EXTRACTA MATHEMATICAE Vol. **26**, Núm. 2, 271 – 294 (2011)

Some Topological Invariants and Biorthogonal Systems in Banach Spaces

Piotr Koszmider *

Institute of Mathematics, Polish Academy of Sciences
ul. Śniadeckich 8, 00-956 *Warszawa, Poland*
P.Koszmider@impan.pl

Presented by Jesús M.F. Castillo Received September 27, 2011

Abstract: We consider topological invariants on compact spaces related to the sizes of discrete subspaces (spread), densities of subspaces, Lindelöf degree of subspaces, irredundant families of clopen sets and others and look at the following associations between compact topological spaces and Banach spaces: a compact K induces a Banach space $C(K)$ of real valued continuous functions on K with the supremum norm; a Banach space X induces a compact space B_{X^*}, the dual ball with the weak* topology. We inquire on how topological invariants on K and B_{X^*} are linked to the sizes of biorthogonal systems and their versions in $C(K)$ and X respectively. We gather folkloric facts and survey recent results like that of Abad-Lopez and Todorcevic that it is consistent that there is a Banach space X without uncountable biorthogonal systems such that the spread of B_{X^*} is uncountable or that of Brech and Koszmider that it is consistent that there is a compact space where spread of K^2 is countable but $C(K)$ has uncountable biorthogonal systems.

Key words: cardinal invariants, biorthogonal systems, weak star topology, irredundance.

AMS *Subject Class.* (2010): 46B, 46E15, 54, 03E.

1. Introduction

It is well known that there are intimate links between Banach spaces and topological compact spaces. On one hand, compact spaces provide an important and big class of Banach spaces of the form $C(K)$, that is, of all real valued continuous functions on a compact Hausdorff K with the supremum norm (where, for example, one can isometrically embed all Banach spaces). On the other hand any Banach space X produces a compact space, namely the dual unit ball B_{X^*} with the weak* topology, that is, the topology whose subbasic sets are of the form $[x, I] = \{x^* \in B_{X^*} : x^*(x) \in I\}$, where $x \in X$ and I is an open interval in $\mathbb{R}$. One could consider these constructions as parallel to the Stone duality type constructions (see [18]) when we replace a

* The author would like to thank Jesús Castillo and Manuel González for organizing a stimulating conference at Castro Urdiales in February 2011.

Boolean algebra by a Banach space. As we do not have the full duality in this case, $B_{C(K)^*}$ is not K, nor X is $C(B_{X^*})$, and we have only the canonical embeddings, however, many interactions do have their Stone duality type analogues.

When a compact space K is not metrizable, or when a Banach space X is not separable, many phenomena appear which establish nontrivial correspondences between classes of compact spaces and Banach spaces. To be more precise, let $\mathcal{C}$ be a class of compact spaces and $\mathcal{X}$ be a class of Banach spaces. We say that $\mathcal{C}$ and $\mathcal{X}$ are associated if and only if $C(K) \in \mathcal{X}$ whenever $K \in \mathcal{C}$ and $B_{X*} \in \mathcal{C}$ whenever $X \in \mathcal{X}$. Thus, for example, WCG Banach spaces are associated with Eberlein compacts, Asplund generated spaces are associated with Radon-Nikodym compacts, etc. In other words, investigating the geometry of $C(K)$ looking at the topological properties of K, or investigating the geometry of a general Banach space X with the help of the topological properties of B_{X^*} is quite fruitful in the nonseparable context.

Definition 1.1. If X is a Banach space and X^* is its dual, then $(x_i, x_i^*)_{i \in I} \subseteq X \times X^*$ is called a biorthogonal system if and only if for each $i, j \in I$ we have $x_i^*(x_i) = 1$ and $x_i^*(x_j) = 0$ if $i \neq j$.

The importance of biorthogonal systems in the theory of Banach spaces is related to the fact that any kind of a basic sequence in a Banach space X must be the X part of a biorthogonal system, because taking its coordinate should be a linear functional satisfying the above definition (see, e.g., [34, 35, 13]).

In this note we are motivated by a growing amount of results related to cardinalities of biorthogonal systems and their versions in nonseparable Banach spaces. Our purpose is to compare them in a partial survey with a well-established body of results concerning certain topological invariants on Ks in the case of Banach spaces of the form $C(K)$ and on B_{X^*} in the case of a general Banach space X. By a topological invariant we will mean a way of associating a cardinal number to a topological space, in most cases a compact space. In a similar way we define a Banach space invariant. The simplest examples of invariants are the weight $w(K)$ of a compact space K and the density character $\mathrm{dens}(X)$ of the Banach space X. Since we have the equalities $w(K) = \mathrm{dens}(C(K))$ and $\mathrm{dens}(X) = w(B_{X^*})$, they are as nicely associated as they can. We will consider two groups of invariants, the first, emerging in Banach space theory is related to the cardinalities of biorthogonal systems. The other group of topological invariants are sometimes called versions of independence (see [6]), and include spread, hereditary density and hereditary

Lindelöf degree but also the irredundance is very relevant and we consider it.

To be more specific we need to recall a long list of definitions of these invariants and introduce some invariants related to the cardinalities of biorthogonal systems and their versions. First, one can define several weaker versions of biorthogonal systems (see [12]):

Definition 1.2. Suppose $0 < \varepsilon < 1$. If X is a Banach space and X^* is its dual, then $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*$ is called an ε-biorthogonal system if and only if for each $i, j \in I$ we have $x_i^*(x_i) = 1$ and $|x_i^*(x_j)| < \varepsilon$ if $i \neq j$. Any ε-biorthogonal system is also called an almost biorthogonal system.

Now we are in the position to define several invariants on Banach spaces:

Definition 1.3. Let X be a Banach space. Define the following:

- $\mathrm{biort}(X) = \sup\{\,|I| :$ there is a biorthogonal system $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*\}$,
- $\mathrm{biort}^{\varepsilon}(X) = \sup\{\,|I| :$ there is an ε-biorthogonal system $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*\}$,
- $\mathrm{abiort}(X) = \sup\{\,|I| :$ there is an almost biorthogonal system $(x_i, x_i^*)_{i\in I} \subseteq X \times X^*\}$.

Almost biorthogonal and ε-biorthogonal systems were introduced in [12] where their relationships with more geometric phenomena in Banach spaces were investigated.

If K is compact, then $M(K)$ denotes the Banach space of Radon measures on K, i.e., countably additive, regular, signed Borel measures on K with the variation norm. Here we should keep in mind the Riesz representation theorem which says that Radon measures on K isometrically represent all linear functionals on Banach spaces of the form $C(K)$, that is there is an isometry between $M(K)$ and $C(K)^*$ which associates the integration with respect to a given measure to a given measure. Pointwise measures concentrated on $x \in K$ will be denoted by δ_x. In the case of Banach spaces of the form $C(K)$ we introduce the following:

Definition 1.4. Let K be a compact space, $n \in \mathbb{N}$ and $\mu \in M(K)$. We say that μ is n-supported if and only if there are $x_1, \ldots, x_n \in K$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that

$$\mu = a_1\delta_{x_1} + \cdots + a_n\delta_{x_n}\,.$$

We say that a biorthogonal system $(f_i, \mu_i)_{i\in I} \subseteq C(K)\times M(K)$ is n-supported if and only if all measures μ_i are n-supported. We define

$$\mathrm{biort}_n(C(K)) = \sup\left\{ |I| \ : \ \begin{array}{c} \text{there is a biorthogonal system} \\ (f_i, \mu_i)_{i\in I} \subseteq C(K)\times M(K) \\ \text{which is } n\text{-supported} \end{array} \right\}.$$

One kind of 2-supported biorthogonal systems appear more often than others, we will follow [7] with the terminology concerning these systems:

Definition 1.5. Let K be a compact space, κ a cardinal and $x_\alpha, y_\alpha \in K$ for all $\alpha < \kappa$. A biorthogonal system in the Banach space $C(K)$ of the form $(f_\alpha, \delta_{x_\alpha} - \delta_{y_\alpha})_{\alpha<\kappa}$ is called a nice biorthogonal system. The supremum of cardinalities of nice biorthogonal systems will be denoted by $\mathrm{nbiort}_2(K)$.

