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Composition Operators between μ-Bloch Spaces

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Abstract: In this paper we study continuity, boundedness from below and compactness of composition operators between μ-Bloch spaces for very general weights μ.

Key words: Bloch space, composition operators.

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1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}$. The Bloch space, $\mathcal{B}$, consist of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

It is known that $\mathcal{B}$ is a Banach space with the norm $\|f\| := |f(0)| + \|f\|_{\mathcal{B}}$ (see, e.g., [1]). In the last decade, many authors have studied different classes of Bloch type spaces, where the typical weight function, $v(z) = 1-|z|^2$, $(z \in \mathbb{D})$, is replaced by a bounded continuous positive function μ defined on $\mathbb{D}$. More precisely, a function $f \in H(\mathbb{D})$ is called a μ-Bloch function, denoted as $f \in \mathcal{B}^{\mu}$, if

$$\|f\|_{\mu} := \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

If $\mu(z) = v(z)^{\alpha}$ with $\alpha > 0$, $\mathcal{B}^{\mu}$ is just the α-Bloch space (see [26]). It is readily seen that B^{μ} is a Banach space with the norm $\|f\|_{\mathcal{B}^{\mu}} := |f(0)| + \|f\|_{\mu}$. The $\mathcal{B}^{\mu}$ spaces appear in the literature in a natural way when one study properties of some operators in certain spaces of holomorphic functions; for instance,

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if $\mu_1(z) = v(z)\log\frac{2}{v(z)}$ with $z \in \mathbb{D}$, Attele in [2] proved that the Hankel operator induced by a function f in the Bergman space is bounded if and only if $f \in B^{\mu_1}$. The space B^{μ_1} is also known as the *Log-Bloch space* or the *weighted Bloch space*. Quite recently Stević in [18] introduced, the so called, logarithmic Bloch type space with $\mu(z) = v(z)^\alpha \ln^\beta \frac{e}{v(z)}$, $\alpha > 0$ and $\beta \geq 0$, where some properties of this spaces are studied. Another Bloch type space, using Young's functions, have been recently introduced by Ramos-Fernández in [17].

There is a big interest in the investigation of Bloch type spaces and various concrete linear operators $L : X \to Y$, where at least one of the spaces X and Y is Bloch. For some other recent results in the area see, for example, [1]–[26] and a lot of references therein. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two linear subspaces of $H(\mathbb{D})$. If ϕ is a holomorphic self-map of $\mathbb{D}$, such that $f \circ \phi$ belongs to $\mathcal{H}_2$ for all $f \in \mathcal{H}_1$, then ϕ induces a linear operator $C_\phi : \mathcal{H}_1 \to \mathcal{H}_2$ defined as

$$C_\phi(f) := f \circ \phi,$$

called the *composition operator* with *symbol* ϕ. Composition operators has been studied by numerous authors in many subspaces of $H(\mathbb{D})$ and in particular in Bloch type spaces.

In [14], Madigan and Matheson characterized continuity and compactness for composition operators on the classical Bloch space $\mathcal{B}$. In turn, their results have been extended by Xiao [22] to the α-Bloch spaces and by Yoneda [24] to the *Log*-Bloch space. On the other hand, Gathage, Zheng and Zorboska [10] characterized closed range composition operators on the Bloch space. This result has been extended by Chen and Gauthier [6] to α-Bloch spaces. Also, in [25], Zhang and Xiao have characterized boundedness and compactness of weighted composition operators that act between μ-Bloch spaces on the unit ball of $\mathbb{C}^n$. In this case it is required that μ be a *normal* function. The results of Zhang and Xiao have been extended by Chen and Gauthier [7] to the μ-Bloch spaces being μ a positive and non-decreasing continuous function such that $\mu(t) \to 0$ as $t \to 0$ and $\mu(t)/t^\delta$ is decreasing for small t and for some $\delta > 0$. Recently, Chen, Stević and Zhou (see [8]) have studied composition operators between Bloch type spaces in the polydisc. While Giménez, Malavé and Ramos-Fernández [11] have extended those results to certain μ-Bloch spaces, where the weight μ can be extended to non vanishing complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1,1)$. Ramos-Fernández in [17] have extended all the results mentioned above to the Bloch-Orlicz spaces. In this paper we study properties such as continuity, boundedness from below and compactness of composition operators acting between μ-Bloch spaces, for very general weights μ.

