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On Sequentially Right Banach Spaces

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Abstract: In this paper, we study the recently introduced class of sequentially Right Banach spaces. We introduce a stronger property (RD) and compare these two properties with other well-known isomorphic properties of Banach spaces such as property (V) or the Dieudonné property. In particular, we show that there is a sequentially Right Banach space without property (V). This answers a question of A.M. Peralta, I. Villanueva, J.D.M. Wright and K. Ylinen. We also generalize a result of A. Pełczyński and prove that every sequentially Right Banach space has weakly sequentially complete dual. Finally, it is shown that if K is a scattered compact Hausdorff space then the space $C(K,X)$ of X-valued continuous functions on K is sequentially Right (resp. has property (RD)) if and only if X has the same property.

Key words: Weakly compact operator, Right topology, Dunford-Pettis property, Property (V), Dieudonné property.

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1. Introduction

In [26], A. M. Peralta, I. Villanueva, J. D. M. Wright and K. Ylinen proved that for a given Banach space X there is a locally convex topology on X, called by them the "Right topology", such that every operator T from X into a Banach space Y is weakly compact if and only if it is Right-to-norm continuous. This topology is obtained as the restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X. It is the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$-compact subsets of X^*. In general, the Right topology is stronger than the weak topology and weaker than the norm topology, thus compatible with the dual pair $\langle X, X^*\rangle$. Every Right-to-norm continuous operator is surely Right-to-norm sequentially continuous. A simple look at the identity operator on ℓ_1 reveals, however, that the converse is not true. Authors in [26] call Right-to-norm sequentially continuous operators *pseudo weakly*

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compact and Banach spaces, on which every pseudo weakly compact operator is weakly compact, *sequentially Right*. They have shown that every Banach space possessing property (V) is sequentially Right (see [26, Corollary 15]) and in the subsequent papers [25] and [38] they asked whether the converse holds. We provide a negative answer to this question.

In fact, we study relations of pseudo weakly compact operators and sequentially Right Banach spaces with respect to several other well-known classes of operators and isomorphic properties of Banach spaces. Among these properties are the *Dunford-Pettis property*, the *Reciprocal Dunford-Pettis property*, the *Dieudonné property* and the aforementioned *Pełczyński's property* (V). We also introduce a new *property* (RD) which is an analogue of the Dieudonné property and is (at least formally) stronger than the property of being sequentially Right. A Banach space X is said to have property (RD) if every operator T from X into a Banach space Y which maps Right-Cauchy sequences into Right-convergent sequences is weakly compact. We improve the result of [26] and show that property (V) actually implies property (RD). Characterizations of property (RD) and sequential Rightness are provided. We generalize the result of A. Pełczyński [24, Corollary 5] and show that every sequentially Right Banach space has weakly sequentially complete dual.

We also take an interest in topological behaviour of the Right topology. Two most important special cases are in the centre of our attention. It is shown that the sequential coincidence of the Right topology with the weak one is just another characterization of the Dunford-Pettis property. Multiple characterizations are also given for the sequential coincidence of the Right topology with the norm topology.

Finally, we show that if K is a scattered compact Hausdorff space, then $C(K, X)$, the Banach space of all continuous functions from K to a Banach space X, is sequentially Right (resp. has property (RD)) if and only if X has the same property.

2. Preliminaries

Throughout this paper, we follow standard notation as in [8] or [17]. The term *operator* means a bounded linear map, all Banach spaces are over real numbers. For a Banach space X, we denote by B_X its closed unit ball.

Let X be a Banach space. Given a Banach space Y, an operator $T : X \to Y$ is called *completely continuous* (*cc*) if it maps weakly Cauchy sequences into norm convergent sequences. Banach space X has the *Dunford-Pettis property* (DP) if, for any Banach space Y, every weakly compact operator $T : X \to Y$

is completely continuous. This is equivalent to saying that for any weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, $\lim_n x_n^*(x_n) = 0$ (see, e.g., [7, Theorem 1]). X is said to have the *Reciprocal Dunford-Pettis property* (RDP) if, for any Banach space Y, every completely continuous operator $T : X \to Y$ is weakly compact. Examples of Banach spaces with (RDP) trivially include all reflexive spaces while, on the other hand, an infinite-dimensional reflexive space can never possess (DP). $C(K)$ spaces are known to enjoy both (DP) and (RDP). We refer to [7] for more information on the Dunford-Pettis property.

We say that an operator $T : X \to Y$ is *weakly completely continuous* (*wcc*) if it sends weakly Cauchy sequences into weakly convergent sequences. Let us denote by $\mathcal{B}_1(X)$ the subspace of X^{**} formed by all $\sigma(X^{**}, X^*)$-limits of weakly Cauchy sequences in X. In case X is a $C(K)$ space, $\mathcal{B}_1(X)$ is precisely the space of all bounded Baire-one functions on K ([14, p. 160]). X is said to have the *Dieudonné property* (D) if, for any Banach space Y, every *wcc* operator $T : X \to Y$ is weakly compact. This happens if and only if every operator $T : X \to Y$, such that $T^{**}(\mathcal{B}_1(X)) \subset Y$, satisfies $T^{**}(X^{**}) \subset Y$ (see, e.g., [11, Proposition 9.4.9]). Clearly, any weakly compact operator is *wcc* and also any *cc* operator is *wcc*. So the Dieudonné property implies (RDP). It follows from Rosenthal's ℓ_1-theorem ([29]) that all spaces not containing ℓ_1 have property (D). The identity operator on $L_1([0, 1])$ is an example of a *wcc* operator which is not *cc*, since L_1 is weakly sequentially complete space without the Schur property (see, e.g., [17, pp. 16–18]). To the best of our knowledge, it is still unknown, whether (D) and (RDP) are equivalent.

A series $\sum_n x_n$ in X is called *weakly unconditionally Cauchy* (*wuC*) if $\sum_n |x^*(x_n)| < \infty$ for every $x^* \in X^*$. We say that an operator $T : X \to Y$ is *unconditionally converging* (*uc*) if it sends every *wuC* series into an unconditionally convergent series. This is the same as saying that X does not contain a subspace isomorphic to c_0 on which T is an isomorphism (see, e.g., [7, p. 37]). Banach space X is said to have *Pełczyński's property* (V) if, for any Banach space Y, every *uc* operator $T : X \to Y$ is weakly compact. Using the Orlicz-Pettis theorem ([8, p. 24]), it is easy to see that every *wcc* operator is *uc*. Therefore, every Banach space with property (V) has property (D). The converse does not hold generally (see, e.g., Example 3.34 below). Examples of Banach spaces with property (V) include all reflexive spaces, $C(K)$ spaces ([24, Theorem 1]), L_1-preduals ([18]) and C^*-algebras ([27]). For more information on these and other isomorphic properties of Banach spaces, we refer to [30].

The relative topology induced on X by restricting the Mackey topology $\tau(X^{**}, X^*)$ will be termed the *Right$_X$ topology* (or simply Right if the space X is obvious). Let us recall that the Mackey topology $\tau(X^{**}, X^*)$ is the finest locally convex topology for the dual pair $\langle X^{**}, X^* \rangle$. It is the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$-compact subsets of X^*. Since X^* is a Banach space, it follows from the theorem of Krein (see, e.g., [31, Chapter IV, Theorem 11.4]) that the closed absolutely convex hull of a relatively weakly compact subset of X^* is weakly compact. So $\tau(X^{**}, X^*)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$-compact subsets of X^*. The space X^{**} is complete in the $\tau(X^{**}, X^*)$-topology (see [32, Proposition 1.1]). In reflexive spaces, $\tau(X^{**}, X^*)$-topology agrees with the norm topology. For more information on topological vector spaces, we refer to [11] or [31].

A linear map between Banach spaces is bounded if and only if it is Right-to-Right continuous ([26, Lemma 12]). An operator $T : X \to Y$ is called *pseudo weakly compact* (*pwc*) if it transforms Right-null sequences into norm-null sequences. Banach space X is said to be *sequentially Right* (SR) if, for any Banach space Y, every *pwc* operator $T : X \to Y$ is weakly compact. The following theorem has been proved in [26].