Now we recall definitions of several topological invariants. Suppose K is an infinite compact space, then we consider the following topological invariants of K:

- $w(K) = \inf\{|\mathcal{B}| \ : \ \mathcal{B}$ is an open basis for $K\}$,
- $d(K) = \inf\{|D| \ : \ D \subseteq K$ where D is dense in $K\}$,
- $L(K) = \inf\{\kappa :$ every open cover of K has a subcover of cardinality $\kappa\}$,
- $\mathrm{ind}(K) = \sup\{\kappa \ :$ there is a continuous surjection $\phi : K \to [0,1]^\kappa\}$,
- $s(K) = \sup\{|D| \ : \ D \subseteq K$ where D is discrete in $K\}$,
- $hd(K) = \sup\{d(X) \ : \ X \subseteq K\}$,
- $hL(K) = \sup\{L(X) \ : \ X \subseteq K\}$.

We call them weight, density, Lindelöf degree, independence, spread, hereditary density, hereditary Lindelöf degree respectively. Most of these functions can be redefined in a uniform language of versions of independence (see [6]). For Banach space theoretic aspects of the independence itself see [27]. We left the tightness as we will not consider it in this note, however it is an invariant which is extensively used in the context of the dual ball of a Banach spaces (see, e.g., [11]) also in relation to biorthogonal systems (see [38]). We recall some well-known inequalities concerning the above invariants, most of them can be found in [15]:

Theorem 1.6. *Suppose K is a compact space, then the following hold:*

- $\mathrm{ind}(K) \leq s(K) \leq hd(K), \quad hL(K) \leq w(K)$,

- $w(K) \leq 2^{s(K)}$,
- $hd(K) \leq s(K)^+$,
- $hd(K) \leq s(K^2)$ (*see* [17]).

Of course we leave many deep inequalities and equalities with other functions like $h\pi w$ or $h\pi\chi$ as we will not make use of them in this note, for some of them see [15]. However one important thing will be:

PROPOSITION 1.7. *Suppose that K is a regular topological space. $hd(K)$ is the supremum of cardinalities of left-separated sequences in K, that is sequences $\{x_\alpha : \alpha < \kappa\}$ for which there are open $U_\alpha \subseteq K$ satisfying $x_\alpha \in U_\alpha$ and $x_\alpha \notin U_\beta$ for $\alpha < \beta < \kappa$. $hL(K)$ is the supremum of cardinalities of right-separated sequences in K, that is sequences $\{x_\alpha : \alpha < \kappa\}$ for which there are open $U_\alpha \subseteq K$ satisfying $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\alpha < \beta < \kappa$.*

Unexplained terminology and notation should be fairly standard. If A is a Boolean algebra, then K_A denotes its Stone space, that is the space of ultrafilters on A with the topology whose subbasic sets are of the form $[a] = \{u \in K_A : a \in u\}$. All totally disconnected compact space are the Stone spaces of some Boolean algebras and this class plays an important role in the theory of Banach spaces of the form $C(K)$. All Banach spaces considered here are infinite dimensional, over the reals and all compact spaces are assumed to be infinite. K usually stands for a compact space, X for a Banach space and κ for an infinite cardinal. CH stands for the continuum hypothesis. Given a Banach space X, when talking about its dual space X^* as a topological space, we will always consider the weak* topology mentioned above.

LEMMA 1.8. *Suppose that X is an infinite dimensional Banach space and ψ is one of the cardinal invariants s, hL, hd. Then $\psi(X^*) = \psi(B_{X^*})$.*

Proof. All of the functions are suprema of sizes of some sets in the space by Proposition 1.7. So $\psi(X^*) \geq \psi(B_{X^*})$. Subsets of discrete sets are again discrete. Subsequences of left- or right-separated sequences are again left or right-separated respectively. $X^* = \bigcup_{n\in N} nB_{X^*}$ and multiplying by $1/n$ is a homeomorphism of nB_{X^*} onto B_{X^*} for each n. Any subset A of X^* can be divided into countably many parts $A_n = X \cap nB_{X^*}$ and each of these parts has a homeomorphic copy $\frac{1}{n}A_n$ in B_{X^*}, and the supremum over sizes of $\frac{1}{n}A_n$ is equal to the size of A. ∎

The recent results concerning the subject matter of this note are included in [3, 4, 7, 19, 22, 38]. The last two papers consider the Banach spaces of the form $C(K)$ as a lateral theme, go far beyond biorthogonal systems and investigate various kinds of uncountable basic sequences and other related topics. These exciting new advances (see also [5]) are not, however, the subject of this survey.

Many of the constructions mentioned in this paper are not absolute, that is the usual axioms of mathematics (ZFC) are not sufficient to carry them out. Many follow from additional axioms which were shown to be equiconsistent with ZFC (they do not lead to contradiction if ZFC does not lead itself) but another group was established only using the method of forcing. The readers less familiar with these maters should consult when needed, for example, the textbook [20]. On the level of ω and ω_1 this lack of absolutness is known to be unavoidable as it is shown in most cases that some other axioms or forcing arguments imply the nonexistence of the constructions. However it is still unknown if one can construct (in ZFC) Banach spaces (of the form $C(K)$) where any of the functions $\mathrm{biort}(X)$, $s(B_{X^*})$, $s(K^\omega) = hd(K^\omega)$ are different.

All proofs of this note are natural enough that they can be considered as a folclore. However we think that in the case of interactions of two disciplines, in this case, Banach space theory and set-theoretic topology, simple facts linking apparently unrelated topics may be known only to isolated groups of reseachers. The aim of this note is to propose breaking this isolation and sharing some list of open problems and simple links among them and the published results.

2. Biorthogonality and the topological weight

The most basic topological invariant is the weight of the space. So, we start by asking what is the relation between the weight of K and the biorthogonality of $C(K)$ or the weight of B_{X^*} and the biorthogonality of a Banach space X. It is well known that $w(B_{X^*}) = \mathrm{dens}(X)$ and $w(K) = \mathrm{dens}(C(K))$, so trivially we have $\mathrm{biort}(X) \leq w(B_{X^*})$ and $\mathrm{biort}(C(K)) \leq w(K)$ and we are asking if there could be big Banach spaces with small biorthogonality.

Theorem 2.1. *Let X be an infinite dimensional Banach space. Then* $\mathrm{dens}(X) \leq 2^{\mathrm{biort}(X)}$.

Proof. Suppose that $\mathrm{dens}(X) > 2^\kappa$. We will construct a biorthogonal system of size κ^+ by transfinte induction using the Hahn-Banach theorem.

First note that if $\mathcal{F} \subseteq X^*$ separates the points of X (i.e., for distinct $x, y \in X$ we have $f(x) \neq f(y)$ for some $f \in \mathcal{F}$, equivalently for every $x \in X \setminus \{0\}$ there is $f \in \mathcal{F}$ such that $f(x) \neq 0$) then it is of cardinality bigger than κ. This is due to the fact that if $\mathcal{F}$ separates the points, then the function F which sends $x \in X$ into $\mathbb{R}^{\mathcal{F}}$ defined by $F(x)(f) = f(x)$ is injective, which would give that $\operatorname{dens}(X) \leq |X| \leq |\mathbb{R}^{\mathcal{F}}| = (2^\omega)^\kappa = 2^\kappa$. contradicting the hypothesis about the density of X.

This fact implies that given a family $\mathcal{F} \subseteq X^*$ of cardinality not bigger than κ, the norm closed linear subspace of X defined as

$$Y = \bigcap \{\ker(f) \, : \, f \in \mathcal{F}\}$$

must have density bigger than κ because if not, then $\mathcal{F}$ together with κ functionals separating the points of Y would separate the the points of X.