The essential norm of a continuous linear operator T is the distance from T to the compact operators, that is, $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$. Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact. The essential norm of a composition operator on the Bloch space was calculated by Montes-Rodríguez in [15]. Similar results for the essential norms of weighted composition operators between weighted Banach spaces of analytic functions were obtained by Montes-Rodríguez in [16], and by Contreras and Hernández-Díaz in [9], in particular, formulas for the essential norm of weighted composition operators on the α-Bloch spaces are obtained (see also the paper of McCluer and Zhao in [13]). Recently have appeared many extensions of the above results, for instance, we can mention the paper of Yang and Zhou in [23] and a lot of references therein.

Let us explain the organization of the paper. In Section 2, we summarize preliminaries on spaces H_v^∞, associated weight and essential weights. Thus, in Section 3, inspired by the results in [5], we characterize continuity and compactness of the composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$, in fact, if we denote by $\|\delta_z\|$ the norm of the evaluation functional at z acting on the weighted Banach space of analytic functions $H_{\mu_1}^\infty$, then we have the following results:

- The operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous if and only if
$$\sup_{z\in\mathbb{D}} \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| < \infty.$$
- The composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and
$$\lim_{|\phi(z)|\to 1^-} \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| = 0.$$

Finally, in Section 4, we characterize composition operators $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ with closed range in term of certain sampling sets for the space $\mathcal{B}^{\mu_1}$.

2. The associated weight

Let $v : \mathbb{D} \to \mathbb{R}_+$ be an arbitrary weight, that is, v is a bounded, continuos and strictly positive function. A function $f \in H(\mathbb{D})$ belongs to the space H_v^∞ if
$$\|f\|_{H_v^\infty} := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty. \tag{2.1}$$

It is known (see [3]) that H_v^∞ is a Banach space with the norm defined in (2.1). The space H_v^∞ is connected with the study of growth conditions of analytic functions and were studied in detail in [3, 4]. When $v(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, we get the Korenblum spaces $\mathcal{A}^{-\alpha}$. The relation between the μ-Bloch space and the space H_v^∞ is evident, in fact, $f \in \mathcal{B}^\mu$ if and only if $f' \in H_\mu^\infty$ and

$$\|f\|_\mu = \|f'\|_{H_\mu^\infty}.$$

From the relation (2.1), it is clear that, for $z \in \mathbb{D}$ fixed, there exists a constant K_z, depending on z and v, such that

$$|f(z)| \leq K_z \|f\|_{H_v^\infty},$$

for all $f \in H_v^\infty$. This means that the evaluation functional at z, denoted as δ_z, is continuous on H_v^∞ and we can define the *associated weight* with v, denoted as $\widetilde{v}$, by

$$\widetilde{v}(z) = \frac{1}{\|\delta_z\|} = \frac{1}{\sup\{|f(z)| : \|f\|_v \leq 1\}},$$

where $z \in \mathbb{D}$ and $\|\delta_z\|$ denotes the norm of the evaluation functional at z. In [3], it is shown that $\widetilde{v}$ satisfies the following useful properties:

1. $\widetilde{v}$ is a weight and $0 < v(z) < \widetilde{v}(z)$ for all $z \in \mathbb{D}$,
2. for every $z \in \mathbb{D}$, there exists $f_z \in H_v^\infty$ such that $\|f\|_{H_v^\infty} \leq 1$ and
$$\widetilde{v}(z)|f_z(z)| = 1,$$
3. H_v^∞ is isometrically equal to $H_{\widetilde{v}}^\infty$ and $\|f\|_v = \|f\|_{\widetilde{v}}$ for all $f \in H_{\widetilde{v}}^\infty$.