Theorem 2.1. ([26, Corollary 5]) *Let $T : X \to Y$ be an operator. Then the following assertions are equivalent:*

(i) *T is Right-to-norm continuous,*

(ii) *$T \restriction_{B_X}$ is Right-to-norm continuous,*

(iii) *T is weakly compact,*

(iv) *$T^{**} : X^{**} \to Y^{**}$ is $\tau(X^{**}, X^*)$-to-norm continuous.*

Clearly, every weakly compact operator is *pwc*. The converse does not hold, as the identity operator on ℓ_1 shows ([26, Example 8]). In fact, no infinite-dimensional Schur space can be sequentially Right. Since every *pwc* operator is *uc* ([26, Proposition 14]), every Banach space with property (V) is sequentially Right ([26, Corollary 15]).

We say that an operator $T : X \to Y$ is *Right completely continuous* (*Rcc*) if it maps Right-Cauchy sequences into Right-convergent sequences. Let us denote by $\mathcal{R}_1(X)$ the subspace of X^{**} formed by all $\tau(X^{**}, X^*)$-limits of Right-Cauchy sequences in X. Clearly, $\mathcal{R}_1(X) \subset \mathcal{B}_1(X)$. We call a set $K \subset X^*$ an *R-set* if for any Right-null sequence (x_n) in X one has $\lim_n \sup_{x^* \in K} x^*(x_n) = 0$.

Banach space X is said to have the *Right Dieudonné property* (RD) if, for any Banach space Y, every *Rcc* operator $T : X \to Y$ is weakly compact.

3. Main results

For better clarity, we start with the scheme of classification of operators we will shortly establish:

$$\begin{array}{ccccc} \text{weakly compact} & \searrow\ \nearrow & \textit{pwc} & \searrow & \\ & & & & \textit{Rcc} \to \textit{uc} \\ \textit{cc} & \nearrow\ \searrow & \textit{wcc} & \nearrow & \end{array}$$

For Banach space properties we will have:

$$\begin{array}{ccccc} & \nearrow & \text{(SR)} & \searrow & \\ \text{(V)} \to \text{(RD)} & & & & \text{(RDP)} \\ & \searrow & \text{(D)} & \nearrow & \end{array}$$

The following lemma will be used implicitly throughout this paper without further mentioning.

Lemma 3.1. *Let (X, τ_X) and (Y, τ_Y) be two topological vector spaces. Then a linear map $T : X \to Y$ maps τ_X-null sequences into τ_Y-null sequences if and only if T maps τ_X-Cauchy sequences into τ_Y-Cauchy sequences.*

In particular, if τ_1 and τ_2 are two vector topologies on X, then every τ_1-null sequence in X is τ_2-null if and only if every τ_1-Cauchy sequence in X is τ_2-Cauchy.

Proof. Notice first that a sequence (x_n) in a topological vector space is Cauchy if and only if for every increasing sequence of natural numbers $j_n < k_n < j_{n+1}$, the sequence $(x_{k_n} - x_{j_n})$ converges to zero. Suppose T maps τ_X-null sequences into τ_Y-null sequences. Let (x_n) be a τ_X-Cauchy sequence in X. If $j_n < k_n < j_{n+1}$ is an arbitrary increasing sequence of natural numbers, then $(x_{k_n} - x_{j_n})$ converges to zero in X and hence $(T(x_{k_n}) - T(x_{j_n}))$ converges to zero in Y. It follows that $(T(x_n))$ is a τ_Y-Cauchy sequence.

On the other hand, suppose T maps τ_X-Cauchy sequences into τ_Y-Cauchy sequences and let (x_n) be τ_X-null. Then the sequence $0, x_1, 0, x_2, \ldots$ is also τ_X-null, hence it is τ_X-Cauchy and so the sequence $0, T(x_1), 0, T(x_2), \ldots$ is τ_Y-Cauchy. By the observation in the beginning of the proof, $(T(x_n) - 0) = (T(x_n))$ is τ_Y-null.

The special case follows by considering the identity map $T : (X, \tau_1) \to (X, \tau_2)$. ∎

Proposition 3.2. *Let X, Y be Banach spaces and $T : X \to Y$ an operator. Then the following assertions hold:*

(i) *If T is completely continuous, then it is pseudo weakly compact.*

(ii) *If T is pseudo weakly compact, then it is Right completely continuous.*

Proof. Assertion (i) is trivial. As for (ii), since T is *pwc*, every Right-Cauchy sequence in X is mapped into norm-Cauchy and therefore norm-convergent sequence in Y. Since Right_Y topology is weaker than norm, the assertion follows. ∎

Corollary 3.3. *Every Banach space with property* (RD) *is sequentially Right. Every sequentially Right Banach space has property* (RDP).

Proposition 3.4. *Let X be a Banach space. The following assertions hold:*

(a) *For any Banach space Y, an operator $T : X \to Y$ is Rcc if and only if $T^{**}(\mathcal{R}_1(X)) \subset Y$.*

(b) *X has property* (RD) *if and only if, for any Banach space Y, any operator $T : X \to Y$ such that $T^{**}(\mathcal{R}_1(X)) \subset Y$, satisfies $T^{**}(X^{**}) \subset Y$.*

Proof. This proposition is a special case of more general [11, Proposition 9.4.9]. We refer the reader to its proof. ∎

Corollary 3.5. *Every wcc operator is Rcc. Every Banach space with property* (RD) *has property* (D).

Proof. The assertions follow from the characterizations given in Proposition 3.4, since $\mathcal{R}_1(X) \subset \mathcal{B}_1(X)$ and T being a *wcc* operator is equivalent to $T^{**}(\mathcal{B}_1(X)) \subset Y$. ∎

Corollary 3.6. *Let X be a Banach space such that $\mathcal{R}_1(X) = X^{**}$. Then the space X has property* (RD).

Proof. Follows immediately from Proposition 3.4(b). ∎

Remark 3.7. The condition in Corollary 3.6 is satisfied if, for example, the unit ball $B_{X^{**}}$ is metrizable in the $\tau(X^{**}, X^*)$ topology. Characterizations of such Banach spaces can be found in [32].

Proposition 3.8. *Let X and Y be two Banach spaces. If Y is a quotient space of X (in particular, if Y is a complemented subspace of X or if Y is isomorphic to X) and X is sequentially Right (resp. has property* (RD)*), then Y has the same property.*

Proof. Let $q : X \to Y$ be the quotient map from X to Y. Suppose X is sequentially Right (resp. has property (RD)). Then for any *pwc* (resp. *Rcc*) operator $T : Y \to Z$, where Z is a Banach space, $T \circ q$ is a *pwc* (resp. an *Rcc*) and thus a weakly compact operator on X. Since q is a quotient map, by the open mapping theorem T is weakly compact. ∎

The following notion was introduced by N. J. Kalton in [19]. We shall say that a closed subspace Y of a Banach space X is *locally complemented* in X if there is a constant λ such that for each finite-dimensional subspace $F \subset X$ there exists an operator $r_F : F \to Y$ such that $\|r_F\| \leq \lambda$ and $r_F(x) = x$ for every $x \in Y \cap F$.

The Principle of Local Reflexivity ([20, Theorem 3.1]) states that every Banach space is locally complemented in its bidual. Also, every Banach space is locally complemented in its ultraproducts (see [19, Theorem 4.1]). It follows from [20, Theorem 4.1] and the subsequent remark that a Banach space is an $\mathcal{L}_\infty$-space if and only if it is locally complemented in every Banach space containing it. The following proposition is well-known.

Proposition 3.9. ([19, Theorem 3.5]) *Let X be a Banach space and let $Y \subset X$ be its closed subspace. The following assertions are equivalent:*

- (i) *Y is locally complemented in X.*
- (ii) *Y^{**} is complemented in X^{**} under its natural embedding.*
- (iii) *There is an extension operator $T : Y^* \to X^*$ such that for every $y^* \in Y^*$ one has $T(y^*) \restriction_Y = y^*$.*

Proposition 3.10. *Let X be a Banach space and let $Y \subset X$ be its closed subspace. Then Right_Y is finer than $\mathrm{Right}_X \restriction_Y$ and both topologies coincide if there is a weakly continuous extension map $T : Y^* \to X^*$, i.e., a weakly continuous map T such that for every $y^* \in Y^*$ one has $T(y^*) \restriction_Y = y^*$ (in particular, if Y is locally complemented in X).*

Proof. We denote by $i : Y \to X$ the natural inclusion. Since every operator is Right-to-Right continuous by [26, Lemma 12], i is Right_Y-to-Right_X continuous. Hence, Right_Y is finer than $\mathrm{Right}_X \restriction_Y$.