Suppose we have constructed a part of the biorthogonal system $(x_\alpha, x^*_\alpha)_{\alpha<\lambda} \subseteq X \times X^*$ for some $\lambda < \kappa^+$. As $\bigcap\{\ker(x^*_\alpha) \, : \, \alpha < \lambda\}$ must have density bigger than κ, there is $x_\lambda \in X \setminus \{0\}$ such that $x^*_\alpha(x_\lambda) = 0$ for all $\alpha < \lambda$ and x is not in the norm closure of $\{x_\alpha \, : \, \alpha < \lambda\}$ which has density at most κ. Using the Hahn-Banach theorem we find a continuous functional $x^*_\lambda \in X^*$ such that $x^*_\lambda(x_\lambda) = 1$ and $x^*_\lambda(x_\alpha) = 0$ for $\alpha < \lambda$. This way we continue the construction up to κ^+. ∎

The above result was known some decades ago to W. Johnson ([24]). For Banach spaces of the form $C(K)$ we have the following:

THEOREM 2.2. *Suppose K is a compact space, then*

$$\operatorname{dens}(C(K)) \leq 2^{\operatorname{biort}_1(C(K))}.$$

Proof. By Proposition 3.2 $\operatorname{biort}_1(C(K)) = s(K)$. So, use the well known inequality for compact spaces $w(K) \leq 2^{s(K)}$ (see [15, 7.7]) ∎

By, now, all known examples of Banach spaces where the density is not equal to the biorthogonality are not absolute.

THEOREM 2.3. *The following are consistent:*

(1) (Kunen) *There is a compact K such that* $\operatorname{dens}(C(K)) = \omega_1 = 2^\omega$ *but* $\operatorname{biort}(C(K)) = \omega$,
(2) (Brech, Koszmider) *There is a compact K such that* $\operatorname{dens}(C(K)) = \omega_2 = 2^\omega$ *but* $\operatorname{biort}(C(K)) = \omega$,

(3) (Todorcevic) *Whenever* $|A| > \omega$, *then* $\mathrm{nbiort}_2(C(K_A)) > \omega$

(4) (Todorcevic) *Whenever* $\mathrm{dens}(X) > \omega$, *then* $\mathrm{biort}(X) > \omega$.

(1) Was first proved by K. Kunen (see [25]) using the Ostaszewski type space. Ostaszewski's original construction assumed $\diamondsuit$ (see [20]) like a construction by S. Shelah of a Banach space X with $\omega = \mathrm{biort}(X) < \mathrm{dens}(X) = \omega_1$ ([33]). However Kunen used a weaker assumption of CH. This assumption can be weakened further to $\mathfrak{b} = \omega_1$ as was shown by Todorcevic ([36, 2.4]).

(2) was constructed in [3] using forcing. (3) and (4) are results obtained in [38] assuming Martin's axiom and the negation of CH and Martin's Maximum respectively. The natural questions in the context of the above results are:

Question 2.4. Is it consistent that for an arbitrary Banach space X we have

$$\mathrm{biort}(X) = \mathrm{dens}(X)?$$

Or there is an absolute example of a Banach space satisfying $\mathrm{biort}(X) < \mathrm{dens}(X)$?

Question 2.5. Is it consistent that there is a Banach space X such that $\mathrm{biort}(X) = \omega$, and $\mathrm{dens}(X) > \omega_2$?

One could try to construct a an exemple as above similar to the Kunen space or the example from [3], i.e., of the form $C(K)$ where K is scattered and K^n is hereditarily separable for all $n \in \mathbb{N}$ (see section on hereditary density). Then the problem becomes more difficult than a well-known open problem if there is a thin very-tall Boolean algebra of height ω_3. However the example could be very different.

Question 2.6. Assume Martin's axiom and the negation of CH. Let A be a Boolean algebra and K be a compact space. Does any of the following statements follow:

- Whenever $|\mathcal{A}| < 2^\omega$, then $\mathrm{biort}(C(K_\mathcal{A})) = |\mathcal{A}|$?
- Whenever $w(K) > \omega$, then $\mathrm{biort}(C(K)) > \omega$?
- Whenever $w(K) < 2^\omega$, then $\mathrm{biort}(C(K)) = \mathrm{dens}(C(K))$?

Question 2.7. Does Martin's axiom with the negation of CH imply that every nonseparable Banach space has an uncountable biorthogonal system?

3. Biorthogonality and spread

When one looks at the definition of the biorthogonal system in the context of the weak* topology, the first thing one notes is the fact that the X^* part of the system forms a discrete set of the dual, in terms of our cardinal invariants this is the following:

Proposition 3.1. *Suppose X is a Banach space. Then*

$$\mathrm{biort}(X) \leq s(B_{X^*}) .$$

Proof. The weak* open sets $U_\alpha = \{x^* : x^*(x_\alpha) > 1/2\}$ separate x^*_α from the remaining x^*_βs. ∎

On the $C(K)$ level we have the following:

Proposition 3.2. *Suppose that K is a compact space, then*

$$\mathrm{biort}_1(C(K)) = s(K) .$$

Proof. Suppose $(x_i : i \in I)$ is a discrete subspace of K. This means that $x_i \notin \overline{\{x_j : j \neq i\}}$, so we can find a continuous function $f_i : K \to [0,1]$ such that $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for all $j \in I \setminus \{i\}$. So $(f_i, \delta_{x_i})_{i\in I}$ forms a 1-supported biorthogonal system.

Now suppose that $(f_i, a_i\delta_{x_i})_{i\in I}$ is a biorthogonal system. Then none of a_is can be 0, hence $f_i(x_j) = 0$ and $f_i(x_i) = 1/a_i$. consider open sets $U_i = \{x \in K : f_i(x) > 1/2a_i\}$. We see that $x_i \in U_i$ and $x_j \notin U_i$ for $j \neq i$, that is $\{x_i : i \in I\}$ is discrete. ∎

An argument similar to that from the proof of Proposition 3.1 can be applied for almost biorthogonal systems, so putting together various functions we obtain the following:

Corollary 3.3. *Let K be a compact space, X be a Banach space, $n \in \mathbb{N} \setminus \{0\}$ and $0 < \varepsilon < 1$. Then*

(1) $s(K) \leq \mathrm{biort}_n(C(K)) \leq \mathrm{biort}_{n+1}(C(K)) \leq \mathrm{biort}(C(K))$,

(2) $\mathrm{biort}(X) \leq \mathrm{biort}^\varepsilon(X) \leq \mathrm{abiort}(X) \leq s(B_{X^*})$.

So, the spread of K is a lower bound of $\mathrm{biort}(C(K))$ and $s(B_{X^*})$ is an upper bound of $\mathrm{biort}(X)$. The split interval "[0, 1]" satisfies $s(\text{"}[0,1]\text{"}) = \omega$ and $\mathrm{biort}(\text{"}[0,1]\text{"}) = 2^\omega$ (see [9]), so the lower bound is not too tight. However by now, there is no absolute example where $\mathrm{biort}(X)$ and $s(B_{X^*})$ are different. Although we have the following:

Theorem 3.4. (Lopez-Abad, Todorcevic) *It is consistent that there is a Banach space such that* $\omega = \mathrm{biort}(X) < s(B_{X^*}) = \omega_1$.

Proof. Use any of [22, Examples 4.1] or [22, Examples 4.2], where a Banach space X is constructed such that $\omega = \mathrm{biort}(X) < a\,\mathrm{biort}(X) = \mathrm{dens}(X) = \omega_1$. Now apply Corollary 3.3. ∎

Proposition 3.5. *Let K be a compact space and $n \in \mathbb{N}$, then*

$$s(K^n) \leq \mathrm{biort}^{1-\frac{1}{n}}(C(K))\,.$$

In particular, we have that $s(K^n) \leq \mathrm{abiort}(C(K))$ *for every* $n \in \mathbb{N}$.

