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Transfer Operators on Complex Hyperbolic Spaces

Abdelhamid Boussejra, Tahani Taoufiq

Department of Mathematics, Faculty of Sciences, University Ibn Tofail,
Kenitra, Morocco, a.boussejra@gmail.com.

Department of Mathematics, University of Sciences and Technics,
Lille 1, France, Tahani@math.univ-lille1.fr

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Abstract: Let $\mathbb{B}^n$ be the unit ball in the n-dimensional complex space and let Δ be the Bergman Laplacian on it. For $\lambda \in \mathbb{C}$ such that $|\Re(i\lambda)| < n$ we give explicitly the transfer operator from the space of holomorphic functions $\mathbb{B}^n$ onto an eigenspace $E^+_\lambda(\mathbb{B}^n)$ of Δ. This answers a question raised by Eymard in [2]. As application, for $\lambda = -i\eta$ with $0 < \eta < n$, we get that the classical Hardy space $H^2(\mathbb{B}^n)$ is isometrically isomorphic to the space

$$H^2_\lambda(\mathbb{B}^n) = \left\{ F \in E^+_\eta(\mathbb{B}^n) : \sup_{0<r<1} \left(1-r^2\right)^{\frac{\eta-n}{2}} \left[\int_{\partial\mathbb{B}^n} |F(r\theta)|^2 d\theta \right]^{\frac{1}{2}} < \infty \right\}.$$

Consequently $H^2_\lambda(\mathbb{B}^n)$ is a Banach space.

Key words: Transfer operator, Hardy spaces.

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1. Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$ and let $\Delta = 4(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}$ be the hyperbolic laplacian on D. For $\lambda \in \mathbb{C}$ and $f \in A'(\partial D)$ the space of hyperfunctions on the circle $\partial D = \{z \in \mathbb{C} : |z| = 1\}$, we define the Poisson transform by

$$P_\lambda f(z) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1-|z|^2}{|1-ze^{-i\theta}|^2} \right)^{\frac{i\lambda+1}{2}} f(e^{i\theta}) d\theta. \tag{1.1}$$

Then it is well known that if λ satisfies $\frac{i\lambda+1}{2} \notin \mathbb{Z}^-$, then P_λ is an isomorphism from $A'(\partial D)$ onto the eigenspace $E_\lambda(D)$ of Δ associated to the eigenvenvalue $-(\lambda^2+1)$. In particular, for $\lambda = -i$, the space $E_{-i}(D)$ consists of harmonic functions on D.

Therefore one can define formally an operator τ_λ such that the following diagram

$$\begin{tikzcd} & A'(\partial D) \arrow[dl, "P_{-i}"'] \arrow[dr, "P_\lambda"] & \\ E_{-i}(D) \arrow[rr, "\tau_\lambda"'] & & E_\lambda(D) \end{tikzcd}$$

is commutative.

In [2] P. Eymard gave the operator τ_λ explicitly. More precisely we have

THEOREM 1.1. ([2]) *Let $\lambda \in \mathbb{C}$ such that $|\Re(i\lambda)| < 1$. For $F \in E_{-i}(D)$, set*

$$\tau_\lambda F(z) = \frac{1}{\Gamma\left(\frac{i\lambda+1}{2}\right)\Gamma\left(\frac{1-i\lambda}{2}\right)} \int_0^1 F(tz) \left[\frac{t(1-|z|^2)}{(1-t)(1-t|z|^2)}\right]^{\frac{i\lambda+1}{2}} \frac{dt}{t}. \qquad (1.2)$$

Then the operator τ_λ is an isomorphism from $E_{-i}(D)$ onto $E_\lambda(D)$.

As it is known the Poincaré disk D is a one dimensional (the unique) Riemannian symetric space of the noncompact type and the above Poisson transform (1.1) can be defined for general Riemannian symmetric space X.

For f in $A'(B)$ the space of all hyperfunctions on the Furstenberg boundary B define the Poisson transform P_λ by

$$P_\lambda f(x) = \int_B P_\lambda(x,b) f(b)\, db, \qquad (1.3)$$

where $P_\lambda(x,b) = e^{(i\lambda+\rho)A(x,b)}$ is the Poisson kernel of X, cf [3] for more details.