Suppose there is a weakly continuous extension map $T : Y^* \to X^*$. Let $(y_\alpha) \subset Y$ be a Right_X-null net and let K be an arbitrary weakly compact subset of Y^*. Then $T(K)$ is weakly compact in X^* and we have

$$\lim_\alpha \sup_{y^* \in K} |y^*(y_\alpha)| = \lim_\alpha \sup_{y^* \in K} |T(y^*)(y_\alpha)| = \lim_\alpha \sup_{x^* \in T(K)} |x^*(y_\alpha)| = 0.$$

Hence, (y_α) is Right_Y-null, which was to show.

If Y is locally complemented in X, then an extension operator is given by Proposition 3.9. ∎

EXAMPLE 3.11. By a result of F. J. Murray [22], there is a non-complemented subspace Y in $X = \ell_p, 1 < p \neq 2$. Using the reflexivity of both spaces and Proposition 3.9 we can see that Y is not locally complemented in X, yet $\mathrm{Right}_Y = \mathrm{Right}_X \restriction_Y$.

An example of spaces $Y \subset X$ such that Right_Y is strictly finer than $\mathrm{Right}_X \restriction_Y$ is easily provided, using Proposition 3.17 below, by any infinite-dimensional reflexive subspace of a non-Schur Dunford-Pettis space (e.g., $Y = \ell_2$ and $X = C([0,1])$).

COROLLARY 3.12. *Let X, Y be Banach spaces and let $T : X \to Y$ be a pwc (resp. Rcc) operator. Then for every closed subspace $Z \subset X$, $T \restriction_Z : Z \to Y$ is pwc (resp. Rcc).*

Proof. Let (z_n) be a Right_Z-Cauchy sequence in Z. Then, by Proposition 3.10, (z_n) is Right_X-Cauchy and the assumption of the operator T finishes the proof. ∎

COROLLARY 3.13. *A Banach space X is sequentially Right (resp. has property* (RD)*) if every separable subspace has the same property.*

Proof. Let $T : X \to Y$ be a *pwc* (resp. *Rcc*) operator and let (x_n) be a sequence in B_X. We need to show that there is a subsequence such that $(T(x_{n_k}))$ is weakly convergent in Y. Put $Z := \overline{\mathrm{span}}\{x_n : n \in \mathbb{N}\}$. Then Z is a separable subspace of X and by Corollary 3.12, $T \restriction_Z$ is *pwc* (resp. *Rcc*). Using the assumption, $T \restriction_Z$ is weakly compact and therefore there exists the sought subsequence. It follows that T is weakly compact. ∎

Remark 3.14. Corollary 3.13 cannot be reversed. Indeed, consider ℓ_1 as a subspace of $C([0,1])$. By [24, Theorem 1], $C(K)$ spaces have property (V). Corollary 3.20 below shows that property (V) implies property (RD). However, ℓ_1 does not even possess property (RDP).

LEMMA 3.15. *Let X be a Banach space and (x_n^{**}) a w^*-null sequence in X^{**}. The following assertions are equivalent:*

(i) $x_n^{**} \to 0$ *in the* $\tau(X^{**}, X^*)$ *topology.*

(ii) $\lim_n x_n^{**}(x_n^*) = 0$ *for every weakly null sequence* (x_n^*) *in* X^*.

(iii) $\lim_n x_n^{**}(x_n^*) = 0$ *for every weakly Cauchy sequence* (x_n^*) *in* X^*.

(iv) *The operator* $T : X^* \to c_0$ *given by* $T(x^*) = (x_n^{**}(x^*))_{n\in\mathbb{N}}$ *is completely continuous.*

In case $(x_n^{**}) \subset X$, *the statements above are equivalent to* $x_n^{**} \overset{\mathrm{Right}_X}{\to} 0$ *and the operator* T *in* (iv) *moreover satisfies* $T^*(\ell_1) \subset X$.

Proof. Suppose (i) holds and let (x_n^*) be a weakly null sequence in X^*. Since $\{x_n^* : n \in \mathbb{N}\}$ is a relatively weakly compact subset of X^*, (x_n^{**}) converges to zero uniformly on $\{x_n^* : n \in \mathbb{N}\}$. This proves (ii).

For (ii) $\Rightarrow$ (iii), let (x_n^*) be a weakly Cauchy sequence in X^*. If (iii) does not hold then by passing to a subsequence if necessary we may assume that $|x_n^{**}(x_n^*)| > \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. Since (x_n^{**}) is w^*-null, there is an increasing sequence of natural numbers (k_n) such that $|x_{k_n}^{**}(x_{k_{n-1}}^*)| < \frac{\varepsilon}{2}$. Now $(x_{k_n}^* - x_{k_{n-1}}^*)$ is weakly null in X^*, but

$$|x_{k_n}^{**}(x_{k_n}^* - x_{k_{n-1}}^*)| = |x_{k_n}^{**}(x_{k_n}^*) - x_{k_n}^{**}(x_{k_{n-1}}^*)| > \frac{\varepsilon}{2},$$

which is a contradiction.

Let T be defined as in (iv). Assuming (iii), it is easy to show that (x_n^{**}) converges to zero uniformly on every weakly Cauchy sequence in X^*. Let (x_n^*) be such a sequence. Then $\lim_n \sup_k |x_n^{**}(x_k^*)| = 0$. A quick computation now shows that $(T(x_n^*))$ is norm Cauchy in c_0. Thus T is completely continuous.

Finally, to prove (iv) $\Rightarrow$ (i), let K be a weakly compact subset of X^* and let T be as in (iv). Since T is completely continuous, $T(K)$ is norm compact in c_0. By a well-known characterization of compact sets in c_0, $\lim_{n\to\infty} \sup_{x^*\in K} |(T(x^*))_n| = 0$. So x_n^{**} converges uniformly to zero on K. Assertion (i) now follows.

The last statement follows immediately from the definition of the Right_X topology and the fact that the operator T^* maps $(t_n) \in \ell_1$ to $\sum_n t_n x_n^{**}$. ∎

For any Banach space X, the Right_X topology is the weakest locally convex topology τ that makes every weakly compact operator, with X as its domain, τ-to-norm continuous. Indeed, suppose τ is a locally convex topology on X

that is strictly weaker than Right_X. Then there is a semi-norm p on X which is continuous with respect to the Right_X topology, but not with respect to τ (see, e.g., [31, p. 48]). According to [25, Proposition 2.2], we can assume p is of the form $p(x) = \|T(x)\|$, $x \in X$, where $T : X \to Y$ is an operator into a reflexive space Y. Clearly, T is weakly compact, but not τ-to-norm continuous. For sequential continuity we can state the following.

PROPOSITION 3.16. *Let X be a Banach space and let τ be a locally convex topology on X compatible with the duality $\langle X, X^* \rangle$ and weaker than Right_X. Then the following assertions are equivalent:*

(i) *For any Banach space Y, every weakly compact operator $T : X \to Y$ is τ-to-norm sequentially continuous.*

(ii) *Topologies τ and Right_X coincide sequentially on X.*

Proof. By [16, Theorem 2], every weakly compact operator $T : X \to Y$ is τ-to-norm sequentially continuous if and only if for any weakly null sequence (x_n^*) in X^* and any τ-null sequence (x_n) in X we have $\lim_n x_n^*(x_n) = 0$. Using Lemma 3.15 and the fact that τ is stronger than $\sigma(X, X^*)$, it is the same as saying that every τ-null sequence is Right-null. This completes the proof. ∎

PROPOSITION 3.17. (cf. [14, Proposition 1 bis]) *For a Banach space X, the following assertions are equivalent:*

(i) *X has the Dunford-Pettis property.*

(ii) *Topologies $\sigma(X, X^*)$ and Right_X coincide sequentially.*

(iii) *Every (relatively) $\sigma(X, X^*)$-compact subset of X is (relatively) Right_X-compact.*

(iv) *For any Banach space Y, every pseudo weakly compact operator $T : X \to Y$ is completely continuous.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is just a restatement of Proposition 3.16 with $\tau = \sigma(X, X^*)$. Equivalence of (ii) and (iii) follows from the fact that both topologies $\sigma(X, X^*)$ and Right_X are angelic ([28, Definition 0.2]), in particular, from the fact that every subset of X is (relatively) compact if and only if it is (relatively) sequentially compact in the respective topologies (see [28, Theorem 1.2]). Trivially, (ii) $\Rightarrow$ (iv) and using Theorem 2.1, (iv) $\Rightarrow$ (i). ∎

As a direct consequence we have:

COROLLARY 3.18. *Let X be a Banach space with the Dunford-Pettis property. Then the following assertions hold:*

(a) *For any Banach space Y, an operator $T : X \to Y$ is pseudo weakly compact if and only if it is completely continuous.*

(b) *X is sequentially Right if and only if it has property* (RDP).

