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On Invariant Submanifolds of LP-Sasakian Manifolds

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Abstract: The object of the present paper is to find a necessary and sufficient condition for an invariant submanifold of an LP-Sasakian manifold to be totally geodesic. An illustrative example is given to support the obtained result.

Key words: LP-Sasakian manifold, invariant submanifolds, totally geodesic.

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1. Introduction

The notion of Lorentzian almost para-contact manifolds was introduced by K. Matsumoto [6]. In subsequent times, a large number of geometers studied Lorentzian almost para-contact manifold and their different classes, viz., Lorentzian para-Sasakian manifolds and Lorentzian special para-Sasakian manifolds [7], [8], [9], [12]. A beautiful example of a five-dimensional Lorentzian para-Sasakian manifold has been given by Matsumoto, Mihai and Rosaca [8]. In brief, Lorentzian para-Sasakian manifolds are called LP-Sasakian manifolds. The study of LP-Sasakian manifolds is of prime importance due to its relevant applications in the theory of relativity.

In modern analysis, the geometry of submanifolds have become a subject of growing interest for its significant application in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system [5]. Also, the notion of geodesics plays an important role in the theory of relativity [8]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds have also importance in physical sciences. The study of geometry of invariant submanifolds was initiated by A. Bejancu and N. Papaghuic [1]. Again N. Papaghuic has worked on semi-invariant submanifolds of LP-Sasakian manifolds. On the other hand, a number of works on the geometry of submanifolds of LP-Sasakian manifolds have been carried out by U. C. De and collaborators [2], [3],

[4]. B. Prasad [13] also studied semi-invariant submanifolds of LP-Sasakian manifolds. Recently, in [14] the first author of the present paper studied submanifolds of Sasakian manifolds.

In [3], it is proved that a submanifold of an LP-Sasakian manifold is invariant if and only if $B(X,\xi)=0$, where B is the second fundamental form of the submanifold. They also obtained some necessary and sufficient conditions for such submanifolds to be totally umbilical or minimal. In order to enquire into under what condition an invariant submanifold will become totally geodesic, the authors of the paper [10], have obtained some necessary and sufficient conditions under which an invariant submanifold becomes totally geodesic. But in the present paper, interestingly, we prove that every three-dimensional invariant submanifold of an LP-Sasakian manifold is totally geodesic and the converse is also true.

The present paper is organized as follows: In Section 2, we give some preliminaries which have been used later. Section 3 is devoted to prove that a three-dimensional submanifold of an LP-Sasakian manifold is invariant if and only if it is totally geodesic. Section 4 contains an illustrative example to support the results obtained in Section 3.

2. Preliminaries

Let $\tilde{M}$ be an n-dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞-vector valued linear function ϕ, a C^∞-vector field ξ, an one form η and a Lorentzian metric g of type $(0,2)$ such that for each $p\in\tilde{M}$, the tensor $g_p : T_p\tilde{M}\times T_p\tilde{M}\to\mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,+,\dots,+)$ where $T_p\tilde{M}$ denotes the tangent vector space of $\tilde{M}$ at p and $\mathbb{R}$ is the field of real numbers, which satisfies

$$\phi^2X = X+\eta(X)\xi,\qquad \eta(\xi)=-1, \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y),\qquad g(X,\xi)=\eta(X), \tag{2.2}$$

for all vector fields X,Y tangent to $\tilde{M}$. Such structure (ϕ,ξ,η,g) is termed as Lorentzian para-contact [6]. In Lorentzian para-contact structure the following relations hold:

$$\phi\xi=0,\qquad \eta(\phi X)=0,$$

$$\operatorname{rank}\phi=n-1.$$

A Lorentzian para-contact manifold $\tilde{M}$ is called Lorentzian para-Sasakian manifold or LP-Sasakian manifold if [6]

$$(\tilde{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

$$\tilde{\nabla}_X \xi = \phi X, \tag{2.4}$$

for all X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to g. In an LP-Sasakian manifold, we also have that the curvature tensor $\tilde{R}$ of the manifold $\tilde{M}$ is given by

$$\tilde{R}(\xi, X)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.5}$$

Let M be a submanifold immersed in an n-dimensional Riemannian manifold $\tilde{M}$, we denote by the same symbol g the induced metric on M. Let TM be the tangent space of M and $T^{\perp}M$ is the set of all vector fields normal to M. Then Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2.6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.7}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in $T^{\perp}M$. The second fundamental form B and A_N are related by

$$g(A_N X, Y) = g(B(X,Y), N). \tag{2.8}$$

It is also noted that $B(X,Y)$ is bilinear, and since $\nabla_{fX} Y = f\nabla_X Y$, for a C^{∞} function f on a manifold we have

$$B(fX, Y) = fB(X,Y). \tag{2.9}$$

Let us now recall the following:

Definition 2.1. Let M be a submanifold of an n-dimensional LP-Sasakian manifold $\tilde{M}$. The submanifold M of $\tilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M, at every point of M and ϕX is tangent to M for any vector field X tangent to M, at every point on M, that is, $\phi TM \subset TM$ at every point on M.