In the case of a Riemannian symmetric space of rank one, Helgason [3] showed that P_λ is an isomorphism from $A'(B)$ onto an eigenspace $E_\lambda(X)$ of the Laplace-Beltrami operator Δ for λ running some subset Λ of $\mathbb{C}$. In particular for $\lambda = -i\rho$, the eigenspace $E_\rho(X)$ consists of harmonic functions with respect to Δ.

Henceforth, it is natural to look as in the case of the unit disk D for an explicit expression of the operator τ_λ from the space of harmonic functions $E_{-i\rho}(X)$ onto the eigenspace $E_\lambda(X)$, when $\lambda \in \Lambda$.

The operator τ_λ -called the transfer operator by Eymard- has been given explicitly in the case of n-dimensional real hyperbolic space by Sami [6].

One of the difficulties we run into when trying to extend Eymard [2] and Sami [6] results to the case of the complex hyperbolic space is, in our point of view, the relative complexity of the harmonic functions in the complex case rather than in the real one.

A natural question raised by Eymard [2] is, by which class of functions might be the harmonic functions replaced. In this paper, we propose the class of holomorphic functions as substitute.

The organization of this paper is as follows. In Section 2 we recall some results on the Poisson transform on the complex hyperbolic space. In Section 3 we state and prove the main result of our paper. In Section 4 some applications of our result is given. In particular we show that the operator τ_λ is a topological isomorphism from the classical Hilbert space H^2 square integrable holomorphic functions on the complex unit ball onto a class of eigenfunctions of Δ.

2. The Poisson transform

Let $\mathbb{B}^n = \{z \in C^n : |z| < 1\}$ be the bounded realization of the n-dimensional complex hyperbolic space and let $\partial\mathbb{B}^n = \{\omega \in C^n : |\omega| = 1\}$ be the unit sphere of the n-dimensional complex space $\mathbb{C}^n$ with the normalized measure $d\sigma$ on it. The Laplace-Beltrami operator of the complex hyperbolic space $\mathbb{B}^n$ is given by

$$\Delta = \sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \bar{z}_j},$$

where δ_{ij} denotes the Kronecker symbol. For λ a complex number, let $E_\lambda(\mathbb{B}^n)$ be the space of all eigenfunctions of Δ in $\mathbb{B}^n$ with eigenvalue $-(\lambda + n)^2$. Let $A'(\partial\mathbb{B}^n)$ be the space of all hyperfunctions on $\partial\mathbb{B}^n$. Then the Poisson transform is the map

$$P_\lambda : A'(\partial\mathbb{B}^n) \longrightarrow E_\lambda(\mathbb{B}^n),$$

defined by

$$P_\lambda f(z) = \int_{\partial\mathbb{B}^n} \left(\frac{1 - |z|^2}{|1 - z\bar{\omega}|^2} \right)^{\frac{i\lambda + n}{2}} f(\omega) d\sigma(\omega),$$

where $z\overline{\omega} = \sum_{j=1}^{n} z_j \overline{\omega_j}$.

Theorem 2.1. ([3]) *Let $\lambda \in \mathbb{C}$ such that $\frac{i\lambda+n}{2} \notin \mathbb{Z}^-$. Then P_λ is an isomorphism from $A'(\partial\mathbb{B}^n)$ onto $E_\lambda(\mathbb{B}^n)$.*

Now, let $H(p,q)$ denote the space of restrictions to $\partial\mathbb{B}^n$ of harmonic polynomials $h_{pq}(z,\overline{z})$ which are homogeneous of degree p in z and degree q in $\overline{z}$. Then we have $L^2(S) = \oplus_{p,q \geq 0} H(p,q)$. We denote by h^j_{pq} $(1 \leq j \leq d(p,q))$ an orthonormal basis of $H(p,q)$. Note that $H(p,0)$ consists of holomorphic polynomials. To each $f \in A'(\partial\mathbb{B}^n)$ we associate its expansion into spherical harmonics h^j_{pq}

$$f = \sum_{p,q\geq 0}^{+\infty} \sum_{j=1}^{d(p,q)} a_{pq,j} h^j_{pq},$$

with

$$\sum_{p,q=0}^{+\infty} \sum_{j=1}^{d(p,q)} |a_{pq,j}|^2 r^{p+q} < \infty,$$

for every $r \in [0,1[$.

Below, we recall a result on the action of the Poisson transform on $H(p,q)$ which will be useful in the sequel.

