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On a Generalization of Hankel Operators via Operator Equations

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Abstract: In this paper, the notion of (λ, μ)-Hankel operators on the space H^2 is introduced. Along with discussion of some of its properties, the paper also presents a result for (λ, μ)-Hankel operators which is similar to the classical theorem of Kronecker known for Hankel operators.

Key words: Analytic Toeplitz operator, λ-Hankel operator, spectrum, essential spectrum.

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1. Historical background

The Hardy space H^2 of analytic functions in the open unit disk $\mathbb{D}$ is defined as

$$H^2 = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

It is clear that the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of H^2 is identified with the vector $f = (a_0, a_1, a_2, \ldots)$ of ℓ^2, the Hilbert space of all one-sided square summable complex sequences and vice-versa. Let $\{e_n\}_{n=0}^{\infty}$ denote the standard basis of ℓ^2, where e_n is the sequence consisting of value zero at every place except the n^{th} place, where its value is 1. The space ℓ^2 can be identified with the Hardy space H^2 and we write $e_n = e_n(z) = z^n$. Let μ denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$ (the boundary of $\mathbb{D}$) and L^2 the Hilbert space of all complex-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\int |f|^2 d\mu < \infty.$$

It is customary to identify the functions of H^2 with the space of their boundary functions (see [1, 11]). The boundary functions correspond to those functions in L^2 whose negative Fourier coefficients vanish. With this identification, H^2

is a closed subspace of L^2. We denote the orthogonal complement of H^2 in L^2 by $H^{2\perp}$, that is, $L^2 = H^2 \oplus H^{2\perp}$. A function $f \in L^2$ is said to be analytic if $f \in H^2$ and co-analytic if $f \in L^2 \ominus zH^2$. The space of all essentially bounded measurable functions on $\mathbb{T}$ is denoted by L^∞ and the space of all bounded analytic functions on $\mathbb{T}$ by H^∞. For basic facts and details about these spaces we refer to [5], [7], [10] and [11].

The unilateral shift U on H^2 is the multiplication operator induced by $\phi(z) = z$, i.e., given by $Uf(z) = zf(z)$. This on ℓ^2 act as the unilateral forward shift given by $Ue_n = e_{n+1}, n \geq 0$, whose adjoint is the backward shift defined as $U^*e_n = e_{n-1}$ for $n \geq 1$ and $U^*e_0 = 0$.

A Hankel operator H is a bounded linear operator on H^2 whose matrix with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$ is constant along each diagonal perpendicular to the main one. In fact, the definition of Hankel operators on H^2 turns out as (see Nehari Theorem in [14]): A Hankel operator H on H^2 is defined as $H = PJM_\phi$ for some $\phi \in L^\infty$, where P is the orthogonal projection of L^2 to H^2, J is the operator on L^2 given by $Jf(z) = f(\overline{z})$ and M_ϕ is the multiplication operator defined as $M_\phi f(z) = \phi(z)f(z)$. In this terminology H is said to be induced by the symbol $\phi \in L^\infty$ and is denoted as $H = H_\phi$.

An intimate class to the class of Hankel operators is the class of Toeplitz operators introduced by Toeplitz [13], which have the matrices with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$ constant along each diagonal parallel to the main one. It can be shown [4] that a Toeplitz operator T on H^2 is always expressed as $T = PM_\phi$ for some $\phi \in L^\infty$ and represented as $T = T_\phi$. Toeplitz operator $T = T_\phi$ is called analytic (co-analytic) if the symbol ϕ is analytic (co-analytic).

It is also seen that (see [4, 11]) Hankel and Toeplitz operators are defined in term of operator equations as follows:

An operator H on H^2 is Hankel if and only if it satisfies the equation

$$U^*H = HU.$$

An operator T on H^2 is Toeplitz if and only if it satisfies the equation

$$U^*TU = T.$$

These characterizations of Hankel and Toeplitz operators lead to the study of various generalizations. Douglas [6] and Pták [9] discussed the solutions of the equations $S^*XT = X$ and $S^*X = XT$ respectively when S and T are contractions. Barría and Halmos [3] in 1982 focussed the attention of mathematicians towards a new direction by proposing the operator equation

$U^*XU = \lambda X$ for an arbitrary complex number λ. In particular, when $\lambda = 1$, the solutions of this operator equation are nothing but the Toeplitz operators. In 1984, this equation was solved completely by Sun [12]. Motivated by the work of Barría and Halmos, Avendaño [1] described the solutions of the equations $\lambda U^*X = XU$ and $U^*X - XU = \lambda X$. The solutions of latter equation are named as λ-Hankel operators by Avendaño [1]. Clearly a 0-Hankel operator is a Hankel operator. The following result of Avendaño [1] is quite useful in our study.