(c) *For any Banach space Y, an operator $T : X \to Y$ is Right completely continuous if and only if it is weakly completely continuous.*

(d) *X has property* (RD) *if and only if it has property* (D).

(e) $\mathcal{R}_1(X) = \mathcal{B}_1(X)$.

Remark 3.19. While condition (a) of Corollary 3.18 actually implies the Dunford-Pettis property (see Proposition 3.17 (iv)), this is not true for conditions (b)–(e). Indeed, just consider an arbitrary infinite-dimensional reflexive space.

The next corollary improves [26, Proposition 14 and Corollary 15].

COROLLARY 3.20. *Every Rcc operator is uc. Every Banach space with property* (V) *has property* (RD).

Proof. Let $T : X \to Y$ be an *Rcc* operator between two Banach spaces. Suppose T is not unconditionally converging. Then there is an injection $I : c_0 \to X$ such that $T \circ I$ is an isomorphism (see, e.g., [8, p. 54]). Let us denote by (e_n) the unit vector basis in c_0. The sequence $(\sum_{k=1}^{n} e_k)_n$ is weakly Cauchy but not weakly convergent in c_0. So the isomorphism $T \circ I$ is not a *wcc* operator. Since c_0 has the Dunford-Pettis property (see, e.g., [8, p. 113]), by Corollary 3.18(c) $T \circ I$ is not an *Rcc* operator. This, however, contradicts the assumption. The second statement is immediate. ∎

PROPOSITION 3.21. *Let X be a Banach space and let $K \subset X^*$ be a bounded subset. The following assertions are equivalent:*

(i) *K is an R-set.*

(ii) *The $\sigma(X^*, X)$-closed absolutely convex hull of K is an R-set.*

(iii) *Every completely continuous operator $T : X^* \to c_0$ such that $T^*(\ell_1) \subset X$ maps K into a relatively compact subset of c_0.*

(iv) *For every $\varepsilon > 0$ there is an R-set $K_\varepsilon \subset X^*$ such that*

$$K \subset K_\varepsilon + \varepsilon B_{X^*}.$$

Proof. Let us start with (i) $\Rightarrow$ (ii). We denote by A the $\sigma(X^*, X)$-closed absolutely convex hull of K. It is easily seen that A is an R-set if and only if every countable subset of A is an R-set. It is also easy to see that an absolutely convex hull of an R-set is an R-set. Without loss of generality, we may assume that K is absolutely convex. Suppose (ii) does not hold. Then there is a Right-null sequence (x_n) in X and a sequence (x_n^*) in A such that $x_n^*(x_n) > \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. For every n, since x_n^* is in the w^*-closure of K, there is $y_n^* \in K$ such that $y_n^*(x_n) > \varepsilon$. Since $\{y_n^* : n \in \mathbb{N}\}$ is not an R-set, neither is K. Converse implication (ii) $\Rightarrow$ (i) is trivial.

(i) $\Leftrightarrow$ (iii): Lemma 3.15 shows there is one to one correspondence between Right-null sequences in X and completely continuous operators $T : X^* \to c_0$ with $T^*(\ell_1) \subset X$. Indeed, if (x_n) is a Right-null sequence in X, then the corresponding operator T is defined as in Lemma 3.15(iv). Conversely, if T is such an operator and (e_n) the unit basis in ℓ_1 then $(T^*(e_n))$ defines the Right-null sequence in X corresponding to T (again by Lemma 3.15).

If K is an R-set and T as in (iii), then by the observation above $(T^*(e_n))$ is a Right-null sequence in X. Hence

$$0 = \lim_n \sup_{x^* \in K} |\langle T^*(e_n), x^* \rangle| = \lim_n \sup_{x^* \in K} |\langle e_n, T(x^*) \rangle|.$$

By the well-known characterization of compact sets in c_0, $T(K)$ is relatively compact.

If, on the other hand, we suppose (iii) is true and (x_n) is a Right-null sequence in X, then the corresponding operator T maps K into a relatively compact set. Again, the characterization of compact subsets of c_0 gives the uniform convergence of (x_n) to zero on K. Thus (iii) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (iii): Suppose (iv) holds. Let T be as in (iii). Then, for every $\varepsilon > 0$,

$$T(K) \subset T(K_\varepsilon) + \varepsilon T(B_{X^*}) \subset T(K_\varepsilon) + \varepsilon \|T\| B_{c_0},$$

and $T(K_\varepsilon)$ is relatively compact. Hence, $T(K)$ is relatively compact (see, e.g., [12, p. 275]).

The implication (i) $\Rightarrow$ (iv) is obvious. ∎

Proposition 3.22. *Let X and Y be Banach spaces and let $T : X \to Y$ be an operator. The following assertions are equivalent:*

(i) *T is pseudo weakly compact.*

(ii) *$T^*(B_{Y^*})$ is an R-set.*

Proof. Assume (i) holds. Let (x_n) be a Right-null sequence in X. Then

$$\lim_n \sup_{x^* \in T^*(B_{Y^*})} |\langle x^*, x_n \rangle| = \lim_n \sup_{y^* \in B_{Y^*}} |\langle y^*, T(x_n) \rangle| = \lim_n \|T(x_n)\| = 0.$$

This implies (ii). The argument above can be reversed to obtain (ii) $\Rightarrow$ (i). ∎

Remark 3.23. An analogue of the Gantmacher's theorem (see, e.g., [21, Theorem 3.5.13]) does not hold for pseudo weakly compact operators. Consider the identity operator $i : c_0 \to c_0$. The space c_0 and all of its duals have the Dunford-Pettis property (see, e.g., [7, p. 19]). Using Corollary 3.18(a), the identity operator on a Banach space with the Dunford-Pettis property is *pwc* if and only if the space is Schur. Thus we see immediately that i is not *pwc*, while $i^* : \ell_1 \to \ell_1$ is, and again both i^{**} and i^{***} are not *pwc*. We remark that the space ℓ_∞^* is not a Schur space, because its predual contains ℓ_1 (see [7, p. 23]). The only conclusion in this direction is a consequence of Corollary 3.12: *An operator T is pwc if T^{**} is pwc.*

A set U in a Hausdorff topological vector space (X, τ) is called sequentially open if for every sequence $(x_n) \subset X$ converging to a point $x \in U$, x_n belongs to U eventually. I.e., if the complement of U is sequentially closed. The space X is said to be *C-sequential* if every convex sequentially open subset of X is open. We refer to [35] and [37] for more information on C-sequential spaces.

We say that the topological vector space X is a *Ck-space* if for each convex set $A \subset X$, the set A is open in X provided that $A \cap K$ is open in K for any compact subset K of X.

Lemma 3.24. *Let (X, τ) be a Hausdorff topological vector space such that the class of compact subsets of X coincides with the class of sequentially compact subsets of X. Then X is C-sequential if and only if X is a Ck-space.*

Proof. Suppose first that X is C-sequential. Let A be a convex subset of X such that $A \cap K$ is open in K for every compact $K \subset X$. To prove that A is open, it suffices to show that A is sequentially open. Let $(x_n) \subset X$ be a sequence converging to some $x \in A$. Then the set $L = \{x, x_1, x_2, \ldots\}$ is compact in X. By the assumption, $A \cap L$ is open in L and thus there is $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$. So A is sequentially open and hence open in X. This shows that X is a Ck-space.

