EXTRACTA MATHEMATICAE Vol. 26, Núm. 1, 131 – 143 (2011)

The TKK Construction of a Cheng-Kac Jordan Superalgebra of Characteristic 3

Elisabete Barreiro *

CMUC, Department of Mathematics, University of Coimbra,
3001-454 *Coimbra, Portugal,* *mefb@mat.uc.pt*

Presented by Consuelo Martínez   Received June 30, 2010

Abstract: The classification of simple Lie superalgebras in prime characteristic is far from being known. In this paper a simple Lie superalgebra in characteristic 3, obtained as a Cheng-Kac superalgebra, is studied and its dimension is found.

Key words: Jordan superalgebras, Lie superalgebras, Cheng-Kac superalgebras, Tits-Kantor-Koecher construction.

AMS *Subject Class.* (2010): 17B25, 17B50, 17C70.

Introduction

Superalgebras appeared for the first time in a context of algebraic topology and homological algebra. However they endured a bigger development when they proved to be useful to capture the supersymmetry of elemental particles.

Simple associative superalgebras over algebraically closed fields of arbitrary characteristic were classified by C.T.C. Wall in 1963.

The classification of simple Lie superalgebras over an algebraically closed field of zero characteristic is due to V. Kac in 1977 [6]. Using this result and the Tits-Kantor-Koecher construction the same author got also, with some additions due to I. Kantor [9], the classification of finite dimensional simple Jordan superalgebras over an algebraically closed field of zero characteristic [7].

In the case of prime characteristic, the situation in Jordan superalgebras differs from the one in Lie superalgebras. While the classification of Jordan superalgebras has been completed, very little is known for Lie superalgebras. Some such superalgebras have been constructed by A. Elduque [2, 3].

The Jordan case has been studied by M. Racine and E. Zelmanov [12] (in case that the even part is assumed to be semisimple) and by C. Martinez and

*The author is supported by CMUC/FCT.

E. Zelmanov [11] for unital simple Jordan superalgebras with a non-semisimple even part. In the first case, the obtained superalgebras correspond to the ones in zero characteristic, with some new examples in characteristic 3. However in the second case, the obtained results, and the underlying ideas, follow the lines of a previous paper about some infinite dimensional Jordan superalgebras, the so called superconformal algebras (see [8]). Cheng-Kac superalgebras play an essential role in this classification.

As in any algebraic structure theory, to gain a better understanding of the Cheng-Kac superalgebras we shall start by easy structures. So in this paper a simple Cheng-Kac superalgebra in characteristic 3 is considered and a basis is found. Its even and odd part turn out to be 48-dimensional.

1. Preliminaries

Throughout this section, we consider superalgebras over an arbitrary field $\mathbb{K}$. We begin by collecting some notions needed in the sequel.

Definition 1.1. 1. A *superalgebra* $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra over a field $\mathbb{K}$ such that $A_i \cdot A_j \subseteq A_{i+j}$, $\forall i, j \in \{\bar{0}, \bar{1}\}$. Elements in $A_{\bar{0}} \cup A_{\bar{1}}$ are called homogeneous elements of the superalgebra, even elements if they belong to $A_{\bar{0}}$ and odd if they belong to $A_{\bar{1}}$. Given an element $a \in A_{\bar{0}} \cup A_{\bar{1}}$, $|a|$ will denote its parity (0 or 1).

2. If $\mathcal{V}$ is a variety of algebras defined by homogeneous identities (see [5] and [13]), a superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is a $\mathcal{V}$- superalgebra if its *Grassmann enveloping algebra* $G(A) = A_{\bar{0}} \otimes G(V)_{\bar{0}} + A_{\bar{1}} \otimes G(V)_{\bar{1}}$ lies in $\mathcal{V}$.

Example 1.2. 1. A superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a *Lie superalgebra* if the multiplication satisfies the following two properties: $\forall x, y, z \in L_{\bar{0}} \cup L_{\bar{1}}$

(i) *super skew-symmetry*:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

(ii) *super Jacobi identity*:

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] \quad = \quad 0.$$

2. A superalgebra $J = J_{\bar{0}} \oplus J_{\bar{1}}$ is a *Jordan superalgebra* if the multiplication satisfies the properties: $\forall x, y, z \in J_{\bar{0}} \cup J_{\bar{1}}$

(i) *super symmetry*:

$$xy = (-1)^{|x||y|} yx,$$

(ii) *super Jordan identity*:

$$((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x \\ = (xy)(zt) + (-1)^{|y||z|}(xz)(yt) + (-1)^{|t|(|y|+|z|)}(xt)(yz).$$

The *Jordan triple product* is defined by

$$\{x, y, z\} = (xy)z + x(yz) - (-1)^{|x||y|} y(xz), \quad \forall x, y, z \in J_{\bar{0}} \cup J_{\bar{1}}.$$

We recall the fruitful *Kantor-Koecher-Tits construction* relating Lie and Jordan (super)algebras.

