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Topologies Corresponding to Continuous Representability of Preorders

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Abstract: A topology τ on a fixed nonempty set X is said to satisfy the weakly continuous representation property if every weakly continuous not necessarily total preorder $\precsim$ on the topological space (X, τ) admits a continuous order preserving function. Such a property generalizes the well known continuous representation property of a topology τ on a set X (according to which every continuous total preorder $\precsim$ on (X, τ) admits a continuous order preserving function). In this paper I present some results concerning the topologies which satisfy the weakly continuous representation property.

Key words: Weakly continuous preorder, weakly continuous representation property, continuous representation property.

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1. Introduction

In this paper I present some results concerning a topology τ on a fixed nonempty set X which satisfies the property according to which every weakly continuous preorder $\precsim$ on the topological space (X, τ) is representable by a continuous order preserving function $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$, where τ_{nat} is the *natural topology* on the real line $\mathbb{R}$. Such a property will be referred to as the *weakly continuous representation property*. It generalizes the well known *continuous representation property* of a topology τ on a set X (according to which every *continuous* total preorder $\precsim$ on (X, τ) admits a continuous order preserving function).

We recall that a preorder $\precsim$ on a topological space (X, τ) is said to be *weakly continuous* if any two points $(x, y) \in \prec$ are separated by some continuous increasing function $u_{xy} : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$.

The existence of a continuous or at least (upper) semicontinuous order preserving function for a not necessarily total preorder was extensively studied in the literature concerning the applications of mathematics to economics and social sciences since the seminal papers of Aumann [2] and Peleg [2].

Weakly continuous preorders were first introduced by Herden and Pallack [23] and then studied by other authors (see e.g. Bosi and Herden [6, 7], Bosi, Caterino and Ceppitelli [4] and Bosi and Isler [8]) not only in order to guarantee the existence of a continuous order preserving function but also in connection with the validity of the *Continuous Analogue* of the *Szpilrajn Theorem* and the *Dushnik-Miller Theorem*. Indeed, a continuous order preserving function u for a preorder $\precsim$ on a topological space (X, τ) does not characterize $\precsim$ (unless the preorder is *total*) but nevertheless it provides a "natural" *refinement* of $\precsim$ by means of a continuous total preorder $\lesssim$ on (X, τ) that admits a continuous representation. We recall that a topology τ on a set X is said to satisfy the *Continuous Analogue* of the *Szpilrajn Theorem* if for every weakly continuous preorder $\precsim$ on (X, τ) there exists a continuous total preorder $\lesssim$ which refines $\precsim$ (i.e., $\precsim \subset \lesssim$ and $\prec \subset <$). Further, a topology τ on X is said to satisfy the *Continuous Analogue* of the *Dushnik-Miller Theorem* if every weakly continuous preorder $\precsim$ on (X, τ) is the intersection of all the total continuous preorders $\lesssim$ that *refine* $\precsim$.

In order to characterize all the topologies τ on a fixed nonempty set X satisfying the weakly continuous representation property, I introduce the concept of a *preorderable in the weak sense topology* on a set X (this is a topology τ' on X such that there exists a weakly continuous preorder $\precsim$ on (X, τ') and in addition we have that τ' is the coarsest topology on X with respect to which $\precsim$ is weakly continuous). The concept of a preorderable in the weak sense topology generalizes the notion of a *preorderable topology* (i.e., the *order topology* corresponding to some total preorder). I show that a topology τ on a set X satisfies the weakly continuous representation property if and only if all preorderable in the weak sense subtopologies τ' of τ are such that the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf. This result appears as a slight generalization of Theorem 5.1 in Campión, Candeal, Induráin [12], according to which a topology τ on a set X satisfies the weakly continuous representation property if and only if all preorderable subtopologies τ' of τ are second countable (or equivalently all preorderable subtopologies τ' of τ are such that the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf).

Finally, I present a proof of the fact that a completely regular T_1 topology on a set X which in addition satisfies the continuous representation property is necessarily normal.

2. Notation and preliminaries

In the sequel we shall denote by $\precsim$ a preorder (i.e., a *reflexive* and *transitive* binary relation) on a nonempty set X. The *strict part* of $\precsim$ is indicated by $\prec$ (i.e., for all $(x, y) \in X \times X$, $x \prec y$ if and only if $x \precsim y$ and not $(y \precsim x)$). If $(X, \precsim)$ is a preordered set, then a subset D of X is said to be *decreasing* if, for all $x, y \in X$, $x \precsim y$ and $y \in D$ imply $x \in D$.