Assume now that X is a Ck-space. Let U be a convex sequentially open subset of X. Consider a compact subset $K \subset X$ such that $K \not\subset U$. We want to show that $U \cap K$ is open in K, or equivalently, that $(X \setminus U) \cap K$ is closed in K. Let (x_n) be a sequence in $(X \setminus U) \cap K$. Since K is sequentially compact in X, there is a subsequence (x_{n_k}) converging to a point $x \in K$. Since the set $X \setminus U$ is sequentially closed, $x \in X \setminus U$. This shows that $(X \setminus U) \cap K$ is sequentially compact in X and hence compact in X. Since X is a Hausdorff space, $(X \setminus U) \cap K$ is a closed set in X (see, e.g., [12, Theorem 3.1.8]). So we have shown that $U \cap K$ is open in K for every compact set $K \subset X$. Since X is a Ck-space, U is open. ∎

Theorem 3.25. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is sequentially Right.*

(ii) *Every pseudo weakly compact operator $T : X \to \ell_\infty$ is weakly compact.*

(iii) *Every R-subset of X^* is relatively $\sigma(X^*, X^{**})$-compact.*

(iv) *(X, Right_X) is C-sequential.*

(v) *(X, Right_X) is a Ck-space.*

Proof. (ii) $\Rightarrow$ (i): Suppose there is a Banach space Y and an operator $T : X \to Y$ which is *pwc* but not weakly compact. Then there is an operator $U : Y \to \ell_\infty$ such that $U \circ T$ is not weakly compact (see [8, Chapter VII, Exercise 6]). Obviously, $U \circ T$ is *pwc*. But this contradicts (ii).

(i) $\Rightarrow$ (iii): Let K be an R-set in X^*. Denote by $B(K)$ the space of all bounded real valued functions on K with the norm $\|f\| = \sup_{x^* \in K} |f(x^*)|$. The operator $T : X \to B(K)$, defined by $Tx(x^*) = x^*(x)$, for any $x \in X$ and $x^* \in K$, is easily seen to be *pwc*, since K is an R-set. By the assumption (i), T is weakly compact, and therefore also T^* is weakly compact (see, e.g., [21, Theorem 3.5.13]). For any $x^* \in K$, if we define $F \in B(K)^*$ by $F(f) = f(x^*)$, then $\|F\| = 1$ and $T^*(F) = x^*$. So $K \subset T^*(B_{B(K)^*})$, but the latter set is relatively weakly compact. Hence, K is relatively weakly compact. (Cf. the proof of [24, Proposition 1].)

(iii) $\Rightarrow$ (ii): Let $T : X \to \ell_\infty$ be a *pwc* operator. By Proposition 3.22, $T^*(B_{\ell_\infty^*})$ is an R-set, and so by (iii) it is relatively weakly compact. Hence T^*, and therefore T, is weakly compact.

The equivalence (i) $\Leftrightarrow$ (iv) follows from Theorem 2.1 and the fact that a topological vector space X is C-sequential if and only if, for any Banach space

Y, every sequentially continuous operator $T : X \to Y$ is continuous (see [35, Theorem 2]).

The equivalence (iv) $\Leftrightarrow$ (v) is a consequence of Lemma 3.24 and the fact that the topology Right_X is angelic (see [28, Theorem 1.2]). ∎

The next corollary generalizes [24, Corollary 5] stating that every Banach space with property (V) has weakly sequentially complete dual.

COROLLARY 3.26. *If X is a sequentially Right Banach space, then X^* is weakly sequentially complete.*

Proof. Let (x_n^*) be a weakly Cauchy sequence in X^*. Using Lemma 3.15, it is easy to show that any Right-null sequence in X converges to zero uniformly on $K := \{x_n^* : n \in \mathbb{N}\}$. Thus K is an R-set. Since X is sequentially Right, by Theorem 3.25, K is relatively weakly compact. Hence (x_n^*) is weakly convergent in X^*. This shows that X^* is weakly sequentially complete. ∎

In the previous paragraphs we have seen that coinciding of the Right topology with the weak one sequentially is just another characterization of the Dunford-Pettis property. Now we take a look at the other extreme: the norm topology. Let X be a Banach space. Using Theorem 2.1, we can clearly see by looking at the identity operator that the Right topology coincides with the norm toplogy on X if and only if X is reflexive. According to J. Borwein ([4]), Banach space X is called *sequentially reflexive* provided the Mackey topology $\tau(X^*, X)$ coincides sequentially with the norm topology on X^*. A result of P. Ørno ([23]) says that *X is sequentially reflexive if and only if X contains no copy of ℓ_1*.

PROPOSITION 3.27. *A Banach space X is reflexive if and only if it is sequentially Right and X^* is sequentially reflexive.*

Proof. The necessity is trivial. Let us show sufficiency. From the definition, if X^* is sequentially reflexive then the Right_X topology coincides sequentially with the norm topology. Hence the identity operator on X is *pwc*. Since X is also sequentially Right, the identity is weakly compact and therefore X is reflexive. ∎

In spite of the fact that, by Rosenthal's ℓ_1-theorem ([29]), the next corollary is a weaker version of Corollary 3.26, we demonstrate an alternative proof using Proposition 3.27.

Corollary 3.28. *Let X be a sequentially Right Banach space. Then either*

(i) *X is reflexive, or*

(ii) *X^* contains a copy of ℓ_1.*

Proof. By Proposition 3.27, a non-reflexive sequentially Right Banach space cannot have sequentially reflexive dual. Using the result of P. Ørno [23], X^* must contain a copy of ℓ_1. ∎

Example 3.29. Although non-containment of ℓ_1 in X^* characterizes sequential coincidence of $\tau(X^{**}, X^*)$ and the norm topology on X^{**}, this condition is too strong to characterize sequential coincidence of the Right_X and the norm topology on X. This example shows there is a Banach space which contains (even complemented) copy of ℓ_1 in its dual, yet the Right and norm topologies coincide sequentially.

Indeed, the first Bourgain-Delbaen space X constructed in [5] is an infinite-dimensional Schur space whose dual is weakly sequentially complete (and as such contains ℓ_1 by [29]). In fact, the dual space X^* is isomorphic to $M([0,1])$, the Banach space of Radon measures on $[0,1]$.

Since, of course, X as a Schur space is not sequentially Right, this also shows that Corollary 3.26 cannot be reversed.

Remark 3.30. There is, in general, no connection between 'sequential Rightness' of a Banach space X and its bidual X^{**}. The classical chain of sequence spaces c_0, ℓ_1, ℓ_∞, ℓ_∞^*, shows that both can have the same sequential Rightness. The space from Example 3.29 is not sequentially Right, but its bidual is isomorphic to a $C(K)$ space (see, e.g., [17, p. 20]) and thus has even property (V) ([24, Theorem 1]). This example also shows that a locally complemented subspace of a sequentially Right Banach space need not be sequentially Right. Finally, the Banach space $X = (\sum \oplus \ell_1^n)_{c_0}$ has property (V) though its bidual $X^{**} = (\sum \oplus \ell_1^n)_{\ell_\infty}$ contains a complemented copy of ℓ_1 (see [30, p. 389]).

Proposition 3.31. *Let X be a Banach space. The following assertions are equivalent:*

(i) *The Right_X topology coincides sequentially with the norm topology on X.*

(ii) *Every (relatively)* Right_X*-compact subset of* X *is (relatively) norm-compact.*

(iii) *For any Banach space* Y, *every operator* $T : X \to Y$ *is pseudo weakly compact.*

(iv) B_{X^*} *is an R-set.*

(v) *Every bounded subset of* X^* *is an R-set.*

(vi) *For any Right-null sequence* (x_n) *in* X *and any bounded sequence* (x_n^*) *in* X^* *one has* $\lim_n x_n^*(x_n) = 0$.

(vii) *Every completely continuous operator* $T : X^* \to c_0$ *such that* $T^*(\ell_1) \subset X$ *is compact.*

Proof. (i) $\Leftrightarrow$ (ii): Assume (i). If $K \subset X$ is a (relatively) Right_X-compact set, then, by [28, Theorem 1.2], K is (relatively) sequentially Right_X-compact. Using the assumption (i), K is (relatively) sequentially norm-compact and hence (relatively) norm-compact. The converse is obvious, since every Right_X-null sequence is relatively Right_X-compact.