PROPOSITION 1.1. *Let A be an operator on H^2 with $\|A\| < 1$. Then an operator X on H^2 is a solution of the equation $AX = XU$ if and only if X is compact and has the following form*

$$X = \sum_{n=0}^{\infty} (A^n \phi) \otimes e_n$$

for some $\phi \in H^2$.

Motivated by the study of these mathematicians, we consider the operator equation $\mu U^*X - XU = \lambda X$ for complex numbers λ and μ and call an operator X satisfying this equation as (λ, μ)-Hankel operator.

For the last few years, many interesting results have been obtained about various generalizations of Hankel operators. A complete characterization of compact Hankel operators is given by Hartman [8]. One of the earliest results on Hankel operators is Kronecker's theorem that describes the Hankel operators of finite rank as the Hankel operators with rational symbols. We refer to [10], [11], [13] and the references therein to provide a nice survey over the historical growth, details and applications of these operators. This paper identifies the complex numbers λ and μ for which the equation $\mu U^*X - XU = \lambda X$ admits of a non-zero solution. It presents some properties of the solutions of this equation and also provides a theorem similar to the Kronecker's theorem for (λ, μ)-Hankel operators. Connections between (λ, μ)-Hankel operators and λ-Hankel operators are derived which also provide alternate method to prove the main results of the paper. Solutions of the equations on some other setting of U and its adjoint are also discussed and at the end we present some problems that are yet to be answered.

Throughout the paper, the word operator is used in reference to a bounded linear transformation on a Hilbert space. For an operator A on a Hilbert space, the symbols $Ker(A)$ and $Ran(A)$ are respectively used to denote the kernel and range spaces of A.

2. On a generalization of Hankel operator

Let K_ω denote the reproducing kernel function of H^2 at $\omega \in \mathbb{D}$, that is,

$$K_\omega(z) = \frac{1}{1-\bar{\omega}z} = \sum_{n=0}^{\infty} \bar{\omega}^n z^n.$$

Then for each $f \in H^2$, $f(z) = \langle f, K_z \rangle$ and it is easy to verify that $U^* K_z e_n = \overline{z} K_z e_n$ for each n. Given two functions f and g in H^2, $f \otimes g$ denotes the rank one operator on L^2 defined as

$$(f \otimes g)(h) = \langle h, g \rangle f$$

for $h \in L^2$. Then $\|f \otimes g\| = \|f\|_2 \|g\|_2$ and $(f \otimes g)^* = g \otimes f$. Also, if S and T are operators on H^2, then $S(f \otimes g)T = Sf \otimes T^* g$.

Now, in this section we try to investigate the solutions of some equations that involve U, the shift operator and its adjoint. The solutions of one of such equations are defined as (λ, μ)-Hankel operators and we formulate the definition as:

Definition 2.1. For a fixed pair (λ, μ) of complex numbers, an operator X on H^2 is said to be (λ, μ)-Hankel operator if it is a solution of the equation $\mu U^* X - XU = \lambda X$.

It is customary to talk of many other symmetric ways to the equation $\mu U^* X - XU = \lambda X$, like, the equation $\mu U^* X - \lambda XU = X$. The only solution to the latter equation for $\lambda = \mu = 0$ is the zero operator, however the existence of non-zero (λ, μ)-Hankel operators can be seen in this situation. This allows us to talk of the equation $\mu U^* X - XU = \lambda X$ for having a wider range of non-zero solutions.

It is clear that $(0, \lambda)$-Hankel operators are the solutions of the equation $\lambda U^* X = XU$ discussed by Avendaño [1]. Also, the class of $(0, 1)$-Hankel operators is the class of Hankel operators and the class of $(\lambda, 1)$-Hankel operators is precisely the class of λ-Hankel operators.

We point out that reversing the order of U and its adjoint in the equation $\mu U^* X - XU = \lambda X$ results in only trivial solutions.