A weight v is called *essential* if there exists a constant $C > 0$ such that $\widetilde{v}(z) \leq Cv(z)$ for all $z \in \mathbb{D}$. For instance, if $v(z) = 1/M(f, |z|)$ for some analytic function $f \in H(\mathbb{D})$, then $v = \widetilde{v}$, where $M(f, r) = \sup_{|z|=r} |f(z)|$. The following are examples of essential weights (see [3] for a reasonable amount of examples of essential weights):

- $v_\alpha(z) = (1-|z|)^\alpha$ with $\alpha > 0$,
- $v(z) = \exp(-1/(1-|z|)^\alpha)$, with $\alpha > 0$,
- $v(z) = 1/\max\{1, -\log(1-|z|)\}$.

In general, is not easy to calculate theassociated weight $\widetilde{v}$; however, in [3, Section 3] Bierstedt, Bonet and Taskinen give some estimations of $\widetilde{v}$.

Finally, we like to comment that some of the properties of composition operators acting on H_v^∞ spaces has been studied by Bonet, Domańki, Lindström and Taskinen in [5], Contreras and Hernández-Díaz in [9] and Wolf in [20].

3. Continuity and compactness of composition operators between μ-Bloch spaces

In this section, we study continuity and compactness of composition operators between μ-Bloch spaces. Throughout this section, ϕ is a holomorphic self-map of $\mathbb{D}$ and C_ϕ denote its associated composition operator. μ_1 and μ_2 are weight functions defined on $\mathbb{D}$ and $\widetilde{\mu}_1$ is the associated weight of μ_1; $\mathcal{B}^{\mu_1}$ and $\mathcal{B}^{\mu_2}$ are their respective μ-Bloch spaces. Also, $\|\delta_z\|$ denotes the norm of the evaluation functional at z on the weighted Banach space of analytic functions $H^\infty_{\mu_1}$. With this notations we have the following results.

3.1. Continuity. The following, generalize many results about continuity of composition operators acting on Bloch type spaces. A similar result was obtained recently by Wolf in [21], while this article was under review.

Theorem 3.1. *The operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous if and only if*

$$\sup_{z\in\mathbb{D}} \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| < \infty. \tag{3.1}$$

Proof. Suppose first that

$$L = \sup_{z\in\mathbb{D}} \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| < \infty.$$

Then, for each $f \in \mathcal{B}^{\mu_1}$, since $\widetilde{\mu}_1(s)\|\delta_s\| = 1$ for all $s \in \mathbb{D}$, we have the following estimate

$$\begin{aligned}\|f\circ\phi\|_{\mu_2} &= \sup_{z\in\mathbb{D}} \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)|\,\widetilde{\mu}_1(\phi(z))|f'(\phi(z))|\\ &\le L\|f\|_{\widetilde{\mu}_1} = L\|f\|_{\mu_1}.\end{aligned}$$

Also, since μ_1 is continuous and positive on the compact set $[0,\phi(0)]$, there exists a constant $K_{\mu_1,\phi} > 0$, depending on μ_1 and $\phi(0)$, such that

$$\int_0^{\phi(0)} \frac{|ds|}{\mu_1(s)} \le K_{\mu_1,\phi}. \tag{3.2}$$

Hence, we have

$$|f(\phi(0))| \le |f(0)| + \int_0^{\phi(0)} |f'(s)||ds| \le |f(0)| + K_{\mu_1,\phi}\|f\|_{\mu_1}.$$

We conclude that

$$|f(\phi(0))| + \|f \circ \phi\|_{\mu_2} \leq |f(0)| + (L + K_{\mu_1,\phi})\|f\|_{\mu_1}$$

and the composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous.