Proposition 2.1. ([3]) *Let $\lambda \in \mathbb{C}$ and let f in $H(p,q)$. Then we have*

$$P_\lambda f(z) = \phi_{\lambda,p,q}(|z|) f\left(\frac{z}{|z|}\right),$$

where $\phi_{\lambda,p,q}(|z|)$ is the generalized spherical function associated to the complex hyperbolic space $\mathbb{B}^n$ given by:

$$\phi_{\lambda,p,q}(|z|) = \frac{\left(\frac{i\lambda+n}{2}\right)_p \left(\frac{i\lambda+n}{2}\right)_q}{(n)_{p+q}} |z|^{p+q} \left(1-|z|^2\right)^{\frac{i\lambda+n}{2}}$$
$$ {}_2F_1\left(\frac{i\lambda+n}{2}+p, \frac{i\lambda+n}{2}+q, p+q+n; |z|^2\right).$$

In above $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochammer symbol and ${}_2F_1(a,b,c;x)$ is the classical Gauss hypergeometric function, see [4].

3. The Transfer formula

In this section we state and prove the main result of this paper. For this, let $A'_+(\partial\mathbb{B}^n)$ denote the subspace of $A'(\partial\mathbb{B}^n)$ consisting of all hyperfunctions f such that

$$f = \sum_{p=0}^{+\infty} \sum_{j=1}^{d(p,0)} a_{p,j} h^j_{p0}.$$

For f in $A'(\partial\mathbb{B}^n)$, we define the Cauchy integral of f by

$$\mathcal{C}f(z) = \int_{\partial\mathbb{B}^n} (1 - z\bar{\omega})^{-n} f(\omega) d\sigma(\omega).$$

Let $O(\mathbb{B}^n)$ be the space of all holomorphic functions on the unit ball $\mathbb{B}^n$. Then it is well known that the Cauchy transform $\mathcal{C}$ is an isomorphism from $A'_+(\partial\mathbb{B}^n)$ onto $O(\mathbb{B}^n)$.

Next, let $E^+_\lambda(\mathbb{B}^n) = P_\lambda(A'_+(\partial\mathbb{B}^n))$. From Proposition 2.1, we get

Corollary 3.1. *Let $\lambda \in \mathbb{C}$ and let F be a $\mathbb{C}$-valued function on $\mathbb{B}^n$.Then we have $F \in E^+_\lambda(\mathbb{B}^n)$ if and only if there exists a sequence of complex numbers a_{pj} such that*

$$F(z) = \sum_{p=0}^{+\infty} \sum_{j=1}^{d(p,0)} a_{pj}\phi_{\lambda,p,0}(|z|)h^j_{p0}(z), \tag{3.1}$$

in $C^\infty(\mathbb{B}^n)$.

From now on we will denote the function $\phi_{\lambda,p,0}$ by $\phi_{\lambda,p}$.

Now, we can state the main result of this paper. By $\Gamma(\cdot)$ we denote the usual Euler Gamma function.

Theorem 3.1. *Let λ be a complex number such that $|\Re(i\lambda)| < n$ Then, the operator τ_λ given by*

$$[\tau_\lambda F](z) = \frac{\Gamma(n)}{\Gamma\left(\frac{i\lambda+n}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)} \int_0^1 F(tz)\left[\frac{t(1-|z|^2)}{(1-t)(1-t|z|^2)}\right]^{\frac{i\lambda+n}{2}} (1-t)^{n-1}\frac{dt}{t}.$$

is an isomorphism from $O(\mathbb{B}^n)$ onto the eigenspace $E^+_\lambda(\mathbb{B}^n)$. Moreover the following diagram

$$\begin{tikzcd} & A'_+(\partial\mathbb{B}^n) \arrow[dl, "\mathcal{C}"'] \arrow[dr, "P_\lambda"] & \\ O(\mathbb{B}^n) \arrow[rr, "\tau_\lambda"'] & & E^+_\lambda(\mathbb{B}^n) \end{tikzcd}$$

is commutative.