DEFINITION 1.3. ([10]) A Jordan (super)pair $P = (P^-, P^+)$ is a pair of vector (super)spaces with a pair of trilinear operations

$$\{\ ,\ ,\ \} : P^- \times P^+ \times P^- \to P^-, \qquad \{\ ,\ ,\ \} : P^+ \times P^- \times P^+ \to P^+$$

that satisfies the following identities:

$$\begin{aligned} &(\text{P.1})\ \{x^\sigma, y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, z^{-\sigma}\}, x^\sigma\}, \\ &(\text{P.2})\ \{\{x^\sigma, y^{-\sigma}, x^\sigma\}, y^{-\sigma}, u^\sigma\} = \{x^\sigma, \{y^{-\sigma}, x^\sigma, y^{-\sigma}\}, u^\sigma\}, \\ &(\text{P.3})\ \{\{x^\sigma, y^{-\sigma}, x^\sigma\}, z^{-\sigma}, \{x^\sigma, y^{-\sigma}, x^\sigma\}\} \\ &\qquad = \{x^\sigma, \{y^{-\sigma}, \{x^\sigma, z^{-\sigma}, x^\sigma\}, y^{-\sigma}\}, x^\sigma\}, \end{aligned}$$

for every $x^\sigma, u^\sigma \in P^\sigma$, $y^{-\sigma}, z^{-\sigma} \in P^{-\sigma}$, $\sigma = \pm$.

Let $L = L_{-1} + L_0 + L_1$ be a $\mathbb{Z}$-graded Lie (super)algebra. Then (L_{-1}, L_1) is a Jordan (super)pair with respect to the trilinear operations

$$\{x^\sigma, y^{-\sigma}, z^\sigma\} = [[x^\sigma, y^{-\sigma}], z^\sigma]; \quad x^\sigma, z^\sigma \in L_{\sigma 1}, y^{-\sigma} \in L_{-\sigma 1}, \sigma = \pm.$$

For an arbitrary Jordan (super)pair $P = (P^-, P^+)$, there exists a unique Z-graded Lie (super)algebra $K = K_{-1} + K_0 + K_1$ such that $(K_{-1}, K_1) \simeq P$, $K_0 = [K_{-1}, K_1]$ and for every 3-graded Lie (super)algebra $L = L_{-1} + L_0 + L_1$, an arbitrary homomorphism of the Jordan pairs $P \to (L_{-1}, L_1)$ uniquely extends to a homomorphism of Lie (super)algebras $K \to L$. We will refer to

$K = K(P)$ as the Tits-Kantor-Koecher (in short TKK) construction of the pair P.

If J is a Jordan superalgebra, let us consider J^- and J^+ two copies of J. Then (J^-, J^+) is a Jordan superpair with the trilinear operations defined, in a natural way, via the triple product of J. That is

$$\{x^\sigma, y^{-\sigma}, z^\sigma\} = \{x, y, z\}^\sigma,\ \sigma = \pm.$$

The Lie superalgebra $K = K(J^-, J^+)$ is called the TKK-construction of J. Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a simple finite-dimensional Jordan superalgebra. Let us consider $L = K(J)$ its TKK-construction.

The Cheng-Kac Jordan superalgebras appeared in the classification of simple finite dimensional Jordan superalgebras in prime characteristic and with non-semisimple even part [11], which is inspired by a classification of some infinite dimensional Jordan superalgebras over arbitrary fields of characteristic zero that come associated to superconformal algebras. Namely, Cheng-Kac Jordan superalgebras $JCK(\mathbb{K}[t^{-1}, t], d/dt)$ which are connected via the Tits-Kantor-Koecher construction with exceptional Cheng-Kac superconformal algebras CK_6 discovered by S-J. Cheng and V. Kac (see [1], [4] and [8]). This superalgebra was also, simultaneously and independently, constructed by Grozman *et al* [4].

Let us recall how the Cheng-Kac Jordan superalgebras are defined. Consider an associative commutative algebra Z with unity and a derivation $d : Z \longrightarrow Z$ in Z. The *Cheng-Kac Jordan superalgebra* is the Jordan superalgebra $JCK(Z, d) = J = J_{\bar{0}} + J_{\bar{1}}$, whose even and odd parts are free Z-modules of rank 4 defined, respectively, by $J_{\bar{0}} = Z + \sum_{i=1}^{3} Zw_i$ and $J_{\bar{1}} = Zx + \sum_{i=1}^{3} Zx_i$, and the multiplication obeys to the following rules:

1. Multiplication on $J_{\bar{0}}$:
   $w_1^2 = w_2^2 = -w_3^2 = 1$ and $w_i \cdot w_j = 0$, $\forall i, j \in \{1, 2, 3\}$ $(i \neq j)$.
2. Module action of $J_{\bar{0}}$ over $J_{\bar{1}}$:

$$\begin{cases} xf \cdot g = x(fg) \\ xf \cdot w_j g = x_j(fd(g)) \\ x_i f \cdot g = x_i(fg) \\ x_i f \cdot w_j g = x_{i\times j}(fg), \qquad \forall f, g \in Z, \forall i, j \in \{1, 2, 3\}, \end{cases}$$

   where we consider the notation $x_{1\times 2} = -x_{2\times 1} = x_3$, $x_{1\times 3} = -x_{3\times 1} = x_2$, $-x_{2\times 3} = x_{3\times 2} = x_1$ and $x_{i\times i} = 0$, $\forall i \in \{1, 2, 3\}$.

