $\mathcal{F}(f)$:

$\mathcal{F}(A) \to \mathcal{F}(C)$. It is said contravariant if, however, $\mathcal{F}(f) : \mathcal{F}(C) \to \mathcal{F}(A)$.

Thus, what EM means, translated to the Banach space world is:

► *Banach space constructions must be understood and formulated as functors.*

As it is formulated by Mityagin and Švarc [22]:

> "The essence of the matter is that nearly every natural construction of a new Banach space from a given Banach space generates a certain (covariant o contravariant) functor".

This is the foundational rule for any categorical (Banach space or not) work. So let us start labelling **Ban** the category of Banach spaces, whose objects are Banach spaces and whose arrows are the linear continuous operators. It will be often necessary to work with the related category $\mathbf{Ban}_1$ whose objects are Banach spaces but whose arrows are only the linear continuous contractions (operators having norm at most 1). Here it comes the first scholium:

SCHOLIUM. Naively speaking, a good category to work within is one in which one can safely work with diagrams. Working with diagrams is simple, extremely nice; and powerful too, sometimes. To work with diagrams there are some rules to follow; basically, that the drawings have to be *complete*, which means that they start with 0 and end with 0. For instance, if we have an operator $T : W \to X$ the completion will be a diagram like

$$W \xrightarrow{T} X$$

which, when completed should give

$$0 \longrightarrow K \xrightarrow{i} W \xrightarrow{T} X \xrightarrow{q} C \longrightarrow 0.$$

Do the objects K, i, C, q exist? Well, K is the kernel of T and $i : \ker T \to W$ is the canonical inclusion. When W is a Banach space and T s a linear continuous operator then $\ker T$ is a Banach space so, up to here, things work. But C should be the cokernel of T, namely $C = X/\operatorname{Im} T$. And the image of an operator is not necessarily closed, so $X/\operatorname{Im} T$ is not necessarily a Banach space. And things have stopped to work.

Thus, the category of Banach spaces is a soft place to work since it is not a "good" category –Abelian category is the proper name– and, nevertheless, a lot of categorical (and homological) constructions can be used. Indeed, the previous difficulty suggests two possible lines of action:

- To widen up the ambient category to admit semi-Banach (in general, non-Hausdorff spaces) spaces. Indeed, it is the Hausdorff character what makes diagrams to be, in general, incomplete. This option has a a price: loosing (in principle) tools as the open mapping theorem.
- To continue working within the category of Banach spaces but reducing the class of allowable operators in order to be able to complete diagrams. Since operators with cokernel are those having closed range (strict morphisms in the language of Raikov), the are essentially two types of operators one wants to see in a diagram: isomorphic embeddings and quotient maps. These are the maps that appear in an exact sequence
$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0.$$
Recall that an exact sequence is a diagram as above in which the kernel of each arrow coincides with the image (cokernel) of the preceding. So, j must be injective, q must be surjective –thus a quotient map– whose kernel is $j(Y)$, which must be closed making thus j an into isomorphism. This type of categories are called exact in Quillen's sense.

End of the Scholium

To definitely loose fear to categorical thinking, we state now a couple of categorical Banach space problems claiming life:

Problem 1. Study the category having as objects Banach spaces and Lipschitz maps as morphisms.

Problem 2. Construct the very much needed category of Banach spaces and multilinear maps.

See in [84] the construction of an operative category of exact sequences of Banach spaces.

A *Banach functor* is a functor $\mathcal{F} : \mathbf{B} \to \mathbf{B}$ acting in a certain subcategory of Banach spaces. Most often it will be moreover required that $\mathcal{F}$ be linear, with the meaning that $\mathcal{F}(\lambda T + S) = \lambda\mathcal{F}(T) + \mathcal{F}(S)$; also, it will also be sometimes required that it is norm decreasing, with the meaning $\|\mathcal{F}(T)\| \leq \|T\|$. A few important examples of linear Banach functors are:

- The identity covariant functor $\imath : \mathbf{B} \to \mathbf{B}$ which is defined in any subcategory $\mathbf{B}$ of $\mathbf{Ban}$.

- The contravariant *duality functor* $\mathcal{D}$ defined by $\mathcal{D}(X) = X^*$ for a Banach space X and $\mathcal{D}(T) = T^*$ for an operator T.
- More generally, given a Banach space Y the contravariant $\mathfrak{L}^Y$ functor defined by $\mathfrak{L}^Y(X) = \mathfrak{L}(X, Y)$ and $\mathfrak{L}^Y(T) = T^\circ$ with he meaning $T^\circ(S) = ST$. The choice $Y = \mathbb{R}$ gives the duality functor.
- Given a Banach space X the covariant $\mathfrak{L}_X$ functor defined by $\mathfrak{L}_X(Y) = \mathfrak{L}(X, Y)$ and $\mathfrak{L}_X(T) = T_*$ with he meaning $T_*(S) = TS$. The choice $X = \mathbb{R}$ gives the identity.
- Given a Banach space X the covariant $\otimes_X$ functor defined by $\otimes_X(Y) = X\widehat{\otimes}_\pi Y$ and $\otimes_X(T) = 1_X \otimes T$.
- Semadeni's covariant *Banach-Mazur* functor [28] (see below) that assigns to a Banach space X the space $C(B_{X^*})$; and to a norm one operator $T : X \to Y$ the operator $T^{*\circ}$ defined by $T^{*\circ}(f) = fT^*$.
- The covariant functors (see [17, 18]) assigning to a Banach space X the space $\ell_p(X)$ of p-summable sequences with the natural induced operators.
- One can equally define the Grothendieck-Pietsch functors that assign to a Banach space X the space $\ell_p^w(X)$ of weakly p-summmable sequences on X.
- If $\mathbf{B}$ is a certain subcategory of **Ban** sometimes it is useful to consider the "forgetful" functor $\square : \mathbf{B} \to \mathbf{Ban}$ that simply "forgets" whatever additional structure the objects or morphisms of $\mathbf{B}$ may have.

Now, the meaning of EM is that "right" Banach space constructions are Banach functors. It is important here to remark that while there is not *a theory of X* there cannot be *a categorical theory of X*. Nevertheless, once the theory develops up to reach a certain stage of sophistication ... then one realizes that what one is actually needing to know is something different. And, from the categorical point of view, this extra "different thing" is the uncovering of the functor behind. Let us exhibit deep Banach space problems related to the knowledge of who is and who is not a Banach functor.

3.1. Banach space constructions as Banach functors. To have a concrete example in mind, consider the well known construction: "Every Banach space can be embedded into a Banach space of continuous functions". From the classical point of view that is what there is. And of course that it is an important result. The claim now is: that is not *all* there is. In this case, (a part of) what the EM says is that a correspondence lies on a categorical level only when it is a functor. Which exactly means, in this case, to know how to assign an operator $C(\tau) : C(B_{X^*}) \to C(B_{Y^*})$ to an operator $\tau : X \to Y$ in such a way that the basic rules are respected. When τ is a norm one operator then $C(\tau)$ can be defined as $C(\tau)(f) = f\tau^*$. Thus, the correspondence establishes a functor when acting $\mathbf{Ban}_1 \to \mathbf{Ban}_1$ –although Banach space tricks yield that to every operator $\tau : X \to Y$ corresponds an "extension operator" $T : C(B_{X^*}) \to C(B_{Y^*})$ given by $\|\tau\| C(\frac{\tau}{\|\tau\|})$. Of course, additivity has been lost. Accepted this restriction, the correspondence that assigns $X \to C(B_{X^*}$ and $\tau \to C(\tau)$ establishes a Banach functor. Here we have a few more examples of Banach space constructions and the categorical questions they suggest.

- On the covariant side, apart of the example previously considered, probably the simplest construction is that associating to a Banach space X the injective space $l_\infty(B_{X^*})$. It was Semadeni [28] the first one to recognize a functor here.
- After the Bourgain-Delbaen paper [76] showing the existence of a Schur $\mathcal{L}_\infty$ space, Bourgain and Pisier entered the categorical approach in [77] showing that any separable Banach space X can be embedded into a separable $\mathcal{L}_\infty$ Banach space $\mathcal{L}_\infty(X)$ in such a way that $\mathcal{L}_\infty(X)/X$ has the Radon-Nikodym and Schur properties. For nonseparable spaces, Abad and Todorcevic [73] obtain the analogue construction: namely, every Banach space X can be embedded into some $\mathcal{L}_\infty$ space $\mathcal{L}_\infty(X)$ in such a way that $\mathcal{L}_\infty(X)/X$ has the Schur and Radon-Nikodym properties. The following question is still unsettled:

  Problem 3. Does the Bourgain-Pisier correspondence $X \rightsquigarrow \mathcal{L}_\infty(X)$ establish a functor?

  The meaning of an affirmative answer is the following deep result: operators $\tau : X \to Y$ can be extended to operators $T : \mathcal{L}_\infty(X) \to \mathcal{L}_\infty(Y)$. Very few things are currently known about the extension of $\mathcal{L}_\infty$-valued operators outside of the case of operators valued on $C(K)$ or Lindenstrauss spaces. In [87] it is shown that Lindenstrauss-valued operators on X can be extended to the Bourgain-Pisier space $\mathcal{L}_\infty(X)$. It is not known if this holds for the Abad-Todorcevic construction.

- Zippin showed in [102] that every separable Banach space X can be embedded into a separable Banach space $Z(X)$ with BAP in such a way that the quotient space $Z(X)/X$ also has BAP and every $C(K)$-valued operator on X can be extended to $Z(X)$.

  Problem 4. Does Zippin's construction establish a Banach functor?

- Pisier [99] solved Grothendieck's problem of topologies by showing that every separable Banach space X of cotype 2 can be embedded into a cotype 2 space $\mathcal{P}(X)$ such that $\mathcal{P}(X) \otimes_\varepsilon \mathcal{P}(X) = \mathcal{P}(X) \otimes_\pi \mathcal{P}(X)$.

  Problem 5. Does Pisier's construction establish a Banach functor?