Now, suppose that there exists a constant $L > 0$ such that $\|f \circ \phi\|_{\mu_2} \leq L\|f\|_{\mu_1}$ for all function $f \in \mathcal{B}^{\mu_1}$ with $f(0) = 0$ and let us fix $z \in \mathbb{D}$. By definition of associated weight, for $a = \phi(z) \in \mathbb{D}$, there exists a function $f_a \in H(\mathbb{D})$ such that $\sup_{w\in\mathbb{D}} \mu_1(w)|f_a(w)| \leq 1$ and $\widetilde{\mu}_1(a)|f_a(a)| = 1$, hence the function g_a given by

$$g_a(w) = \int_0^w f_a(s)ds,$$

with $w \in \mathbb{D}$ belongs to $\mathcal{B}^{\mu}_1$ and satisfies $g_a(0) = 0$. Thus, applying the hypothesis with $f = g_a$, we have $\|g_a \circ \phi\|_{\mu_2} \leq L$; that is,

$$\sup_{w\in\mathbb{D}} \mu_2(w)|g_a'(\phi(w))||\phi'(w)| \leq L.$$

This last, implies that

$$\mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| \leq L.$$

The proof of the theorem is complete. ∎

As an immediate consequence of Theorem 3.1, we have the following result, which has been obtained by many authors for various type of weight μ.

Corollary 3.2. *If μ_1 is a essential weight, then the composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu_2(z)}{\mu_1(\phi(z))}|\phi'(z)| < \infty.$$

Proof. It follows from the fact that $\widetilde{\mu}_1(s)\|\delta_s\| = 1$ for all $s \in \mathbb{D}$ and $\widetilde{\mu}_1 \sim \mu_1$. ∎

Example 3.3. If $\mu_1(z) = (1 - |z|)^\alpha$ and $\mu_2(z) = (1 - |z|)^\beta$ with $\alpha, \beta > 0$, we have the result of Xiao in [22]. If $\mu_1(z) = \mu_2(z) = (1 - |z|) \log\left(\frac{2}{1-|z|}\right)$, we have the result of Yoneda in [24].

3.2. Compactness. Now, we are going to characterize compactness of composition operators that act between μ-Bloch spaces. Our goal is to obtain genuine extensions of the results in [17, 11, 14]. In [19], Tjani showed the following result.

Lemma 3.4. *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

1. *The point evaluation functionals on X are continuous.*
2. *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
3. *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Observe that for $z \in \mathbb{D}$ fixed, since μ_1 is positive and continuous on the compact set $[0, z]$, there exists a constant $K_{\mu_1,z} > 0$ such that

$$|f(z)| \leq |f(0)| + \int_0^z \frac{|ds|}{\mu_1(s)} \leq K_{\mu_1,z}\|f\|_{\mathcal{B}^{\mu_1}}$$

and the point evaluation functionals on $\mathcal{B}^{\mu_1}$ are continuous. Thus, as a consequence of Lemma 3.4, we have the following result.

Lemma 3.5. *The composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is compact if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{B}^{\mu_1}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\phi(f_n)\|_{\mu_2} \to 0$ as $n \to \infty$.*

Next we establish our criterion for the compactness of $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$. It generalizes a result of Madigan and Matheson in [14].

Theorem 3.6. *The composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and*

$$\lim_{|\phi(z)| \to 1^-} \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| = 0. \tag{3.3}$$

Proof. Let us suppose first that $\phi \in \mathcal{B}^{\mu_2}$ and (3.3) holds. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}^{\mu_1}$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then, by Lemma 3.5, it is suffices to show that $\|C_\phi(f_n)\|_{\mu_2} \to 0$ as $n \to \infty$. To this end, we set $K = \sup_n \|f_n\|_{\mu_1} = \sup_n \|f_n\|_{\widetilde{\mu}_1}$. Then, for $\epsilon > 0$ we can find an $r \in (0, 1)$ such that

$$\mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)| < \frac{\epsilon}{K},$$

for any $z \in \mathbb{D}$ satisfying $r < |\phi(z)| < 1$. Hence, we have

$$\begin{aligned}\mu_2(z)|(f_n \circ \phi)'(z)| &= \mu_2(z)\|\delta_{\phi(z)}\||\phi'(z)|\widetilde{\mu}_1(\phi(z))|f_n'(\phi(z))| \\ &\leq \frac{\epsilon}{K}K = \epsilon\end{aligned}$$

whenever $r < |\phi(z)| < 1$. Here, we have used the fact that $\|\delta_s\|\widetilde{\mu}_1(s) = 1$ for all $s \in \mathbb{D}$.