3. Multiplication on $J_{\bar{1}}$:

$$\begin{cases} xf \cdot xg = d(f)g - fd(g) \\ xf \cdot x_jg = -w_j(fg) \\ x_if \cdot xg = w_i(fg) \\ x_if \cdot x_jg = 0, \qquad \forall f,g \in Z, \forall i,j \in \{1,2,3\}. \end{cases}$$

We have the following result concerning simple Cheng-Kac Jordan superalgebras.

PROPOSITION 1.4. *The Cheng-Kac Jordan superalgebra $JCK(Z,d)$ is simple if and only if Z does not contain proper d-invariant ideals.*

2. TKK construction of a Cheng-Kac Jordan superalgebra

In particular, we are interested in the Cheng-Kac Jordan superalgebra $JCK(Z,d)$ of characteristic 3, with $Z = \mathbb{Z}_3[t]/t^3$ and $d : Z \longrightarrow Z$ defined by $d(t) = 1$. The aim of this section is to present the Tits-Kantor-Koecher construction of the Cheng-Kac Jordan superalgebra $JCK(\mathbb{Z}_3[t]/t^3, d)$, namely a exceptional superconformal algebra CK_6. The Jordan superalgebra $J = JCK(\mathbb{Z}_3[t]/t^3, d)$ has dimension 24 over $\mathbb{Z}_3$, where $\dim_{\mathbb{Z}_3} J_{\bar{0}}$ and $\dim_{\mathbb{Z}_3} J_{\bar{1}}$ are both 12. Being a basis of the even part constituted by

$$J_{\bar{0}} = Z(1,w_1,w_2,w_3) = \mathbb{Z}_3(1,w_1,w_2,w_3,t,tw_1,tw_2,tw_3,t^2,t^2w_1,t^2w_2,t^2w_3),$$

and a basis of odd part formed by

$$J_{\bar{1}} = Z(x,x_1,x_2,x_3) = \mathbb{Z}_3(x,x_1,x_2,x_3,tx,tx_1,tx_2,tx_3,t^2x,t^2x_1,t^2x_2,t^2x_3).$$

The multiplication in the Cheng-Kac Jordan superalgebra J appears in a straightforward way, following the rules mentioned above. We display the multiplication in the tables listed below.

Even part $J_{\bar{0}}$:

	1	w_1	w_2	w_3	t	tw_1	tw_2	tw_3	t^2	t^2w_1	t^2w_2	t^2w_3
1	1	w_1	w_2	w_3	t	tw_1	tw_2	tw_3	t^2	t^2w_1	t^2w_2	t^2w_3
w_1	w_1	1	0	0	tw_1	t	0	0	t^2w_1	t^2	0	0
w_2	w_2	0	1	0	tw_2	0	t	0	t^2w_2	0	t^2	0
w_3	w_3	0	0	-1	tw_3	0	0	$-t$	t^2w_3	0	0	$-t^2$
t	t	tw_1	tw_2	tw_3	t^2	t^2w_1	t^2w_2	t^2w_3	0	0	0	0
tw_1	tw_1	t	0	0	t^2w_1	t^2	0	0	0	0	0	0
tw_2	tw_2	0	t	0	t^2w_2	0	t^2	0	0	0	0	0
tw_3	tw_3	0	0	$-t$	t^2w_3	0	0	$-t^2$	0	0	0	0
t^2	t^2	t^2w_1	t^2w_2	t^2w_3	0	0	0	0	0	0	0	0
t^2w_1	t^2w_1	t^2	0	0	0	0	0	0	0	0	0	0
t^2w_2	t^2w_2	0	t^2	0	0	0	0	0	0	0	0	0
t^2w_3	t^2w_3	0	0	$-t^2$	0	0	0	0	0	0	0	0

Module action of $J_{\bar{0}}$ over $J_{\bar{1}}$:

	1	w_1	w_2	w_3	t	tw_1	tw_2	tw_3	t^2	t^2w_1	t^2w_2	t^2w_3
x	x	0	0	0	tx	x_1	x_2	x_3	t^2x	$-tx_1$	$-tx_2$	$-tx_3$
x_1	x_1	0	x_3	x_2	tx_1	0	tx_3	tx_2	t^2x_1	0	t^2x_3	t^2x_2
x_2	x_2	$-x_3$	0	$-x_1$	tx_2	$-tx_3$	0	$-tx_1$	t^2x_2	$-t^2x_3$	0	$-t^2x_1$
x_3	x_3	$-x_2$	x_1	0	tx_3	$-tx_2$	tx_1	0	t^2x_3	$-t^2x_2$	t^2x_1	0
tx	tx	0	0	0	t^2x	tx_1	tx_2	tx_3	0	$-t^2x_1$	$-t^2x_2$	$-t^2x_3$
tx_1	tx_1	0	tx_3	tx_2	t^2x_1	0	t^2x_3	t^2x_2	0	0	0	0
tx_2	tx_2	$-tx_3$	0	$-tx_1$	t^2x_2	$-t^2x_3$	0	$-t^2x_1$	0	0	0	0
tx_3	tx_3	$-tx_2$	tx_1	0	t^2x_3	$-t^2x_2$	t^2x_1	0	0	0	0	0
t^2x	t^2x	0	0	0	0	t^2x_1	t^2x_2	t^2x_3	0	0	0	0
t^2x_1	t^2x_1	0	t^2x_3	t^2x_2	0	0	0	0	0	0	0	0
t^2x_2	t^2x_2	$-t^2x_3$	0	$-t^2x_1$	0	0	0	0	0	0	0	0
t^2x_3	t^2x_3	$-t^2x_2$	t^2x_1	0	0	0	0	0	0	0	0	0