Proof. Given a discrete set in K^n of cardinality κ for some cardinal κ, we will construct an $(1-\frac{1}{n})$-biorthogonal system of the same cardinality. We may assume that n is minimal such that there is a discrete set in K^n of cardinality κ.

Suppose that for $\alpha < \kappa$ the points $(x_1^\alpha, \ldots, x_n^\alpha)$s form a discrete set in K^n as witnessed by open neighbourhoods $U_1^\alpha \times \cdots \times U_n^\alpha \subseteq K^n$, i.e., $(x_1^\alpha, \ldots, x_n^\alpha) \in U_1^\alpha \times \cdots \times U_n^\alpha$ and for every distinct $\alpha, \beta < \kappa$ we have that $x_i^\beta \notin U_i^\alpha$ for some $i \leq n$. By the minimality of n, we may assume that the points $x_1^\alpha, \ldots, x_n^\alpha$ are distinct for each $\alpha < \kappa$, and so, we may assume that the sets $U_1^\alpha, \ldots, U_n^\alpha$ are pairwise disjoint for each $\alpha < \kappa$. Consider functions $f_i^\alpha : K \to [0,1]$ such that $f_i^\alpha(x) = 0$ if $x \notin U_i^\alpha$ and $f_i^\alpha(x_i^\alpha) = 1$. Let

$$f_\alpha = f_1^\alpha + \cdots + f_n^\alpha, \qquad \mu_\alpha = \frac{1}{n}(\delta_{x_1^\alpha} + \cdots + \delta_{x_n^\alpha})\,.$$

It follows that $\mu_\alpha(f_\alpha) = 1$ and $|\mu_\alpha(f_\beta)| \leq 1 - \frac{1}{n}$ if $\alpha \neq \beta$, as required. ∎

Question 3.6. Is there (consistently) a compact space K such that

$$\mathrm{biort}(C(K)) < s(K^n)$$

for some $n \in \mathbb{N}$?

QUESTION 3.7. Is there (consistently) a compact space K such that

$$s(K^n) < \mathrm{biort}(C(K))$$

for all $n \in \mathbb{N}$?

Here we have some positive result:

THEOREM 3.8. (BRECH, KOSZMIDER [4]) *For every $n \in \mathbb{N}$ it is consistent that there is a compact space K such that*

$$\omega = hd(K^n) = s(K^n) < \mathrm{biort}(C(K)) = \omega_1 \,.$$

QUESTION 3.9. Suppose K is compact. Is it true that

$$\mathrm{biort}_n(C(K)) = s(K^n)$$

for all (some) $n > 1$?

QUESTION 3.10. Is it consistent that $s(B_{X^*}) = \mathrm{biort}(X)$ for every Banach space X?

4. BIORTHOGONALITY AND HEREDITARY DENSITY

For many decades the only example of a nonseparable $C(K)$ space without uncountable biorthogonal systems was the Kunen space. Here K is scattered and such that K^n is hereditarily separable for every $n \in K$. The fact that K is scattered gives that the dual space is isometric to $l_1(K)$ and hence is accessible to our sight. Now quite useful is the following:

LEMMA 4.1. *Suppose that K is a scattered compact space. Then*

$$hd(K^\omega) = hd\big(B^\omega_{C(K)^*}\big) \,.$$

In particular K^n is hereditary separable for each $n \in \mathbb{N}$, if and only if $(C(K)^*)^n$ and so $B^n_{C(K)^*}$ is hereditary separable for every $n \in \mathbb{N}$.

Proof. We note that by Proposition 1.7, we have that

$$hd(Z^\omega) = \sup\big\{hd(Z^n) \,:\, n \in \mathbb{N}\big\}$$

for a regular space Z. There is a homeomorphic embedding of K into $B_{C(K)^*} \subseteq C(K)^*$ sending $x \in K$ to δ_x, so the backward implication is clear. For the forward implication, using Proposition 1.7 given a left-separated sequence $\mu_\alpha = (\mu_\alpha^1, \dots, \mu_\alpha^n)_{\alpha<\kappa}$ of a regular length κ in the n-th power of $B_{X^*} \subseteq X^* = l_1(K)$, we will produce a left-separated sequence of length κ in some finite power of K. This is enough since regular cardinals are unbounded in singular cardinals.

Let $a_j^{\alpha i} \in \mathbb{R}$ and $x_j^{\alpha i} \in K$ be such that

$$\mu_\alpha^i = \sum_{j\in\mathbb{N}} a_j^{\alpha i} \delta_{x_j^{\alpha i}} .$$

We may assume that the neighbourhoods as in Proposition 1.7 which witness the fact that μ_αs form a left-separated sequence are of the form

$$U_\alpha = \left\{(\mu^1, \dots, \mu^n) : \int f_\alpha^{il} \,\mathrm{d}\mu^i \in I_{il} \quad \forall\, i \le n \text{ and } \forall\, l \le k\right\}$$

for some $f_\alpha^{il} \in C(K)$ and some open intervals $I_{il} \subseteq \mathbb{R}$ for $i \le n$ and $l \le k$ for some $k \in \mathbb{N}$. The integer k and the interval I_{il} may be fixed for all αs because the same values will be repeated κ many times by the regularity of κ. By the same argument we may assume that the norms of all functions f_α^{il} are bounded by some positive real M.

Using the fact that sets of reals have strong condensation points and regularity of κ we may assume that there are open subintervals J_{il} of I_{il} and there is $\varepsilon > 0$ satisfying the following for all valued of indices:

$$\Big(\int f_\alpha^{il} \,\mathrm{d}\mu_\alpha^i - 2\varepsilon\, , \int f_\alpha^{il} \,\mathrm{d}\mu_\alpha^i + 2\varepsilon\Big) \subseteq J_{il}\, ,$$

$$\big(\min(J_{il}) - 2\varepsilon\, , \,\max(J_{il}) + 2\varepsilon\big) \subseteq I_{il}\, .$$

Let U'_α be sets defined as U_αs with I_{il} replaced by J_{il}. In particular U'_αs witness the fact that μ_αs form a left-separated sequence. Again using the regularity of κ (in fact only the part that $cf(\kappa) \neq \omega$) we may assume that there are $j_i \in \mathbb{N}$ such that $\sum_{j>j_i} |a_j^{\alpha i}| < \varepsilon/M$ for all $\alpha < \kappa$ and $i \le n$. Now define $\nu_\alpha^i = \sum_{j\le j_i} a_j^{\alpha i} \delta_{x_j^{\alpha i}}$. It is clear that for each $i \le n$ and each $l \le k$ we have

$$\left|\int f_\alpha^{il} \,\mathrm{d}\mu_\alpha^i - \int f_\alpha^{il} \,\mathrm{d}\nu_\alpha^i\right| \le \sum_{j>j_i} \big|a_j^{\alpha i} f_\alpha^{il}(x_j^{\alpha i})\big| \le \varepsilon\, .$$

So, if we define $\nu_\alpha = (\nu_\alpha^1, \dots, \nu_\alpha^n)$, in particular we have that U_αs witness the fact that ν_αs form a left-separated sequence. Finally define

$$x_\alpha = \big(x_1^{\alpha 1}, \dots, x_{j_1}^{\alpha 1}, \dots, x_1^{\alpha n}, \dots, x_{j_n}^{\alpha n}\big)$$

and

$$V_\alpha = V_1^{\alpha 1} \times \dots, \times V_{j_1}^{\alpha 1} \times \dots \times V_1^{\alpha n} \times \dots \times V_{j_n}^{\alpha n},$$

where given $\alpha \leq \kappa$, $i \leq n$ and $j \leq j_i$ whenever $y \in V_j^{\alpha i}$ then $|f_\alpha^{il}(x_j^{\alpha i}) - f_\alpha^{il}(y)| < \varepsilon/j_i$. for each $l \leq k$. Now if $x = (x_1^1, \dots, x_{j_1}^1, \dots, x_1^n, \dots, x_{j_n}^n) \in V_\alpha$, then for each $i \leq n$ and $l \leq k$ we have

$$\left| \sum_{j \leq j_i} a_j^{\alpha i} f_\alpha^{il}(x_j^{\alpha i}) - \sum_{j \leq j_i} a_j^{\alpha i} f_\alpha^{il}(x_j^i) \right| \leq \varepsilon,$$

because $\sum_{j=0}^\infty |a_j^{\alpha i}| \leq 1$ because μ_α is in the dual unit ball, so

$$\big(a_1^{\alpha 1}\delta_{x_1^1} + \dots + a_{j_1}^{\alpha 1}\delta_{x_{j_1}^1}, \dots, a_1^{\alpha n}\delta_{x_1^n} + \dots + a_{j_n}^{\alpha n}\delta_{x_{j_n}^n}\big) \in U_\alpha .$$