On the other hand, since $\phi \in \mathcal{B}^{\mu_2}$ and $\widetilde{\mu}_1$ is continuous and positive on the compact set $\{w \in \mathbb{D} : |w| \leq r\}$, we can find a constant $C > 0$, depending only on r and μ_1, such that

$$\sup_{|\phi(z)| \leq r} \frac{\mu_2(z)}{\widetilde{\mu}_1(\phi(z))}|\phi'(z)| \leq C\|\phi\|_{\mu_2}.$$

Thus, since $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ and $\widetilde{\mu}_1$ is bounded on the compact set $|s| \leq r$, we have $\sup_{|s|\leq r} \widetilde{\mu}_1(s)|f_n'(s)| \to 0$, as $n \to \infty$. Hence, for the given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned}\sup_{|\phi(z)|\leq r} \mu_2(z)|(f_n \circ \phi)'(z)| &= \sup_{|\phi(z)|\leq r} \frac{\mu_2(z)}{\widetilde{\mu}_1(\phi(z))}|\phi'(z)|\widetilde{\mu}_1(\phi(z))|f_n'(\phi(z))| \\ &\leq C\|\phi\|_{\mu_2}\epsilon\end{aligned}$$

whenever $n \geq N$. Finally, since $f_n \circ \phi(0) \to 0$ as $n \to \infty$, we conclude that

$$\|f_n \circ \phi\|_{\mu_2} = |f_n \circ \phi(0)| + \sup_{z\in\mathbb{D}} \mu_2(z)|(f_n \circ \phi)'(z)| < (1 + C\|\phi\|_w)\epsilon$$

whenever $n \geq N$, which means that $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is a compact operator.

To prove the converse, suppose that there exists an $\epsilon_0 > 0$ such that

$$\sup_{|\phi(z)|\geq r} \frac{\mu_2(z)}{\widetilde{\mu}_1(\phi(z))}|\phi'(z)| \geq \epsilon_0$$

for any $r \in (0,1)$. Then, given a sequence of real numbers $\{r_n\} \subset (0,1)$ such that $r_n \to 1$ as $n \to \infty$, we can find a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\phi(z_n)| > r_n$ and

$$\frac{\mu_2(z_n)}{\widetilde{\mu}_1(w_n)}\,|\phi'(z_n)| \geq \frac{1}{2}\,\epsilon_0,$$

where $w_n = \phi(z_n)$. By taking a subsequence, if necessary, we may suppose that $w_n \to w_0 \in \partial\mathbb{D}$. Also, since $|w_n| \to 1$ as $n \to \infty$, we can find an increasing sequence $\{\alpha(n)\}$ of positive integers such that $\alpha(n) \to \infty$ as $n \to \infty$ and $|w_n|^{\alpha(n)} \geq \frac{1}{2}$ for all $n \in \mathbb{N}$.

Now, for $n \in \mathbb{N}$, we choose a function $f_n \in H(\mathbb{D})$ such that

$$\sup_{w\in\mathbb{D}} \mu_1(w)|f_n(w)| \leq 1$$

and $\widetilde{\mu}_1(w_n)|f_n(w_n)| = 1$ and we set

$$g_n(z) = \int_0^z s^{\alpha(n)} f_n(s)ds,$$

with $z \in \mathbb{D}$. We can see that $\{g_n\}$ is a bounded sequence in $\mathcal{B}^{\mu_1}$, in fact,

$$\|g_n\|_{\mu_1} = \sup_{z\in\mathbb{D}} \mu_1(z)|g_n'(z)| = \sup_{z\in\mathbb{D}} \mu_1(z)|z|^{\alpha(n)}|f_n(z)| \leq 1$$

for all $n \in \mathbb{N}$. Furthermore, because of the factor $z^{\alpha(n)}$, the sequences $\{g_n'\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, therefore, since

$$g_n(z) = \int_0^z g_n'(s)ds$$

for all $z \in \mathbb{D}$, we can see that $\{g_n\}$ is a sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$ and satisfying

