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The Hitchhiker Guide to Categorical Banach Space Theory. Part I.

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Abstract: What has category theory to offer to Banach spacers? In this survey-like paper we will focus on some of the five basic elements of category theory –namely, i) The definition of category, functor and natural transformation; ii) Limits and colimits; iii) Adjoint functors; plus a naive presentation of Kan extensions– to support the simplest answer "tools that work and a point of view that helps to understand problems, even if one does not care at all about categories". Homology will be treated in a second part.

Key words: Categorical Banach space theory, universal constructions, duality and adjointness.

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