Definition 2.2. A submanifold of an LP-Sasakian manifold is called totally geodesic if $B(X,Y) = 0$, for any $X, Y \in TM$.

3. Invariant submanifolds of LP-Sasakian manifolds

Proposition 3.1. *Let M be an invariant submanifold of an LP-Sasakian manifold $\tilde{M}$. Then there exist two differentiable orthogonal distributions D and $D^{\perp}$ on M such that*

$$TM = D \oplus D^{\perp} \oplus < \xi >,$$

and

$$\phi(D) \subset D^{\perp}, \quad \phi(D^{\perp}) \subset D.$$

Proof. For an invariant submanifold M, ξ is tangent to M. Hence, we can write $TM = D^1 \oplus < \xi >$. Let $X_1 \in D^1$. Now $g(X_1, \phi X_1) = 0$ and $g(\xi, \phi X_1) = 0$. So, ϕX_1 is orthogonal to X_1 and ξ. Consequently, it is possible to write $D^1 = D \oplus D^{\perp}$, where $X_1 \in D \subset D^1$ and $\phi X_1 \in D^{\perp} \subset D^1$. For $\phi X_1 \in D^{\perp}$, we note that

$$\phi(\phi X_1) = \phi^2 X_1 = X_1 + \eta(X_1)\xi = X_1 \in D.$$

Let $\phi X_1 = X_2 \in D^{\perp}$. Hence, for $X_1 \in D$, $\phi X_1 \in D^{\perp}$ and for $X_2 \in D^{\perp}$, $\phi X_2 \in D$. Hence, the proposition follows. ∎

Proposition 3.2. *For an invariant submanifold M of an LP-Sasakian manifold $\tilde{M}$, we have [3] for the two differentiable tangent vector fields X, Y of M*

$$B(X, \xi) = 0, \tag{3.1}$$

$$B(X, \phi Y) = \phi B(X, Y) = B(\phi X, Y). \tag{3.2}$$

Proposition 3.3. ([10]) *An invariant submanifold of an LP-Sasakian manifold is also LP-Sasakian.*

Let us consider a three-dimensional invariant submanifold M of an LP-Sasakian manifold $\tilde{M}$. Then, M is LP-Sasakian. Now it is obvious that $B(X, Y)$ satisfies

$$\phi^2\big(B(X, Y)\big) = B(X, Y) + \eta\big(B(X, Y)\big)\xi. \tag{3.3}$$

Let $X_1, Y_1 \in D$. Then we have from (3.2)

$$B(X_1, \phi Y_1) = \phi B(X_1, Y_1).$$

Therefore in view of (3.2) and (3.3)

$$\phi B(X_1, \phi Y_1) = \phi^2 B(X_1, Y_1) = B(X_1, Y_1) + \eta\big(B(X_1, Y_1)\big)\xi. \tag{3.4}$$

Since $B(X_1, Y_1) \in T^{\perp}M$, $B(X_1, Y_1)$ is orthogonal to $\xi \in TM$. Hence, we obtain $\eta\big(B(X_1, Y_1)\big) = 0$. Thus, in view of (3.2) and (3.4) it follows that

$$B(\phi X_1, \phi Y_1) = B(X_1, Y_1). \tag{3.5}$$

Let $\phi X_1 = X_2$, $\phi Y_1 = Y_2$. We note that $X_2 = \phi X_1 \in D^{\perp}$ and $Y_2 = \phi Y_1 \in D^{\perp}$. Therefore,

$$B(X_2, Y_2) = B(X_1, Y_1), \tag{3.6}$$

for $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^{\perp}$. Since B is bilinear, for $X_1, Y_1 \in D$ and $X_2, Y_2 \in D^{\perp}$, it follows that

$$B(X_1 + X_2 + \xi, Y_1) = B(X_1, Y_1) + B(X_2, Y_1) + B(\xi, Y_1), \tag{3.7}$$

$$B(X_1 + X_2 + \xi, -Y_2) = -B(X_1, Y_2) - B(X_2, Y_2) - B(\xi, Y_2), \tag{3.8}$$

$$B(X_1 + X_2 + \xi, \xi) = B(X_1, \xi) + B(X_2, \xi) + B(\xi, \xi). \tag{3.9}$$

Keeping in mind that $B(X, \xi) = 0$, for $X \in TM$ and using (3.7), (3.8), and (3.9) we get, by virtue of (3.6)

$$B(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = B(X_2, Y_1) - B(X_1, Y_2).$$