$$\begin{aligned}\|C_\phi(g_n)\|_{\mu_2} &\geq \mu_2(z_n)|g_n'(w_n)||\phi'(z_n)| \\ &= \mu_2(z_n)|w_n|^{\alpha(n)}|f_n(w_n)||\phi'(z_n)| \\ &= \frac{1}{2}\,\frac{\mu_2(z_n)}{\widetilde{\mu}_1(w_n)}\,|\phi'(z_n)| > \frac{1}{4}\,\epsilon_0 > 0,\end{aligned}$$

where, we have used the fact that $|f_n(w_n)| = 1/\widetilde{\mu}_1(w_n)$. Therefore, $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is not a compact operator. This completes the proof of the theorem. ∎

As an immediate consequence, we have.

Corollary 3.7. *If μ_1 is an essential weight, then the composition operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and*

$$\lim_{|\phi(z)|\to 1^-} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| = 0.$$

Example 3.8. If $\mu_1(z) = \mu_2(z) = 1 - |z|$, we get the result of Madigan and Matheson in [14]. When $\mu_1(z) = (1 - |z|^2)^\alpha$ and $\mu_2(z) = (1 - |z|^2)^\beta$ with $\alpha, \beta > 0$, we obtain a criterion for the compactness of composition operators between α-Bloch type spaces, similar result was found by Montes-Rodríguez in [16], later, independently, by Xiao in [22] and by McCluer and Zhao in [13]. If $\mu_1(z) = \mu_2(z) = (1 - |z|) \log(2/(1 - |z|))$ we obtain the result of Yoneda in [24].

4. Composition operators with closed range between μ-Bloch spaces

In this section, we characterize the composition operators $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ with closed range in terms of certain sampling sets for the μ-Bloch space. The purpose here is to generalize the results in [6, 10, 11, 17] for the μ-Bloch spaces. Recall (see [12]) that a subset G of the unit disk $\mathbb{D}$ is said to be a *sampling set* for the Korenblum space $\mathcal{A}^{-\alpha}$ if there exists a positive constant $L > 0$ such that

$$\sup_{z\in G} (1 - |z|^2)^\alpha |f(z)| \geq L\|f\|_{\mathcal{A}^{-\alpha}}$$

for all $f \in \mathcal{A}^{-\alpha}$. Observe that $v(z) = (1 - |z|^2)^\alpha = \widetilde{v}(z)$ and for this reason, we introduce the following definition.

Definition 4.1. Let v be a weight defined on $\mathbb{D}$. A subset G of the unit disk $\mathbb{D}$ is said to be a *sampling set* for $\mathcal{B}^v$ if there exists a positive constant $L > 0$ such that

$$\sup_{z\in G} \widetilde{v}(z)|f'(z)| \geq L\|f\|_v \tag{4.1}$$

for all $f \in \mathcal{B}^v$.

Remark 4.2. When v is an essential weight, we can replace, in the preceding definition, the weight $\widetilde{v}$ by v.

Let μ_1 and μ_2 two weights. For $\varepsilon > 0$, let us denote

$$\Omega_\varepsilon := \left\{ z \in \mathbb{D} : \frac{\mu_2(z)}{\widetilde{\mu}_1(\phi(z))} |\phi'(z)| \geq \varepsilon \right\}.$$

With this notation, we have the following result.

THEOREM 4.3. *Let $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ be a continuous composition operator. $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is bounded below if and only if there exists $\varepsilon > 0$ such that $G_\varepsilon = \phi(\Omega_\varepsilon)$ is a sampling set for $\mathcal{B}^{\mu_1}$.*