Proof. Let F be a holomorphic function on $\mathbb{B}^n$. Then

$$F(r\theta) = \sum_{p=0}^{+\infty} \sum_{j=1}^{d(p,0)} a_{pj} r^p h^j_{p0}(\theta),$$

in $C^\infty([0,1[\times\partial\mathbb{B}^n)$. Moreover $F=\mathcal{C}f$ where $f\in A'_+(\mathbb{B}^n)$ is given by

$$f=\sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,0)} a_{pj}h_{p0}^j.$$

Now we have

$$[\tau_\lambda F](r\theta)=\frac{\Gamma(n)}{\Gamma\big(\frac{i\lambda+n}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)}\int_0^1\sum_{p\geq 0}\sum_{j=1}^{d(p,0)} a_{pj}h_{p,0}^j(\theta)(tr)^p \left[\frac{t(1-r^2)}{(1-t)(1-tr^2)}\right]^{\frac{i\lambda+n}{2}}(1-t)^{n-1}\frac{dt}{t}. \tag{3.2}$$

Since for each fixed $\theta\in\partial\mathbb{B}^n$ and $r\in[0,1[$, the series $\sum_{p\geq 0}(tr)^p f_{p,0}(\theta)$ converges uniformly in $t\in[0,1[$ we can reverse the order of sum and integration in (3.2) to get

$$[\tau_\lambda F](r\theta)=\frac{\Gamma(n)}{\Gamma\big(\frac{i\lambda+n}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)}\sum_{p\geq 0}\sum_{j=1}^{d(p,0)} a_{pj}h_{p,0}^j \int_0^1 (tr)^p\left[\frac{t(1-r^2)}{(1-t)(1-tr^2)}\right]^{\frac{i\lambda+n}{2}}(1-t)^{n-1}\frac{dt}{t}. \tag{3.3}$$

Next, since $|\Re(i\lambda)|<n$, we can use the following integral representation of the hypergeometric function (see [4])

$$F(a,b,c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 s^{a-1}(1-s)^{c-a-1}(1-sx)^{-b}dt, \tag{3.4}$$

to see that

$$\frac{\Gamma(n)}{\Gamma\big(\frac{i\lambda+n}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)}\int_0^1 (tr)^p\left[\frac{t(1-r^2)}{(1-t)(1-tr^2)}\right]^{\frac{i\lambda+n}{2}}(1-t)^{n-1}\frac{dt}{t}=\phi_{\lambda,p}(r),$$

from which we obtain

$$[\tau_\lambda F](r\theta)=\sum_{p=0}^{+\infty}\phi_{\lambda,p}(r)\sum_{j=1}^{d(p,0)} a_{p,j}h_{p0}^j(\theta),$$

and since this later sum is none other than $P_\lambda f$, the result follows. This finishes the proof of Theorem 3.1. ∎

4. Application

In this section we use the explicit expression of the transfer operator τ_λ to derive some growth condition on the class of eigenfunctions in $E^+_\lambda(\mathbb{B}^n)$. For this, let us fix some notations. As usual $L^p(\partial\mathbb{B}^n)$ is the space of measurable complex function f on $\partial\mathbb{B}^n$ such that

$$\|f\|_p = \left(\int_{\partial\mathbb{B}^n} |f(\theta)|^p d\sigma(\theta)\right)^{\frac{1}{p}} < \infty.$$

For a $\mathbb{C}$-valued function F on $\mathbb{B}^n$ and $0 < r < 1$, we let F_r denote the function defined on $\partial\mathbb{B}^n$ by $F_r(\theta) = F(r\theta)$. Finally, we denote by $K_\lambda(t,r)$ $(r = |z|)$, the kernel of the operator τ_λ given by:

$$K_\lambda(t,r) = \frac{\Gamma(n)}{\Gamma\left(\frac{i\lambda+n}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)} \left[\frac{t(1-r^2)}{(1-t)(1-tr^2)}\right]^{\frac{i\lambda+n}{2}} \frac{(1-t)^{n-1}}{t}.$$

We first establish a Lemma which will be needed in the sequel.