A pair $(x, y) \in \prec$ is said to be a *jump* of a preordered set $(X, \precsim)$ if

$$]x, y[:= \{z \in X \mid x \prec z \prec y\} = \emptyset.$$

A preorder $\precsim$ is said to be *dense* if it has no jumps.

A preorder $\precsim$ on X is said to be *total* if, for all $x, y \in X$, either $x \precsim y$ or $y \precsim x$.

If $(X, \precsim)$ is a preordered set, then a function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ is said to be *order preserving* (or a *utility function*) on $(X, \precsim)$ if it is *increasing* (i.e., $x \precsim y$ implies $u(x) \leq u(y)$ for all $x, y \in X$) and in addition $x \prec y$ implies $u(x) < u(y)$ for all $x, y \in X$.

If further τ is a topology on X, then the triplet $(X, \precsim, \tau)$ is said to be a *topological preordered space*.

A total preorder $\precsim$ on a topological space (X, τ) is said to be *continuous* if

$$L_{\prec}(x) = \{z \in X : z \prec x\},\ G_{\prec}(x) = \{z \in X : x \prec z\}$$

are both open subsets of X for all $x \in X$.

We recall that the *order topology* $\tau^{\precsim}$ associated with a total preorder $\precsim$ on a set X is the topology generated by the sets $L_{\prec}(x)$ and $G_{\prec}(x)$ with $x \in X$.

A preorder $\precsim$ on a topological space (X, τ) is said to be *weakly continuous* if for every pair $(x, y) \in \prec$ there exists a continuous increasing function $u_{xy} : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ such that $u_{xy}(x) < u_{xy}(y)$. Equivalently, a preorder $\precsim$ on a topological space (X, τ) is weakly continuous if and only if for every pair $(x, y) \in \prec$ there exists a continuous increasing function $u_{xy} : (X, \precsim, \tau) \longrightarrow ([0, 1], \leq, \tau_{nat})$ such that $u_{xy}(x) = 0$ and $u_{xy}(y) = 1$.

Herden and Pallack[23, Lemma 2.2] proved that a total preorder is continuous if and only if it is weakly continuous.

It is well known that it is possible to characterize weak continuity of a preorder by using the concept of a decreasing scale introduced by Burgess and Fitzpatrick [10] (see also Johnson and Mandelker [24], Alcantud et al. [1] and Herden [19] where a generalization of the concept of a decreasing scale has been introduced). A family $\mathcal{G} = \{G_r : r \in S\}$ of open decreasing subsets of X

is said to be a *decreasing scale* in a topological preordered space $(X, \precsim, \tau)$ if the following conditions are satisfied:

(i) S is a dense subset of $[0,1]$ such that $1 \in S$ and $G_1 = X$;

(ii) For every $r_1, r_2 \in S$ with $r_1 < r_2$ we have $\overline{G_{r_1}} \subset G_{r_2}$.

The following proposition, that is presented for reader's convenience and appears as a consequence of the Nachbin-Urysohn approach to mathematical utility theory (see Nachbin [25]), provides a characterization of weak continuity of a preorder in terms of suitable separation properties referred to a decreasing scale. Its proof is omitted.

PROPOSITION 2.1. *Let $(X, \precsim, \tau)$ be a topological preordered space. Then the following conditions are equivalent:*

(i) *The preorder $\precsim$ on (X, τ) is weakly continuous;*

(ii) *For every pair $(x, y) \in \prec$ there exists a decreasing scale $\mathcal{G} = \{G_r : r \in S\}$ in $(X, \precsim, \tau)$ such that $x \in G_{r_1}$ and $y \notin G_{r_2}$ for some $r_1, r_2 \in S \setminus \{1\}$ such that $r_1 < r_2$;*

(iii) *For every pair $(x, y) \in \prec$ there exists a decreasing scale $\mathcal{G} = \{G_r : r \in S\}$ in $(X, \precsim, \tau)$ such that $x \in G_r$ and $y \notin G_r$ for all $r \in S \setminus \{1\}$.*

It is now easy to prove, for example, that a preorder $\precsim$ on a topological space (X, τ) is weakly continuous provided that $\precsim$ is dense and for all pairs $(x, y) \in \prec$ the set $L_{\prec}(y)$ is an open subset of X and $\overline{L_{\prec}(x)} \subset L_{\prec}(y)$ (see the concept of a *spacious (pre)order* introduced by Peleg [26]).