- The paper [74] shows how to produce, given a Banach space X a James-Tree-like space $JT(X)$ some of whose properties depend on X. Still unsettled is the question of to what extent this is a functor.

  Problem 6. Does such constructions have a functorial character?

- On the contravariant side, each Banach space X is a quotient of $\ell_1(B_X)$. The construction is obviously functorial.
- In [91] Gowers-Maurey solved the classical unconditional basic sequence problem with the construction of a *Hereditarily Indecomposable* space (in short H.I.). Recall that a Banach space X is said to be H.I. if no infinite dimensional subspace $Y \subset X$ can be decomposed as $Y = A \oplus B$ with infinite dimensional A, B. A series of papers of Argyros et alt. produced many variations of that construction leading to a great variety of H.I. spaces. If it is clearly impossible to embed a space X into an H.I. space, they show that every separable space not containing ℓ_1 is a quotient of an H.I. space.

  Problem 7. Does there exist a functorial character behind such constructions?

A general pattern for this type of problems is that of envelopes, which we consider now.

3.2. Banach envelopes. Read in purely Banach space terms, the general situation is that every Banach space X can be naturally embedded into a space of continuous functions $C(B_{X^*})$ in such a way that the embedding $\delta_X : X \to C(B_{X^*})$ has the universal property that every $C(K)$-valued operator defined on X can be extended through δ_X to the whole $C(B_{X^*})$. Does there exist a similar universal embedding for other classes of Banach spaces or operators? In categorical terms: do other similar constructions provide Banach functors?

$\mathcal{L}_\infty$-spaces are though to be the local version of $C(K)$-spaces. The following subclasses of $\mathcal{L}_\infty$-spaces have appeared in the literature:

1. Lindenstrauss spaces (denoted $\mathcal{L}$); i.e., spaces that are $\mathcal{L}_{1+\varepsilon}$-spaces for all $\varepsilon > 0$.
2. Separably injective (Θ) and universally separably injective (Θ^u) spaces. Recall that a Banach space E is said to be separably injective if for every separable Banach space X and each subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. If some extension T exists with $\|T\| \leq \lambda\|t\|$ we say that E is λ-separably injective. A Banach space E is said to be universally separably injective –see [75]– if for every Banach space X and each separable subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : Y \to X$. If some extension T exists with $\|T\| \leq \lambda\|t\|$ we say that E is universally λ-separably injective.
3. Lindenstrauss-Pełczyński spaces ($\mathcal{LP}$). Recall from [85, 86] that a Banach space E is said to be a Lindenstrauss-Pełczyński space if all operators from subspaces of c_0 into E can be extended to c_0. If some extension exists verifying $\|\widehat{T}\| \leq \lambda\|T\|$ we shall say that E is an $\mathcal{LP}_\lambda$ space.
4. $\mathcal{L}_\infty$-spaces ($\mathcal{L}_\infty$).

Given a class $\mathcal{A}$ of Banach spaces, the $\mathcal{A}$-envelope of X is a space $\mathcal{A}(X) \in \mathcal{A}$ and an embedding $\delta : X \to \mathcal{A}(X)$ with the property that every $\mathcal{A}$-valued operator defined on X can be extended through δ to $\mathcal{A}(X)$.

Problem 8. Let $\mathcal{A}$ denote one of the classes $\mathcal{L}, \Theta, \Theta^u, \mathcal{LP}$ or $\mathcal{L}_\infty$. Given a Banach space X does there exists the $\mathcal{A}$-envelope of X?

The construction of the $\mathcal{L}$-envelope for separable Banach spaces can be seen in [87]. The $1-\Theta$-envelope, as well as the $1-\Theta^u$ envelope can be seen in [75]. It is also possible to construct the $1-\mathcal{LP}$-envelope.

3.3. Banach spaces as functors. But there is more. The EM program considers that, even when a "single" construction is studied, it must be understood as a functor. This means that ℓ_p spaces, Tsirelson's space, Gurarij space, ... are made out of something, and thus the comprehension of the space is not "right", in the Eilenberg-MacLane sense, until the correspondence between the constituents and the final space has been clearly established as an understandable functor. So, what EM also says is:

▶ *Banach spaces themselves must be understood as functors.*

From the classical point of view, when one constructs Tsirelson's space, that's it. One gets a reflexive space without copies of l_p and that is all one wanted to get. From the categorical point of view one needs more. Let us test this point of view against the simplest classical example: spaces of continuous functions on a compact space K.

The categorical point if view here is to understand this construction as a contravariant functor $\mathcal{C}(\cdot)$ from the category **Comp** of compact topological Hausdorff spaces to a category of Banach spaces (in principle, **Ban** or $\mathbf{Ban}_1$): it transforms a compact space K into the Banach space $C(K)$ and a continuous function $f : K \to S$ into the operator $f^\circ : C(S) \to C(K)$ given by $f^\circ(g) = gf$. It is due to Semadeni the clear understanding of the categorial approach. The series of papers [48, 101, 16, 28, 37] establish the nature and properties of the functor

$$C(\cdot) : \mathbf{Comp} \to \mathbf{Ban},$$

as well as its relationships with its natural "dual" (the right meaning of the duality will be explained later)

$$\bigcirc^* : \mathbf{Ban} \to \mathbf{Comp}$$

defined by $\bigcirc^*(X) = B_{X^*}$.

In addition to Semadeni's papers, Pełczyński's monograph [36] also adopts a categorial approach for what concerns the understanding of $C(K)$-spaces. Right from the start Pełczyński sees the connection between continuous functions $\phi : K \to S$ and linear "composition with ϕ" operators $\phi^\circ : C(S) \to C(K)$; namely, the necessity of considering the contravariant functor $C(\cdot) : \mathbf{Comp} \to \mathbf{Ban}$. In fact, Pełczyński is concerned with the problem of the existence of "simultaneous extension operators" $E : C(T) \to C(S)$ when $S \subset T$ is a subcompact; i.e., if there is a linear continuous operator such that the restriction of $E(f)$ to S is again f. Pełczyński defines a linear exave as follows: if $\varphi : S \to T$ is a continuous map between two compact spaces then $\varphi^\circ : C(T) \to C(S)$ denotes the natural dual operator: $\varphi^\circ(f) = f\varphi$. Now, a linear operator $u : C(S) \to C(T)$ is called a linear exave if $\varphi^\circ u \varphi^\circ = \varphi^\circ$. The condition is always satisfied whenever u is either a left of right inverse for φ°. In the first case u is in called a linear averaging operator while in the second case u is a linear extension operator. The work closes in the only way that can really do justice to the author's categorical thinking: with an appendix entitled "category-theoretical approach". In the appendix Pełczyński realizes that what was useful from $C(K)$-spaces was their functorial character. He says that one could have considered a contravariant functor $\mathcal{F}$ -instead of $C(\cdot)$- as starting point, define $\mathcal{F}$-extension and $\mathcal{F}$-averaging operators and that, under some extra hypothesis on the functor $\mathcal{F}$, the theory developed for $C(K)$-space can be reproduced for functors F he calls "of the Banach-Stone type".

Of course that the same idea (Banach spaces must be understood as functors) applies, or should be applied, to other spaces. Sequence spaces, spaces of functions on trees... The EM general approach here, still incipient, would be to precise a category $\mathbf{D}$ of "diagrams" ($\mathbb{N}$, trees, $\dots$) and then establish the adequate functors $\mathbf{D} \to \mathbf{Ban}$ able to generate the desired spaces (say, ℓ_p, James, James-tree, Schreier, Tsirelson, $\dots$).

3.4. Operators as natural transformations. The impact of the categorical approach does not however fades here and makes necessary to reinterpret operators in the same categorical point of view. To see this at work, let us return again to the sample assertion "every Banach space can be embedded into a Banach space of continuous functions". We have explained so far the terms "we can associate with each Banach space a certain space o continuous functions" as a Banach functor and "space of continuous functions" understanding spaces themselves as functors. Still to explain is the "embedding" part.

To start with, the abstract point of view clearly demands that if one wants to construct a category with functors as objects, then a definition for arrows is required.

DEFINITION OF NATURAL TRANSFORMATION. Given two functors $\mathcal{F}, \mathcal{G}$ acting between the same categories $\mathbf{A} \to \mathbf{C}$, a natural transformation $\tau : \mathcal{F} \to \mathcal{G}$ is a correspondence that assigns to each object A in $\mathbf{A}$ an arrow $\tau_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in such a way that give an arrow $f : A \to C$ there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\tau_A} & \mathcal{G}(A) \\ {\scriptstyle \mathcal{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(C) & \xrightarrow[\tau_C]{} & \mathcal{G}(C). \end{array}$$

Natural transformation is arguably the most important notion in mathematics. If functors tell us how things change natural transformations exist to tell us when things change in the same manner. So, turning back to EM and Banach space theory, if one has to understand Banach spaces as functors, then

► *Operators have to be understood as natural transformations.*

3.5. BANACH SPACES OF NATURAL TRANSFORMATIONS; YONEDA'S LEMMA AND EXAMPLES. The definition of Banach natural transformation is due to Mityagin and Švarc [22, 17, 18].

DEFINITION. Let $\mathcal{F}, \mathcal{G} : \mathbf{B} \to \mathbf{B}$ two Banach functors acting on a subcategory $\mathbf{B}$ of $\mathbf{Ban}$. A Banach natural transformation is a natural transformation $\tau : \mathcal{F} \to \mathcal{G}$ such that the quantity

$$\|\tau\| = \sup_X \|\tau_X\|$$

is finite; here the supremum is taken over all the Banach spaces of the category $\mathbf{B}$. Addition and multiplication by scalars can be defined in an obvious way for Banach natural transformations, and it is clear that the quantity above is a complete norm. So the space $[\mathcal{F}, \mathcal{G}]$ of all Banach natural transformations between the Banach functors $\mathcal{F}$ and $\mathcal{G}$ is a Banach space.