Now,

$$TM = D \oplus D^{\perp} \oplus < \xi >,$$

so, $U = X_1 + X_2 + \xi \in TM$ and $V = Y_1 - Y_2 + \xi \in TM$. Thus, the above equation yields

$$B(U, V) = B(X_2, Y_1) - B(X_1, Y_2). \tag{3.10}$$

From the above equation it follows that

$$\phi B(U, V) = B(X_2, \phi Y_1) - B(\phi X_1, Y_2) = B(X_2, Y_2) - B(X_2, Y_2) = 0.$$

The above equation gives $\phi^2 B(U, V) = 0$. Consequently, $B(U, V) = 0$. Now, we are in a position to state the following:

THEOREM 3.1. *Every three-dimensional invariant submanifold of an LP-Sasakian manifold is totally geodesic.*

Next suppose that the submanifold is totally geodesic. If the submanifold is even dimensional, then it is not invariant, because, an invariant submanifold of an LP-Sasakian manifold is also LP-Sasakian and an even dimensional submanifold can not admit LP-Sasakian structure. Suppose that the submanifold is odd dimensional. Then for $X, Y \in TM$, $B(X,Y) = 0$. Now we shall show that $\phi X \notin T^{\perp}M$. If possible, let ϕX has a component, say FX, along $T^{\perp}M$. For $X, Y \in TM$, we note that $A_{FX}Y \in TM$. Let us denote $A_{FX}Y = Z \neq 0$. Here Z is also not a null vector, in general. Now, by virtue of (2.8) we get

$$g(Z,Z) = g(Z,Z) = g(A_{FX}Y, Z) = g\big(B(Y,Z), FX\big) = g(0, FX) = 0.$$

Since Z is non-null and non-zero vector, the above equation yields a contradiction. This shows that ϕX has no component along $T^{\perp}M$. Hence, $\phi X \in TM$. Therefore, the sub manifold is invariant. The above discussion helps us to state the following:

Theorem 3.2. *Every odd dimensional totally geodesic submanifold of an LP-Sasakian manifold is invariant.*

As a direct consequence of Theorem 3.2, we get the following:

Corollary 3.1. *Every three-dimensional totally geodesic submanifold of an LP-Sasakian manifold is invariant.*

Combining Theorem 3.1 and Corollary 3.1, we obtain

Theorem 3.3. *A three-dimensional submanifold of an LP-Sasakian manifold is totally geodesic if and only if it is invariant.*

4. Example

In this section we like to construct an example of a five-dimensional LP-Sasakian manifold and there on an example of three-dimensional invariant submanifold of the manifold.

Let us consider the 5-dimensional manifold $\tilde{M} = \big\{(x,y,z,u,v) \in \mathbb{R}^5\colon (x,y,z,u,v) \neq (0,0,0,0,0)\big\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \quad e_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. Let g be the metric defined by

$$\begin{aligned} &g(e_i, e_j) = 1, \quad \text{for} \quad i = j \neq 3, \\ &g(e_i, e_j) = 0, \quad \text{for} \quad i \neq j, \\ &g(e_3, e_3) = -1. \end{aligned}$$

Here i and j runs from 1 to 5. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any vector field Z tangent to $\tilde{M}$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \phi(e_4) = e_5, \quad \phi(e_5) = e_4.$$

Then, using the linearity of ϕ and g we have

$$\eta(e_3) = -1, \quad \phi^2 Z = Z + \eta(Z) e_3,$$

for any vector fields Z, W tangent to $\tilde{M}$. Thus for $e_3 = \xi$, $\tilde{M}(\phi, \xi, \eta, g)$ defines an almost para-contact metric manifold. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric g. Then we have

$$\begin{aligned} &[e_1, e_2] = -2e_3, \qquad &&[e_1, e_3] = 0, \qquad &&[e_1, e_4] = 0, \\ &[e_1, e_5] = 0, &&[e_2, e_3] = 0, &&[e_2, e_4] = 0, \\ &[e_2, e_5] = 0, &&[e_3, e_4] = 0, &&[e_4, e_5] = -2e_3. \end{aligned}$$