Lemma 4.1. *Let $\lambda \in \mathbb{C}$ such that $0 < \Re(i\lambda) < n$. Then we have*

$$\sup_{o<r<1}\left((1-r^2)^{-\left(\frac{n-\Re(i\lambda)}{2}\right)}\int_0^1 |K_\lambda(t,r)|dt\right) \le \frac{\Gamma\left(\Re(i\lambda)\right)\Gamma(n)\Gamma\left(\frac{n-\Re(i\lambda)}{2}\right)}{\left|\Gamma\left(\frac{n+i\lambda}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)\right|\Gamma\left(\frac{n+\Re(i\lambda)}{2}\right)}.$$

Proof. We have

$$\int_0^1 K_\lambda(t,r)|dt = \frac{\Gamma(n)}{\left|\Gamma\left(\frac{n+i\lambda}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)\right|}(1-r^2)^{\frac{n+\Re(i\lambda)}{2}} \int_0^1 t^{\frac{n+\Re(i\lambda)}{2}-1}(1-t)^{-\frac{n+\Re(i\lambda)}{2}}(1-tr^2)^{\frac{n-\Re(i\lambda)}{2}-1}dt.$$

By using the formula (3.1) we get

$$\int_0^1 |K_\lambda(t,r)|dt = \frac{\Gamma\left(\frac{n+\Re(i\lambda)}{2}\right)\Gamma\left(\frac{n-\Re(i\lambda)}{2}\right)}{\left|\Gamma\left(\frac{n+i\lambda}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)\right|}(1-r^2)^{\frac{n+\Re(i\lambda)}{2}} F\left(\frac{n+\Re(i\lambda)}{2}, \frac{n+\Re(i\lambda)}{2}, n; r^2\right).$$

Next, use the following identity on hypergeometric functions

$$F(a,b,c;x) = (1-x)^{c-a-b}F(c-a, c-b, c; x),$$

to rewrite the above equality as

$$\int_0^1 |K_\lambda(t,r)|dt = \frac{\Gamma\big(\frac{n+\Re(i\lambda)}{2}\big)\Gamma\big(\frac{n-\Re(i\lambda)}{2}\big)}{\big|\Gamma\big(\frac{n+i\lambda}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)\big|}(1-r^2)^{\frac{n-\Re(i\lambda)}{2}} F\left(\frac{n-\Re(i\lambda)}{2}, \frac{n-\Re(i\lambda)}{2}, n; r^2\right).$$

We have

$$F\left(\frac{n-\Re(i\lambda)}{2}, \frac{n-\Re(i\lambda)}{2}, n; r^2\right) \le F\left(\frac{n-\Re(i\lambda)}{2}, \frac{n-\Re(i\lambda)}{2}, n; 1\right),$$

by $\Re(i\lambda) < n$. Next since $\Re(i\lambda) > 0$ we can use the following well known identity (for $\Re(c-a-b) > 0$)

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

to obtain

$$\int_0^1 |K_\lambda(t,r)|dt \le (1-r^2)^{\frac{n-\Re(i\lambda)}{2}} \frac{\Gamma\big(\Re(i\lambda)\big)\Gamma(n)\Gamma\big(\frac{n-\Re(i\lambda)}{2}\big)}{\big|\Gamma\big(\frac{n+i\lambda}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)\big|\Gamma\big(\frac{n+\Re(i\lambda)}{2}\big)},$$

and the lemma follows. ∎

Proposition 4.1. *Let $p \in]1, +\infty[$, λ a complex number such that $0 < \Re(i\lambda) < n$ and let F be a holomorphic function on $\mathbb{B}^n$. Then we have:*

$$\left(\int_{\partial\mathbb{B}^n} |\tau_\lambda F(r\theta)|^p d\sigma(\theta)\right)^{\frac{1}{p}} \le (1-r^2)^{\frac{n-\Re(i\lambda)}{2}} \frac{\Gamma\big(\Re(i\lambda)\big)\Gamma(n)\Gamma\big(\frac{n-\Re(i\lambda)}{2}\big)}{\big|\Gamma\big(\frac{n+i\lambda}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)\big|\Gamma\big(\frac{n+\Re(i\lambda)}{2}\big)} \|F_r\|_p,$$

for every $r \in [0, 1[$.