SCHOLIUM. A point here that Banach spacers overlook is that the previous definition makes sense only if the "space" of all natural transformations between two Banach functors forms a set. A study fairly general and complete of the situation has been performed by Linton [26]. Linton defines a category to be *autonomous* as one in which, roughly speaking, the "space" of natural transformations between two functors forms a set (which is not automatically true) and admits the same natural additional structure as the "spaces" $\mathrm{Hom}(A, B)$. Just for the readers peace of mind: yes, Banach spaces is an autonomous category.

Let us identify a few spaces of natural transformations:

PROPOSITION 3.1.

1. *For a covariant Banach functor $\mathcal{F}$ one has*
   (a) *Svarc* [17], *Mityagin and Svarc* [22, Lemma 2]
   $$[\mathfrak{L}_A, \mathcal{F}] = \mathcal{F}(A)$$
   (b) *Pothoven* [52]
   $$[\otimes_A, \mathcal{F}] = \mathfrak{L}(A, \mathcal{F}(\mathbb{R}))$$
2. *For a contravariant Banach functor $\mathcal{G}$ one has*
   (a) $[\mathfrak{L}^A, \mathcal{G}] = \mathcal{G}(A)$.
   (b) $[\otimes_A(\cdot)^*, \mathcal{G}] = \mathfrak{L}(A, G\mathbb{R})$.

The results (1a) and (2a) can be considered special instances of Yoneda's lemmas [95]. We make the proof for covariant functors, and leave for the reader the contravariant case.

Proof of (1a). The natural equivalence $\eta : F \longrightarrow [\mathfrak{L}_A, F]$ must be defined as follows: for each A the operator $\eta_A : FA \to [\mathfrak{L}_A, F]$ takes points $p \in FA$ and send them into a natural transformation $\eta_A(p) : \mathfrak{L}_A \to F$: for each X one must have an operator $\eta_A(p)_X : \mathfrak{L}(A, X) \to FX$, which is
$$\eta_A(p)_X(T) = FT(p).$$
This is well defined because if $T : A \to X$ then $FT : FA \to FX$.

Proof of (1b). Recall that every element $x \in X$ can be understood as an operator $x : \mathbb{R} \to X$; therefore $\mathcal{F}x$ is an operator $\mathbb{R} \to \mathcal{F}X$. Given an operator $T : X \to \mathcal{F}\mathbb{R}$ the associated natural transformation $\nu(T) : \otimes_X \to \mathcal{F}$ comes defined by
$$\nu(T)_X(a \otimes x) = \mathcal{F}x(Ta).$$

Conversely, given a natural transformation $\eta \in [\otimes_A, \mathcal{F}]$ the associated operator is $T = \eta_{\mathbb{R}}$. ∎

The result (1) in particular yields

$$[\mathfrak{L}_X, \mathfrak{L}_Y] = \mathfrak{L}(Y, X),$$

while the second result had been formulated by Mityagin and Svarc [22] as

$$[\otimes_X, \otimes_Y] = \mathfrak{L}(X, Y),$$

showing that the interpretation of the space X as the functor $\otimes_X$ produces no loss of information.

The basic functor in homology, the functor Ext, allows analogous results. Recall that $\mathrm{Ext}(B, A)$ is the space of exact sequences $0 \to A \to \diamondsuit \to B \to 0$ modulus a certain equivalence relation. This is not a Banach functor although in [78, 79] it is made a "semi-Banach" functor. One has [44, IV, Prop. 10.3] (see also [83])

$$[\mathrm{Ext}(X, \cdot), \mathrm{Ext}(Y, \cdot)] = \mathfrak{L}(Y, X)$$

This results inherits the ideas of Hilton and Rees [15]. There is a true Banach (better, quasi-Banach) space theory result without general counterpart (see [83]):

Proposition 3.2. *Let $0 < p < 1$. Given subspaces A, B of $L_p(0, 1)$ one has*

$$[\mathrm{Ext}(L_p/A, \cdot), \mathrm{Ext}(L_p/B, \cdot)] = \mathfrak{L}(B, A).$$

We check now our test assertion ("operators are natural transformations"):

- The canonical embedding $\delta_X : X \to C(B_{X^*})$ means the existence of a natural transformation $\delta : \imath \to \mathcal{C}(\bigcirc^*)$ between the identity and the Banach-Mazur functor; it assigns to each space X the canonical embedding δ_X. It is clear the commutativity of the squares

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & \mathcal{C}(B_{X^*}) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{*\circ}} \\ Y & \xrightarrow[\delta_Y]{} & \mathcal{C}(B_{Y^*}). \end{array}$$

It was again Semadeni [28] the first one to recognize and study a natural transformation here.

- If the Bourgain-Pisier construction [77] was a functor, it would be natural to expect that the embedding $i_X : X \to \mathcal{L}_\infty(X)$ given by the theorem would be a natural transformation. The same can be said about Zippin construction [102] and Pisier's construction [99]. Related constructions are worth mentioning.
- Let $\mathcal{A}$ be the functor defining the $\mathcal{A}$-envelope. It carries with it a natural transformation $\delta : \imath \to \mathcal{A}$
- On the contravariant side, each Banach space X is a quotient of $\ell_1(B_X)$. These quotients obviously establish natural transformations $\ell_1(\cdot) \to \imath$.

- [**Interpolation theory**] This is one of the theories most suitable, in principle, for a categorical treatment. Indeed, Mityagin and Shvarts [22, 17, 18] already mention interpolation theory as one of the key examples. The paradox appears again when realizing that the existence of a categorical context does not move "interpolation people" to stop working by hand[1]. The categorical context is as follows: let us consider the category **(Ban,Ban)** whose objects are *compatible couples* of Banach spaces, and whose arrows $T : (A_0, A_1) \to (B_0, B_1)$ are operators $T : A_0 + A_1 \to B_0 + B_1$ such that $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are continuous. Although it is rather standard to say that an interpolation method is a functor **(Ban,Ban)** $\to$ **Ban** this is not entirely correct since it overlooks the fact (otherwise required) that the interpolated space is "intermediate" between the intersection space $A_0 \cap A_1$ in the max norm and the sum $A_0 + A_1$ in the inf norm. Each author skips this difficulty his own way. The simplest way, however, is to recall the two interpolation functors already implicitly defined in the definition of "compatible couple": the intersection functor $\Delta(A_0, A_1) = A_0 \cap A_1$ and the sum functor $\Sigma(A_0, A_1) = A_0 + A_1$. In

[1] The following sentence, reproduced here without any permission, reflects the thinking of one expert in interpolation theory: "In my opinion, interpolation theory is what it is, independently of whether some of its results/parts can be described with a more or less categorical language. The "shortcomings" of classical theory are what they are. Sometimes sufficient conditions to get the results are known. I don't feel categorical perspectives serve to amend the shortcomings of classical theory"

this way, what makes a true (at least an interesting) interpolation functor F : **(Ban,Ban)** $\to$ **Ban** is the existence of two injective natural transformations $\Delta \to F$ and $F \to \Sigma$ such that their composition is the canonical inclusion $\Delta \to \Sigma$.

The approaches to interpolation can be summarized as: 1) The classical approach to interpolation, contemplating "interpolation functors" such as described in Chapter 2 of Bergh-Löfstrom [60]. Since the text does not contains the word natural transformation which is, as we have remarked, at the core of the theory, we understand that that exposition does not have, categorically speaking, much to offer. 2) The very general schema presented [55]; although the categorical language is omitted, the approach of the author is pregnant of categorical ideas. 3) The interpretation of Kaijser and Pelletier [92, 93] via a special type of pull-back-push-out diagrams called Doolittle diagrams. Their approach is conceptually very interesting, soundly categorically based and they, moreover, obtain a remarkable duality theorem. 4) A particularly interesting and categorically biased approach has been presented by Carro, Cerdá and Soria in [81] then generalized by Cwikel, Rochberg and Kalton in [88].

4. Universal constructions: limits and colimits

If Banach functor is the categorical way of saying "a correspondence that assigns to certain Banach spaces another Banach spaces", Universal construction is the categorical form of saying "a correspondence that assigns to certain family of Banach spaces another Banach space". The word "universal" contains the meaning that the construction must be compatible with the point of view that contemplates spaces as functors and operators as natural transformations.

There are two types of universal constructions: limits and colimits. The prefix co- is the heart of categorical duality, to be considered later: whenever a statement (definition, theorem, ...) can be formulated in categorical terms, namely, in terms of points (objects) and arrows (i.e., diagrams) then there is a correspondent "dual" statement obtained reversing the arrows. To give an example, if given the situation

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

there is a universal (i.e., well and uniquely defined) construction of an object $\bigtriangledown$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bigtriangledown \end{tikzcd}$$

then the dual construction is the universal construction that allows one to pass from

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ & \bullet \arrow[u] \end{tikzcd}$$

to

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \triangle \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd}.$$

It is a matter of choice which construction $\bigtriangledown$ or $\triangle$ will be called limit and which co-limit. But the natural choice should be that all constructions of one type will be called say limits and all belonging to the other will then be colimits.

Products, pull-backs, subspaces, projective limits ... are all of the same type; as well as co-products, push-outs, quotients, inductive limits. It would therefore be natural to call limits to the first and co-limits to the second. But Banach space tradition is illogical: indeed, tradition gave the names "inductive limit" to a construction of type $\bigtriangledown$, and "product" to a construction of type $\triangle$. However, one of the two should have a "co". Since it is hard to change either the name product or the inductive limit, let us from now on to fix that constructions of type $\triangle$ will be called inverse limits, while those of type $\bigtriangledown$ will be direct limits. This makes Products, pull-backs, subspaces, projective limits ... to be inverse limits; while co-products, push-outs, quotients, inductive limits ... become direct limits.

Direct limit. The definition goes back to Grothendieck [11]. Let D be an abstract diagram made with points and arrows. There is no difficulty in considering D as a category. A functor $\mathcal{F} : D \to \mathbf{Ban}$ means just drawing D with Banach spaces in the place of points and operators in the place of arrows.