The implication (i) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv): If (iii) holds and we consider the identity operator on X then, by Proposition 3.22, B_{X^*} is an R-set.

Since every subset of an R-set is an R-set, the implication (iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (vi): Assume (v) and let (x_n) and (x_n^*) be as in (vi). Then

$$\lim_n |x_n^*(x_n)| \leq \lim_n \sup_{k\in\mathbb{N}} |x_k^*(x_n)| = 0,$$

since $\{x_n^* : n \in \mathbb{N}\}$ is a bounded set in X^* and so, by the assumption, an R-set.

(vi) $\Rightarrow$ (i): If (i) does not hold, then there is a Right-null sequence (x_n) in X which does not converge to zero in norm. Hence there is a sequence (x_n^*) in B_{X^*} such that $(x_n^*(x_n))$ does not converge to zero. This contradicts (vi).

The equivalence of (iv) and (vii) follows from Proposition 3.21, where we put $K := B_{X^*}$. ∎

Corollary 3.32. *A Banach space* X *is a Schur space if and only if* X *has the Dunford-Pettis property and* B_{X^*} *is an R-set.*

Proof. The space X is Schur if and only if the weak and norm topologies coincide sequentially on X, i.e., if and only if the Right_X topology coincides with both the weak and the norm topology sequentially. Combining Proposition 3.17 with Proposition 3.31 yields the requested equivalence. ∎

Remark 3.33. Let us only remark that X^* *is a Schur space if and only if* X *has the Dunford-Pettis property and* X *contains no copy of* ℓ_1 (see [7, p. 23]).

Concerning compactness, it follows from [33, Proposition 3.1] that $B_{X^{**}}$ *is* $\tau(X^{**}, X^*)$*-compact if and only if* X^* *is a Schur space.* The situation is different for the Right_X topology. Indeed, if B_X is Right_X-compact, then it is weakly compact and so X must be reflexive. Since in reflexive spaces the Right_X and norm topologies coincide, X is necessarily finite-dimensional.

Now we return back to the classification of operators and Banach spaces. Here, for the convenience of the reader, we summarize the relations we have already established. There is generally no connection between weakly compact and *cc* operators. The identity operator on ℓ_2 is an example of a weakly compact operator that is not *cc*, the identity on ℓ_1 is a non-weakly compact *cc* operator. That weakly compact operators are both *pwc* and *wcc* has been mentioned in Preliminaries. Every *cc* operator is trivially *wcc*. By Proposition 3.2, every *cc* operator is *pwc* and all *pwc* operators are *Rcc*. Corollary 3.5 states that all *wcc* operators are *Rcc* and Corollary 3.20 that every *Rcc* is *uc*. The identity on L_1 provides an example of a *wcc* operator which is not *pwc*, since L_1 has the Dunford-Pettis property and so we can use Corollary 3.18. What remains is to show that there is a *pwc* operator which is not *wcc* (see Example 3.35 below) and a *uc* operator that is not *Rcc* (see Example 3.34 below).

As for Banach space properties, (V) $\Rightarrow$ (RD) is shown in Corollary 3.20, (RD) $\Rightarrow$ (SR) and (SR) $\Rightarrow$ (RDP) in Corollary 3.3 and (RD) $\Rightarrow$ (D) in Corollary 3.5. (D) $\Rightarrow$ (RDP) is mentioned in Preliminaries. Examples 3.34 and 3.35 below show (RD) $\not\Rightarrow$ (V) and (D) $\not\Rightarrow$ (SR), respectively.

Example 3.34. Let Y be the second Bourgain-Delbaen space constructed in [5]. It is a non-reflexive Banach space with the Dunford-Pettis property that does not contain c_0 or ℓ_1 and its dual is isomorphic to ℓ_1.

Since Y does not contain ℓ_1, it has the Dieudonné property. As a Dunford-Pettis space, it has also property (RD) by Corollary 3.18(d). However, since Y is not reflexive and does not contain c_0, it cannot possess property (V) (see [24, Proposition 8]). This answers the question raised in [25] and [38] whether every sequentially Right Banach space has property (V).

The identity operator $i : Y \to Y$ is clearly *uc*, since Y does not contain a copy of c_0. Since Y is not reflexive and does not contain ℓ_1, it cannot be weakly sequentially complete (by [29]). Hence, i is not *wcc*. By Corollary 3.18(c), i is not *Rcc*.

Example 3.35. In [15], R.C. James constructed a separable non-reflexive Banach space X isomorphic to its bidual. In particular, since X^{**} is separable, neither X nor X^* contains an isomorphic copy of ℓ_1.

Since X does not contain ℓ_1, X has property (D). By Corollary 3.28, X cannot be sequentially Right.

Since the dual space X^* does not contain ℓ_1, the Right_X topology coincides with the norm topology on X sequentially (see the comments preceding Proposition 3.27). The identity operator $i : X \to X$ is therefore *pwc*. However, since X is neither reflexive nor contains ℓ_1, X is not weakly sequentially complete and hence i is not *wcc*.

Remark 3.36. The only loose end left is an example for (SR) $\not\Rightarrow$ (D). As far as we know, the implication (RDP) $\not\Rightarrow$ (D) has been an open problem ever since it was introduced by A. Grothendieck in [14]. The implication (SR) $\not\Rightarrow$ (RD) seems to be analogical.

4. Vector-valued continuous functions

For a compact Hausdorff space K and a Banach space X we denote by $C(K, X)$ the Banach space of all X-valued continuous functions defined on K, endowed with the supremum norm. It is a long-standing open problem whether the space $C(K, X)$ has property (V) (resp. (D), (RDP)) whenever X has the same property (see [30]). For the Dunford-Pettis property this has been shown to be false by M. Talagrand (see [36]). However, if the compact space K is scattered, then $C(K, X)$ *has property* (V) *(resp.* (D), (RDP), (DP)*) if and only if* X *has the same property* (see [6]). Recall that a compact space K is scattered if every subset A of K has a point relatively isolated in A. The aim of this section is to show that the equivalence above holds also for properties (RD) and (SR). We use the same ideas and techniques as in [6].

Let K be a compact Hausdorff space and X a Banach space. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of K. It is well-known that the dual space $C(K, X)^*$ is isometrically isomorphic to the Banach space $M(K, X^*)$ of all regular countably additive X^*-valued measures of bounded variation defined on the σ-algebra $\mathcal{B}$ and equipped with the variation norm $\|m\| = |m|(K)$. In fact, for any Banach space Y and any operator $T : C(K, X) \to Y$, there is a finitely additive set function $m : \mathcal{B} \to L(X, Y^{**})$, from $\mathcal{B}$ to the space of all operators from X to Y^{**}, having finite semi-variation $\widehat{m}(K)$ with $\widehat{m}(K) = \|T\|$ such that

$$T(f) = \int_K f\, dm \quad \text{for every } f \in C(K, X)$$

(see, e.g., [9, p. 182]). This set function m is called the *representing measure of* T. We recall that the semi-variation of m is defined by

$$\widehat{m}(E) = \sup \Big\{ \Big\| \sum_{i=1}^{n} m(E_i)(x_i) \Big\| : E_i \in \mathcal{B},\ E_i \subset E,\ \{E_i\}_{i=1}^{n} \text{ pairwise disjoint}, \\ x_i \in B_X,\ i = 1, \dots, n,\ n \in \mathbb{N} \Big\}, \quad E \in \mathcal{B}$$

(see [2, p. 217]). The semi-variation $\widehat{m}$ is said to be continuous at $\emptyset$ if $\lim_{n\to\infty} \widehat{m}(E_n) = 0$ for every decreasing sequence $E_n \searrow \emptyset$ in $\mathcal{B}$, or equivalently, if there exists a control measure for $\widehat{m}$, that is, a positive countably additive regular Borel measure λ on K such that $\lim_{\lambda(E)\to 0} \widehat{m}(E) = 0$.