This implies that $x_\alpha \notin V_\beta$ for $\alpha < \beta$, because this would give that $\nu_\alpha \in U'_\beta$. Hence x_αs form a left-separated sequence in a finite power of K. ∎

So $B_{C(K)^*}$ for K being the Kunen space has countable spread, and so no uncountable biorthogonal system by Proposition 3.1. For scattered space K we have $hL(K) = w(K)$ (see [23, 25.130]) and so we have $\omega = hd(K) < hL(K) = \omega_1$. It is worthy to note the following:

Lemma 4.2. *Suppose that K is a compact space, then*

$$hL(K^2) = w(K) .$$

In particular if K is nonmetrizable, then $hL(K^2)$ is uncountable.

Proof. It is clear that $hL(K^2) \leq w(K)$. For the opposite inequality consider $\Delta = \{(x,x) : x \in K\} \subseteq K^2$. If $hL(K^2) = \kappa < w(K)$, then $L(K^2 \setminus \Delta) \leq \kappa$ so the open cover of $K^2 \setminus \Delta$ by open sets whose closures in K^2 are disjoint from Δ would have subcover of size κ which would yield that $K^2 \setminus \Delta$ is a union of κ many closed sets and so Δ is an intersection of κ many open sets $(G_\alpha)_{\alpha<\kappa}$.

Now it is enough to consider a version of a proof of Sneider's theorem (see [26, 5.3]). Consider a family $\mathcal{B}$ of open sets of K such that for every $\alpha < \kappa$ there are $B_1, \dots B_k \in \mathcal{B}$ such that

$$\Delta \subseteq \overline{B_1} \times \overline{B_1} \cup \dots \cup \overline{B_k} \times \overline{B_k} \subseteq G_\alpha .$$

By the compactness, one may assume that $|\mathcal{B}| \leq \kappa$. Note that $\mathcal{B}$ is a basis for K as it is a pseudobasis. Indeed, otherwise, if $\bigcap\{B \in \mathcal{B} : x \in B\} \ni y$ for some $y \neq x$, then $(x,y) \in G_\alpha$ for all $\alpha < \kappa$, contradicting that fact that $\Delta = \bigcap_{\alpha<\kappa} G_\alpha$. Hence $w(K) \leq \kappa$ as required. ∎

Recall that a regular space X is called a (strong) S-space, if and only if X^n is hereditarily separable (for each $n \in \mathbb{N}$) for $n = 1$ while X is not hereditarily Lindelöf. So Kunen's space is a strong S-space. Martin's axiom with the negation of CH implies that there are no strong S-spaces or no compact S-spaces (see [28]). So, quite natural is the following general question:

QUESTION 4.3. Does the existence of a nonseparable Banach space without uncountable biorthogonal systems imply the existence of a (strong) compact S-space?

In the case of the Kunen space both K and $B_{C(K)^*}$ are strong S-spaces, so they are the natural candidates, however we have two consistent counterexamples. In the proof of the first one we will need the following

LEMMA 4.4. *Let X be a Banach space considered with the weak topology and let X^* be its dual considered with weak* topology. The following hold for every $n \in \mathbb{N}$ $hL(X^n) = \kappa$ if and only if for every $n \in \mathbb{N}$ $hd(X^{*n}) = \kappa$.*

Proof. We will use Proposition 1.7 and will see that using the hypothesis one can obtain left-separated sequences in finite powers of X^* of a regular length κ from right-separated sequences in finite powers of X of a length κ and vice versa. This is enough as regular cardinals are unbounded in singular cardinals. Suppose $x_\alpha = (x_\alpha^{1^*}, \dots, x_\alpha^{n*})$s for $\alpha < \kappa$ form a left-separated sequence in $(X^*)^n$, we may assume that it is witnessed as in Proposition 1.7 by open sets U_α of the form

$$U_\alpha = \Big\{ \big(x^{1^*}, \dots, x^{n*}\big) \ : \ x^{i^*}(x_\alpha^{ij}) \in I_\alpha^{ij} \quad \forall\, i \le n \text{ and } \forall\, l \le m \Big\}$$

for some $x_\alpha^{ij} \in X$, open interval I_α^{ij} and $j \le m$ for some $m \in \mathbb{N}$ and all $i \le n$. The integer m is fixed for all $i \le n$ because we can take one which works for all $i \le n$, it is fixed for all α because κ is assumed to be uncountable regular, so the same integer is repeated κ many times.

Consider the dual open sets in X^{nm} defined as

$$V_\alpha = \Big\{ \big(x^{11}, \dots x^{1m}, \dots, x^{n1}, \dots x^{nm}\big) \ : \ x^{i^*}(x_\alpha^{ij}) \in I_\alpha^{ij} \quad \forall\, i \le n \text{ and } \forall\, l \le m \Big\}.$$

Define $y_\alpha = (x_\alpha^{11}, \dots x_\alpha^{1m}, \dots, x_\alpha^{n1}, \dots x_\alpha^{nm})$. We see that $x_\alpha \in U_\beta$ if and only if $y_\beta \in V_\alpha$ for $\alpha, \beta < \kappa$. So, U_αs witnesses that $\{x_\alpha : \alpha < \kappa\}$ is left-separated if and only if V_αs witnesses that $\{y_\alpha : \alpha < \kappa\}$ is right-separated. The other direction is analogous. ∎

THEOREM 4.5. (LOPEZ-ABAD, TODORCEVIC) *It is consistent that there is a nonseparable Banach space X such that B_{X^*} (nor any of its power) is not a strong S-space but* $\mathrm{abiort}(X) = \mathrm{biort}(X) = \omega$.

Proof. Consider either of [22, Examples 5.1] or [22, Examples 5.2], which have this property that for some $n \in \mathbb{N}$ the power X^n is not hereditarily Lindelöf with respect to the weak topology, so by Lemma 4.4 some power of X^* is not hereditarily separable with respect to the weak* topology. ∎

THEOREM 4.6. (KOSZMIDER, LOPEZ-ABAD, TODORCEVIC) *It is consistent there is a nonseparable $C(K)$ without biorthogonal systems such that K is hereditarily separable and hereditarily Lindelöf an so is not an S-space.*

Proof. The space constructed in [19] is hereditarily Lindelöf and $C(K)$ has no biorthogonal systems. In [22, Section 8] it is proved that the space of [1] has the same properties. The squares of both of theses spaces are strong S-spaces. ∎

We have some partial positive results however:

PROPOSITION 4.7. *If K is nonmetrizable compact space such that $C(K)$ has no uncountable almost biorthogonal systems, then K^2 is a strong S-space, in particular, there exists a compact strong S-space.*

Proof. Let K be nonmetrizable and such that $C(K)$ has no uncountable almost biorthogonal systems. It follows from Proposition 3.5 that $s(K^n)$ is countable for all $n \in \mathbb{N}$. By [17] $hd(L) \leq s(L^2)$ holds for any compact space, and so we have that $hd(K^n)$ is countable for every n. On the other hand $hL(K^2)$ must be uncountable by Lemma 4.2.

