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On the Degrees of the Non-faithful Irreducible Characters in Finite Groups

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Abstract: In this paper, we consider the degrees of the non-faithful irreducible characters of finite groups. We classify finite groups in which non-faithful nonlinear irreducible characters admit distinct degrees. Also we study finite groups whose non-faithful nonlinear irreducible characters are of degree a prime p and classify all of the p-groups with this property.

Key words: Minimal normal subgroups, non-faithful characters, Frobenius groups.

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1. Introduction

Suppose that G is a finite group and let $\mathrm{cd}(G)$ be the set of the character degrees of G. It is a natural question which information about the structure of G may be derived whenever $\mathrm{cd}(G)$ is known. This question has been considered by many authors from different aspects. For example, Isaacs in [5, Theorem 12.15] showed that G is solvable if $|\mathrm{cd}(G)| \leq 3$ (the alternating group A_5 implies that this is the best bound). Also Isaacs and Passman in [6, 7] and Noritzcsh in [8] have studied groups with only two and three character degrees, respectively. In this paper, we define the set $\mathrm{cd}_{\mathrm{nf}}(G)$ of all non-faithful character degrees of G. More precisely,

$$\mathrm{cd}_{\mathrm{nf}}(G) = \big\{\chi(1) \,:\, \chi \in \mathrm{Irr}(G)\,,\ \ker\chi \neq 1\big\}.$$

Here $\mathrm{Irr}(G)$ is the set of the all of the irreducible characters of G. If G is a simple group, then it is easy to see that $\mathrm{cd}_{\mathrm{nf}}(G) = \{1\}$. Also for $n \geq 5$, $\mathrm{cd}_{\mathrm{nf}}(S_n)$ is a singleton where S_n is the symmetric group on n words. Thus solvability of groups may not be obtained even if $|\mathrm{cd}_{\mathrm{nf}}(G)| = 1$. In 1968, Seitz in [9] characterized groups with only one nonlinear irreducible character. Indeed if G is such a group, then G is either an extra-special 2-group or a

Frobenius group of order $p^m(p^m-1)$, where p^m is a prime power. Also in this case, the Frobenius kernel and complement of G are both abelian. Berkovich, Chillag and Herzog in [2], dramatically generalized Seitz's results. In fact they prove that if G is a non-abelian group in which distinct nonlinear characters have distinct degrees, then either G has exactly one nonlinear irreducible character or it is a Frobenius group of order 72. We denote this group by Φ_{72} and remark that this is the *SmallGroup* (72,41), the 41th group of order 72, in the library of GAP [10]. Following [2], we call a group with distinct nonlinear irreducible characters, a D-group. In this paper, we are interested in the situation that a certain condition (such as distinct character degrees condition) is imposed on the set $\mathrm{cd}_{\mathrm{nf}}(G)$, instead of the whole $\mathrm{cd}(G)$. Namely, we consider two certain problems of this type. For the purposes of this paper, we say that a group G is a D_{nf}-group if non-faithful nonlinear irreducible characters of G have distinct degrees. Evidently, every D-group is D_{nf}-groups. We prove that:

Theorem A. *Let G be a finite group. Then the non-faithful nonlinear irreducible characters of G admit distinct degrees if and only if one of the following holds:*

(i) *G has at most one non-faithful nonlinear irreducible character.*

(ii) *G contains a unique minimal normal subgroup N and $G/N \cong \Phi_{72}$.*

We mention that groups with only one nonlinear non-faithful irreducible character have been studied by Iranmanesh and the second author in [4]. Theorem A may be also viewed as a dual of the results of [3], where the authors studied finite p-groups whose faithful irreducible characters admit distinct degrees. Observe that groups that satisfy Theorem A (ii), have two nonlinear non-faithful irreducible characters. The following is an immediate consequent of Theorem A and [4, Corollary 3.3].

Corollary B. *Let G be a nilpotent group. Then all of the non-faithful nonlinear irreducible characters of G admit distinct degrees if and only if one of the following holds:*

(a) *G is an extra-special 2-group,*

(b) *$|G| = 16$ and G is of class 3,*

(c) *$G \cong \mathbb{Z}_p \times E$, where p is an odd prime and E is an extra-special 2-group.*

Throughout the paper, all groups are finite. A monolith is a group with a unique minimal normal subgroup. Unexplained notations are standard.

2. Preliminaries

In this section, we state many facts which are vital to prove the main results of this paper. The following lemma may be verified by GAP.

Lemma 2.1. *Let* $G = \Phi_{72}$. *Then* G *is a monolith,* $|G'| = 18$ *and the unique minimal normal subgroup of* G *is of order* 9.