Proof. Let F be a holomorphic function on $\mathbb{B}^n$ and let $\phi \in L^q(\partial\mathbb{B}^n)$, with $\frac{1}{q} + \frac{1}{p} = 1$. We have:

$$\left|\int_{\partial\mathbb{B}^n} \tau_\lambda F(r\theta)\overline{\phi(\theta)}d\sigma(\theta)\right| = \left|\int_{\partial\mathbb{B}^n} \left[\int_0^1 K_\lambda(t,r)F(tr\theta)dt\right] \overline{\phi(\theta)}d\sigma(\theta)\right|$$

By using the Fubini Theorem we get

$$\left|\int_{\partial\mathbb{B}^n} \tau_\lambda F(r\theta)\overline{\phi(\theta)}d\sigma(\theta)\right| = \left|\int_0^1 \left[\int_{\partial\mathbb{B}^n} F(tr\theta)\overline{\phi(\theta)}d\sigma(\theta)\right] K_\lambda(t,r)dt\right|,$$

We have

$$\left|\int_{\partial\mathbb{B}^n} F(tr\theta)\overline{\phi(\theta)}d\theta\right| \leq \|F_{tr}\|_p\|\phi\|_q,$$

where for $s \in [0,1[$, $\|F_s\|_p = \left(\int_{\partial\mathbb{B}^n} |F(s\theta)|^p d\sigma(\theta)\right)^{\frac{1}{p}}$.
Thus

$$\left|\int_{\partial\mathbb{B}^n} \tau_\lambda F(r\theta)\overline{\phi(\theta)}d\sigma(\theta)\right| \leq \left(\int_0^1 \|F_{tr}\|_p|K_\lambda(t,r)|dt\right)\|\phi\|_q.$$

Since for F a holomorphic function on $\mathbb{B}^n$, the function $s \to \|F_s\|_p$ is increasing in $s \in [0,1[$ we obtain

$$\left|\int_{\partial\mathbb{B}^n} \tau_\lambda F(r\theta)\overline{\phi(\theta)}d\sigma(\theta)\right| \leq \|\phi\|_q\|F_r\|_p \int_0^1 |K_\lambda(t,r)|dt.$$

Taking the supremum over all $\phi \in L^q(\partial\mathbb{B}^n)$ with$\|\phi\|_q \leq 1$, we get

$$\left|\int_{\partial\mathbb{B}^n} \tau_\lambda F(r\theta)\overline{\phi(\theta)}d\sigma(\theta)\right| \leq \|F_r\|_p \int_0^1 |K_\lambda(t,r)|dt.$$

Next, use the estimate in Lemma 4.1 to get the right hand side estimate in Proposition 4.1 and the proof is finished. ∎

Let p be a real number with $p > 1$ and let

$$H^p(\mathbb{B}^n) = \left\{F \in O(\mathbb{B}^n) : \|F\|_p = \sup_{0<r<1}\|F_r\|_p < \infty\right\},$$

be the classical Hardy space on the unit complex ball of $\mathbb{C}^n$. Then using the above proposition we get the following characterization of eigenfunctions $\Psi \in E^+_\lambda(\mathbb{B}^n)$ which are transform by τ_λ of functions on the Hardy space $H^p(\mathbb{B}^n)$. More precisely

Corollary 4.1. *Let $\lambda \in \mathbb{C}$ such that $0 < \Re(i\lambda) < n$ and let $\Psi \in E^+_\lambda\mathbb{B}^n)$ such that $\Psi = \tau_\lambda F$ with $F \in H^p(\mathbb{B}^n)$. Then the eigenfunction Ψ satisfies the following growth condition of Hardy type:*

$$\sup_{0<r<1}(1-r^2)^{-\left(\frac{n-\Re(i\lambda)}{2}\right)}\left[\int_{\partial\mathbb{B}^n}|\Psi(r\theta)|^p d\sigma(\theta)\right]^{\frac{1}{p}} < \infty.$$

Moreover we have

$$\sup_{0<r<1}(1-r^2)^{-\left(\frac{n-\Re(i\lambda)}{2}\right)}\left[\int_{\partial\mathbb{B}^n}|\Psi(r\theta)|^p d\sigma(\theta)\right]^{\frac{1}{p}} \leq \frac{\Gamma\big(\Re(i\lambda)\big)\Gamma(n)\Gamma\big(\frac{n-\Re(i\lambda)}{2}\big)}{\big|\Gamma\big(\frac{n+i\lambda}{2}\big)\Gamma\big(\frac{n-i\lambda}{2}\big)\big|\Gamma\big(\frac{n+\Re(i\lambda)}{2}\big)}\|F\|_p.$$