A total preorder $\precsim$ on a topological space (X, τ) is said to be *upper semicontinuous* if

$$L_{\prec}(x) = \{z \in X : z \prec x\}$$

is an open subset of X for all $x \in X$.

Let X be any nonempty set. A topology τ on X is said to satisfy

1. the *Semicontinuous Representation Property* (**SRP**) if every upper semicontinuous total preorder $\precsim$ on (X, τ) admits an upper semicontinuous order preserving function $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$;
2. the *Continuous Representation Property* (**CRP**) if every continuous total preorder $\precsim$ on (X, t) admits a continuous order preserving function $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$;

3. the *Weakly Continuous Representation Property* (**WCRP**) if every weakly continuous preorder $\precsim$ on (X,τ) admits a continuous order preserving function $u : (X,\precsim,\tau) \longrightarrow (\mathbb{R},\leq,\tau_{nat})$.

We have that CRP is implied by both SRP and WCRP. The Semicontinuous Representation Property has been recently studied by Campión, Candeal and Induráin [11, 12] (see also Bosi and Herden [5]). The Continuous Representation Property was first studied by Herden [21] (see also Herden and Pallack [22] and Campión, Candeal, Induráin and Mehta [13]). Some results concerning the Weakly Continuous Representation Property are found in Bosi and Herden [6, 7].

3. Properties of topologies satisfying the weakly continuous representability property

The following theorem summarizes the validity of the above representation properties in the case of metrizability.

THEOREM 3.1. *Let τ be a metrizable topology on a set X. Then the following conditions are equivalent:*
(i) *τ is second countable;*
(ii) *τ satisfies SRP;*
(iii) *τ satisfies CRP;*
(iv) *τ satisfies WCRP.*

The proof is omitted for the sake of brevity. We just notice that in the proof we may use the fundamental theorem presented by Estévez and Hervés [18] (ensuring that a metrizable topology τ on a set X satisfies CRP if and only if τ is second countable), Theorem 3.1 in Bosi, Caterino and Ceppitelli [4] according to which a second countable topology satisfies WCRP) and Corollary 4.5 in Bosi and Herden (stating the equivalence of second countability and SRP in the metrizable case).

We say that a topology τ on a set X is a *preorderable in the weak sense topology* if there exists a weakly continuous preorder $\precsim$ on (X,τ) and in addition we have that τ is the coarsest topology on X with respect to which $\precsim$ is weakly continuous. In this case we denote by $\tau_w^{\precsim}$ such a topology on X.

Observe that in the particular case when $\precsim$ is a total preorder on X, we have that the order topology $\tau^{\precsim}$ on X associated with $\precsim$ is a preorderable in the weak sense topology. Hence, it is clear that a *preorderable topology* (i.e., a topology τ on X such that τ is the order topology $\tau^{\precsim}$ of some total preorder

$\precsim$ on X) is a preorderable in the weak sense topology.

The reader may recall that a topology τ on a set X is said to be a *hereditarily Lindelöf-topology* if for every subset A of X and every open covering $\mathcal{C}$ of A there exists a countable covering $\mathcal{C}' \subset \mathcal{C}$ of A. It is clear that if τ is a second countable topology on a set X (or, more generally, τ has a *countable network weight*), then the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf.

The following theorem is inspired by the proof of Theorem 3.1 in Bosi, Caterino and Ceppitelli [4]. It provides a characterization of topologies satisfying the weakly continuous representation property. In order to prove the theorem, we need the following lemma whose very simple proof is omitted.