So, given a Banach functor $\mathcal{F} : D \to \mathbf{Ban}$, a Banach space $L(\mathcal{F})$ will be called the *direct limit* of $\mathcal{F}$ through D if there is a family of operators $\alpha_d : \mathcal{F}(d) \to L(\mathcal{F})$, $d \in D$, making commutative the whole diagram in $\mathbf{Ban}$ in such a way that for every other Banach space X and family of morphisms β_d with the same property, there is a unique operator $\alpha : L(\mathcal{F}) \to X$ making the whole diagram (i.e., $\alpha\alpha_d = \beta_d$) commutative.

Inverse limit. Let D and $\mathcal{F}$ be as before. A Banach space $L(\mathcal{F})$ will be called the *inverse limit* of $\mathcal{F}$ through D if there is a family of operators $\alpha_d : L(\mathcal{F}) \to \mathcal{F}(i)$, $d \in D$, making commutative the whole diagram in $\mathbf{Ban}$ in such a way that for every other Banach space X and family of morphisms β_d with the same property, there is a unique operator $X \to \alpha : L(\mathcal{F})$ making the whole diagram commutative (i.e., $\alpha_d\alpha = \beta_d$).

Probably the most important result in this regard is [27]:

Theorem 4.1. (Semadeni-Zidenberg) *Every diagram in* $\mathbf{Ban}_1$ *admits limits and colimits.*

This result is not trivial even although Grothendieck had already proved in [11] that in a suitable abelian category –recall that $\mathbf{Ban}$ is not abelian– limits and colimits exist. On the other hand, the reason to work with the category $\mathbf{Ban}_1$ in the Semadeni-Zidenberg theorem is simple: one cannot expect that even the simplest infinite diagram have limits in $\mathbf{Ban}$ since the fact that operators have norm prevents it. When working with finite diagrams no essential difference exists between $\mathbf{Ban}$ and $\mathbf{Ban}_1$. A partial converse of this result is:

Proposition 4.1. *Every Banach space can be represented as a direct limit of* ℓ_1^n *spaces.*

Proof. Indeed, the proof is easy: let FIN(X) be the set of all finite dimensional subspaces of X. Take for each $X_d \in$ FIN(X) a quotient norm one operator $q_d : \ell_1^d \to X_d$. Without too much precision we are using also d for the dimension of X_d and ℓ_1^d, so that q_d has the form $q_d(e_j) = x_j$ for a certain normalized Hamel basis $(x_j)$of X_d. Use now as index set the set of couples $(\ell_1^d, q_d)_{X_d \in \mathrm{FIN(X)}}$ with the partial order $(\ell_1^d, q_d) \le (\ell_1^e, q_e)$ if $d \le e$ and $q_{e|\ell_1^d} = q_d$. It is almost obvious that

$$X = \lim_{\to} (\ell_1^d, q_d)_{X_d \in \mathrm{FIN(X)}}.$$

Given a Banach space Z and a compatible collection of operators $r_d : \ell_1^d \to Z$, simply define $q : X \to Z$ as $q(x) = r_d(x)$ for $d = (\ell_1^1, q_x)$ with $q_x(e_1) = x$.

4.1. Products and coproducts. The set-theorethic definition of product $\prod_{i \in I} X_i$ of spaces is

$$\prod_{i \in I} X_i = \{f : I \to \bigcup X_i : f(i) \in X_i\},$$

whose existence obviously depends on the axiom of choice. The categorical definition of the product space is: it is (the unique, up to isomorphisms) object $\prod_{i \in I} X_i$ admitting arrows $\pi_j : \prod_{i \in I} X_i \to X_j$ with the property that for every object O and arrows $p_j : O \to X_j$ there is a un unique arrow $P : O \to \prod_{i \in I} X_i$ such that $\pi_i P = p_i$ for all i.

The simplest infinite diagram is obviously

$$\cdots \bullet \cdots$$

i.e., just points, no arrows (except identities). If a two minutes break does not clearly brings into the reader's mind the unquestionable idea that the inverse limit of this diagram is the product, he can stop reading.

The existence of that product in a specific category, in particular in Banach spaces is a different thing. In the category of sets the product set was defined above. In the category of vector spaces the underlying set is the same and the vector space structure is quite naturally defined. If one tries to transport the same construction to Banach spaces then one soon realizes that when there is an infinite number of spaces involved there is no Banach space structure on $\prod_{i\in I} X_i$ making continuous the projections π_j. When there is only a finite number of factors then the natural norm is

$$\|(x_i)_I\|_\infty = \sup_{i\in I} \|x_i\|_{X_i}.$$

The problem with infinite terms is how to handle the universal property with respect to an infinite sequence of operators ... with norms tending to infinity. There are ways to circumvent this difficulty, but since they do not bring news to Banach spacers we will omit that part. When limits exist with respect to bounded families of operators we will simply say that there exist restricted limits; another way is to work in $\mathbf{Ban}_1$, but we prefer to keep this notation for the situations that truly require to work with norm one operators. In particular, restricted products exist in Banach spaces: the product of a family (X_i) is the Banach space $\ell_\infty(X_i)$, the essential point being that to each bounded family of operators $T_i : X \to X_i$ corresponds an operator $T : X \to \ell_\infty(X_i)$. In the Banach space language,

$$\mathfrak{L}\big(X, \ell_\infty(X_i)\big) = \ell_\infty\big(\mathfrak{L}(X, X_i)\big).$$

The dual notion i.e., the direct limit of the diagram

$$\cdots\bullet\cdots$$

is that of co-product. It must be obvious now that, in the restricted sense, it corresponds to the ℓ_1-vector sum. The universal property here means that for every uniformly bounded family of operators $T_i : X_i \to Y$ there exists a unique operator $T : \ell_1(X_i) \to Y$ whose restriction to each X_i is T_i. The operator is $Tx = \sum T_i x_i$. In classical Banach space terms

$$\mathfrak{L}\big(\ell_1(X_i), X\big) = \ell_\infty\big(\mathfrak{L}(X_i, X)\big).$$

4.2. Inductive and projective limits. The second simplest infinite diagrams are

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

whose direct limit is called usually inductive limit and whose inverse limit is just the selection of the first left element; and

$$\cdots\bullet \longrightarrow \bullet \longrightarrow \bullet$$

whose inverse limit is usually called projective limit and its direct limit is just the picking of the last right element.

The explicit form of the projective limit of a projective spectrum $\tau_n : X_n \to X_{n-1}$, $n \geq 0$, is the subspace of the product formed by those elements with "linked" coordinates; precisely ...

$$\lim_{\leftarrow} \tau_n X_n = \{x \in \ell_\infty(X_n) : \forall n \in \mathbb{N} \quad \tau_n x_n = x_{n-1}\}.$$

The inductive limit in $\mathbf{Ban}_1$ of an inductive system $\langle X_n, i_n \rangle$ can be obtained as follows: consider the vector space

$$X = \{(x_n)_n \in \ell_\infty(X_n) : \exists \mu \in \mathbb{N} : i_n(x_n) = x_{n+1},\ \forall n > \mu\},$$

endowed with the seminorm $\|(x_n)_n\| = \lim \|x_n\|$. If $K = \ker \|\cdot\|$ then $\lim_{to}\langle X_n, i_n\rangle$ is the completion of the quotient X/K together with the family of isometries $I_n : X_n \to \mathfrak{X}$ defined as $I_n(x) = [(0, 0, \dots, x, x, \dots)]$. The inductive limit can be described as $\lim_{to}\langle X_n, i_n\rangle = \overline{\cup I_n(X_n)}$. If the connecting morphisms are into isometries then the (Banach) inductive limits is plainly the completion of the union (endowed with the unique norm one is considering). The Banach inductive limit, in general, does not coincide with locally convex inductive limit, and locally convex inductive limits of Banach spaces are very far from being Banach or Fréchet spaces.

Inductive limits were brought to the hard-core of Banach space theory apparently by Pisier [99] and skilfully exploited, among others, by Bourgain and Delbaen in [76] and Bourgain and Pisier in [77] to obtain involved constructions of $\mathcal{L}_\infty$-spaces. Semadeni has studied inductive and projective limits. In [16] presents ad-hoc constructions attempting to define the limit of a sequence of linear metric spaces.

4.3. Generalized pull-back and push-out. The direct limit for the diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

is usually called push-out. Push-outs exist in Banach spaces, and it was Kislyakov [94] who re-discovered this construction in the Banach space setting. The push-out of

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \\ {\scriptstyle S}\big\downarrow & & \\ M & & \end{array}$$

is the Banach space PO $= (M \oplus_1 X)/\overline{\Delta}$ where $\Delta = \{(Sy, -jy) \in M \oplus_1 X\}$ together with the operators $u_S : X \to PO$ and $u_j : M \to PO$ which are the restrictions to M and X of the quotient map $M \oplus_1 X \to PO$. They obviously verify that $u_S j = u_j S$, and have the universal property that given two operators $\alpha : M \to E$ and $\beta : X \to E$ such that $\alpha S = \beta T$ there exists a unique arrow $\gamma : PO \to E$ such that $\gamma u_j = \alpha$ and $\gamma u_S = \beta$.

If the starting diagram is, instead of two arrows with a common point, a family of arrows with a common point –something we can represent as $\bullet \lhd\!-$ its direct limit will be called the generalized push-out. Thus, if we instead consider the diagram given by given a family $(f_j)_{j\in J}$ of operators $f_j : Y \to X_j$, their generalized push-out PO is the quotient $\ell_1(X_j)/\Delta$, where Δ is the closed span in $\ell_1(X_j)$ of the vectors (x_j) such that all its coordinates are null except a couple of indices a, b for which one has $x_a = f_a(y)$ and $x_b = -f_b(y)$. Whenever f_j is an embedding, the operator $u_j : X_j \to \mathrm{PO}$ given by $u_j(x_j) = (0, \ldots, x_j, \ldots 0 \ldots) + \Delta$ is also an embedding. For all i, j one has $u_i f_i = u_j f_j$. If all the maps f_j are into isometries then also the u_j are into isometries. We will use the following visual notation: the family (f_j) will be denoted $Y \blacktriangleleft X_j$ and their push-out $X_j \blacktriangleright \mathrm{PO}$. In this form, $Y \blacktriangleleft X_j \blacktriangleright \mathrm{PO}$ suggests that all the squares

$$\begin{array}{ccc} Y & \xrightarrow{f_j} & X_j \\ {\scriptstyle f_i}\big\downarrow & & \big\downarrow{\scriptstyle u_j} \\ X_i & \xrightarrow[u_i]{} & \mathrm{PO} \end{array}$$

are commutative. This generalized push-out is useful when the construction of many successive push-outs is required.