Now we will prove the converse of the above result in the case of $p = 2$. For this we introduce, for $\lambda \in \mathbb{C}$, the following Hardy type space $H^2_\lambda(\mathbb{B}^n)$ for eigenfunctions of the Laplacian Δ of the complex hyperbolic space:

$$H^2_\lambda(\mathbb{B}^n) = \Big\{ \Psi \in E^+_\lambda(\mathbb{B}^n) : \\ \|\Psi\|_{\lambda,2} = \sup_{0<r<1} (1-r^2)^{-\left(\frac{n-\Re(i\lambda)}{2}\right)} \left[\int_{\partial\mathbb{B}^n} |\Psi(r\theta)|^2 d\sigma(\theta) \right]^{\frac{1}{2}} < \infty \Big\}.$$

THEOREM 4.1. *Let $\lambda \in \mathbb{C}$ such that $0 < \Re(i\lambda) < n$. Then the transfer operator τ_λ is a topological isomorphism from the Hardy $H^2(\mathbb{B}^n)$ onto $H^2_\lambda(\mathbb{B}^n)$. Moreover we have*

$$\left| \frac{\Gamma(n)\Gamma(i\lambda)}{\Gamma^2\left(\frac{i\lambda+n}{2}\right)} \right| \|F\|_2 \le \|\tau_\lambda F\|_{\lambda,2} \le \frac{\Gamma\big(\Re(i\lambda)\big)\Gamma(n)\Gamma\left(\frac{n-\Re(i\lambda)}{2}\right)}{\left|\Gamma\left(\frac{n+i\lambda}{2}\right)\Gamma\left(\frac{n-i\lambda}{2}\right)\right|\Gamma\left(\frac{n+\Re(i\lambda)}{2}\right)} \|F\|_2. \tag{4.1}$$

For the proof of the above theorem we will need the following result giving the asymptotic behavior of the generalized spherical function $\phi_{\lambda,p}(r)$ as r goes to 1^-.

LEMMA 4.2. *Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. Then we have*

$$\phi_{\lambda,p}(r) \sim \frac{\Gamma(n)\Gamma(i\lambda)}{\Gamma^2\left(\frac{n+i\lambda}{2}\right)} (1-r^2)^{\frac{n-i\lambda}{2}},$$

as r goes to 1^-, uniformly in p.

We postpone the proof of this lemma to the end of this section.

Proof of Theorem 4.1. The necessary condition follows from Corollary 4.1. To prove the sufficiency condition, let $\Psi \in H^2_\lambda(\mathbb{B}^n)$. Then, by Corollary 3.1, there exists a sequence of complex numbers a_{pj} such that

$$\Psi(r\theta) = \sum_{p=0}^{+\infty} \sum_{j=1}^{d(p,0)} \phi_{\lambda,p}(r) a_{pj} h^j_{p,0}(\theta).$$

Since $\|\Psi\|_{\lambda,2} < \infty$,we have

$$(1-r^2)^{-\left(\frac{n-\Re(i\lambda)}{2}\right)} \left[\sum_{p=0}^{+\infty} \sum_{j=1}^{d(p,0)} |a_{pj}|^2 |\phi_{\lambda,p}(r)|^2 \right]^{\frac{1}{2}} \le \|\Psi\|_{\lambda,2} < \infty,$$

for every $r \in [0,1[$. Next, using the uniform asymptotic behavior in p of $\phi_{\lambda,p}(r)$, given by Lemma 4.2, we obtain

$$\left|\frac{\Gamma(n)\Gamma(i\lambda)}{\Gamma^2\left(\frac{n+i\lambda}{2}\right)}\right| \left[\sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,o)} |a_{pj}|^2\right]^{\frac{1}{2}} \leq \|\Psi\|_{\lambda,2} < \infty.$$

Therefore the sum $\sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,o)} a_{pj}h_{p0}^j$ defines a function f in the space $L^2(\partial\mathbb{B}^n)$. In fact f is in the Hardy space $H^2(\partial\mathbb{B}^n)$ on the boundary $\partial\mathbb{B}^n$, since

$$H^2(\partial\mathbb{B}^n) = \bigoplus_{p=0}^{+\infty} H(p,0).$$

Next let $F = Cf$. Then F is in the Hardy space $H^2(\mathbb{B}^n)$, since the Cauchy operator maps $H^2(\partial\mathbb{B}^n)$ to the Hardy space $H^2((\mathbb{B}^n)$, see Rudin [5].