LEMMA 3.2. *Let τ'' be a topology on a set X. If τ'' has a countable network weight, then every subtopology τ' of τ'' is such that the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf.*

THEOREM 3.3. *Let τ be a topology on a set X. Then the following conditions are equivalent:*

(i) *τ satisfies WCRP;*

(ii) *For every preorderable in the weak sense subtopology τ' of τ there exists a second countable subtopology τ'' of τ that is finer than τ';*

(iii) *For every preorderable in the weak sense subtopology τ' of τ there exists a subtopology τ'' of τ with a countable network weight that is finer than τ';*

(iv) *All preorderable in the weak sense subtopologies τ' of τ are such that the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf.*

Proof. (i) $\Rightarrow$ (ii). Let τ be a topology on X satisfying WCRP and consider any preorderable in the weak sense subtopology τ' of τ. Assume that τ' is induced by a weakly continuous preorder $\precsim$ on (X, τ) (i.e., we have that $\tau' = \tau_w^{\precsim}$). Let $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ be a continuous order preserving function on $(X, \precsim, \tau)$. Consider the total preorder $\lesssim$ on X defined by

$$x \lesssim y \Leftrightarrow u(x) \leq u(y) \quad (x, y \in X),$$

and let $\tau'' = \tau^{\lesssim}$ be the order topology associated to $\lesssim$. We claim that τ'' is a second countable subtopology of τ that is finer than τ'. Observe that from the Debreu Open Gap Lemma (see e.g. Bridges and Mehta [9, Lemma 3.13]), since there exists a utility function u on the totally preordered set $(X, \lesssim)$ there also exists a continuous utility function u'' on the totally preordered topological space $(X, \lesssim, \tau'')$. Since $\lesssim$ is (continuously) representable we have that τ'' is second countable (see Bridges and Mehta [9, Proposition 1.6.11]). It is clear that τ'' is coarser than τ from the definition of the total preorder $\lesssim$ and the continuity of the function u on the topological space (X, τ). Further, we have that $\precsim$ is weakly continuous on (X, τ'') since u'' is continuous on (X, τ'') and we have that, for all $x, y \in X$,

$$x \precsim y \Rightarrow u(x) \leq u(y) \Rightarrow x \lesssim y \Rightarrow u''(x) \leq u''(y),$$
$$x \prec y \Rightarrow u(x) < u(y) \Rightarrow x < y \Rightarrow u''(x) < u''(y).$$

Hence, we have that τ'' is finer than τ' from the definition of τ'.

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (iv). Consider any preorderable in the weak sense subtopology τ' of τ. Since there exists a subtopology τ'' of τ that has a countable network weight and is finer than τ', this implication is a consequence of Lemma 3.2.

(iv) $\Rightarrow$ (i). Assume that every preorderable in the weak sense subtopology τ' of τ is such that the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf. In order to show that τ satisfies WCRP, consider any weakly continuous preorder $\precsim$ on (X, τ). Then we have that $\precsim$ is in particular weakly continuous on $(X, \tau_w^{\precsim})$. For every pair $(x, y) \in X \times X$ such that $x \prec y$ there exists a continuous increasing function u_{xy} on $(X, \precsim, \tau_w^{\precsim})$ such that $u_{xy}(x) < u_{xy}(y)$. It is not restrictive to assume that u_{xy} takes values in $[0,1]$. Define for every pair $(x, y) \in X \times X$ such that $x \prec y$

$$A_{u_{xy}}(x) := u_{xy}^{-1}\Big([0, \frac{u_{xy}(x) + u_{xy}(y)}{2}[\Big),\ B_{u_{xy}}(y) := u_{xy}^{-1}\Big(]\frac{u_{xy}(x) + u_{xy}(y)}{2}, 1]\Big).$$

Then the family $C := \{A_{u_{xy}(x)} \times B_{u_{xy}(y)}\}_{(x,y) \in \mathcal{R}_S}$ is an open cover of $\prec$ $(\subset X \times X)$. Since the topology $\tau_w^{\precsim} \times \tau_w^{\precsim}$ on $X \times X$ is hereditarily Lindelöf, there exists a countable subfamily C' of C which also covers $\prec$, and therefore there exists a countable family $\{u_n\}_{n \in \mathbb{N}}$ of continuous increasing functions on $(X, \precsim, \tau_w^{\precsim})$ such that for every $(x, y) \in X \times X$ with $x \prec y$ there exists some $n \in \mathbb{N}$ such that $u_n(x) < u_n(y)$. Hence, $u := \sum_{n=0}^{\infty} 2^{-n} u_n$ is a continuous utility function on the topological preordered space $(X, \precsim, \tau_w^{\precsim})$. Since $\tau_w^{\precsim}$ is coarser than τ, we have that u is also a continuous utility function on the topological preordered space $(X, \precsim, \tau)$ and the proof is complete. ∎

Corollary 3.4. *Let τ be a topology on a set X such that the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf. Then τ satisfies WCRP.*

The following proposition provides some conditions on the product topology $\tau \times \tau$ on $X \times X$ implying that τ satisfies WCRP.