Theorem 2.2. *If X is an operator satisfying $\mu UX - XU^* = \lambda X$ for a pair (λ, μ) of complex numbers then $X = 0$.*

Proof. If $(\lambda, \mu) = (0, 0)$ then we have $XU^* = 0$ and hence $X = 0$. Suppose that $(\lambda, \mu) \neq (0, 0)$. Now we apply [1, Theorem 2.1] on the facts $(\mu U - \lambda I)X = XU^*$ and $Ker(\mu U - \lambda I) = \{0\}$ to conclude that $X = 0$. ∎

It is also seen that there is no non-zero (λ, μ)-Hankel operator, when $\mu = 0$ and $|\lambda| \geq 1$. We need the following lemma to prove this.

Lemma 2.3. *If X is an operator on H^2 satisfying $XU = \lambda X$ for some complex number $|\lambda| = 1$ then $X = 0$.*

Proof. $XU = \lambda X$ implies $Xe_{n+1} = \lambda^{n+1} Xe_0$ for each $n \geq 0$. If possible $Xe_0 \neq 0$. Let a_k be a non-zero Fourier coefficient of Xe_0. Then the sequence formed by the k^{th} row of the matrix of X does not converge to zero. This contradicts the boundedness of X. This proves $Xe_0 = 0$ and hence $X = 0$. ∎

Theorem 2.4. *If X is an operator satisfying $\mu U^* X - XU = \lambda X$ for complex numbers $|\lambda| \geq 1$ and $\mu = 0$, then $X = 0$.*

Proof. Let $-XU = \lambda X$. If $|\lambda| > 1$ then $-\lambda$ does not belong to the spectrum of U. Now $U + \lambda I$ is an invertible operator and hence $X(U + \lambda I) = 0$ provides $X = 0$. If $|\lambda| = 1$ then we use the above lemma to conclude $X = 0$. ∎

Now we make a look over the equations $\mu UX - XU = \lambda X$. If $|\lambda| + |\mu| < 1$ then on substituting $A = \mu U - \lambda I$ in Proposition 1.1, we have that an operator X is a solution of $\mu UX - XU = \lambda X$ if and only if X is compact and has the following form

$$X = \sum_{n=0}^{\infty} (\mu e^{i\theta} - \lambda)^n \phi \otimes e_n$$

for some $\phi \in H^2$. However, for the study of the case when $|\lambda| + |\mu| > 1$, we need the following lemma, the ideas for which we got through the ideas suggested to Avendaño [1] by Peter Rosenthal.

Lemma 2.5. *Suppose the numbers λ and $0 \neq \mu$ are such that $|\lambda| + |\mu| > 1$. If $f \in H^2$ is such that $\|(\mu U - \lambda)^n f\| \leq K$ for all n and some fixed $K > 0$ then $f = 0$.*

Proof. We choose $\epsilon > 0$ (which is always possible) such that the set $A^{\epsilon}_{(\lambda,\mu)} = \{\theta \in [0, 2\pi) : |\mu e^{i\theta} - \lambda| > 1 + \epsilon\}$ has positive measure. Then for each n,

$$K^2 \geq \|(\mu U - \lambda)^n f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\mu e^{i\theta} - \lambda|^{2n} |f(e^{i\theta})|^2 d\theta$$

$$\geq \frac{1}{2\pi}\int_{A^{\epsilon}_{(\lambda,\mu)}} |\mu e^{i\theta} - \lambda|^{2n} |f(e^{i\theta})|^2 d\theta \geq \frac{1}{2\pi}(1+\epsilon)^{2n}\int_{A^{\epsilon}_{(\lambda,\mu)}} |f(e^{i\theta})|^2 d\theta$$

This yields that $\int_{A^{\epsilon}_{(\lambda,\mu)}} |f(e^{i\theta})|^2 d\theta = 0$ and so $f(e^{i\theta}) = 0$ for $\theta \in A^{\epsilon}_{(\lambda,\mu)}$. Since $A^{\epsilon}_{(\lambda,\mu)}$ has positive measure and $f \in H^2$ hence it is possible only if $f = 0$. ∎

THEOREM 2.6. *Suppose the numbers λ and $0 \neq \mu$ are such that $|\lambda| + |\mu| > 1$. Then the only solution of $\mu UX - XU = \lambda X$ is the zero solution.*