Odd part $J_{\bar{1}}$:

	x	x_1	x_2	x_3	tx	tx_1	tx_2	tx_3	t^2x	t^2x_1	t^2x_2	t^2x_3
x	0	$-w_1$	$-w_2$	$-w_3$	-1	$-tw_1$	$-tw_2$	$-tw_3$	t	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$
x_1	w_1	0	0	0	tw_1	0	0	0	t^2w_1	0	0	0
x_2	w_2	0	0	0	tw_2	0	0	0	t^2w_2	0	0	0
x_3	w_3	0	0	0	tw_3	0	0	0	t^2w_3	0	0	0
tx	1	$-tw_1$	$-tw_2$	$-tw_3$	0	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	$-t^2$	0	0	0
tx_1	tw_1	0	0	0	t^2w_1	0	0	0	0	0	0	0
tx_2	tw_2	0	0	0	t^2w_2	0	0	0	0	0	0	0
tx_3	tw_3	0	0	0	t^2w_3	0	0	0	0	0	0	0
t^2x	$-t$	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	t^2	0	0	0	0	0	0	0
t^2x_1	t^2w_1	0	0	0	0	0	0	0	0	0	0	0
t^2x_2	t^2w_2	0	0	0	0	0	0	0	0	0	0	0
t^2x_3	t^2w_3	0	0	0	0	0	0	0	0	0	0	0

In the following we pay some attention to the Kantor-Koecher-Tits construction of J, providing the ultimate Lie superalgebra with a explicitly basis. Take the Lie superalgebra $K(J) = J^- + [J^-, J^+] + J^+$ from the Kantor-Koecher-Tits construction, where J^- and J^+ are just two copies of J, being the multiplication inside J^- and J^+ zero. While, $[J^-, J^+]$ is formally the linear span of elements $[X^-, Y^+]$ (with $X^- \in J^-$ and $Y^+ \in J^+$) that act as transformations of J^- and J^+, via the Jordan triple product of J. We must be careful in the way we determine the elements in $[J^-, J^+]$, since we are working in superalgebras. So we have to apply the used rules, namely, if x, y are odd elements of J and a, b are even elements of J, we have $[x^-, y^+] = [y^+, x^-]$, $[a^-, b^+] = -[b^+, a^-]$, $[a^-, x^+] = -[x^+, a^-]$. Then, for instance, if $z \in J$ we get

$$
\begin{aligned}
&[x^-, y^+](z^+) = [y^+, x^-](z^+) = \{y, x, z\}^+ \\
&[a^-, b^+](z^+) = -[b^+, a^-](z^+) = -\{b, a, z\}^+ \\
&[a^-, x^+](z^+) = -[x^+, a^-](z^+) = -\{x, a, z\}^+.
\end{aligned}
$$

In our case, $J = JCK(\mathbb{Z}_3[t]/t^3, d)$, $p = 3$. The even and odd components of the Lie superalgebra $K(J)$ are performed in the following way:

$$
\begin{aligned}
K(J)_{\bar{0}} &= \underbrace{J^-_{\bar{0}}}_{\dim 12} + \Big([J^-_{\bar{0}}, J^+_{\bar{0}}] + [J^-_{\bar{1}}, J^+_{\bar{1}}]\Big) + \underbrace{J^+_{\bar{0}}}_{\dim 12} \\
K(J)_{\bar{1}} &= \underbrace{J^-_{\bar{1}}}_{\dim 12} + \Big([J^-_{\bar{0}}, J^+_{\bar{1}}] + [J^-_{\bar{1}}, J^+_{\bar{0}}]\Big) + \underbrace{J^+_{\bar{1}}}_{\dim 12}
\end{aligned}
$$

So in order to find the dimension of the even part of $K(J)$ we need to find a basis in $[J^-_{\bar{0}}, J^+_{\bar{0}}] + [J^-_{\bar{1}}, J^+_{\bar{1}}]$. Similarly, to find the dimension of the odd part $K(J)_{\bar{1}}$ we need to study $[J^-_{\bar{0}}, J^+_{\bar{1}}] + [J^-_{\bar{1}}, J^+_{\bar{0}}]$. We have used a Mathematica software program to help. The program determines all elements $[X^-, Y^+]$ (considered as linear transformations), where X, Y run over a basis of J. These results are presented in Tables 1 to 4.

Concerning to the even part $[J^-_{\bar{0}}, J^+_{\bar{0}}] + [J^-_{\bar{1}}, J^+_{\bar{1}}]$, we obtain a basis with 21 elements inside $[J^-_{\bar{0}}, J^+_{\bar{0}}]$:

$[1^-, 1^+]$	$[1^-, w_1^+]$	$[1^-, w_2^+]$	$[1^-, w_3^+]$	$[1^-, t^+]$	$[1^-, tw_1^+]$
$[1^-, tw_2^+]$	$[1^-, tw_3^+]$	$[1^-, t^{2+}]$	$[1^-, t^2w_1^+]$	$[1^-, t^2w_2^+]$	$[1^-, t^2w_3^+]$
$[w_1^-, w_2^+]$	$[w_1^-, w_3^+]$	$[w_1^-, tw_2^+]$	$[w_1^-, tw_3^+]$	$[w_1^-, t^2{w_2}^+]$	$[w_1^-, t^2w_3^+]$
$[w_2^-, w_3^+]$	$[w_2^-, tw_3^+]$	$[w_2^-, t^2w_3^+]$			

and 3 elements in $[J^-_{\bar{1}}, J^+_{\bar{1}}]$:

$$[x^-, x^+] \quad [x^-, tx^+] \quad [x^-, t^2x^+]$$

We present explicitly the elements of this basis in Tables 1 and 2.