The inverse limit of the diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \uparrow \\ & & \bullet \end{array}$$

is called the pull-back. The pull-back construction exists in Banach spaces and, given a diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Z \\ & & \Big\uparrow{\scriptstyle T} \\ & & W \end{array}$$

the pull-back space comes defined as the Banach space $\mathrm{PB} = \{(x, w) : qx = Tw\} \subset X \oplus_\infty W$ endowed with the relative product topology, together with the operators $\mathrm{PB} \to X$ and $\mathrm{PB} \to W$, restrictions of the canonical projections of $X \oplus_\infty W$ into, respectively, X and W, and $i(y) = (jy, 0)$. Pull-backs were reinvented in the locally convex setting by Dierolf [56, 89].

5. Functors and limits

If functors and natural transformations is everything there is, and all Banach space constructions are limits, the obliged question is: *How things respect Banach space constructions?*; or, what is the same, *How to determine the behaviour of functors with respect to limits?* In other words, when do functors and limits commute?

Let us present an example. Consider as a sampler the functor $\mathcal{C}(\cdot)$: **Comp** $\to$ **Ban**. Since the functor $\mathcal{C}(\cdot)$ is contravariant, the question is to which extent limits of compact spaces are transformed into co-limits of Banach spaces, and conversely. The problem was considered by Semadeni in [37]: he first proves that $C(\cdot)$ transforms direct limits into inverse limits. Let us give an example:

Observe first that the category **Comp** of compact spaces admits push-outs: indeed if $K \xrightarrow{f_i} A_i, i = 1, 2$ then the push-out is the compact space disjoint union of A_1 and A_2 under the identification $a_1 = a_2$ if there exists

$k \in K$ such that $a_i = f_i(k)$. Let now I be an uncountable discrete set and let $\gamma : \beta\mathbb{N} - \mathbb{N} \to \alpha I$ be a continuous map; consider the canonical inclusion $i : \beta\mathbb{N} - \mathbb{N} \to \beta\mathbb{N}$ and construct the push-out diagram as compact spaces the category **Comp**

$$\begin{array}{ccc} \beta\mathbb{N} - \mathbb{N} & \xrightarrow{i} & \beta\mathbb{N} \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ \alpha I & \longrightarrow & POK. \end{array}$$

The push-out-compact space POK is the quotient of the disjoint union $\alpha I \bigcup \beta\mathbb{N}$ by the equivalence relation $p \sim \mathcal{U}$ if and only if $p = \gamma(\mathcal{U})$. Since $\mathcal{C}(\cdot)$ transforms direct limits into inverse limits, it transforms push-out into pull-backs. One therefore gets the pull-back diagram

$$\begin{array}{ccc} C(\beta\mathbb{N}) & \xrightarrow{i^*} & C(\beta\mathbb{N} - \mathbb{N}) \\ \uparrow & & \uparrow{\scriptstyle\gamma^*} \\ C(\mathrm{PO}K) & \longrightarrow & C(\alpha I). \end{array}$$

If, moreover, I has the cardinal of the continuum then, assuming CH, γ can be chosen surjective (Probably the simplest way is to proceed as follows: enumerate the nodes of he dyadic tree; now, if $\mathfrak{b}$ is the uncountable set of its branches and $\mathcal{U}$ is a free ultrafilter then define

$$\gamma(\mathcal{U}) = \begin{cases} r, & r \in \mathfrak{b}; r \in \mathcal{U}; \\ \infty, & \text{otherwise.} \end{cases}$$

With this choice the compact POK turns out to be a classification of the elements of $\beta\mathbb{N}$ in three types of ultrafilters: type 1 is formed by the principal ultrafilters (identified to no point in αI); type 2 are those containing a branch of the dyadic tree (all free ultrafilters containing a given branch are thus identified); and type 3 is the identification of all free ultrafilters that do not contain any branch. Since γ is surjective, γ^* an into isomorphism, and thus one gets a pull pull-back diagram in **Ban**

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(\beta\mathbb{N}) & \xrightarrow{i^*} & C(\beta N - \mathbb{N}) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle\gamma^*} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & C(\mathrm{PO}K) & \longrightarrow & C(\alpha I) & \longrightarrow & 0. \end{array}$$

whose lower sequence cannot split (otherwise $C(\mathrm{PO}K)$ would be isomorphic to $c_0(I)$ and every operator $c_0(I) \to \ell_\infty$ has separable range, something that fails in this example since γ^* has nonseparable range). Thus, one gets a counterexample for the 3-space problem for weakly compactly generated spaces: $C(\mathrm{PO}K)$ cannot be WCG since copies of c_0 in WCG spaces are complemented.

Returning to the behavior of the functor $\mathcal{C}(\cdot)$ against limits, Semadeni obtains in [37] a deep result showing that when the diagram has the property that given two points r, s there exists another point t such that $t \geq \{r, s\}$ then $\mathcal{C}(\cdot)$ also transforms inverse limits into direct limits.

Proposition 5.1. *Given a projective system of compact spaces (K_α) one has*

$$C(\lim_{\leftarrow} K_\alpha) = \lim_{\rightarrow} C(K_\alpha)$$

A particular case of this result when the linking operators are surjective had been obtained by Pełczyński in [36]. The "filtering" condition is necessary since the functor $C(\cdot)$ does not send pull-backs into push-outs (see [54]). The result is especially interesting since projective families of compact spaces always have nonempty inverse limit, as it was proved by Steenrod [90]. Such property was used by Tabor and Yost in [100] to show that a function $f : D \to E$ defined on a convex set D of a normed space with values in a finite dimensional Banach space such that its Cauchy difference function $\Delta f(x, y) = f(x + y) - f(x) - f(y)$ is Lipschitz on $\{(x, y) \in D \times D : x + y \in D\}$ can be approximated by a true Lipschitz function.

5.1. Adjointness.

A beautiful technique to decide when a functor commutes with limits is provided by the notion of adjointness. Two examples will make clear the meaning of adjunction.

Example 1. Our first example is the classical identity:

$$\mathfrak{L}\left(A \widehat{\otimes}_\pi X, Y\right) = \mathfrak{L}\big(X, \mathfrak{L}(A, Y)\big).$$

The proof is clear identifying both terms with the space of continuous bilinear forms $A \times X \to Y$. After checking the corresponding naturality assumptions, it yields

$$\mathfrak{L}\left(\otimes_A X, Y\right) = \mathfrak{L}\big(X, \mathfrak{L}_A(Y)\big).$$

Thus, if one says that a covariant functor $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ is the left adjoint of a covariant functor $\mathcal{G} : \mathbf{B} \to \mathbf{A}$ (and, consequently, G is called a right adjoint for F), something written as $\mathcal{F} \dashv \mathcal{G}$, if for every object A of $\mathbf{A}$ and B of $\mathbf{B}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{B}}\big(\mathcal{F}(A), B\big) = \mathrm{Hom}_{\mathbf{A}}\big(A, \mathcal{G}(B)\big).$$

Thus, what we have is that $\otimes_A$ is the left adjoint of $\mathfrak{L}_A$; i.e., $\otimes_A \dashv \mathfrak{L}_A$.

Example 2. Everything is contained in the following result [28, Prop. 1]:

Proposition 5.2. *For every compact space K and every norm one operator $X \to C(K)$ there is a unique continuous map $f : K \to B_{X^*}$ making commutative the diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "T"'] & C(B_{X^*}) \arrow[dl, "C(f)"] \\
C(K) &
\end{tikzcd}$$

If $\Delta : K \to C(K)^*$ is the natural embedding then the correspondence is $f = T^*\Delta$. The proposition thus establishes a correspondence between operators $T : X \to C(K)$ in $\mathbf{Ban_1}$ and continuous functions $f : K \to B_{X^*}$ in $\mathbf{Comp}$; namely (let us agree that $\mathbf{func}(K, K')$ is the space of continuous functions between the compact spaces K and K')

$$\mathfrak{L}\big(X, C(K)\big) \longrightarrow \mathbf{func}(K, B_{X^*}).$$

The inverse correspondence

$$\mathfrak{L}\big(X, C(K)\big) \longleftarrow \mathbf{func}(K, B_{X^*})$$

is $f \to f^*\delta_X$ where $\delta_X : X \to C(B_{X^*})$ is the canonical embedding. These two maps are inverse one of the other in a deep sense: there is a *natural* isomorphism:

$$\mathfrak{L}\big(X, C(K)\big) = \mathbf{func}(K, B_{X^*}).$$

Two contravariant functors $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ and $\mathcal{G} : \mathbf{B} \to \mathbf{A}$ are called *adjoint on the right* if for every object A of $\mathbf{A}$ and B of $\mathbf{B}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{A}}\big(A, \mathcal{G}(B)\big) = \mathrm{Hom}_{\mathbf{B}}\big(B, \mathcal{F}(A)\big).$$

Thus, what Semadeni shows in Proposition 5.2 is that the functors $C(\cdot)$ and $\bigcirc^*$ are adjoints on the right.