Proof. If X satisfies the equation $\mu UX - XU = \lambda X$ then we have $(\mu U - \lambda I)^n X = XU^n$ for each $n \geq 0$. It means $(\mu U - \lambda I)^n X e_0 = X e_n$ for all n. Thus Xe_0 satisfies the hypothesis of Lemma 2.5 with $K = \|X\|$. Hence $Xe_0 = 0$, which implies $Xe_n = 0$ for $n \geq 0$. This completes the proof. ∎

If X is an operator on H^2 that satisfies $Xe_m = (-\lambda)^m Xe_0$ for $m \geq 0$ and $Xe_0 \in \overline{[e_0]}$, the closed linear subspace generated by e_0, for some complex number λ (certainly $|\lambda| < 1$) then it can be seen that X is a solution of the equation $\mu U^* X - XU = \lambda X$ for each complex number μ. This observation motivates to find the pairs (α, β) and (λ, μ) of complex numbers so that we have unique operator (the zero operator) satisfying both the equations $\mu U^* X - XU = \lambda X$ and $\beta U^* X - XU = \alpha X$. The following theorem provides a sufficient condition for this.

THEOREM 2.7. *Let (α, β) and (λ, μ) be distinct pairs of complex numbers.*

1) *If $|\lambda - \alpha| \geq |\mu - \beta|$ then a non-zero (λ, μ)-Hankel operator can not be (α, β)-Hankel.*
2) *If $|\lambda - \alpha| < |\mu - \beta|$ and X is both (λ, μ)-Hankel and (α, β)-Hankel then it is a compact operator of the form $X = \sum_{n=0}^{\infty} \gamma^n e_n \otimes \phi$ for some $\phi \in H^2$ and $|\gamma| < 1$.*

Proof. If X satisfies $\mu U^* X - XU = \lambda X$ and $\beta U^* X - XU = \alpha X$ then $\hat{\beta} U^* X = \hat{\alpha} X$, where $\hat{\alpha} = \lambda - \alpha$ and $\hat{\beta} = \mu - \beta$. First we consider part (1), i.e., when $|\hat{\alpha}| \geq |\hat{\beta}|$. We consider the following two subcases.

Subcase (i): Let $\hat{\beta} = 0$. Then $\hat{\alpha} \neq 0$ and hence $\hat{\alpha} X = 0$, which implies $X = 0$.
Subcase (ii): Let $\hat{\beta} \neq 0$ and $\hat{\alpha} \neq 0$. Then we have $U^* X = \gamma X$, where $\gamma = \frac{\hat{\alpha}}{\hat{\beta}}$ satisfies $|\gamma| \geq 1$. Now by applying Theorem 2.4, we get $X^* = 0$ and so $X = 0$.

Now we consider part (2), i.e., $|\hat{\alpha}| < |\hat{\beta}|$. Then X satisfies $\overline{\gamma} X^* = X^* U$, where $\gamma = \frac{\hat{\alpha}}{\hat{\beta}}$. Hence, by applying Proposition 1.1, we get that $X = \sum_{n=0}^{\infty} \gamma^n e_n \otimes \phi$ for some $\phi \in H^2$. ∎

Now we discuss some basic properties of (λ, μ)-Hankel operators. It is evident that the class of (λ, μ)-Hankel operators is a subspace of $\mathfrak{B}(H^2)$, the space of all bounded operators on H^2. Moreover, it is easy to verify that the class of (λ, μ)-Hankel operators is closed in the strong operator topology. The following result shows that adjoint of a (λ, μ)-Hankel operator is an operator of same type for some other values of λ and μ.

THEOREM 2.8. *The adjoint of a (λ, μ)-Hankel operator, $\mu \neq 0$, is a $(\frac{-\bar{\lambda}}{\bar{\mu}}, \frac{1}{\bar{\mu}})$-Hankel operator.*

Proof. Proof follows on taking the adjoint of the operators on both sides in the equation $\mu U^* X - XU = \lambda X$. ∎

An immediate result that follows by using Theorem 2.8 and Theorem 2.7 is the following.