PROPOSITION 4.8. (TODORCEVIC, DZAMONJA-JUHASZ) *Suppose that K is a compact space, then*

$$hd(K) \geq \mathrm{nbiort}_2(C(K))\,.$$

Here in [38], Todorcevic strengthened a previous result of Lazar from [21] showing the above for $hd(K) = \omega_1$, it was then generalized by Dzamonja and Juhasz in [7].

QUESTION 4.9. Suppose that K is an nonmetrizable compact space such that $C(K)$ has no uncountable biorthogonal systems is some finite power of K is a (strong) S-space? Is K^2 an S-space?

5. Biorthogonality and irredundance

The irredundance of Boolean algebras is a well known and investigated invariant of Boolean algebras (see [23]). As an introduction to its relation with topological invariants and biorthogonal systems we propose somewhat general discussion concerning irredundance in various structures

Definition 5.1. Suppose $\mathcal{S}$ is a class of structures with fixed families of substructures. Let $S \in \mathcal{S}$ and $R \subseteq S$. We say that R is irredundant if and only if for each $r \in R$ there is a substructure of S containing $R \setminus \{r\}$ and not containing r. The irredundance $\mathrm{irr}_{\mathcal{S}}(S)$ of S is the supremum of cardinalites of irredundant subsets of S.

We will consider these notions for Boolean algebras with subalgebras ($\mathcal{S} = BA$), Banach spaces with closed linear subspaces ($\mathcal{S} = BaS$) and Banach algebras of the form $C(K)$ with closed subalgebras ($\mathcal{S} = BaA$). Note that in all these cases the existence of substructures containing $R \setminus \{r\}$ and not containing r means that the substructure generated by $R \setminus \{r\}$ does not contain r. So if A is a Boolean algebra then $R \subseteq A$ is irredundant if and only if none of $r \in \mathcal{R}$ belongs to the Boolean algebra generated by the remaining elements i.e., $R \setminus \{r\}$, or if K is a compact Hausdorff space $\mathcal{F} \subseteq C(K)$ is irredundant if and only if none of $f \in \mathcal{F}$ belongs to the closed subalgebra of $C(K)$ generated by $\mathcal{F} \setminus \{f\}$.

Proposition 5.2. *Suppose $\mathcal{A}$ is a Boolean algebra and K is a compact space, then*

$$\mathrm{irr}_{BA}(A) \leq \mathrm{irr}_{BaA}(C(K_A)) \leq \mathrm{irr}_{BaS}(C(K_A)),$$
$$\mathrm{irr}_{BaA}(C(K)) \leq \mathrm{irr}_{BaS}(C(K)).$$

Proof. If B is a subalgebra of a Boolean algebra A, and $a \in A \setminus B$, then $\chi_{[a]}$ does not belong to the closed Banach subalgebra generated by $\chi_{[b]}$s for $b \in B$. Clearly any Banach subalgebra is a Banach subspace. ∎

Below we will see that two of the above functions are equal to invariants previously considered. First note the following proposition which explains why we are talking about the irredundance:

Proposition 5.3. *Suppose X is a Banach space and $(x_\alpha)_{\alpha<\kappa} \subseteq X$. There is $(x_\alpha^*)_{\alpha<\kappa} \subseteq X^*$ such that $(x_\alpha, x_\alpha^*)_{\alpha<\kappa}$ is a biorthogonal system if and only* *if for every $\alpha \in \kappa$ the vector x_α does not belong to the norm closed linear subspace generated by $\{x_\beta : \beta \neq \alpha\}$. In particular*

$$\mathrm{irr}_{BaS}(X) = \mathrm{biort}(X)\,.$$

Proof. In one direction one considers $\ker(x^*_\alpha)$ as a closed subspace which contains x_βs for $\beta \neq \alpha$. In the other direction one uses the Hahn-Banach theorem to extend the functional that chooses the x_α's coordinate in

$$\overline{\mathrm{span}\left(\{x_\alpha : \alpha \neq \beta\}\right)} \oplus \mathbb{R}x_\alpha$$

to the entire space X obtaining x^*_α. Here we used the fact the the above direct sum is a closed subspace of X which follows from the fact that the second factor is one-dimensional.

THEOREM 5.4. *Suppose K is a compact Hausdorff space. $\mathcal{F} \subseteq C(K)$ is irredundant if and only if for each $f \in \mathcal{F}$ there are $x_f, y_f \in K$ such that $f(x_f) - f(y_f) > 0$ and $f(x_g) - f(y_g) = 0$ for distinct $f, g \in \mathcal{F}$. In particular*

$$\mathrm{irr}_{BaA}(C(K)) = \mathrm{nbiort}_2(K)$$

for every compact space K.

Proof. It is clear that if for each $f \in \mathcal{F}$ there are $x_f, y_f \in K$ such that $f(x_f) - f(y_f) > 0$ and $f(x_g) - f(y_g) = 0$ for distinct $f, g \in \mathcal{F}$, then all functions of the closed subalgebra generated by $\mathcal{F} \setminus \{f\}$ do not separate x_f from y_f, i.e., f does not belong to it.

Now suppose that f is not in the closed subalgebra $\mathcal{A}$ generated by $\mathcal{F} \setminus \{f\}$, and let us construct x_f and y_f as required. Consider the equivalence relation E on K defined by xEy if and only if $g(x) = g(y)$ for all $g \in \mathcal{A}$. Let L be the quotient space K/E and $\phi : K \to L$ be the quotient map. Note that for each $g \in \mathcal{A}$ there is a well-defined $[g] : L \to \mathbb{R}$ such that $[g] \circ \phi = g$, so by the properties of the quotient topology (see [8, 2.4.2]) $[g] \in C(L)$. Note that $\{[g] : g \in \mathcal{A}\}$ is a closed subalgebra of $C(L)$ which separates the points of L and contains the constant functions and hence, by the Weierstrass-Stone theorem it is the entire $C(L)$.

If f is constant on every equivalence class of E, then there were $[f] : L \to \mathbb{R}$ satisfying $[f] \circ \phi = f$, and then we would, the same way, have that $[f] \in C(L)$ and so $[f] = [g]$ for some $g \in \mathcal{A}$, which would give $f = g$ contradicting the hypothesis about f. So f is nonconstant on some equivalence class of E and so there are $x_f, y_f \in K$ as required

To obtain a nice biorthogonal system from an irredundant set of functions via the first part the theorem just multiply $f \in \mathcal{F}$ by $1/(f(x_f) - f(y_f))$. ∎

Hence, we do not need special notation either for irr_{BaS} nor for irr_{BaA} and so we will use only irr_{BA} which will be simply denoted as irr from this point on. The link between irredundant sets in Boolean algebras and biorthogonal systems and spread of the square was first indicated in the literature in the following:

THEOREM 5.5. (HEINDORFF) *Suppose $\mathcal{A}$ is a Boolean algebra and K is a compact space, then*

$$\mathrm{irr}(\mathcal{A}) \leq \mathrm{nbiort}_2(K_{\mathcal{A}}), \qquad \mathrm{nbiort}_2(K) \leq s(K^2).$$

Proof. The first inequality follows from Proposition 5.2 and Theorem 5.4. For the second consider a nice biorthogonal system $(f_\alpha, \delta_{x_\alpha} - \delta_{y_\alpha})_{\alpha<\kappa}$. Consider open $V_\alpha, U_\alpha \subseteq K$ such that $|f_\alpha(x) - f_\alpha(x_\alpha)| < 1/2$ for $x \in V_\alpha$ and $|f_\alpha(x) - f_\alpha(y_\alpha)| < 1/2$ for $x \in V_\alpha$. Note that $x_\beta \in V_\alpha$ and $y_\beta \in U_\alpha$ would give that $|f_\alpha(x_\beta) - f_\alpha(y_\beta)| > 0$ contradicting the biorthogonality. So $V_\alpha \times U_\alpha$ witnesses that $(x_\alpha, y_\alpha)_{\alpha<\kappa}$ is a discrete subspace of K^2. ∎

The above result was used to conclude that any strong S-space has countable irredundance, this applies to the Kunen space as well as to that of [3]. However, the first construction of countably irredundant and uncountable Boolean algebra (assuming $\diamondsuit$) was due to Rubin [31] and seems not to have applications in Banach spaces.