Analogously, two contravariant functors $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ and $\mathcal{G} : \mathbf{B} \to \mathbf{A}$ are called *adjoint on the left* if for every object A of $\mathbf{A}$ and B of $\mathbf{B}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{A}}\big(\mathcal{G}(B), A\big) = \mathrm{Hom}_{\mathbf{B}}\big(\mathcal{F}(A), B\big).$$

Observe that $\mathcal{C}(\cdot)$ and $\bigcirc^*$ are NOT adjoints on the left: a continuous function $f : B_{X^*} \to K$ does not induce an operator $C(K) \to X$ and an operator $T : C(K) \to X$ does not induce a continuous map $B_{X^*} \to K$. At least, not in general. Here is where the choice of the categories where the functors act shows its foremost importance. Semadeni introduces in the book [54] a certain category **Bcf** of "Bocoff" spaces which are defined as those Banach spaces "of type $C(K)$" together with the operators having the form $C(f) = f^*$. Then $\mathcal{C}(\cdot)$ is a well-defined contravariant functor **Comp** $\to$ **Bcf**. Semadeni considers the *Gelfand functor* $\mathcal{G}$: **Bcf** $\to$ **Comp** associating to each X the Gelfand spectrum; i.e., the compact space of all multiplicative functionals with norm lesser than or equal to one endowed with the weak*-topology. Proposition 5.2 still establishes that the functors $\mathcal{C}(\cdot) : \mathbf{Comp} \to \mathbf{Bcf}$ and $\mathcal{G} : \mathbf{Bcf} \to \mathbf{Comp}$ are adjoint on the right while it is easy to check they are also adjoint on the left. So one has natural equivalences

$$\mathbf{func}(K, \mathcal{G}(X)) = \mathrm{Hom}_{\mathbf{Bcf}}\big(X, C(K)\big),$$

and

$$\mathbf{func}\big(G(X), K\big) = \mathbf{Bcf}\big(C(K), X\big).$$

Adjointness was introduced by Kan in [13], as an attempt to formulate a duality theory for functors.

Other examples one can consider:

- Consider the contravariant functor $\mathfrak{L}^A$. One has the well-known identity:
$$\mathfrak{L}\big(X, \mathfrak{L}(Y, A)\big) = \mathfrak{L}\big(Y, \mathfrak{L}(X, A)\big).$$
When $A = \mathbb{R}$ this equation is just the duality identity
$$\mathfrak{L}(X, Y^*) = \mathfrak{L}(Y, X^*).$$
In both cases, the identities mean that the contravariant functor $\mathfrak{L}^A$ is adjoint on the right of itself.
- Consider the covariant Banach-Mazur functor $\mathcal{C}\bigcirc^* : \mathbf{Ban_1} \longrightarrow \mathbf{Ban_1}$. As it is, it has no left or right adjoints. Nevertheless, as a functor
$$\mathcal{C}\bigcirc^* : \mathbf{Ban_1} \longrightarrow \mathbf{Bcf}$$
has an obvious right adjoint: the "forgetful" functor $\Box : \mathbf{Bcf} \longrightarrow \mathbf{Ban_1}$ that simply "forgets" the additional structure. It is clear that $\mathcal{C}\bigcirc^* \dashv \Box$ since there is a natural equivalence
$$\mathfrak{L}\big(C(B_{X^*}), C(K)\big) = \mathfrak{L}\big(X, C(K)\big).$$

The paper [28] establishes many other adjointness results among those and related functors such as the Stone-Cech compactification functor $\beta : \mathbf{Set} \to \mathbf{Comp}$, the Banach-Mazur functor, etc.

5.2. Adjoints and limits. The adjointness relations are important and relevant to the problem of studying the behavior of functors regarding limits because:

Proposition 5.3. *For covariant functors:*

- *Right adjoints preserve inverse limits.*
- *Left adjoints preserve direct limits.*

The proof for this result is very simple with a bit of extra abstraction effort able to give us the right definition of limit. Given a category $\mathbf{C}$ and a diagram D understood as a category, a clean sense can be given to $\mathbf{C}^D$: the objects are the functors $D \to \mathbf{C}$ with the corresponding natural transformations as morphisms. In this setting there is an obvious covariant "diagonal" functor $\Delta : \mathbf{C} \to \mathbf{C}^D$: the image of an object is the functor $D \to \mathbf{C}$ sending all elements to X and all arrows to the identity of X. One obviously has (thirty seconds thought):

Proposition 5.4.

- $\Delta \dashv \lim_{\leftarrow}$,
- $\lim_{\rightarrow} \dashv \Delta$.

This can be considered the true definition of direct and inverse limit constructions: inverse limits are right adjoints of Δ while direct limits are left adjoints of Δ. Moreover, it provides an immediate proof for Proposition 5.3, since, equally obvious is that if $\mathcal{F} \dashv \mathcal{G}$ and $\mathcal{F}' \dashv \mathcal{G}'$, and the compositions make sense, $\mathcal{F}'F \dashv \mathcal{G}G'$. Now, if one has

$$\begin{aligned} \lim_{\rightarrow} &\dashv \Delta \\ \mathcal{F} &\dashv \mathcal{G} \end{aligned}$$

then

$$\mathcal{F} \lim_{\rightarrow} \dashv \Delta\mathcal{G}.$$

While on the other hand, since

$$\begin{aligned} \lim_{\rightarrow} &\dashv \Delta \\ \mathcal{F}^D &\dashv \mathcal{G}^D \end{aligned}$$

then

$$\lim_{\rightarrow} \mathcal{F}^D \dashv \mathcal{G}^D\Delta.$$

Finally, since $\mathcal{G}^D\Delta = \Delta\mathcal{G}$, the uniqueness of adjoints immediately yields

$$\lim_{\rightarrow} \mathcal{F}^D = \mathcal{F} \lim_{\rightarrow}.$$

For contravariant functors one has:

Proposition 5.5. *For contravariant functors:*

- *Adjoints on the right transform direct limits into inverse limits.*
- *Adjoints on the left transform inverse limits into direct limits.*

The first immediate consequence is that the $\mathcal{C}(\cdot)$ functor, which is adjoint on the right of the Alaoglu functor, transforms direct limits of compact spaces into inverse limits of Banach spaces as Semadeni proved. Another immediate consequence, since the duality functor is adjoint on the right of itself, is that the dual of a direct limit of Banach spaces is the inverse limit of the duals (the converse is false: $L_\infty(0,1) = \lim_{\leftarrow} L_p(0,1)$ while $\lim_{\rightarrow} L_p(0,1) = L_1(0,1)$ and $L_\infty(0,1)^* \neq L_1(0,1)$).

5.3. Adjointess is not enough for Banach space theory. Fuks [29, Section 6] established a wonderful result for covariant Banach functors.

Proposition 5.6. *Let S, T be two covariant Banach functors such that $S \dashv T$. Then there is a Banach space A such that S is naturally equivalent with $\otimes_A$ and T is naturally equivalent to $\mathfrak{L}_A$.*

Proof. Since $S \dashv T$ one has

$$\mathfrak{L}(S\mathbb{R}, X) = \mathfrak{L}(\mathbb{R}, TX) = TX.$$

So, setting $A = S(\mathbb{R})$, we have $T = \mathfrak{L}_A$. On the other hand, since $X = \lim_{\rightarrow} \ell_1^n$ and left adjoints commute with direct limits (applied to S, to $\otimes$ and recalling that ℓ_1^n is a (finite) direct limit):

$$\begin{aligned} SX &= S\lim_{\rightarrow} \ell_1^n = \lim_{\rightarrow} S(\ell_1^n) = \lim_{\rightarrow} \ell_1^n(S\mathbb{R}) = \lim_{\rightarrow} (\ell_1^n \otimes S\mathbb{R}) \\ &= (\lim_{\rightarrow} \ell_1^n) \otimes S\mathbb{R} = X \otimes S\mathbb{R}. \end{aligned}$$

∎

Our proof is different from his. For contravariant Banach functors one also has:

Proposition 5.7. *Let $\mathcal{F}, \mathcal{G}$ be two contravariant Banach functors adjoint on the right. Then there is a Banach space X such that both $\mathcal{F}$ and $\mathcal{G}$ are naturally equivalent to $\mathfrak{L}^X$.*

Proof. The adjointness relation $\mathfrak{L}(A, \mathcal{G}(B)) = \mathfrak{L}(B, \mathcal{F}(A))$ provides, for the choice $A = \mathbb{R}$, that $G(B) = \mathfrak{L}(B, \mathcal{F}(\mathbb{R}))$. We set $X = \mathcal{F}(\mathbb{R})$. The choice $B = \mathbb{R}$ yields $\mathcal{F}(A) = \mathfrak{L}(\mathbb{R}, \mathcal{F}(A)) = \mathfrak{L}(A, \mathcal{G}(\mathbb{R})) = \mathfrak{L}(A, \mathcal{F}(\mathbb{R}))$, which concludes the proof. ∎

A closely related result has been obtained by Semadeni and Wiweger [49]: *A covariant linear Banach functor that preserves cokernels and coproducts is naturally equivalent to some $\otimes_A$. If it preserves kernels and products it is naturally equivalent to some $\mathfrak{L}_A$.* The Semadeni-Wiweger theorem is formally weaker and essentially equivalent to Fuks theorem since left adjoints preserve direct limits and right adjoints preserve inverse limits.

This important result has, apart from its theoretical interest, applications to Banach space theory affairs:

- To prove that a correspondence commutes with a limit process, do as follows:
  - First be sure that the correspondence is functorial; i.e., defines a Banach functor.
  - Be sure that the limit process considered is a limit.
  - If all that happens –and quite strange if not– then show that the required Banach functor has an adjoint.
- To understand a certain correspondence F, do as follows:
  - Be sure that the correspondence is functorial and thus defines a Banach functor.
  - That implies to know if it is covariant or contravariant.
  - Try to determine if it has an adjoint. In that case, the correspondence F must be –in the covariant case– either tensorization with some Banach space A or taking operators $\mathfrak{L}(A,\cdot)$ on some Banach space.

A example of this appears in Nel's papers [96, 97]. In the author's words "Topological algebras $[X;A]$ of all continuous functions $X \to A$ are familiar objects of study. Many situations are known in which the restricted functors $[-;A]$ have left adjoints $(-,A)$, where (B,A) is usually the æspectral spaceÆ formed by all continuous algebra homomorphisms $B \to A$. This paper was partly motivated by questions about the companion covariant functors $[X;-]$. When do they also have left adjoints $X\Box-$ (say)? And when they do, what do the representing objects $X\Box A$ look like?"

