Several Open Problems in Operator Theory

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Abstract: We report on the meeting *Operators in Banach spaces* recently held in Castro Urdiales as a homage to Pietro Aiena, and we collect the questions proposed by the participants during the Open Problems Session.

Key words: operator theory, open problems.

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The conference "Operators on Banach spaces" was held in Castro Urdiales, a town located in the west of Cantabria (a region in the north of Spain), during the second week of June 2013. It was the tenth edition of a series of meetings about Banach spaces and operator theory organised by the Universities of Extremadura and Cantabria.

Operator theory is the research field of Pietro Aiena, to whom this meeting paid a well-deserved homage. He began his research career with several papers on Fredholm theory around 1980, and soon he concentrated his efforts in the study of the spectral theory of multipliers. Around 1990 he discovered the usefulness of local spectral theory, and he contributed to its development and successfully applied it to the study of the spectrum of special classes of operators on Banach spaces. A good part of this research until 2002 can be found in his monograph [5], and he has continued doing research on this topic since then. He was the supervisor of 7 Ph. D. Theses, and MathSciNet shows more than 90 items authored by him.

The invited speakers of the conference were José Bonet (Valencia), Cristina Câmara (Lisboa), Robin Harte (Dublin), Francisco L. Hernández (Madrid),

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Teresa Malheiro (Braga), Martin Mathieu (Belfast), Alfonso Montes Rodríguez (Sevilla), Vladimír Müller (Praha) and Florian Vasilescu (Lille). There were also several sessions of short communications and plenty of discussion time. Among the attendants we could meet many friends of Pietro Aiena, including mathematicians from Canada, México, Japan, Poland, Germany, Belgium, France, Italy and Spain. Moreover Jesús Ramón Guillén Ruiz and Pedro Leonardo Peña Duarte, from Universidad de los Andes (Mérida), represented the mathematicians from Venezuela whose Ph. D. Theses have been supervised by Pietro Aiena.

The venue of the conference was the Centro Cultural y de Congresos "La Residencia", close to the fishing port of Castro Urdiales, which is the customary venue of the activities of the Centro Internacional de Encuentros Matemáticos (CIEM), a joint initiative of the University of Cantabria and the City Council of Castro Urdiales.

The organisers of the conference were Jesús M. F. Castillo (Badajoz), Manuel González (Santander) and Javier Pello (Madrid). They acknowledge financial support from the Universities of Cantabria and Palermo, and from the town council of Castro Urdiales.

For additional information we refer to the official web page of the conference:

http://www.ciem.unican.es/encuentros/banach/2013/

Open problems session

Here we collect the questions that were posed and discussed during the Open Problems Session of the conference. We thank the participants for their interest and enthusiasm, which provided a stimulating working environment.

1. Hypercyclic and chaotic operators on Fréchet spaces
José Bonet (Universidad Politécnica de Valencia)

We begin by introducing the concepts involved in this part. We refer to [13] for additional information.

Definition 1. Let T be an operator acting on a Fréchet space X.

The operator T is called *hypercyclic* if there exists a point $x \in X$ such that the set $\{T^n x : n \in \mathbb{N}\}$ is dense in X.

A point $x \in X$ is called a *periodic point of T* if there is some $n \geq 1$ such that $T^n x = x$.

The operator T is called *chaotic* if it is hypercyclic and has a dense set of periodic points.

Hypercyclic operators behave in a different way when they are defined on a Banach space or on a non-normable Fréchet space in the following sense:

1. Let X be a Banach space. Given an operator $T \in L(X)$ and a complex number λ such that $\|\lambda T\| \leq 1$, obviously λT is not hypercyclic.

2. Let D be the differentiation operator, which is defined on the space of entire functions $H(\mathbb{C})$ by $Df = f'$. Then λD is hypercyclic for any $\lambda \in \mathbb{C} \setminus \{0\}$.

QUESTION 1. Suppose that E is a non-normable, separable Fréchet space. Is there $T \in L(E)$ such that λT is hypercyclic for any $\lambda \in \mathbb{C} \setminus \{0\}$?

Some partial results were given by Frerick and Peris.