Proof. Let us suppose first that there exists $\varepsilon > 0$ such that $G_\varepsilon = \phi(\Omega_\varepsilon)$ is a sampling set for $\mathcal{B}^{\mu_1}$. In this case, we can find a constant $L > 0$ such that

$$\|f\|_{\mu_1} \leq L \sup_{z \in G_\varepsilon} \widetilde{\mu}_1(z)|f'(z)|$$

for all functions $f \in \mathcal{B}^{\mu_1}$. Hence, we have that

$$\begin{aligned} \|f\|_{\mu_1} &\leq L \sup_{z \in \Omega_\varepsilon} \widetilde{\mu}_1(\phi(z))|f'(\phi(z))| \\ &= L \sup_{z \in \Omega_\varepsilon} \frac{\widetilde{\mu}_1(\phi(z))}{\mu_2(z)|\phi'(z)|} \mu_2(z)|(f \circ \phi)'(z)| \\ &\leq \frac{L}{\varepsilon} \|f \circ \phi\|_{\mu_2}\,, \end{aligned}$$

and since

$$|f(0)| \leq |f(\phi(0))| + K_{\mu_1,\phi}\|f\|_{\mu_1},$$

where $K_{\mu_1,\phi}$ is the constant in (3.2), we conclude that

$$|f(0)| + \|f\|_{\mu_1} \leq |f(\phi(0))| + (1 + K_{\mu_1,\phi})\,\frac{L}{\varepsilon}\,\|f \circ \phi\,\|_{\mu_2}$$

and the operator $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is bounded below.

To prove the converse, suppose that $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is bounded below. For any non constant function $g \in \mathcal{B}^{\mu_1}$, we set

$$f(z) = \frac{1}{\|g\|_{\mu_1}} \big(g(z) - g(\phi(0))\big)$$

and we have that $f(\phi(0)) = 0$ and $\|f\|_{\mu_1} = 1$. Hence, by hypothesis, there exists a constant $K > 0$ (not depending on g), such that $\|C_\phi(f)\|_{\mathcal{B}^{\mu_2}} \geq K\|f\|_{\mathcal{B}^{\mu_1}}$; this last, implies that

$$\|C_\phi(f)\|_{\mu_2} = \sup_{z \in \mathbb{D}} \mu_2(z)|f'(\phi(z))||\phi'(z)| \geq K.$$

Thus, by definition of supremum, we can find $z_f \in \mathbb{D}$, such that

$$\mu_2(z_f)|f'(\phi(z_f))||\phi'(z_f)| \geq \frac{K}{2},$$

which, in turn, implies that

$$\frac{\mu_2(z_f)}{\widetilde{\mu}_1(\phi(z_f))}\,|\phi'(z_f)|\,\widetilde{\mu}_1(\phi(z_f))\,|f'(\phi(z_f))| \geq \frac{K}{2}. \tag{4.2}$$

Thus, since $\widetilde{\mu}_1(\phi(z_f))|f'(\varphi(z_f))| \leq 1$, it must be

$$\frac{\mu_2(z_f)}{\widetilde{\mu}_1(\phi(z_f))}|\phi'(z_f)| \geq \frac{K}{2}.$$

Therefore, putting $\varepsilon := \frac{K}{2}$, we have $z_f \in \Omega_\varepsilon$.

Now, since $C_\phi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous, Theorem 3.1 implies that there is a constant $M > 0$, such that

$$\frac{\mu_2(z_f)}{\widetilde{\mu}_1(\phi(z_f))}\,|\phi'(z_f)| \leq M.$$

From (4.2) we conclude that

$$\widetilde{\mu}_1(\phi(z_f))|f'(\phi(z_f))| \geq \frac{K}{2M}.$$

Finally, since $\phi(z_f) \in G_\varepsilon$, it must be

$$\sup_{z\in G_\varepsilon} \widetilde{\mu}_1(z)|f'(z)| \geq \frac{K}{2M}.$$

That is,

$$\sup_{z\in G_\varepsilon} \widetilde{\mu}_1(z)\frac{|g'(z)|}{\|g\|_{\mu_1}} \geq \frac{K}{2M}$$

and therefore G_ε is a sampling set for $\mathcal{B}^{\mu_1}$. The proof of the theorem is complete. ∎

Example 4.4. If $\mu_1(z) = \mu_2(z) = 1 - |z|^2$, we have the result of Ghatage, Zheng and Zorboska in [10]. If $\mu_1(z) = (1 - |z|)^\alpha$ and $\mu_2(z) = (1 - |z|)^\beta$ with $\alpha, \beta > 0$, we obtain the result of Chen and Gauthier in [6].

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