Nel creates a functor $\Box$ (actually, two: one he calls the tensorization $\Box$ and the co-tensorization $[\cdot;\cdot]$ such that $\Box \dashv [\cdot;\cdot]$). If it is true that in other categories those functors might require definition (which is what he does in the paper [96]), in Banach spaces the problem is to find the left adjoint of a covariant functor $\mathfrak{L}_A$, which Fuks result solves: necessarily there must exist a Banach space A such that $\Box = \otimes_A$ and $[\cdot;\cdot] = \mathfrak{L}_A$. Now, passing to the applications to Banach spaces (see the end of [96] and [97]): the choice $A = l_1(I)$ plus the well-known identities $l_1(I)\widehat{\otimes}_\pi X = l_1(I, X)$ and $\mathfrak{L}(l_1(I), X) = l_\infty(I, X)$ yields

$$\mathfrak{L}\big(l_1(I, X), Y\big) = \mathfrak{L}\big(X, l_\infty(I, Y)\big).$$

The choice $A = L_1(\mu)$ and the well known identity $L_1(\mu)\widehat{\otimes}_\pi X = L_1(\mu, X)$ yields

$$\mathfrak{L}\big(L_1(\mu, X), Y\big) = \mathfrak{L}\big(X, \mathfrak{L}(L_1(\mu), Y)\big),$$

which is the main theorem of [97], together with the observation that $\mathfrak{L}(L_1(\mu), Y) = \mathcal{L}_\infty(\mu, Y)$ if and only if Y has the Radon-Nikodym property.

6. Duality

Duality is one of the basic notions in mathematics. And it is not simple to formulate the adequate notion of duality corresponding to a problem. In Banach space theory one is often satisfied with the knowledge that the dual of X is X^* and that some arrows "reverse" under duality: embeddings transform into quotients and vice-versa. This duality had a lot of success and applies with greater or minor variations to other categories: topological groups, Banach algebras, operator spaces, ...

There are however further notions of duality. In fact, the "categorical duality" we have encountered so far roughly says that reversing arrows in a good diagram yields a good diagram. So, the categorical dual of pull-back is push-out and viceversa; and this is so even though we have not proved that the Banach space dual of the pull-back (resp. push-out) is the push-out (resp. pull-back) of the duals.

The categorical study of Banach space duality starts with Dixmier's paper [5]. Willing to attack one of Banach's questions, Dixmier tackles in [5] the problem of characterizing dual spaces. Naively speaking, everybody knows that a predual of a Banach space X is a Banach space

V for which there is an isomorphism $T : X \to V^*$. Dixmier [5, Thm, 17] characterizes a predual V of X as a minimal vector subspace V of X^* with characteristic 1. Recall that the *characteristic* of $V \subset X^*$ is the greatest number $r \geq 0$ such that $V \cap B_{X^*}$ is weakly dense in rB_{X^*}; the subspace V is *minimal* if it is norm-closed, weakly dense in E^* and no norm-closed subspace of V is weakly dense in X^*. This characterization can be, loosely speaking, rephrased as *if the unit ball of X is weak*-compact, then X is a dual space* (of course this is nonsense since the problem is: weak* with respect to what?). Ng [51] made this formulation sound by showing that *if there is a Hausdorff locally convex topology τ on X making its unit ball compact then X is a dual space; i.e., there is a Banach space V such that X is isomorphic to V^*.*

A few important aspects of the problem are however overlooked by this approach. One of them is the *form* in which X and V are in duality; another is if two different subspaces $V_1, V_2 \subset X^*$ have to be considered different preduals. For instance, the first point is essential when considering the dual of a twisted sum as a twisted sum of the duals. Instead of considering $V \subset X^*$, let us consider an into isomorphism $\tau : V \to X^*$ (this is what we will call a *position* of V in X^*). Two preduals of X, say V_1 and V_2, have to be considered different if they occupy different positions in X^*, namely, they correspond to different embeddings $\tau_1 : V_1 \to X^*$ and $\tau_2 : V_2 \to X^*$. And this must be taken into account even if V_1, V_2 are isomorphic or even isometric.

About the form of the duality between V and X, the problem was reconsidered in a much more homological setting by Dieudonné [6] to conclude that, in the same form that to know when X is the dual space of V one just takes the canonical inclusion $\delta_V : V \to V^{**}$ to check that the restriction $\delta^*_{V|X}$ is an isomorphism, we say that V is a predual of X if there is an embedding $\tau : V \to X^*$ such that $\tau^*{}_{|X}$ is an isomorphism. The duality between V and X is thus clear: $< v, x > = < \tau v, x >$.

A more categorical point of view was adopted by Linton [72]. Define the category $\mathbf{B}^*$ (which plays the role of "duals of Banach spaces") whose objects are pairs (X, μ) where X is a Banach space and $\mu : X^{**} \to X$ a contractive linear projection onto X making commutative the diagram

$$\begin{tikzcd} X^{****} \arrow[r, "\mu^{**}"] \arrow[d, "\delta_{X^*}{}^*"'] & X^{**} \arrow[d, "\mu"] \\ X^{**} \arrow[r, "\mu"'] & X \end{tikzcd}$$

A morphism $f : (X, \mu) \to (Y, \eta)$ in $\mathbf{B}^*$ is a linear operator $f : X \to Y$ making commutative the diagram

$$\begin{tikzcd} X^{**} \arrow[r, "f^{**}"] \arrow[d, "\mu"'] & Y^{**} \arrow[d, "\eta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In this way, Linton shows that there is an equivalence of categories established by the functor $\mathcal{F} : \mathbf{B}^{op} \longrightarrow \mathbf{B}^*$ that sends $X \to (X^*, \delta_X)$ and $T : X \to Y$ to $T^* : Y^* \to X^*$.

Moreover, once adopted the idea that Banach space are functors, the existence of a duality theory for Banach spaces generates the necessity to develop a duality theory for functors. Kan's notion of adjointness is a try, although it can be seen more as a way to establish when two categories are one dual of the other (contravariant case) of equivalent (covariant case) (see [95]). In any case, Fuks theorem shows that the adjointness notion is too restrictive to work with Banach functors; and thus the problem is to find a wider notion of duality. Actually, Fuks developed his duality theory for functors with the following purpose in mind [14]:

> B. Eckmann and P. Hilton have obvserved in [12] that there is a duality between certain concepts and theorems in homotopy theory. [...] in the Eckmann-Hilton theory cohomology groups are dual to homotopy groups, the wedge spaces is dual to the cartesian product, and cofibrations are dual to Serre fibrations. If in any theorem of homotopy theory concepts are replaced by their duas, then the resulting theorem is as a rule true. [...] However, the Eckmann-Hilton duality suffers from essential defects. No precise definition of the duality has

> yet been given, and in each separate case the definition of the dual object must be given from intuitive considerations. If a particular theorem of homotopy theory is given, then in the best case a dual theorem can be formulated. But even if an assertion can be dualized, the dual assertion does not follow from the original, and requires an independent proof. Also, there exist theorems that cannot be dualized [...].

Fuks' still elementary formulation begins defining two functors: the functor $\Omega_A(X) = X^A$ is a covariant $\mathrm{Hom}(A, \cdot)$-like functor; the functor $\Sigma_A(X) = X\sharp A$ is a tensor-like functor (a quotient of the product $X \times A$ under some identification of points). Then, given a covariant functor $\mathcal{F}$ Fuks declares the dual functor of $\mathcal{F}$ to be

$$D\mathcal{F}(A) = [\mathcal{F}, \Sigma_A].$$

This is enough to show that $D\Omega_A = \Sigma_A$ and $D\Sigma_A = \Omega_A$. When applied to the Eckmann-Hilton homotopy theory one of the constructions of that theory turns out to be a Ω_A-like functor and the other in "duality" with the previous one a Σ_A-like construction. So Fuks theory pays off.

Of course that this theory has problems with the definitions and with its applicability to other categories (Banach spaces in particular), but the following remarkable comment of Fuks is pointing out where things go: "It is clear that in the Eckmann-Hilton duality one can speak not only of dual concepts, but also of dual functors".

Svarc [17] translates Fuks ideas to "a wide class of categories" which includes, in particular, abelian groups, topological spaces (with or without base points), Banach spaces, sets, partially ordered sets and lattices. He is however aware that **Ban** does not satisfy the axioms formulated at the begining of the paper, but

> A duality can also be constructed for Banach functors by virtue of the fact that the category of unit spheres in Banach spaces satisfies our requirements.

With this in mind, Svarc identifies Ω_A as $\mathrm{Hom}(A, \cdot)$ and Σ_A as the tensor-like functor $\otimes_A$ defined as "the symmetric functor for which $\mathrm{Hom}(X \otimes Y, Z) = \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$. His next paper [18] is devoted to work out the notions for Banach spaces. He sets the notion of Banach functor, the Banach space structure in the set of natural transformations between two Banach functors, properly defines $\Omega_A = \mathfrak{L}(A, \cdot)$ and $\Sigma_A = A\widehat{\otimes}_\pi \cdot)$ to finally formulate the definition of dual functors as follows: given a Banach functor $\mathcal{F}$, its dual functor $D\mathcal{F}$ is defined as the one assigning to the Banach space A the space

$$D\mathcal{F}A = [\mathcal{F}, \otimes_A]$$

and given an arrow $\phi : X \to Y$ then $D\mathcal{F}(\phi)$ "is the map induced by $\otimes(\phi)$". In addition to state $D\mathfrak{L}_A = \otimes_A$ and $D\otimes_A = \mathfrak{L}_A$, Svarc also considers the ℓ_p functors: if $\ell_p(\cdot)$ denotes the covariant Banach functor $X \to \ell_p(X)$ then $D\ell_p(\cdot) = \ell_{p^*}(\cdot)$. The theory culminates with the major paper in the series, the Mitjagin-Svarc paper [22] in which the topic of Banach functors and their duality theory comes to age.