Taking $e_3 = \xi$ and using Koszul's formula for g, it can be easily calculated that

$$\begin{aligned} &\tilde{\nabla}_{e_1} e_5 = 0, \quad &&\tilde{\nabla}_{e_1} e_4 = 0, \quad &&\tilde{\nabla}_{e_1} e_3 = e_2, \quad &&\tilde{\nabla}_{e_1} e_2 = -e_3, \quad &&\tilde{\nabla}_{e_1} e_1 = 0, \\ &\tilde{\nabla}_{e_2} e_5 = 0, &&\tilde{\nabla}_{e_2} e_4 = 0, &&\tilde{\nabla}_{e_2} e_3 = e_1, &&\tilde{\nabla}_{e_2} e_2 = 0, &&\tilde{\nabla}_{e_2} e_1 = e_3, \\ &\tilde{\nabla}_{e_3} e_5 = e_4, &&\tilde{\nabla}_{e_3} e_4 = e_5, &&\tilde{\nabla}_{e_3} e_3 = 0, &&\tilde{\nabla}_{e_3} e_2 = e_1, &&\tilde{\nabla}_{e_3} e_1 = e_2, \\ &\tilde{\nabla}_{e_4} e_5 = -e_3, &&\tilde{\nabla}_{e_4} e_4 = 0, &&\tilde{\nabla}_{e_4} e_3 = e_5, &&\tilde{\nabla}_{e_4} e_2 = 0, &&\tilde{\nabla}_{e_4} e_1 = 0, \\ &\tilde{\nabla}_{e_5} e_5 = 0, &&\tilde{\nabla}_{e_5} e_4 = e_3, &&\tilde{\nabla}_{e_5} e_3 = e_4, &&\tilde{\nabla}_{e_5} e_2 = 0, &&\tilde{\nabla}_{e_5} e_1 = 0. \end{aligned}$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \phi X$. Hence, it is an LP-Sasakian manifold.

Let f be an isometric immersion from M to $\tilde{M}$ defined by $f(x, y, z) = (x, y, z, 0, 0)$.

Let $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any vector field Z tangent to M. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Then, using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = Z + \eta(Z)e_3,$$

for any vector fields Z, W tangent to M. Thus for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost para-contact metric manifold.

Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then we have

$$[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

Taking $e_3 = \xi$ and using Koszul's formula for the metric g, it can be easily calculated that

$$\begin{aligned}
&\nabla_{e_1} e_3 = e_2, && \nabla_{e_1} e_2 = -e_3, && \nabla_{e_1} e_1 = 0,\\
&\nabla_{e_2} e_3 = e_1, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_1 = e_3,\\
&\nabla_{e_3} e_3 = 0, && \nabla_{e_3} e_2 = e_1, && \nabla e_3 e_1 = e_2.
\end{aligned}$$

We see that the (ϕ, ξ, η, g) structure satisfies the formula $\nabla_X \xi = \phi X$, $\eta(\xi) = -1$. Hence $M(\phi, \xi, \eta, g)$ is a three-dimensional LP-Sasakian manifold. It is obvious that the manifold M under consideration is a submanifold of the manifold $\tilde{M}$.

Let us take $D =< e_1 >$, $D^\perp =< e_2 >$. Then clearly we see that $TM = D \oplus D^\perp \oplus < \xi >$. For any $X \in D$ and $Y \in D^\perp$, we can write $X = \lambda e_1$ and $Y = \mu e_2$, where λ, μ are two scalars. Now, $\phi(\lambda e_1) = \lambda\phi(e_1) = \lambda e_2 \in D^\perp \subset TM$. Similarly, $\phi(\mu e_2) \in D \subset TM$. Hence the submanifold is invariant.

Let $U = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in TM$ and $V = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 \in TM$, where λ_i and μ_i are scalars, $i = 1, 2, 3$.

Then

$$\begin{aligned} B(U,V) = {} & \lambda_1\mu_1 B(e_1,e_1) + \lambda_1\mu_2 B(e_1,e_2) + \lambda_1\mu_3 B(e_1,e_3) \\ & + \lambda_2\mu_1 B(e_2,e_1) + \lambda_2\mu_2 B(e_2,e_2) + \lambda_2\mu_3 B(e_2,e_3) \\ & + \lambda_3\mu_1 B(e_3,e_1) + \lambda_3\mu_2 B(e_3,e_2) + \lambda_3\mu_3 B(e_3,e_3). \end{aligned}$$

From the values of $\tilde{\nabla}_{e_i} e_j$ and $\nabla_{e_1} e_j$ calculated before and from the relation $B(e_i, e_j) = \tilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j$, we see that $B(U,V) = 0$, for all $U, V \in TM$. Hence, the sub manifold is totally geodesic.

The above arguments tell us that the submanifold M under consideration agrees with Theorem 3.3 which is the main result of the present paper.

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