With respect to the odd part we conclude that $[J^-_{\bar 0}, J^+_{\bar 1}] + [J^-_{\bar 1}, J^+_{\bar 0}]$ has a basis, with all 24 elements in $[J^-_{\bar 0}, J^+_{\bar 1}]$:

$[1^-, x^+]$	$[1^-, x_1^+]$	$[1^-, x_2^+]$	$[1^-, x_3^+]$	$[1^-, tx^+]$	$[1^-, tx_1^+]$
$[1^-, tx_2^+]$	$[1^-, tx_3^+]$	$[1^-, t^2x^+]$	$[1^-, t^2x_1^+]$	$[1^-, t^2x_2^+]$	$[1^-, t^2x_3^+]$
$[w_1^-, x^+]$	$[w_1^-, x_1^+]$	$[w_1^-, tx^+]$	$[w_1^-, tx_1^+]$	$[w_1^-, t^2x^+]$	$[w_1^-, t^2x_1^+]$
$[w_2^-, x^+]$	$[w_2^-, tx^+]$	$[w_2^-, t^2x^+]$	$[w_3^-, x^+]$	$[w_3^-, tx^+]$	$[w_3^-, t^2x^+]$

A detailed description of these elements are in Tables 3 and 4.

Remark. It is clear that $J^-_{\bar 0} + [J^-_{\bar 0}, J^+_{\bar 0}] + J^+_{\bar 0}$ is a proper 45-dimensional ideal of the even part of $K(J)$. So $K(J)_{\bar 0}$ is not a simple Lie algebra.

In order to simplify the tables, we establish the following abbreviation. We note that some elements $[a^-, b^+]$ of these bases satisfy one of the following two conditions: for arbitrary element c in the basis of $J_{\bar 0}$ and basis of $J_{\bar 1}$ there exists a suitable d in the basis of $J_{\bar 0}$ and basis of $J_{\bar 1}$ such that

A. $[a^-, b^+](c^+) = d^+$ and $[a^-, b^+](c^-) = d^-$;

B. $[a^-, b^+](c^+) = d^+$ and $[a^-, b^+](c^-) = -d^-$.

So in the tables, we write "equal" if $[a^-, b^+]$ satisfies A and "sym" if $[a^-, b^+]$ verifies condition B.

Table 1: Elements of the basis of $[J_{\bar 0}^-, J_{\bar 0}^+] + [J_{\bar 1}^-, J_{\bar 1}^+]$

$[1^-, 1^+]$ (sym)	$[1^-, w_1^+]$ (sym)	$[1^-, w_2^+]$ (sym)	$[1^-, w_3^+]$ (sym)	$[1^-, t^+]$ (sym)	$[1^-, tw_1^+]$ (sym)	$[1^-, tw_2^+]$ (sym)	$[1^-, tw_3^+]$ (sym)	$[1^-, t^{2+}]$ (sym)	$[1^-, t^2w_1^+]$ (sym)	$[1^-, t^2w_2^+]$ (sym)	$[1^-, t^2w_3^+]$ (sym)	$[w_1^-, w_2^+]$ (equal)
-1	$-w_1$	$-w_2$	$-w_3$	$-t$	$-tw_1$	$-tw_2$	$-tw_3$	$-t^2$	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	0
$-w_1$	-1	0	0	$-tw_1$	$-t$	0	0	$-t^2w_1$	$-t^2$	0	0	$-w_2$
$-w_2$	0	-1	0	$-tw_2$	0	$-t$	0	$-t^2w_2$	0	$-t^2$	0	w_1
$-w_3$	0	0	1	$-tw_3$	0	0	t	$-t^2w_3$	0	0	t^2	0
$-t$	$-tw_1$	$-tw_2$	$-tw_3$	$-t^2$	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	0	0	0	0	0
$-tw_1$	$-t$	0	0	$-t^2w_1$	$-t^2$	0	0	0	0	0	0	$-tw_2$
$-tw_2$	0	$-t$	0	$-t^2w_2$	0	$-t^2$	0	0	0	0	0	tw_1
$-tw_3$	0	0	t	$-t^2w_3$	0	0	t^2	0	0	0	0	0
$-t^2$	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	0	0	0	0	0	0	0	0	0
$-t^2w_1$	$-t^2$	0	0	0	0	0	0	0	0	0	0	$-t^2w_2$
$-t^2w_2$	0	$-t^2$	0	0	0	0	0	0	0	0	0	t^2w_1
$-t^2w_3$	0	0	t^2	0	0	0	0	0	0	0	0	0
$-x$	0	0	0	$-tx$	$-x_1$	$-x_2$	$-x_3$	$-t^2x$	tx_1	tx_2	tx_3	0
$-x_1$	0	$-x_3$	$-x_2$	$-tx_1$	0	$-tx_3$	$-tx_2$	$-t^2x_1$	0	$-t^2x_3$	$-t^2x_2$	$-x_2$
$-x_2$	x_3	0	x_1	$-tx_2$	tx_3	0	tx_1	$-t^2x_2$	t^2x_3	0	t^2x_1	x_1
$-x_3$	x_2	$-x_1$	0	$-tx_3$	tx_2	$-tx_1$	0	$-t^2x_3$	t^2x_2	$-t^2x_1$	0	0
$-tx$	0	0	0	$-t^2x$	$-tx_1$	$-tx_2$	$-tx_3$	0	t^2x_1	t^2x_2	t^2x_3	0
$-tx_1$	0	$-tx_3$	$-tx_2$	$-t^2x_1$	0	$-t^2x_3$	$-t^2x_2$	0	0	0	0	$-tx_2$
$-tx_2$	tx_3	0	tx_1	$-t^2x_2$	t^2x_3	0	t^2x_1	0	0	0	0	tx_1
$-tx_3$	tx_2	$-tx_1$	0	$-t^2x_3$	t^2x_2	$-t^2x_1$	0	0	0	0	0	0
$-t^2x$	0	0	0	0	$-t^2x_1$	$-t^2x_2$	$-t^2x_3$	0	0	0	0	0
$-t^2x_1$	0	$-t^2x_3$	$-t^2x_2$	0	0	0	0	0	0	0	0	$-t^2x_2$
$-t^2x_2$	t^2x_3	0	t^2x_1	0	0	0	0	0	0	0	0	t^2x_1
$-t^2x_3$	t^2x_2	$-t^2x_1$	0	0	0	0	0	0	0	0	0	0