COROLLARY 2.9. *If X is a self adjoint (λ, μ)-Hankel operator then either $X = 0$ or X is a compact operator given by $X = \sum_{n=0}^{\infty} \gamma^n e_n \otimes \phi$ for some $\phi \in H^2$ and complex number γ.*

Proof. Using Theorem 2.8, if X is self adjoint operator then it is (λ, μ)-Hankel as well as (α, β)-Hankel operator, where $\alpha = -\frac{\overline{\lambda}}{\bar{\mu}}$ and $\beta = \frac{1}{\bar{\mu}}$. Now we apply Theorem 2.7 to complete the proof. ∎

THEOREM 2.10. *The kernel of a (λ, μ)-Hankel operator, $\mu \neq 0$, is an invariant subspace of U and the closure of its range is an invariant subspace of U^*.*

Proof. Let X be satisfying $\mu U^* X - XU = \lambda X$. Then for each $f \in Ker(X)$, $XUf = (\mu U^* - \lambda I) X f = 0$ so that $U(Ker(X)) \subseteq Ker(X)$.

Now X^* is $(\frac{-\bar{\lambda}}{\bar{\mu}}, \frac{1}{\bar{\mu}})$-Hankel operator and by the arguments used earlier, $Ker(X^*)$ is invariant under U. Thus $Ker(X^*)^{\perp} = \overline{Ran(X)}$ is invariant subspace of U^*. ∎

It is interesting to see that a non-zero (λ,μ)-Hankel operator cannot be invertible. In fact, in the next result, it is shown that both, the spectrum and the essential spectrum of a non-zero (λ,μ)-Hankel operator contain 0.

THEOREM 2.11. *Let X be a non-zero (λ,μ)-Hankel operator. Then $0 \in \sigma(X) \cap \sigma_e(X)$, where $\sigma(X)$ and $\sigma_e(X)$ denote the spectrum and the essential spectrum of X respectively.*

Proof. Suppose X is a non-zero (λ,μ)-Hankel operator. Then $XU = (\mu U^* - \lambda I)X$. Now invertibility of X implies U and $\mu U^* - \lambda I$ are similar, which is not feasible. Hence X cannot be invertible or $0 \in \sigma(X)$.

Again, if we suppose that $\sigma_e(X)$ does not contain 0, then X is essentially invertible (i.e. $X + \mathfrak{K}(H^2)$ is invertible in the Calkin algebra $\mathfrak{B}(H^2)/\mathfrak{K}(H^2)$, where $\mathfrak{K}(H^2)$ is the collection of all compact operators on H^2). Now $XU = (\mu U^* - \lambda I)X$ gives that U and $\mu U^* - \lambda I$ are essentially similar. This is not possible being $\sigma_e(U) = \sigma_e(U^*) = \mathbb{T}$, the unit circle. This proves that $0 \in \sigma_e(X)$. ∎

THEOREM 2.12. *Let X be a (λ,μ)-Hankel operator. If T is a analytic Toeplitz operator then XT is a (λ,μ)-Hankel operator.*

Proof. T, being analytic Toeplitz, satisfies $UT = TU$. A simple computation verifies that $\mu U^*(XT) - (XT)U = (\mu U^* X - XU)T = \lambda(XT)$. This completes the proof. ∎

If T' is a co-analytic Toeplitz operator then $U^*T' = T'U^*$. Hence, along the lines of the proof of Theorem 2.12, we can show that the product $T'X$ of T' and a (λ,μ)-Hankel operator X is a (λ,μ)-Hankel operator.

THEOREM 2.13. *Let α,β,λ,μ be complex numbers all of modulus one. If X is a (λ,μ)-Hankel operator then $Y = D_{\frac{1}{\overline{\alpha}\lambda\beta\overline{\mu}}} X D_{\frac{1}{\overline{\alpha}\lambda}}$ is a (α,β)-Hankel operator, where D_a is the diagonal unitary operator defined as $D_a e_n = a^n e_n$ for $n \geq 0$.*