It is mandatory to mention other related papers: Fuks [20] studies the duality of functors in the category of topological spaces with base point; Berman translates in [21] part of the theory to the category of locally convex spaces, and in [34] to the category of nuclear F-spaces; Fuks [29] produces a more topologically oriented paper in which the result about the limitations of Kan's duality appears; the last paper in the series is again Fuks [33] who brings back the duality theory to the category of homotopy types.

6.1. Duality for covariant Banach functors. We already know that the space of natural transformations between two Banach functors is a Banach space. But we should have not forgotten that Banach spaces *are* functors. So the same occurs to these "new" Banach spaces; and tit is important to look at them this way to clearly understand the meaning of the definition.

So, given a Banach functor $\mathcal{F}$ consider the Banach functor $[\mathfrak{L}, \mathcal{F}]$. It assigns to a Banach space A the Banach space $[\mathfrak{L}_A, \mathcal{F}]$ and to an operator $T : A \to B$ the operator $T_* : [\mathfrak{L}_A, \mathcal{F}] \longrightarrow [\mathfrak{L}_B, \mathcal{F}]$ given by $T_*(\nu)_X(\phi) = \nu_X(\phi T)$. In this way the assertions of Mitjagin and Svarc "for every A one has $GA = [\mathfrak{L}_A, \mathcal{F}]$" has to be understood as: there is a natural equivalence between the functors G and $[\mathfrak{L}, \mathcal{F}]$.

Indeed, it is clear that under the natural equivalence the functor $[\mathfrak{L}, \mathcal{F}]$ is simply $\mathcal{F}$: the equality was then proved for spaces, and we check it now for operators: given $T : A \to B$ the operator $T_* : [\mathfrak{L}_A, \mathcal{F}] \longrightarrow [\mathfrak{L}_B, \mathcal{F}]$ is just composition with a natural transformation $[\mathfrak{L}_B, \mathfrak{L}_A] = \mathfrak{L}(A, B)$, i.e., T again.

Dually, one can consider the functor $[\mathcal{F}, \otimes]$ which assigns to a Banach space A the Banach space $[\mathcal{F}A, \otimes_A]$ and to an operator $T : A \to B$ the operator $T^* : [\mathcal{F}, \otimes_A] \longrightarrow [\mathcal{F}, \otimes_B]$ given by $T^*(\nu)_X(\phi) = \nu_X(\phi T)$. Hence, T^* is just right composition with a natural transformation $T \in \mathfrak{L}(A, B) = [\otimes_A, \otimes_B]$.

Definition. The dual functor of $\mathcal{F}$ is defined to be $D\mathcal{F} = [\mathcal{F}, \otimes]$.

Proposition 6.1.

- $D\otimes_A = \mathfrak{L}_A$,
- $D\mathfrak{L}_A = \otimes_A$.

Proof. That the dual of $\otimes_A$ is $\mathfrak{L}_A$ is immediate:

$$D \otimes_A (B) = [\otimes_A, \otimes_B] = \mathfrak{L}(A, B) = \mathfrak{L}_A(B).$$

The dual of $\mathfrak{L}_A$ is calculated, with the help of Proposition 3.1, as:

$$D\mathfrak{L}_A(B) = [\mathfrak{L}_A, \otimes_B] = \otimes_B(A) = \otimes_A(B).$$

∎

6.2. Duality for contravariant Banach functors. For contravariant functors Michor presented in [65] a duality theory modelled upon Fuks' theory for covariant functors. Precisely, the dual of a contravariant functor $\mathcal{G}$ is defined as

$$D\mathcal{G}X = [G, (X\widehat{\otimes_\varepsilon}\cdot)^*].$$

6.3. General duality theory for functors. The Mityagin-Svarc paper [22] concludes with a long section 11 in which a series of problems and lines of research are pointed. The first one deals with the duality process for functors. Precisely:

> The algebraic properties of the duality operator D should be the subject of a deeper investigation. The following two problems are the most interesting:

1. To prove that in the category **Ban** the dual $D\mathcal{F}$ of any functor is reflexive (i.e., $D\mathcal{F} = DDD\mathcal{F}$)
2. Under what conditions is the functor $D(\mathcal{F}\mathcal{G})$ isometric to the functor $D\mathcal{F}D\mathcal{G}$?

(This second problem was treated by Fuks in [20] in the category of topological spaces with distinguished point; his results only partially apply to Banach spaces.)

Recall that given a covariant functor $\mathcal{F}$, the definition of the dual functor immediately yields natural isometries [22, 2. Thm. 1]

$$[\mathcal{F}, D\mathcal{G}] = [\mathcal{G}, D\mathcal{F}].$$

The choice $\mathcal{G} = D\mathcal{F}$ yields $[\mathcal{F}, DD\mathcal{F}] = [D\mathcal{F}, D\mathcal{F}]$. The functor $\mathcal{F}$ is said to be *reflexive* if the morphisms that corresponds to the identity (on the right) is surjective. Mityagin and Svarc observe that when a functor $\mathcal{F}$ is defined on a category of Banach spaces including the reflexive ones then $\mathcal{F}$ is reflexive if and only if $\mathcal{F}(\mathbb{R})$ is reflexive. Michor obtains in [65] the same result for contravariant functors.

The paper [57] gives a first step towards a general duality theory for functors. The paper starts remarking that the relation $\mathcal{F} \to D\mathcal{F}$ establishes itself a contravariant functor

$$D : \mathbf{Ban}^{\mathbf{Ban}} \longrightarrow \mathbf{Ban}^{\mathbf{Ban}}$$

acting in the category $\mathbf{Ban}^{\mathbf{Ban}}$ of Banach functors and natural Banach transformations. This requires to observe that that given $\tau : F \to G$ is a natural transformation then there is a natural transformation $D\tau : DG \to DF$. The simplest way to describe it is as $D\tau : [G, \otimes] \to [F, \otimes]$ is τ^*, in the sense that given X and $\eta \in [G, \otimes_X]$ then $D\tau(\eta) = \eta\tau$.

As a byside product this approach finally formalizes our starting ambition to identify the category **Ban** of [Banach spaces + operators] with that of [Banach functors + Banach natural transformations]. Moreover, it lifts the duality process from **Ban** to $\mathbf{Ban}^{\mathbf{Ban}}$ with a perfect symmetry. In a sense, everything one has to do from now on is to replace Banach spaces by Banach funtors and operators by Banach natural transformations. Thus, the classical duality formula $\mathfrak{L}(X, Y^*) = \mathfrak{L}(Y, X^*)$ becomes $[\mathcal{F}, D\mathcal{G}] = [\mathcal{G}, D\mathcal{F}]$ (allowing identical interpretation in terms of adjointness). This is a good moment for the impatient reader that cannot wait to Part II to see a naive definition of Kan extensions:

Observe the existence of several natural interpretations

$$\delta : \mathbf{Ban} \longrightarrow \mathbf{Ban}^{\mathbf{Ban}}$$

of **Ban** inside $\mathbf{Ban}^{\mathbf{Ban}}$: it could be $\delta(X) = \otimes_X$ since $[\otimes_A, \otimes_B] = \mathfrak{L}(A,B)$; or else $\delta(X) = \mathfrak{L}_X$: in this case the interpretation is of $\mathbf{Ban}^{op}$ since $[\mathfrak{L}_A, \mathfrak{L}_B] = \mathfrak{L}(B,A)$; etc.
Once an interpretation δ has been established, a Kan extension of a Banach functor $\mathcal{F} : \mathbf{Ban} \to \mathbf{Ban}$ is a functor $\mathcal{F}^K : \mathbf{Ban}^{\mathbf{Ban}} \longrightarrow \mathbf{Ban}^{\mathbf{Ban}}$ such that $\mathcal{F}^K\delta = \delta\mathcal{F}$.

The formal appearance of a general duality theory for duality of functors occurs in [68]. The first thing is to define *a duality* for covariant functors as a contravariant Banach functor

$$\mathcal{D} : \mathbf{Ban}^{\mathbf{Ban}} \longrightarrow \mathbf{Ban}^{\mathbf{Ban}}$$

self-adjoint on the right i.e., $[\mathcal{F}, \mathcal{D}\mathcal{G}] = [\mathcal{G}, \mathcal{D}\mathcal{F}]$ naturally and isometrically.

Recovering the idea that a duality between X and Y is something generated by a bilinear form $B : X \times Y \to \mathbb{R}$, one can generate a duality for functors on a Banach category $\mathbf{B}$ with a covariant-covariant bifunctor

$$\mathcal{B} : \mathbf{B} \times \mathbf{B} \longrightarrow \mathbf{Ban}$$

which is symmetric $\mathcal{B}(X,Y) = \mathcal{B}(Y,X)$ naturally and isometrically. The duality $\mathcal{B}$ generates is

$$\mathcal{D}^{\mathcal{B}}\mathcal{F}X = [\mathcal{F}, \mathcal{B}(X, \cdot)].$$

There are several possible $\mathcal{B}$:

- $\mathcal{B}(X,Y) = X\widehat{\otimes_\pi}Y$ (this choice yields Fuks' duality),
- $\mathcal{B}(X,Y) = X\widehat{\otimes_\varepsilon}Y$,
- $\mathcal{B}(X,Y) = \mathfrak{L}_{w^*}(X^*, Y)$,
- $\mathcal{B}(X,Y) = \mathfrak{A}_{w^*}(X^*, Y)$ for an operator ideal $\mathfrak{A}$ such that Schauder's theorem is valid for $\mathfrak{A}$; i.e., $T \in \mathfrak{A} \Rightarrow T^* \in \mathfrak{A}$.

Moreover, all possible dualities can be generated this way: indeed, if $\mathcal{D}$ is a duality then $\mathcal{B}(X,Y) = \mathcal{D}\mathfrak{L}_X(Y)$ is a symmetric bifunctor whose associated duality is precisely $\mathcal{D}$.

Looking at these examples of duality, it was just a matter of time until realizing that our "several examples of bifunctors" are essentially one: a tensor-like product. Thus, the definition of tensor product for functors (and operator ideal of natural transformations) is on its way coming.

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