Table 2: Elements of the basis of $[J_{\bar{0}}^{-}, J_{\bar{0}}^{+}] + [J_{\bar{1}}^{-}, J_{\bar{1}}^{+}]$

$[w_1^-, w_3^+]$ (equal)	$[w_1^-, tw_2^+]$ (equal)	$[w_1^-, tw_3^+]$ (equal)	$[w_1^-, t^2 {w_2}^+]$ (equal)	$[w_1^-, t^2 w_3^+]$ (equal)	$[w_2^-, w_3^+]$ (equal)	$[w_2^-, tw_3^+]$ (equal)	$[w_2^-, t^2 w_3^+]$ (equal)	$[x^-, x^+]$ (equal)	$[x^-, tx^+]$ $J^+ \to J^+$	$[x^-, tx^+]$ $J^- \to J^-$	$[x^-, t^2x^+]$ $J^+ \to J^+$	$[x^-, t^2x^+]$ $J^- \to J^-$
0	0	0	0	0	0	0	0	0	1	-1	$-t$	t
$-w_3$	$-tw_2$	$-tw_3$	$-t^2w_2$	$-t^2w_3$	0	0	0	0	w_1	$-w_1$	$-tw_1$	tw_1
0	tw_1	0	t^2w_1	0	$-w_3$	$-tw_3$	$-t^2w_3$	0	w_2	$-w_2$	$-tw_2$	tw_2
$-w_1$	0	$-tw_1$	0	$-t^2w_1$	$-w_2$	$-tw_2$	$-t^2w_2$	0	w_3	$-w_3$	$-tw_3$	tw_3
0	0	0	0	0	0	0	0	1	$-t$	0	0	$-t^2$
$-tw_3$	$-t^2w_2$	$-t^2w_3$	0	0	0	0	0	w_1	$-tw_1$	0	0	$-t^2w_1$
0	t^2w_1	0	0	0	$-tw_3$	$-t^2w_3$	0	w_2	$-tw_2$	0	0	$-t^2w_2$
$-tw_1$	0	$-t^2w_1$	0	0	$-tw_2$	$-t^2w_2$	0	w_3	$-tw_3$	0	0	$-t^2w_3$
0	0	0	0	0	0	0	0	$-t$	0	t^2	0	0
$-t^2w_3$	0	0	0	0	0	0	0	$-tw_1$	0	t^2w_1	0	0
0	0	0	0	0	$-t^2w_3$	0	0	$-tw_2$	0	t^2w_2	0	0
$-t^2w_1$	0	0	0	0	$-t^2w_2$	0	0	$-tw_3$	0	t^2w_3	0	0
0	$-x_3$	$-x_2$	tx_3	tx_2	0	x_1	$-tx_1$	0	$-x$	0	tx	0
$-x_3$	$-tx_2$	$-tx_3$	$-t^2x_2$	$-t^2x_3$	0	0	0	0	0	x_1	0	$-tx_1$
0	tx_1	0	t^2x_1	0	$-x_3$	$-tx_3$	$-t^2x_3$	0	0	x_2	0	$-tx_2$
$-x_1$	0	$-tx_1$	0	$-t^2x_1$	$-x_2$	$-tx_2$	$-t^2x_2$	0	0	x_3	0	$-tx_3$
0	$-tx_3$	$-tx_2$	t^2x_3	t^2x_2	0	tx_1	$-t^2x_1$	x	0	tx	$-t^2x$	t^2x
$-tx_3$	$-t^2x_2$	$-t^2x_3$	0	0	0	0	0	x_1	tx_1	$-tx_1$	t^2x_1	0
0	t^2x_1	0	0	0	$-tx_3$	$-t^2x_3$	0	x_2	tx_2	$-tx_2$	t^2x_2	0
$-tx_1$	0	$-t^2x_1$	0	0	$-tx_2$	$-t^2x_2$	0	x_3	tx_3	$-tx_3$	t^2x_3	0
0	$-t^2x_3$	$-t^2x_2$	0	0	0	t^2x_1	0	$-tx$	t^2x	$-t^2x$	0	0
$-t^2x_3$	0	0	0	0	0	0	0	$-tx_1$	$-t^2x_1$	0	0	0
0	0	0	0	0	$-t^2x_3$	0	0	$-tx_2$	$-t^2x_2$	0	0	0
$-t^2x_1$	0	0	0	0	$-t^2x_2$	0	0	$-tx_3$	$-t^2x_3$	0	0	0