Lemma 2.2. *Let* G *be a* D_{nf}*-group. Then for every non-trivial normal subgroup* N *of* G *not containing* G', G/N *a* D*-group. Moreover if* $G/N \ncong \Phi_{72}$, *then* $(G/N)'$ *is the unique minimal normal subgroup of* G/N. *In particular,* G/N *possesses a unique nonlinear irreducible character.*

Proof. Let χ_1 and χ_2 be distinct nonlinear irreducible characters of G/N. Since $N \leq \ker \chi_i$ $(i = 1, 2)$, we conclude that χ_1, χ_2 are non-faithful irreducible characters of G. Hence $\chi_1(1) \neq \chi_2(1)$. For the second part, observe that by the main theorem of [2], G/N is an extra-special 2-group or a Frobenius group with abelian kernel and complement. Now it is easy to see that if H belongs to these families of groups, then H' is a unique minimal normal subgroup. The last part is obvious. ∎

Lemma 2.3. *Let* G *be a group and suppose that* G' *and* $Z(G)$ *are minimal normal subgroups of* G. *Then* G *has no other minimal normal subgroup.*

Proof. If L is a minimal normal subgroup of G and $L \neq G'$, then $L \cap G' = 1$. Thus $L \leq Z(G)$. As $Z(G)$ is minimal normal in G, we get $L = Z(G)$. ∎

Lemma 2.4. *Let* G *be a* D_{nf}*-group. Then* $Z(G)$ *is cyclic. Moreover,* $|G|$ *is even unless* G' *is a unique minimal normal subgroup of* G.

Proof. Assume by contradiction that $Z(G)$ is not cyclic. Then by [5, Lemma 2.32], G has no faithful irreducible characters. Hence G is a D-group; while the center of a D-groups is cyclic. This is a contradiction. For the second part, let N be a minimal normal subgroup of G. By our assumption, we may assume that $N \neq G'$. Then by Lemma 2.2, G/N is a D-group. Consequently, $|G|$ is even and the result follows. ∎

PROPOSITION 2.5. *Let G be a nilpotent D_{nf}-group. Then G has at most one non-faithful nonlinear irreducible character.*

Proof. If G' is the unique minimal normal subgroup of G, then the result follows by [5, Lemma 12.3]. So assume that this is not the case. First suppose that G is a 2-group and let N be a minimal normal subgroup of G. Then G/N is extra-special. Hence G'/N is the unique minimal normal subgroup of G/N. If χ is a non-faithful nonlinear irreducible character of G, then $N \leq \ker\chi$. If $N < \ker\chi$, then $G' \leq \ker\chi$, a contradiction. So $\ker\psi = N$ for every non-faithful nonlinear irreducible character ψ of G. As G is a D_{nf}-group, we conclude that χ is the only non-faithful nonlinear irreducible character of G. Now assume that G is not a 2-group. Let N be a minimal normal subgroup of G, and $|N| = p$, a prime, which is greater than 2. If G' is the only subgroup with this property, then $G = A \times P$, where A is an abelian 2-group and P is a p-group. Moreover, P' is the unique minimal normal subgroup of P. So by [5, Lemma 12.3], $\mathrm{cd}(G) = \mathrm{cd}(P) = \{1, m\}$, for an integer $m > 1$. As G is a D_{nf}-group, we conclude that G has at most one nonlinear non-faithful irreducible character and the result follows in this case. So assume that $N \neq G'$. Then G/N is an extra-special 2-group and $G \cong G/N \times \mathbb{Z}_p$. So by [4, Corollary 3.3], G has only one non-faithful nonlinear irreducible character. The proof is completed. ∎

3. MAIN RESULTS

In this section, we prove Theorem A. Before that, we consider a problem posed by Berkovich in [1, Research Problem 94]: study the p-groups all of whose non-faithful nonlinear irreducible characters are of order p. According to our notation, these are just the p-groups with $\mathrm{cd}_{\mathrm{nf}}(G) = \{1, p\}$. First we need the following result of Isaacs and Passman [5, Lemma 12.11].

LEMMA 3.1. *Let G be a non-abelian group. Then $\mathrm{cd}(G) = \{1, p\}$, where p is a prime if and only if one of the followings hold:*

(i) *There exists an abelian $A \trianglelefteq G$ with $|G : A| = p$.*
(ii) $|G : Z(G)| = p^3$.

COROLLARY 3.2. *Let G be a p-group with a cyclic center. Then all of the non-faithful nonlinear irreducible characters of G are of degree p if and only if either of the followings hold:*

(a) *There exists a maximal subgroup* A *of* G *with* $|A'| \leq p$.

(b) *There exists a subgroup* $L \trianglelefteq G$ *with* $|G : L| = p^3$ *and* $\big|[G, L]\big| \leq p$.

If $Z(G)$ is not cyclic, then all of the nonlinear irreducible characters of G are non-faithful. So in this case, $\mathrm{cd}(G)$ and $\mathrm{cd}_{\mathrm{nf}}(G)$ coincide. Thus in Corollary 3.2, the hypothesis that $Z(G)$ is cyclic does not reduce the generality.