Proof. For each complex number a with $|a| = 1$, $D_{\frac{1}{a}}$ satisfies $D_{\frac{1}{a}}U^* = aU^*D_{\frac{1}{a}}$ and $UD_{\frac{1}{a}} = aD_{\frac{1}{a}}U$. Now using these properties, a routine computation shows that

$$\begin{aligned} D_{\frac{1}{\overline{\alpha}\lambda\beta\overline{\mu}}}(\mu U^*X - XU)D_{\frac{1}{\overline{\alpha}\lambda}} &= D_{\frac{1}{\overline{\alpha}\lambda\beta\overline{\mu}}}(\mu U^*X)D_{\frac{1}{\overline{\alpha}\lambda}} - D_{\frac{1}{\overline{\alpha}\lambda\beta\overline{\mu}}}(XU)D_{\frac{1}{\overline{\alpha}\lambda}} \\ &= \overline{\alpha}\lambda(\beta U^*Y - YU) \end{aligned}$$

and
$$D_{\frac{1}{\overline{\alpha}\lambda\beta\overline{\mu}}}(\lambda X)D_{\frac{1}{\overline{\alpha}\lambda}} = \lambda D_{\frac{1}{\overline{\alpha}\lambda\beta\overline{\mu}}}XD_{\frac{1}{\overline{\alpha}\lambda}} = \lambda Y.$$
If X is a (λ,μ)-Hankel operator then $\mu U^*X - XU = \lambda X$ yields that $\overline{\alpha}\lambda(\beta U^*Y - YU) = \lambda Y$ equivalently $\beta U^*Y - YU = \alpha Y$. This completes the proof. ∎

3. Main results

For an operator T on a Hilbert space, we denote the spectrum of T by $\sigma(T)$. We begin this section with an attempt to find the existence of solutions of the equation $\mu U^*X - XU = \lambda X$ for given λ and μ. The main result provides the locations of complex numbers λ and μ for which the operator equation $\mu U^*X - XU = \lambda X$ must admit of a bounded solution. However, in the next section, we discuss an alternate method to handle these situations.

Theorem 3.1. *The operator equation $\mu U^*X - XU = \lambda X$ has a solution in each of the following cases:*

1) $\mu = 0$, $|\lambda| < 1$;
2) $0 < |\mu| \leq 1$, $|\lambda| < 2|\mu|$;
3) $|\mu| > 1$, $|\lambda| < 2$;
4) $|\mu| < 1$, $|\lambda| > 1 + |\mu|$;
5) $|\mu| > 1$, $|\lambda| > 1 + |\mu|$.

In fact, in the cases (4) *and* (5), *the only solution is the zero solution.*

Proof. Case (1): Let $\mu = 0$ and $|\lambda| < 1$. In this case solutions are assured by Proposition 1.1 and are of the form $X = \sum_{n=0}^{\infty}(-\lambda)^n\phi \otimes e_n$ for some $\phi \in H^2$.

Case (2): Let $0 < |\mu| \leq 1$ and $|\lambda| < 2|\mu|$. Pick a complex number a with $|a| < 1$ and $|\mu a - \lambda| < 1$ (this is possible). Now the rank one operator $X = K_{\overline{a}} \otimes K_{\mu a - \lambda}$ satisfies
$$\mu U^*X - XU = \mu a(K_{\overline{a}} \otimes K_{\mu a-\lambda}) - (\mu a - \lambda)(K_{\overline{a}} \otimes K_{\mu a-\lambda}) = \lambda X.$$
Case (3): Let $|\mu| > 1$ and $|\lambda| < 2$. If X is a solution of the equation $\mu U^*X - XU = \lambda X$ then it satisfies the equation $\beta U^*X^* - X^*U = \alpha X^*$, where $\alpha = -\frac{\overline{\lambda}}{\overline{\mu}}$ and $\beta = \frac{1}{\overline{\mu}}$. Now we apply case (2) and find that $X^* = K_{\overline{a}} \otimes K_{\beta a-\alpha}$ for some complex number $a \in \mathbb{D}$. As a consequence $X = K_{\frac{a}{\overline{\mu}}+\frac{\overline{\lambda}}{\overline{\mu}}} \otimes K_{\overline{a}}$.

Case (4): Let $|\mu| < 1$ and $|\lambda| > 1 + |\mu|$. It is clear from case (2) that for $|\mu| < 1$, each complex number λ with $|\lambda| < 2|\mu|$ is an eigen value of the operator $\tau_\mu : \mathfrak{B}(H^2) \mapsto \mathfrak{B}(H^2)$ defined as

$$\tau_\mu(X) = \mu U^* X - XU$$

for each $X \in \mathfrak{B}(H^2)$. Thus

$$\{z \in \mathbb{C} : |z| < 2|\mu|\} \subseteq \sigma(\tau_\mu) \subseteq \{z \in \mathbb{C} : |z| \leq |\mu| + 1\}.$$

It turns out if λ is such that $|\lambda| > 1 + |\mu|$ then it can not be in the spectrum of τ_μ. This provides that the only solution of the equation $\mu U^* X - XU = \lambda X$ is the zero solution.