Now we are left with the invariants irr, nbiort_2, s^2, biort and we ask if the inequalities between them are strict. As before we have no absolute examples but a few consistent examples required considerable work:

THEOREM 5.6. (ROSŁANOWSKI, SHELAH [29]) *It is consistent that there is a Boolean algebra such that*

$$\omega = \mathrm{irr}(A) < s(K_{\mathcal{A}}^2) = \omega_1.$$

THEOREM 5.7. (BRECH, KOSZMIDER [4]) *It is consistent that there is a Boolean algebra A such that*

$$\omega = \mathrm{irr}(A) = \mathrm{nbiort}_2(K_A) = s(K_{\mathcal{A}}^2) < \mathrm{biort}(K_{\mathcal{A}}) = \omega_1.$$

The above examples on the level of ω and ω_1 cannot be obtained in ZFC because we have

THEOREM 5.8. (TODORCEVIC [37, 38]) *Assume Martin's axiom and the negation of the CH. Suppose $\mathcal{A}$ is an uncountable Boolean algebra, then* $\mathrm{irr}(\mathcal{A})$ *is uncountable.*

However we do not know the answer to the following:

QUESTION 5.9. Is any of the following equalities true, where K stands for compact Hausdorff space and A for a Boolean algebra:

(1) $\mathrm{irr}(A) = \mathrm{nbiort}_2(K_A)$?
(2) $\mathrm{nbiort}_2(K_A) = \mathrm{biort}_2(K_A)$?
(3) $\mathrm{nbiort}_2(K_A) = s(K_A^2)$?

Natural questions related to Todorcevic's result are:

QUESTION 5.10. Assume Martin's axiom and the negation of CH. Suppose that K is a nonmetrizable compact space. Does $C(K)$ have an uncountable nice biorthogonal system?

QUESTION 5.11. Does Martin's axiom imply that $\mathrm{irr}(\mathcal{A}) = |\mathcal{A}|$ for any Boolean algebra of cardinality less than continuum?

The above statement is consistent as proved in [37, Proposition 2].

THEOREM 5.12. (BRECH, KOSZMIDER [4]) *For each natural $n > 1$ it is consistent that there is a compact Hausdorff space K_{2n} such that in $C(K_{2n})$ there is no uncountable $(2n-1)$-supported biorthogonal sequence but there are $2n$-supported biorthogonal systems, i.e.,*

$$\omega = \mathrm{biort}_{2n-1}(K_{2n}) < \mathrm{biort}_{2n}(K_{2n}) = \omega_1 .$$

Here the parity of the integers involved plays an important role, and we do not know the answer to the following:

QUESTION 5.13. Is it consistent that there is an integer $n > 1$ and a compact space K such that

$$\mathrm{biort}_{2n}(K) < \mathrm{biort}_{2n+1}(K)?$$

Beyond the cardinals ω and ω_1 most of the important questions are unsolved as even the following is a well known open problem:

QUESTION 5.14. Is there an absolute example of a Boolean algebra $\mathcal{A}$ such that $\mathrm{irr}(\mathcal{A}) < |\mathcal{A}|$?

6. Semibiorthogonality and hereditary Lindelöf degree

In this section we go well beyond versions of biorthogonality which we considered in the previous sections and consider a well-ordered as well as positive version of it:

Definition 6.1. If κ is an ordinal, a transfinite sequence $(x_i, x_i^*)_{i<\alpha} \subseteq X \times X^*$ is called a semi-biorthogonal sequence if and only if $x_i^*(x_i) = 1$ for all $i \in I$ and $x_i^*(x_j) = 0$ if $j < i < \alpha$ and $x_i^*(x_j) \geq 0$ if $i < j < \alpha$.

$$\mathrm{sbiort}(X) = \sup \left\{ |\kappa| \ : \ \begin{array}{c} \text{there is a semibiorthogonal} \\ \text{system } (x_i, \phi_i)_{i\in\kappa} \subseteq X \times X^* \end{array} \right\}.$$

Semibiorthogonal sequences became important after Borwein and Vanderwerff proved in [2] that the existence of a support set and so the so called Rolewicz's problem from [30] is equivalent to the existence of an uncountable semibiorthogonal sequence. Similar procedure which is applied to the definitions of biorthogonal system and biort to obtain the definitions of a semibiorthogonal sequence and sbiort can be applied to other versions of biorthogonal sequences and biort. In particular we will consider sbiort_n for $n \in \mathbb{N}$. There is a relationship between hL and versions of semibiorthogonality, namely:

Theorem 6.2. (Lazar [21]) *Suppose K is a compact Hausdorff space, then*

$$\mathrm{sbiort}_1(C(K)) = hL(K)\,,$$

and so $hL(K) \leq \mathrm{sbiort}(C(K))$.

Proof. We use Proposition 1.7. Suppose $\{x_\alpha : \alpha < \kappa\} \subseteq K$ and $\{U_\alpha : \alpha < \kappa\}$ is a sequence of open subsets of K such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\beta > \alpha$. Let $f_\alpha : K \to [0,1]$ be a continuous function such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha \restriction (K \setminus U_\alpha) = 0$. Then $(f_\alpha, \delta_{x_\alpha})_{\alpha<\kappa}$ forms a 1-supported semibiorthogonal sequence.

Now if $(f_\alpha, a_\alpha\delta_{x_\alpha})_{\alpha<\kappa}$ is semibiorthogonal, then $a_\alpha \neq 0$. Consider $U_\alpha = \{x \in K : f_\alpha(x) > a_\alpha/2\}$. We have that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ if $\beta > \alpha$, hence $\{x_\alpha : \alpha < \kappa\}$ is not Lindelöf, so $hL(K) \geq \mathrm{sbiort}_1(K)$. ∎

Based on this result and the fact that $hL(K) = w(K)$ for K scattered (see [23, 25.130]), using the Kunen space, and the space obtained in [3] we may conclude the following:

COROLLARY 6.3. (1) *CH implies that there is a Banach space X of the form $C(K)$ such that* $\omega = \mathrm{biort}(X) < \mathrm{sbiort}(X) = \omega_1 = 2^\omega$.

(2) *It is consistent that there is a Banach space X of the form $C(K)$ such that* $\omega = \mathrm{biort}(X) < \mathrm{sbiort}(X) = \omega_2 = 2^\omega$.

However there is no hope that $hL(B_{X^*})$ will play a nontrivial role as we have the following:

PROPOSITION 6.4. *Suppose X is a Banach space with* $\mathrm{dens}(X) = \kappa$, *then there is a sequence $(x_\alpha, x^*_\alpha)_{\alpha<\kappa}$ such that $x^*_\alpha(x_\beta) = 0$ for $\beta < \alpha$ and $x^*(x_\alpha) = 1$. In particular*

$$hL(B_{X^*}) = \mathrm{dens}(X)$$

for any Banach space X.

Proof. Given a closed subspace in a Banach space X and a vector which does not belong to it, one can construct a functional which has value zero on the subspace and value one on the vector. This allows us to construct a transfinite sequence as in the proposition. Then weak* open sets $U_\alpha = \{x^* : x^*(x_\alpha) > 1/2\}$ witness the fact that x^*_αs form a right-separated sequence, and so Proposition 1.7 can be used to conclude the proposition. ∎

The above result can be considered as a Banach space version of Lemma 4.2. Most important recent result on semibiorthogonal sequences is the following:

THEOREM 6.5. (KOSZMIDER; LOPEZ-ABAD, TODORCEVIC) *There is a nonmetrizable compact K without uncountable semibiorthogonal sequences. In particular such that* $\mathrm{sbiort}(C(K)) < \mathrm{dens}(C(K))$.