Example 3.3. Consider $G = \mathrm{SL}(2, 3)$, the special linear group of degree n over the Galois field $\mathbb{F}_3$. Then G has a non-abelian maximal subgroup of order 8. Also G is a monolith and $Z(G)$ is the unique minimal normal subgroup of G of order 2. So G satisfies Corollary 3.2. Using GAP [10] one observes that G has a unique non-linear non-faithful irreducible character of degree 3 and three faithful irreducible characters, all of which of degree 2. Hence $\mathrm{cd}_{\mathrm{nf}}(G) = \{1, 3\}$ and $\mathrm{cd}(G) = \{1, 2, 3\}$.

Proof of Theorem A. Assume that G has at least two non-faithful nonlinear irreducible characters. Our goal is to show that G is a monolith with a unique minimal normal subgroup N and $G/N \cong \Phi_{72}$. First assume that G is a monolith with a unique minimal normal subgroup N. If $N = G'$, then by [5, Lemma 12.3], all of the nonlinear irreducible characters of G are faithful. So let $N < G'$ and note that $G/N \cong \Phi_{72}$ by Lemma 2.2. That is, G is in case (ii). Therefore, we may assume that G is not a monolith and seek for a contradiction. To this end, we divide the proof into several steps.

Step 1 : G' is not a minimal normal subgroup of G.

Assume by contradiction that G' is a minimal normal subgroup. Let N be a minimal normal subgroup of G and $N \neq G'$. Observe that N must be a central subgroup. If $G/N \cong \Phi_{72}$, then by Lemma 2.1,

$$|G'| = |G' : G' \cap N| = |(G/N)'| = 18\,,$$

which is a contradiction. Thus $(G/N)'$ is the unique minimal normal subgroup of G/N. If $N < Z(G)$, then $G' \leq Z(G)$, which implies that G is nilpotent. This is a contradiction by Proposition 2.5. Therefore, $N = Z(G)$ and we conclude by Lemma 2.3 that N and G' are the only minimal normal subgroups of G. Since G/N has only one nonlinear irreducible character by Lemma 2.2, we conclude that G has only one non-faithful nonlinear irreducible character. This is a contradiction.

Step 2 : If N is a minimal normal subgroup of G, contained in G', then $G/N \cong \Phi_{72}$.

Let N be a counterexample and suppose that L is a minimal normal subgroup of G with $L \neq N$. Since $G/N \ncong \Phi_{72}$, we conclude that $G' \leq LN$ and we can write:

$$G' = G' \cap NL = N(G' \cap L) = NL\,.$$

If $G/L \cong \Phi_{72}$, then $L < K < G'$ for a normal subgroup K and we conclude that $N \nleq K$. Thus $N \cap K = 1$. This is a contradiction, for $NL = NK = G'$. Now let $N = \ker\chi$ and $L = \ker\psi$. Then $|G : N| = |G : G'| + \chi(1)^2$ and $|G : L| = |G : G'| + \psi(1)^2$. As G is a D_{nf}-group, we get $|L| \neq |N|$. Also note that both G/N and G/L are solvable. So both L and N are prime power groups. Assume that $|N| = p^n$ and $|L| = q^m$, where p^m and q^n are prime powers. If G/N and G/L are Frobenius groups, then $|G : L| = p^n(p^n - 1)$ and $|G : N| = q^m(q^m - 1)$. This implies that $|N| = |L|$ which is a contradiction. A similar argument shows that both G/N and G/L can not be 2-groups. Hence either G/N or G/L is an extra-special 2-group, while the other is a Frobenius group. Let G/N be an extra-special 2-group. Hence $|G : G'|$ is an elementary abelian 2-group. On the other hand, G/G' is isomorphic to the Frobenius complement of G/L. That is, G/G' is cyclic of order $p^n - 1$. Combining these facts, one deduces that $p^n - 1 = 2$. That is, $p = 3$ and $|G| = 12$. However by using GAP, one observes that D_{nf}-groups of order 12 have at most one non-faithful irreducible character. This is the final contradiction.

Step 3 : Proof of the theorem.

Let N be a minimal normal subgroup of G, contained in G'. Then by Step 2, $G/N \cong \Phi_{72}$. Let L be a minimal normal subgroup and $L \neq N$. If $L \cap G' = 1$, then L is a central subgroup of G. Hence NL/N is a central subgroup of G/N which is impossible, for, G/N is a Frobenius group. So we must have $L < G'$. As $|G' : L| = 18$, we get $|N| = |L|$. Also $NL \leq G'$. By Lemma 2.1, G/N has only one non-trivial normal subgroup, strictly contained in G'. Thus $|NL : N| = 9$ or 18. The latter fails, because $|N|$ is a prime power. Therefore, $|N| = |L| = 9$ and we conclude that $|G| = 648$ and $|G'| = 162$. Now by using GAP, we can verify that *SmallGroup* (648,253) is the only D_{nf}-group of order 648 with derived subgroup of order 162. But this group is a monolith, which is a contradiction. ∎

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