Table 3: Elements of the basis of $[J_{\bar{0}}^-, J_{\bar{1}}^+] + [J_{\bar{1}}^-, J_{\bar{0}}^+]$

$[1^-, x^+]$ (sym)	$[1^-, x_1^+]$ (sym)	$[1^-, x_2^+]$ (sym)	$[1^-, x_3^+]$ (sym)	$[1^-, tx^+]$ (sym)	$[1^-, tx_1^+]$ (sym)	$[1^-, tx_2^+]$ (sym)	$[1^-, tx_3^+]$ (sym)	$[1^-, t^2x^+]$ (sym)	$[1^-, t^2x_1^+]$ (sym)	$[1^-, t^2x_2^+]$ (sym)	$[1^-, t^2x_3^+]$ (sym)
$-x$	$-x_1$	$-x_2$	$-x_3$	$-tx$	$-tx_1$	$-tx_2$	$-tx_3$	$-t^2x$	$-t^2x_1$	$-t^2x_2$	$-t^2x_3$
0	0	x_3	x_2	0	0	tx_3	tx_2	0	0	t^2x_3	t^2x_2
0	$-x_3$	0	$-x_1$	0	$-tx_3$	0	$-tx_1$	0	$-t^2x_3$	0	$-t^2x_1$
0	$-x_2$	x_1	0	0	$-tx_2$	tx_1	0	0	$-t^2x_2$	t^2x_1	0
$-tx$	$-tx_1$	$-tx_2$	$-tx_3$	$-t^2x$	$-t^2x_1$	$-t^2x_2$	$-t^2x_3$	0	0	0	0
$-x_1$	0	tx_3	tx_2	$-tx_1$	0	t^2x_3	t^2x_2	$-t^2x_1$	0	0	0
$-x_2$	$-tx_3$	0	$-tx_1$	$-tx_2$	$-t^2x_3$	0	$-t^2x_1$	$-t^2x_2$	0	0	0
$-x_3$	$-tx_2$	tx_1	0	$-tx_3$	$-t^2x_2$	t^2x_1	0	$-t^2x_3$	0	0	0
$-t^2x$	$-t^2x_1$	$-t^2x_2$	$-t^2x_3$	0	0	0	0	0	0	0	0
tx_1	0	t^2x_3	t^2x_2	t^2x_1	0	0	0	0	0	0	0
tx_2	$-t^2x_3$	0	$-t^2x_1$	t^2x_2	0	0	0	0	0	0	0
tx_3	$-t^2x_2$	t^2x_1	0	t^2x_3	0	0	0	0	0	0	0
0	$-w_1$	$-w_2$	$-w_3$	-1	$-tw_1$	$-tw_2$	$-tw_3$	t	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$
w_1	0	0	0	tw_1	0	0	0	t^2w_1	0	0	0
w_2	0	0	0	tw_2	0	0	0	t^2w_2	0	0	0
w_3	0	0	0	tw_3	0	0	0	t^2w_3	0	0	0
1	$-tw_1$	$-tw_2$	$-tw_3$	0	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	$-t^2$	0	0	0
tw_1	0	0	0	t^2w_1	0	0	0	0	0	0	0
tw_2	0	0	0	t^2w_2	0	0	0	0	0	0	0
tw_3	0	0	0	t^2w_3	0	0	0	0	0	0	0
$-t$	$-t^2w_1$	$-t^2w_2$	$-t^2w_3$	t^2	0	0	0	0	0	0	0
t^2w_1	0	0	0	0	0	0	0	0	0	0	0
t^2w_2	0	0	0	0	0	0	0	0	0	0	0
t^2w_3	0	0	0	0	0	0	0	0	0	0	0

Table 4: Elements of the basis of $[J_{\bar{0}}^-, J_{\bar{1}}^+] + [J_{\bar{1}}^-, J_{\bar{0}}^+]$