From Lemma 4.1 and Proposition 4.2 in Bosi and Herden [5] and from Proposition 7.8 in Bosi and Herden [6], we immediately recover the following corollary of Theorem 3.7.

Corollary 3.5. *A topology τ on a set X satisfies WCRP provided that the product topology $\tau \times \tau$ on $X \times X$ satisfies one of the following conditions:*

(i) *$\tau \times \tau$ satisfies SRP;*

(ii) *$\tau \times \tau$ is compact and satisfies the Continuous Analogue of the Szpilrajn Theorem.*

Corollary 3.6. *Let $\tau^{\precsim}$ be the order topology of some total preorder $\precsim$ on a set X. Then the following conditions are equivalent:*

(i) *$\tau^{\precsim}$ is second countable;*

(ii) *$\tau^{\precsim}$ has a countable network weight;*

(iii) *the product topology $\tau^{\precsim} \times \tau^{\precsim}$ on $X \times X$ is hereditarily Lindelöf.*

The following corollary is the *continuous part* of Theorem 5.1 in Campión, Candeal, Induráin [12].

Corollary 3.7. *Let τ be a topology on a set X. Then the following conditions are equivalent:*

(i) *τ satisfies CRP;*

(ii) *all preorderable subtopologies τ' of τ are second countable;*

(iii) *all preorderable subtopologies τ' of τ have a countable network weight;*

(iv) *all preorderable subtopologies τ' of τ are such that the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf.*

We now present an interesting property of completely regular topologies satisfying WCRP.

Proposition 3.8. *Let τ be a completely regular T_1 topology on a set X. Then in order that τ satisfies WCRP it is necessary that τ is normal.*

Proof. Let τ be a completely regular T_1 topology on a set X that satisfies WCRP. In order to show that τ is normal, consider any two disjoint closed subsets A and B of X. We may assume without loss of generality that $X \setminus (A \cup B) \neq \emptyset$. Otherwise, the sets A and B are open and closed and nothing remains to be shown. Let some point $z \in X \setminus (A \cup B) \neq \emptyset$ be arbitrarily chosen. Then we define a preorder $\precsim$ on X by setting

$$\begin{aligned} \precsim \; &:= \; \{(x,x) \mid x \in X\} \cup \{(x,z) \mid x \in A\} \cup \{(z,y) \mid y \in B\} \\ &\cup \; \{(x,y) \mid x \in A \wedge y \in B\}. \end{aligned}$$

Since τ is a completely regular T_1 topology on X there exist for any pair of points $x \in A$ and $y \in B$ continuous functions $u_x : (X,\tau) \longrightarrow ([0,1], \tau_{nat})$ and $u_y : (X,\tau) \longrightarrow ([0,1], \tau_{nat})$ such that $u_x(x) = 0$ and $u_x(B \cup \{z\}) = \{1\}$ and $u_y(A \cup \{z\}) = \{0\}$ and $u_y(y) = 1$. These properties of u_x and u_y respectively imply that both functions u_x and u_y are increasing with respect to $\precsim$ and that $u_x(x) < u_x(z) \leq u_x(y)$ and $u_y(x) \leq u_y(z) < u_y(y)$ for all points $x \in A$ and $y \in B$. Hence, it follows that $\precsim$ is a weakly continuous preorder on (X,τ). Since τ satisfies WCRP there exists a continuous order preserving function $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$. Then $u^{-1}(]-\infty, u(z)[)$ and $u^{-1}(]u(z), +\infty[)$ are disjoint open subsets of X that contain A and B respectively. ∎

The previous theorem may be considered as interesting since it provides an entire class of examples of topologies that satisfy CRP but fail to satisfy WCRP.

Example 3.9. The Niemytzki space (cf. Steen and Seebach [Example 82]) is a separable and connected completely regular Hausdorff space that is not normal. Hence, the Niemytzki space satisfies CRP (since it satisfies the assumptions of the Eilenberg Representation Theorem) but does not satisfy WCRP.

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