It shows that Rolewicz's problem is undecidable. The space of [19] is a version of the split interval, and the example of [22, Section 8] of the form $C(K)$ is from [1]. The question whether assuming only CH there is a nonseparable Banach space X satisfying $\mathrm{sbiort}(X) = \omega$ was posed in [7]. There the authors construct under CH a compact space K having the topological and measure theoretic properties of the space from [19] where there are only uncountable semibiorthogonal sequences of a special kind. Thus the question remains open. A related construction is that of [32].

Another very interesting result showing that the behaviour of sbiort is different than that of biort is the following:

Theorem 6.6. (Todorcevic [38, Theorem 9]) *If a Banach space of the form $C(K)$ has density bigger than ω_1, then there is an uncountable semi-biorthogonal sequence in $C(K)$.*

It is asked in [38, Problem 5] if the above result can be generalized to an arbitrary Banach space. This also generates the following

Question 6.7. Is it true that for every Banach space X (of the form $C(K)$) we have

$$\mathrm{dens}(X) \leq \mathrm{sbiort}(X)^+?$$

We also have some results on finitely supported versions of semibiorthogonal sequences:

Theorem 6.8. (Brech, Koszmider) *Suppose $n \geq 4$ is an even integer. It is consistent that there are compact spaces K such that*

$$\omega = \mathrm{sbiort}_{n-1}(X) < \mathrm{biort}_n(X) = \omega_1.$$

The following result shows again that almost biorthogonal sequences are quite far away from the rest. We can even have a Banach space without support sets but with uncountable almost biorthogonal systems:

Proposition 6.9. (Lopez-Abad, Todorcevic, [22]) *It is consistent that there is a Banach space X such that*

$$\mathrm{sbiort}(X) = \mathrm{biort}(X) = \omega < \mathrm{abiort}(X) = \omega_1\,.$$

In particular that $\mathrm{sbiort}(X) < s(B_{X^*})$.

References

[1] M. Bell, J. Ginsburg, S. Todorcevic, Countable spread of exp$Y$ and λY, *Topology Appl.* **14** (1982), 1 – 12.

[2] J.M. Borwein, J.D. Vanderwerff, Banach spaces that admit support sets, *Proc. Amer. Math. Soc.* **124** (3) (1996), 751 – 755.

[3] C. Brech, P. Koszmider, Thin-very tall compact scattered spaces which are hereditarily separable, *Trans. Amer. Math. Soc.* **363** (1) (2011), 501 – 519.

[4] C. Brech, P. Koszmider, On biorthogonal systems whose functionals are finitely supported, *Fund. Math.* **213** (2011), 43 – 66.

[5] P. Dodos, J. Lopez-Abad, S. Todorcevic, Unconditional basic sequences in spaces of large density, *Adv. Math.* **226** (4) (2011), 3297 – 3308.

[6] E. van Douwen, Cardinal functions on Boolean spaces, in "Handbook of Boolean Algebras", Vol. 2 (J. Donald Monk and Robert Bonnet, ed.), North-Holland Publishing Co., Amsterdam, 1989, 417 – 467.

[7] M. Džamonja, I. Juhász, CH, a problem of Rolewicz and bidiscrete systems, *Topology Appl.* **158** (18) (2011), 2458 – 2494.

[8] R. Engelking, "General Topology", Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[9] C. Finet, G. Godefroy, Biorthogonal systems and big quotient spaces, in "Banach Space Theory", Contemp. Math., 85, Amer. Math. Soc., Providence, RI, 1989, 87 – 110.

[10] M. Fabian et al, "Functional Analysis and Infinite-Dimensional Geometry", CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8, Springer-Verlag, New York, 2001.

[11] R. Frankiewicz, G. Plebanek, C. Ryll-Nardzewski, Between Lindelöf property and countable tightness, *Proc. Amer. Math. Soc.* **129** (1) (2001), 97 – 103.

[12] A.S. Granero, M. Jiménez, A. Montesinos, J. Moreno, A. Plichko, On the Kunen-Shelah properties inBanach spaces, *Studia Math.* **157** (2) (2003), 97 – 120.

[13] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, V. Zizler, "Biorthogonal Systems in Banach Spaces", CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 26. Springer, New York, 2008.

[14] L. Heindorf, A note on irredundantsets, *Algebra Universalis* **26** (2) (1989), 216 – 221.

[15] R. Hodel, Cardinal functions, I, in "Handbook of Set-Theoretic Topology", North-Holland, Amsterdam, 1984, 1 – 61.

[16] I. Juhász, Cardinal functions, in "Recent Progress in General Topology" (Miroslav Hušek and Jan van Mill, ed.), North-Holland Publishing Co., Amsterdam, 1992, 417 – 441.

[17] I. Juhász, Z. Szentmiklóssy, On d-separability of powers and $C_p(X)$, *Topology Appl.* **155** (4) (2008), 277 – 281.

[18] S. Koppelberg, "Handbook of Boolean Algebras", Vol. 1 (J. Donald Monk and Robert Bonnet, ed.), North-Holland Publishing Co., Amsterdam, 1989.

[19] P. Koszmider, On a problem of Rolewicz about Banach spaces that admit support sets, *J. Funct. Anal.* **257** (2009), 2723 – 2741.

[20] K. Kunen, "Set Theory. An Introduction to Independence Proofs", Studies in Logic and the Foundations of Mathematics, 102, North-Holland Publishing Co., Amsterdam-New York, 1980.

[21] A.J. Lazar, Points of support for closedconvex sets, *Illinois J. Math.* **25** (2) (1981), 302 – 305.

[22] J. Lopez-Abad, S. Todorcevic, Generic Banach spaces and generic simplexes, *J. Funct. Anal.* **261** (2) (2011), 300 – 386.

[23] J. Monk, "Cardinal Invariants on Boolean Algebras", Progress in Mathematics, 142. Birkhauser Verlag, Basel, 1996.

[24] J. Mujica, personal e-mail message.

[25] S. Negrepontis, Banach spaces and topology, in "Handbook of Set-Theoretic Topology", North-Holland, Amsterdam, 1984, 1045 – 1142.

[26] T. Przymusiński, Products of normal spaces, in "Handbook of Set-Theoretic Topology", North-Holland, Amsterdam, 1984, 781 – 826.

[27] G. Plebanek, On compact spaces carrying random measures of large Maharam type, *Acta Univ. Carolin. Math. Phys.* **43** (2) (2002), 87 – 99.

[28] J. Roitman, Basic S and L, in "Handbook of Set-Theoretic Topology", North-Holland, Amsterdam, 1984, 295 – 326.

[29] A. Rosłanowski, S. Shelah, More on cardinal invariants of Boolean algebras, *Ann. Pure Appl. Logic* **103** (1-3) (2000), 1 – 37.

[30] S. Rolewicz, On convex sets containing only points of support, *Comment. Math. Special Issue* **1** (1978), 279 – 281.

[31] M. Rubin, A Boolean algebra with few subalgebras, interval Boolean algebras and retractiveness, *Trans. Amer. Math. Soc.* **278** (1) (1983), 65 – 89.

[32] S. Shelah, On uncountable Boolean algebras with no uncountable pairwise comparable or incomparable sets of elements, *Notre Dame J. Formal Logic* **22** (1981), 301 – 308.

[33] S. Shelah, Uncountable constructions for B.A., e.c. groups and Banach spaces, *Israel J. Math.* **51** (4) (1985), 273 – 297.

[34] I. Singer, "Bases in Banach Spaces I", Springer-Verlag, New York-Berlin, 1970.

[35] I. Singer, "Bases in Banach Spaces II", Springer-Verlag, Berlin-New York, 1981.

[36] S. Todorcevic, "Partition Problems in Topology", Contemporary Mathematics, 84, American Mathematical Society, Providence, RI, 1989.

[37] S. Todorcevic, Irredundant sets in Boolean algebras, *Trans. Amer. Math. Soc.* **339** (1) (1993), 35 – 44.

[38] S. Todorcevic, Biorthogonal systems and quotient spaces via Baire category methods, *Math. Ann.* **335** (3) (2006), 687 – 715.