We have

$$\tau_\lambda F(r\theta) = \int_0^1 \sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,o)} a_{pj}h_{p0}^j(\theta)(tr)^p(\theta)K_\lambda(t,r)dt.$$

Thus

$$\tau_\lambda F(r\theta) = \sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,o)} a_{pj}h_{p0}^j(\theta)\int_0^1 (tr)^p K_\lambda(t,r)dt,$$

by the uniform convergence in $t \in [0,1[$ of the series $\sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,o)} a_{pj}h_{p0}^j(\theta)(tr)^p$.

Now, recall that

$$\int_{\partial\mathbb{B}^n} (tr)^p K_\lambda(t,r)dt = \phi_{\lambda,p}(r),$$

from which we obtain

$$[\tau_\lambda F](r\theta) = \sum_{p=0}^{+\infty}\sum_{j=1}^{d(p,o)} a_{pj}\phi_{\lambda,p}(r)h_{p0}^j(\theta).$$

Therefore $\tau_\lambda F = \Psi$ and the proof of Theorem 4.1 is finished. ∎

Letting $\lambda = -i\eta$ with $0 < \eta < n$, in the above theorem we get

Corollary 4.2. *Let $\eta \in \mathbb{R}$ such that $0 < \eta < n$. Then the transfer operator $\tau_{-i\eta}$ is a topological isomorphism from the Hardy space $H^2(\partial\mathbb{B}^n)$ onto $H^2_{-i\eta}(\mathbb{B}^n)$, with*

$$\left|\frac{\Gamma(n)\Gamma(\eta)}{\Gamma\left(\frac{n+\eta}{2}\right)}\right| \|F\|_2 = \|\tau_{-i\eta}F\|_{2,\lambda}.$$

Consequently $H^2_\lambda(\mathbb{B}^n)$ is a Banach space.

Now we give the proof of Lemma 4.2 giving the asymptotic behaviour of the generalized spherical function.

Proof of Lemma 4.2. Recall that

$$\phi_{\lambda,p}(r) = \frac{\left(\frac{i\lambda+n}{2}\right)_p}{(n)_p} r^p \left(1-r^2\right)^{\frac{i\lambda+n}{2}} F\left(\frac{i\lambda+n}{2}+p, \frac{i\lambda+n}{2}, p+n; r^2\right).$$

Then using the following identity on hypergeometric functions (see [4]):

$$\begin{aligned} F(a,b,c;x) = {} & \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1;1-x) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-b-a} \\ & \qquad F(c-a,c-b,c-a-b+1;1-x), \end{aligned}$$

we can rewrite $\phi_{\lambda,p}$ as

$$\begin{aligned} \phi_{\lambda,p}(r) = {} & \frac{\left(\frac{i\lambda+n}{2}\right)_p}{(n)_p} r^p \frac{\Gamma(n+p)\Gamma(-i\lambda)}{\Gamma\left(\frac{n-i\lambda}{2}+p\right)\Gamma\left(\frac{n-i\lambda}{2}\right)} (1-r^2)^{\frac{i\lambda+n}{2}} \\ & \qquad F\left(\frac{n+i\lambda}{2}, \frac{n+i\lambda}{2}+p, i\lambda+1; 1-r^2\right) \\ & + \frac{\Gamma(i\lambda)\Gamma(n)}{\Gamma^2\left(\frac{i\lambda+n}{2}\right)}\left(1-r^2\right)^{\frac{n-i\lambda}{2}} \\ & \qquad F\left(\frac{n-i\lambda}{2}, \frac{n-i\lambda}{2}+p, -i\lambda+1; 1-r^2\right). \end{aligned}$$

and the result follows.

Remark 4.1. Notice that along the lines of the above proof we can established the following result which is interesting in its own:

Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$ and let $\Psi \in E_\lambda(\mathbb{B}^n)$. Then Ψ has an L^2-Poisson integral representation over the boundary $\partial\mathbb{B}^n$ if and only if its satisfies the following growth condition

$$\sup_{0<r<1} \left(1-r^2\right)^{-\left(\frac{n-\Re(i\lambda)}{2}\right)} \left[\int_{\partial\mathbb{B}^n} |\Psi(r\theta)|^2 d\sigma(\theta)\right]^{\frac{1}{2}} < \infty.$$

This will be proved elsewhere, see also [1] for similar results on characterization of L^p-Poisson integrals on rank one symmetric spaces.

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