PROPOSITION 1. [10] Every separable Fréchet space E without a continuous norm admits a continuous linear operator $T \in L(E)$ such that λT is hypercyclic for each $\lambda \neq 0$.

QUESTION 2. Let E be a norm-normable and separable Fréchet space. Is there an infinite-dimensional subspace $\mathcal{A} \subset L(E)$ such that T is hypercyclic for any non-zero operator $T \in \mathcal{A}$?

Aron observed that the answer of Question 2 is positive for the Fréchet spaces $H(\mathbb{C})$ and $\mathcal{C}^\infty(\mathbb{R})$, as follows from results by Godefroy and Shapiro [11]. It also follows from results of Bès and Conejero [6] that this is the case for the Fréchet space ω of all complex sequences.

Bonet, Martínez and Peris [8] proved that there exists a Banach space X that admits no chaotic operator.

QUESTION 3. Which Fréchet spaces admit a chaotic operator?

Recently, de la Rosa, Frerick, Grivaux and Peris [9] proved that Banach spaces with an unconditional basis and Fréchet spaces with an unconditional basis and a continuous norm admit a chaotic operator; thus giving a partial answer to Question 3.

Some of these questions appear in [7].

2. A QUESTION ON THE ROTUNDITY OF BALLS
KAZIMIERZ GOEBEL (MARIA CURIE-SKLODOWSKA UNIVERSITY)

The following questions have over 20 years of history. They were first asked in 1989, at the Fixed Point and Applications Conference in Marseille, Luminy.

Let $(X, \|\cdot\|)$ be a Banach space.

The standard method of measuring *"the rotundity"* of the unit ball in X is through *the modulus of convexity* $\delta_X : [0, 2] \to [0, 1]$ of X

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| \, : \, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}$$

and *the characteristic of convexity*

$$\varepsilon_0(X) := \sup\{\varepsilon \, : \, \delta_X(\varepsilon) = 0\} \, .$$

The modulus of convexity has "two-dimensional character", meaning that

$$\delta_X(\varepsilon) = \inf\{\delta_E(\varepsilon) \, : \, E \subset X \, , \, \dim E = 2\} \, .$$

It is known that the Hilbert space H is the most rotund space among all Banach spaces X, in the sense that

$$\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = \delta_{E_2}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \, ,$$

where E_2 is the two-dimensional Euclidean space.

Now, fix $a \in [0, 2)$ and consider the class $\mathcal{E}_a$ of all two-dimensional spaces $(E, \|\cdot\|)$ having $\varepsilon_0(E) = a$. Which of these spaces is the most rotund?

It can be formulated in the following questions:

QUESTION 4. Given $\varepsilon \in [a, 2)$, what is $\sup\{\delta_E(\varepsilon) \, : \, E \in \mathcal{E}_a\}$?

QUESTION 5. Does there exist a space $E_a \in \mathcal{E}_a$ such that $\delta_E(\varepsilon) \leq \delta_{E_a}(\varepsilon)$ for all $E \in \mathcal{E}_a$?

QUESTION 6. If the answer to the above is positive, is such a space E_a in some sense unique?

For additional information we refer to [12, page 35].

3. SPECTRA OF THE RESTRICTION AND QUOTIENT MAPS ASSOCIATED TO A PARTICULAR SUBSPACE
ROBIN HARTE (TRINITY COLLEGE DUBLIN)

Let X be a complex Banach space and let $T \in L(X)$ be a bounded operator. We consider the point spectrum and the surjective spectrum of T which are defined as follows:

$$\pi(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not injective}\},$$

$$\pi'(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\},$$

and the subspace

$$Y := E_X(T) := \sum_{\lambda \in \mathbb{C}} (\lambda - T)^{-\infty}(0) = \bigcup_{0 \neq p \in \mathcal{P}} p(T)^{-1}(0),$$

where $\mathcal{P}$ is the set of all polynomials.

The set Y is a linear subspace not necessarily closed but could be dense. Let us denote by $T_Y : Y \to Y$ and $T'_Y : X/Y \to X/Y$ the restriction and the quotient operator associated to T and Y.

QUESTION 7. Are the sets $\pi(T)$, $\pi(T_Y)$ and $\pi'(T_Y)$ equal?