$[w_1^-, x^+]$ (equal)	$[w_1^-, x_1^+]$ (equal)	$[w_1^-, tx^+]$ (equal)	$[w_1^-, tx_1^+]$ (equal)	$[w_1^-, t^2x^+]$ (equal)	$[w_1^-, t^2x_1^+]$ (equal)	$[w_2^-, x^+]$ (equal)	$[w_2^-, tx^+]$ (equal)	$[w_2^-, t^2x^+]$ (equal)	$[w_3^-, x^+]$ (equal)	$[w_3^-, tx^+]$ (equal)	$[w_3^-, t^2x^+]$ (equal)
0	0	0	0	0	0	0	0	0	0	0	0
$-x$	$-x_1$	$-tx$	$-tx_1$	$-t^2x$	$-t^2x_1$	0	0	0	0	0	0
0	$-x_2$	0	$-tx_2$	0	$-t^2x_2$	$-x$	$-tx$	$-t^2x$	0	0	0
0	$-x_3$	0	$-tx_3$	0	$-t^2x_3$	0	0	0	x	tx	t^2x
$-x_1$	0	$-tx_1$	0	$-t^2x_1$	0	$-x_2$	$-tx_2$	$-t^2x_2$	$-x_3$	$-tx_3$	$-t^2x_3$
$-tx$	$-tx_1$	$-t^2x$	$-t^2x_1$	0	0	x_3	tx_3	t^2x_3	x_2	tx_2	t^2x_2
$-x_3$	$-tx_2$	$-tx_3$	$-t^2x_2$	$-t^2x_3$	0	$-tx$	$-t^2x$	0	$-x_1$	$-tx_1$	$-t^2x_1$
$-x_2$	$-tx_3$	$-tx_2$	$-t^2x_3$	$-t^2x_2$	0	x_1	tx_1	t^2x_1	tx	t^2x	0
tx_1	0	t^2x_1	0	0	0	tx_2	t^2x_2	0	tx_3	t^2x_3	0
$-t^2x$	$-t^2x_1$	0	0	0	0	$-tx_3$	$-t^2x_3$	0	$-tx_2$	$-t^2x_2$	0
tx_3	$-t^2x_2$	t^2x_3	0	0	0	$-t^2x$	0	0	tx_1	t^2x_1	0
tx_2	$-t^2x_3$	t^2x_2	0	0	0	$-tx_1$	$-t^2x_1$	0	t^2x	0	0
0	1	w_1	t	$-tw_1$	t^2	0	w_2	$-tw_2$	0	w_3	$-tw_3$
-1	0	$-t$	0	$-t^2$	0	w_3	tw_3	t^2w_3	w_2	tw_2	t^2w_2
$-w_3$	0	$-tw_3$	0	$-t^2w_3$	0	-1	$-t$	$-t^2$	$-w_1$	$-tw_1$	$-t^2w_1$
$-w_2$	0	$-tw_2$	0	$-t^2w_2$	0	w_1	tw_1	t^2w_1	1	t	t^2
$-w_1$	t	0	t^2	t^2w_1	0	$-w_2$	0	t^2w_2	$-w_3$	0	t^2w_3
$-t$	0	$-t^2$	0	0	0	tw_3	t^2w_3	0	tw_2	t^2w_2	0
$-tw_3$	0	$-t^2w_3$	0	0	0	$-t$	$-t^2$	0	$-tw_1$	$-t^2w_1$	0
$-tw_2$	0	$-t^2w_2$	0	0	0	tw_1	t^2w_1	0	t	t^2	0
tw_1	t^2	$-t^2w_1$	0	0	0	tw_2	$-t^2w_2$	0	tw_3	$-t^2w_3$	0
$-t^2$	0	0	0	0	0	t^2w_3	0	0	t^2w_2	0	0
$-t^2w_3$	0	0	0	0	0	$-t^2$	0	0	$-t^2w_1$	0	0
$-t^2w_2$	0	0	0	0	0	t^2w_1	0	0	t^2	0	0

References

[1] S-J. Cheng, V. Kac,A new $n = 6$ superconformal algebra, *Comm. Math. Phys.,* **186** (1997), 219 – 231.

[2] A. Elduque,The Tits construction and some simple Lie superalgebras in characteristic 3, *J. Lie Theory,* **18** (2008), 601 – 615.

[3] A. Elduque,Models of some simple modular Lie superalgebras, *Pacific J. Math.,***240** (2009), 49 – 83.

[4] P. Grozman, D. Leites, I. Shchepochkina,Lie superalgebras of string theories, *Acta Math. Vietnam,* **26** (2001), 27 – 63.

[5] N. Jacobson,"Structure and Representation of Jordan Algebras", American Mathematical Society Colloquium Publications, Vol. XXXIX American Mathematical Society, Providence, R.I., 1968.

[6] V.G. Kac,Lie superalgebras, *Adv. Math.,***26** (1977), 8 – 96.

[7] V.G. Kac,Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras, *Comm. Algebra,***5** (1977), 1375 – 1400.

[8] V.G. Kac, C. Martínez, E. Zelmanov, Graded simple Jordan superalgebras of growth one, *Mem. Amer. Math. Soc.,***150** (2001), 1 – 140.

[9] I.L. Kantor, Jordan and Lie superalgebras determined by a Poisson algebra, in "Algebra and analysis", 1989, 55-80. Amer. Math. Soc. Transl. Ser. 2, 151, Amer. Math. Soc., Providence, RI, 1992.

[10] O. Loos, "Jordan Pairs", Lectures Notes in Mathematics, vol. 460, Springer-Verlag, Berlin-New York, 1975.

[11] C. Martínez, E. Zelmanov, Simple finite-dimensional Jordan superalgebras in prime characteristic, *J. Algebra,***236** (2001), 575 – 629.

[12] M. Racine, E. Zelmanov,Simple Jordan superalgebras with semisimple even part, *J. Algebra,***270** (2003), 374 – 444.

[13] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, "Rings that are Nearly Associative", Academic Press, New York, 1982.