The representing measure m determines an extension $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ of T, where $B(\mathcal{B}, X)$ denotes the Banach space of all strongly measurable functions on $\mathcal{B}$ with values in X, i.e., the Banach space of all functions $g : K \to X$ which are the uniform limit of a sequence of $\mathcal{B}$-simple functions, endowed with the supremum norm, given by

$$\widehat{T}(g) = \int_K g\, dm, \quad g \in B(\mathcal{B}, X),$$

with $\|\widehat{T}\| = \|T\|$ (see [2, Theorem 1]). This extension is just the restriction to $B(\mathcal{B}, X)$ of the biadjoint T^{**} of T.

It has been shown in [10, Theorem 3] that if T is unconditionally converging then m is $L(X, Y)$-valued and $\widehat{m}$ is continuous at $\emptyset$. In this case, by [2, Theorem 2], the extension $\widehat{T}$ maps $B(\mathcal{B}, X)$ into Y.

In the following we consider a compact Hausdorff space K and Banach spaces X, Y.

Proposition 4.1. *Let K be metrizable. Then an operator $T : C(K, X) \to Y$ is Rcc if and only if its extension $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ is Rcc.*

Proof. Let $T : C(K, X) \to Y$ be an *Rcc* operator. Then, by Corollary 3.20, T is *uc* and so m is $L(X, Y)$-valued with a control measure λ and $\widehat{T}$ is Y-valued.

Let (g_n) be a Right-Cauchy sequence in $B(\mathcal{B}, X)$ and let $y^{**} \in Y^{**}$ be the $\tau(Y^{**}, Y^*)$-limit of $(\widehat{T}(g_n))$ (recall that by [26, Lemma 12] every operator is Right-Right continuous and so the sequence $(\widehat{T}(g_n))$ is Right-Cauchy in Y, hence $\tau(Y^{**}, Y^*)$-convergent in Y^{**}).

Suppose, for contradiction, that $y^{**} \notin Y$. Since y^{**} is not $\sigma(Y^*, Y)$-continuous, by Grothendieck's completeness theorem ([31, Chapter IV, Theorem 6.2]) it is not $\sigma(Y^*, Y)$-continuous on B_{Y^*}. Hence there exist $\varepsilon > 0$ and a net $(y^*_\alpha) \subset B_{Y^*}$ which is $\sigma(Y^*, Y)$-convergent to zero such that

$$|y^{**}(y^*_\alpha)| > \varepsilon \text{ for all } \alpha. \tag{1}$$

Choose $\delta > 0$, $\delta < \lambda(K)$, so that

$$\widehat{m}(E) < \frac{\varepsilon}{4 \sup \|g_n\|} \text{ for each } E \in \mathcal{B} \text{ with } \lambda(E) < \delta.$$

According to Lusin's theorem, for every $n \in \mathbb{N}$, there exists a compact set $K_n \subset K$ such that $\lambda(K \setminus K_n) < \frac{\delta}{2^n}$ and the restriction $g_n \restriction_{K_n}$ is continuous. Put $K_0 := \bigcap_{n=1}^{\infty} K_n$. Then $\lambda(K \setminus K_0) < \delta$ and $K_0 \neq \emptyset$ since $\delta < \lambda(K)$. Let us denote $f_n := g_n \restriction_{K_0}$ for every $n \in \mathbb{N}$.

We show that (f_n) is Right-Cauchy in $C(K_0, X)$. Consider the restriction operator $r : B(\mathcal{B}, X) \to B(\mathcal{B} \restriction_{K_0}, X)$. Since (g_n) is Right-Cauchy in $B(\mathcal{B}, X)$, (f_n) is Right-Cauchy in $B(\mathcal{B} \restriction_{K_0}, X)$. Every measure $\mu \in M(K_0, X^*) = C(K_0, X)^*$ can be naturally extended to an element of $B(\mathcal{B} \restriction_{K_0}, X)^*$. Using Proposition 3.10, (f_n) is Right-Cauchy in $C(K_0, X)$.

By the Borsuk-Dugundji theorem (see, e.g., [34, Theorem 21.1.4]), there is an extension operator $S : C(K_0, X) \to C(K, X)$, with $\|S\| = 1$, so that $S(f) \restriction_{K_0} = f$ for every $f \in C(K_0, X)$. Since $T \circ S$ is an *Rcc* operator, $(TS(f_n))$ is Right$_Y$-convergent to an element $y \in Y$. Since (y^*_α) is $\sigma(Y^*, Y)$-convergent to zero there exists an index α_0 so that

$$|y^*_\alpha(y)| < \frac{\varepsilon}{6} \text{ for all } \alpha \geq \alpha_0.$$

Let $\alpha \geq \alpha_0$. There is $n \in \mathbb{N}$ verifying

$$|\langle \widehat{T}(g_n) - y^{**}, y^*_\alpha \rangle| < \frac{\varepsilon}{6} \text{ and } |\langle TS(f_n) - y, y^*_\alpha \rangle| < \frac{\varepsilon}{6}.$$

Thus we have

$$\begin{aligned}|y^{**}(y^*_\alpha)| &\le |\langle y^{**} - \widehat{T}(g_n), y^*_\alpha\rangle| + |\langle \widehat{T}(g_n) - TS(f_n), y^*_\alpha\rangle| \\ &\quad + |\langle TS(f_n) - y, y^*_\alpha\rangle| + |\langle y, y^*_\alpha\rangle| \\ &< \frac{\varepsilon}{2} + \|y^*_\alpha\| \|\widehat{T}(g_n) - TS(f_n)\| \\ &\le \frac{\varepsilon}{2} + \Big\| \int_{K\setminus K_0} g_n - S(f_n)\, dm \Big\| \\ &\le \frac{\varepsilon}{2} + 2\|g_n\| \widehat{m}(K \setminus K_0) < \varepsilon.\end{aligned}$$

But this contradicts (1).

Conversely, if $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ is *Rcc* then, by Corollary 3.12, $T : C(K, X) \to Y^{**}$ is *Rcc*. Hence, every Right-Cauchy sequence (f_n) in $C(K, X)$ is mapped into a Right-convergent sequence in Y^{**}. Since Right$_{Y^{**}}$-topology is compatible with the norm topology and $(T(f_n))$ is contained in the closed convex set $Y \subset Y^{**}$, the limit point y of $(T(f_n))$ must be a member of Y (see, e.g., [31, Chapter IV, 3.1]). Now, Proposition 3.10 implies that $T(f_n) \to y$ in the Right$_Y$-topology. ∎

Proposition 4.2. *Let K be metrizable. Then an operator $T : C(K, X) \to Y$ is pseudo weakly compact if and only if its extension $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ is pseudo weakly compact.*

Proof. Let $T : C(K, X) \to Y$ be a *pwc* operator. By Proposition 3.2(ii) and Corollary 3.20, T is *uc*. Let m and λ be as in the proof of Proposition 4.1. Let (g_n) be a Right-null sequence in $B(\mathcal{B}, X)$. Suppose, for contradiction, that $\widehat{T}$ is not *pwc*. Without loss of generality we may assume that there is $\varepsilon > 0$ so that

$$\|\widehat{T}(g_n)\| > \varepsilon \text{ for all } n \in \mathbb{N}. \tag{2}$$

Choose $\delta > 0$, $\delta < \lambda(K)$, verifying

$$\widehat{m}(E) < \frac{\varepsilon}{4 \sup \|g_n\|} \text{ for each } E \in \mathcal{B} \text{ with } \lambda(E) < \delta.$$

Reasoning as in the proof of Proposition 4.1, there exist a non-empty compact set $K_0 \subset K$ with $\lambda(K \setminus K_0) < \delta$ such that $f_n = g_n \restriction_{K_0}$ is continuous for all $n \in \mathbb{N}$ and an isometric extension operator $S : C(K_0, X) \to C(K, X)$. By the same argument as in the proof of Proposition 4.1, (f_n) is Right-null in $C(K_0, X)$. So $TS(f_n) \to 0$ in Y and there exists $n_0 \in \mathbb{N}$ such that

$$\|TS(f_n)\| < \frac{\varepsilon}{2} \text{ for all } n \geq n_0.$$

Thus if $n \geq n_0$ one has

$$\begin{aligned}\|\widehat{T}(g_n)\| &\leq \|\widehat{T}(g_n) - TS(f_n)\| + \|TS(f_n)\| \\ &< \Big\| \int_{K \setminus K_0} g_n - S(f_n)\, dm \Big\| + \frac{\varepsilon}{2} \\ &\leq 2\|g_n\| \widehat{m}(K \setminus K_0) + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

But this contradicts (2). The converse follows from Corollary 3.12. ∎

LEMMA 4.3. ([6, Lemma 6]) *Let K be a metrizable scattered compact space and let $T : C(K, X) \to Y$ be an operator whose representing measure m verifies*

(i) $m(\mathcal{B}) \subset L(X, Y)$,

(ii) $m(E) : X \to Y$ *is weakly compact for each* $E \in \mathcal{B}$,

(iii) $\widehat{m}$ *is continuous at* $\emptyset$.