QUESTION 8. Is $\pi(T'_Y) = \varnothing$?

QUESTION 9. What can we say about $\pi(T)$ and $\pi'(T)$ when Y is dense?

4. SPECTRAL CHARACTERISATIONS OF JORDAN HOMOMORPHISMS
MARTIN MATHIEU (QUEEN'S UNIVERSITY BELFAST)

Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple Banach algebras. A linear operator $T : \mathcal{A} \to \mathcal{B}$ is called

1. *spectrally bounded* if there exists $M \geq 0$ such that for all $a \in \mathcal{A}$

$$r(Ta) \leq Mr(a),$$

where $r(\cdot)$ denotes the spectral radius;

2. a *spectral isometry* if for all $a \in \mathcal{A}$ it holds that $r(Ta) = r(a)$;

3. *unital* if $T1 = 1$.

KAPLANSKY'S QUESTION (1970): If T is unital and surjective such that $\sigma(Ta) = \sigma(a)$ for all $a \in \mathcal{A}$, where $\sigma(\cdot)$ denotes the spectrum, does it follow that T is a "Jordan homomorphism", that is, $T(a^2) = T(a)^2$ for all $a \in \mathcal{A}$?

MATHIEU'S CONJECTURE (2001): Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If T is a unital surjective spectral isometry, then T is a Jordan homomorphism.

PROPOSITION 2. ([15]) Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear mapping. Then T is unital, surjective and spectrally bounded if and only if there are a Jordan automorphism S of $M_n(\mathbb{C})$ and a non-zero complex number γ such that

$$Tx = \gamma\, Sx + (1-\gamma)\, \tau(x), \qquad x \in M_n(\mathbb{C}),$$

where τ denotes the normalised centre-valued trace on $M_n(\mathbb{C})$.

COROLLARY 1. The conjecture holds for $T : \mathcal{A} \longrightarrow \mathcal{M}_n$ with $\mathcal{A}$ a C^*-algebra.

COROLLARY 2. ([16]) The conjecture holds for $T : \mathcal{A} \longrightarrow \mathcal{B}$ with $\dim(\mathcal{B}) < \infty$.

QUESTION 10. Does the above proposition hold if $M_n(\mathbb{C})$ is replaced by a von Neumann factor of type II_1?

A positive answer would imply that the above conjecture holds for all finite von Neumann factors.

5. EXISTENCE OF A UNIVERSAL CLASS OF m-ISOMETRIES
VLADIMÍR MÜLLER (CZECH ACADEMY OF SCIENCES)

A bounded linear operator T acting on a Hilbert space X is called an *m-isometry* if it satisfies the following identity:

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0,$$

where T^* denotes the adjoint operator of T.

We refer to [1, 2, 3, 4] for the study of m-isometries on Hilbert spaces.

QUESTION 11. Is there a universal operator for all m-isometries? Equivalently, is there an m-isometry $A \in L(K)$ such that for any m-isometry T on a separable Hilbert space H there exist a subspace $M \subset K$ invariant for A such that T is unitarily equivalent to $A|M$?

6. OPERATORS ON ℓ_∞ WITH DENSE RANGE
AMIR-BAHMAN NASSERI (BERGISCHE UNIVERSITÄT WUPPERTAL)

Tauberian operators were introduced in [14] as those bounded operators $T : X \to Y$ between Banach spaces X and Y such that the second conjugate satisfies $T^{**-1}(Y) = X$.

The following question is related with that of the existence of tauberian operators $T : L^1(0,1) \to L^1(0,1)$ with non-closed range.

QUESTION 12. Let $T : L^1(0,1) \to L^1(0,1)$ be a bounded operator. Suppose that for all the subspaces M of $L^1(0,1)$ that are isometric to $\ell^1(\mathbb{N})$, the restriction $T|_M$ is an isomorphism (i.e., bounded below). Is T itself an isomorphism?

It is possible to show that the previous question admits an equivalent formulation:

QUESTION 13. Let $T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be a bounded operator with dense range. Is T surjective?

The latter formulation of the question was posed in MathOverflow [17]. Thanks are due to W.B. Johnson for some useful comments.

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