Then T is weakly compact.

THEOREM 4.4. *Suppose that K is scattered. Then $C(K, X)$ is sequentially Right (resp. has property* (RD)*) if and only if X has the same property.*

Proof. The necessity follows from Proposition 3.8, since X can be identified with a complemented subspace of $C(K, X)$.

For the sufficiency, assume that X is sequentially Right (resp. has property (RD)) and $T : C(K, X) \to Y$ is a *pwc* (resp. *Rcc*) operator.

(A) Suppose first that K is metrizable. Since T is *uc*, by [10, Theorem 3] its representing measure m satisfies conditions (i) and (iii) of Lemma 4.3. For each $E \in \mathcal{B}$ we define an operator $\Phi_E : X \to B(\mathcal{B}, X)$ by $\Phi_E(x) = x\chi_E$, $x \in X$, where χ_E is the characteristic function of E on K. It follows from Proposition 4.2 (resp. 4.1) that the operator $m(E) = \widehat{T} \circ \Phi_E : X \to Y$ is *pwc* (resp. *Rcc*) and so, since X is sequentially Right (resp. has property (RD)), $m(E)$ is weakly compact. Therefore, all conditions of Lemma 4.3 are satisfied and thus T is weakly compact.

(B) For a general K, let (f_n) be an arbitrary sequence in the unit ball of $C(K,X)$. The method used in [2, p. 236] shows there is a subspace H of $C(K,X)$ such that $(f_n) \subset H$ and H is isometric to some $C(L,X)$, where L is a compact metric space and a quotient space of K. Since a metrizable quotient space of a scattered space is scattered (see [34, Proposition 8.5.3]), L is scattered. Corollary 3.12 in conjunction with the part (A) of this proof shows that $T\restriction_H$ is weakly compact. So there is a subsequence (f_{n_k}) of (f_n) such that $(T(f_{n_k}))$ is weakly convergent in Y. This shows that T is weakly compact. ∎

Analogues of Propositions 4.1 and 4.2 for *cc*, *wcc* and *uc* operators and *general* compact Hausdorff space K have been shown in [3]. The arguments of [3] cannot be employed here, since, unlike the weak topology, the Right topology is not preserved under subspaces in general. We do not know whether the metrizability assumption in Propositions 4.1 and 4.2 can be dropped completely. In the rest of this paper we show, however, that it is possible under the *Continuum Hypothesis* (CH) or if the weight of K is at most $\aleph_1$.

Let M and N be arbitrary Hausdorff topological spaces and let F be a map from M to non-empty subsets of N. We say that F is *upper semi-continuous* (*usc*) if $\{m \in M : F(m) \cap C \neq \emptyset\}$ is closed for every closed subset C of N. A map $f : M \to N$ is called a selection for F if $f(m) \in F(m)$ for all $m \in M$. The weight $w(M)$ of the topological space M is the smallest cardinality of a base for the topology of M. We denote by $\mathcal{B}(M)$ the σ-algebra of Borel subsets of M. If M is completely regular in addition then $\mathcal{B}_0(M)$ will be the σ-algebra of Baire subsets of M, i.e., the σ-algebra generated by the zero-sets of continuous functions on M. We recall that if M is a normal space then the zero-sets of continuous functions on M are precisely the closed G_δ-subsets of M and if M is a metric space then $\mathcal{B}_0(M) = \mathcal{B}(M)$ (see, e.g., [34, Proposition 6.5.2]).

LEMMA 4.5. *Let (g_n) be a Right-null sequence in $B(\mathcal{B}(K),X)$. Let K_0 be a compact subset of K such that $g_n\restriction_{K_0} \in C(K_0,X)$ for all $n \in \mathbb{N}$. Assume* (CH) *or $w(K_0) \leq \aleph_1$. Then there is a Right-null sequence $(\tilde{f}_n)$ in $C(K,X)$ such that $\|\tilde{f}_n\| \leq \|g_n\|$ and $\tilde{f}_n(t) = g_n(t)$ for every $t \in K_0$ and $n \in \mathbb{N}$.*

Proof. Put $f_n := g_n\restriction_{K_0}$ for all $n \in \mathbb{N}$. We have already shown in the proof of Proposition 4.1 that (f_n) is Right-null in $C(K_0,X)$.

We will continue by employing the method from [2, p. 236]. Let us define the pseudo-metric p (see, e.g., [1, p. 15] for the definition of pseudo-metric) on K_0 by

$$p(t,t') = \sum_{n=1}^{\infty} 2^{-n} \|f_n(t) - f_n(t')\|, \quad t,t' \in K_0.$$

Let L be the set of equivalence classes τ of K_0 under the relation: $t \sim s$ if and only if $p(t,s) = 0$. The continuous mapping $\phi : t \mapsto \tau$ of a point $t \in K_0$ into its equivalence class is a continuous mapping from K_0 onto L and thus L is a compact metric space equipped with the metric $\rho(\tau,\tau') = p(t,t')$, $t \in \tau$, $t' \in \tau'$. The mapping $i : h \mapsto h \circ \phi$ defines an isometric embedding of $C(L,X)$ into $C(K_0,X)$. We denote by H the image of $C(L,X)$ in $C(K_0,X)$ under i. Clearly, $f_n \in H$ for all $n \in \mathbb{N}$.

Now we show that (f_n) is Right-null in H. Consider the multi-valued map $F : L \to 2^{K_0}$ defined by $F(\tau) = \phi^{-1}(\tau)$, $\tau \in L$. Since ϕ is continuous, F is compact-valued and *usc*. If we assume (CH) (resp. $w(K_0) \leq \aleph_1$) then, by [13, Theorem 7] (resp. [13, Theorem 3]), there exists a $\mathcal{B}(L)$-$\mathcal{B}_0(K_0)$-measurable selection φ for F. It is easy to verify that every continuous function $f \in C(K_0,X)$ is a uniform limit of $\mathcal{B}_0(K_0)$-simple functions. Since $\Phi : g \mapsto g \circ \varphi$ defines a bounded linear map from the normed vector space of all $\mathcal{B}_0(K_0)$-simple functions to $B(\mathcal{B}(L),X)$, extending Φ by continuity to all of $B(\mathcal{B}_0(K_0),X)$ and then restricting to $C(K_0,X)$ provides an operator, denoted again by Φ, from $C(K_0,X)$ into $B(\mathcal{B}(L),X)$ such that $\Phi(f) \in C(L,X)$ and $i(\Phi(f)) = f$ for every $f \in H$. Hence, $(\Phi(f_n))$ is Right-null in $B(\mathcal{B}(L),X)$ and so, by the same argument as in the proof of Proposition 4.1, Right-null in $C(L,X)$. Since $f_n = i(\Phi(f_n))$ for all $n \in \mathbb{N}$, (f_n) is Right-null in H.

The theorem of Arens [1, Theorem 4.2] (put $A := K_0, X := K, F := B_H, K := B_X, L := X$ and $q := p$) yields an extension operator $S : H \to C(K,X)$ with $\|S\| = 1$. Defining $\tilde{f}_n := S(f_n)$, $n \in \mathbb{N}$, finishes the proof. ∎

Proposition 4.6. *Assume* (CH) *or* $w(K) \leq \aleph_1$. *Then Propositions* 4.1 *and* 4.2 *hold without the metrizability assumption.*

Proof. The only reason for the metrizability asumption on K in Propositions 4.1 and 4.2 was the Borsuk-Dugundji theorem. We used this theorem only to obtain the conclusion of Lemma 4.5. ∎

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