Case (5): Let $|\mu| > 1$ and $|\lambda| > 1 + |\mu|$. In this case, we consider the operator $\tau_{\frac{1}{\mu}} : \mathfrak{B}(H^2) \mapsto \mathfrak{B}(H^2)$ defined as

$$\tau_{\frac{1}{\mu}}(X) = U^* X - \frac{1}{\mu} XU$$

for each $X \in \mathfrak{B}(H^2)$. Then

$$\sigma(\tau_{\frac{1}{\mu}}) \subseteq \left\{ z \in \mathbb{C} : |z| \leq \frac{1}{|\mu|} + 1 \right\}.$$

Now $|\frac{\lambda}{\mu}| > \frac{1}{|\mu|} + 1$ and hence $(\tau_{\frac{1}{\mu}} - \frac{\lambda}{\mu} I)(X) = 0$ implies that $X = 0$. This gives that $\mu U^* X - XU = \lambda X$ means that $X = 0$. ∎

An immediate observation to the cases (4), (5) of the above theorem is the following.

Corollary 3.2. *There are no non-zero (λ, μ)-Hankel operators in each of the following cases:*

1) $|\mu| < 1$, $|\lambda| \geq 2$;
2) $|\mu| > 1$, $|\lambda| \geq 2|\mu|$.

Now, observe that, given $\mu \neq 0$ and $\lambda \in \mathbb{C}$ one can find $a \in \mathbb{D}$ such that $\mu a - \lambda \in \mathbb{D}$ if and only if $|\lambda| < 1 + |\mu|$. We have been suggested by the referee about this observation, which provide the following result.

Theorem 3.3. *If $\mu \neq 0$ and $|\lambda| < 1 + |\mu|$, then there are non-zero (λ, μ)-Hankel operators.*

Proof. Choose $a \in \mathbb{D}$ such that $\mu a - \lambda \in \mathbb{D}$. Then $K_{\overline{a}} \otimes K_{\mu a - \lambda}$ is a non-zero (λ, μ)-Hankel operator. ∎

At the end of this section, we present a result for (λ, μ)-Hankel operators similar to the classical theorem of Kronecker, which states that a Hankel matrix is of finite rank if and only if its symbol is a rational function. Theorem 4.6 presents an alternative proof for this result. Now we drop the assumption of bounded on using the word operator i.e. the operator X satisfying $\mu U^* X - XU = \lambda X$ is not necessarily assumed to be bounded.

THEOREM 3.4. *Let X be a (λ, μ)-Hankel operator. Then the matrix representation of X with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ of H^2 has finite rank if and only if Xe_0 is a rational function.*

Proof. Let X be a (λ, μ)-Hankel operator. Then $Xe_n = (\mu U^* - \lambda)^n \phi$ for each $n \geq 0$, (hence X, as a densely defined operator on the polynomials, is uniquely determined by Xe_0) where $\phi = Xe_0$. A simple computation shows that

$$\begin{aligned}\langle Xe_m, e_n \rangle &= \langle XU^m e_0, e_n \rangle = \langle Xe_0, (\mu U - \lambda)^m e_n \rangle \\ &= \sum_{k=0}^{m} d_{m,k} (-\lambda)^{m-k} \mu^k \langle Xe_0, e_{n+k} \rangle\end{aligned}$$

where $d_{m,k} = \frac{m(m-1)\cdots(m-k+1)}{k(k-1)\cdots 2\cdot 1}$. Also, $Xe_n = (\mu U^* - \lambda)^n \phi$ implies that the columns of the matrix of X are just the vectors $(\mu U^* - \lambda)^n \phi$. Hence, the matrix of X is of finite rank at most N if and only if there exist constant numbers $a_0, a_1, a_2, \ldots, a_N$, not all zero, such that

$$\sum_{n=0}^{N} a_n (\mu U^* - \lambda)^n \phi = 0.$$

Now

$$\begin{aligned}\sum_{n=0}^{N} a_n (\mu U^* - \lambda)^n \phi &= \sum_{n=0}^{N} a_n \sum_{k=0}^{N} d_{n,k} (-\lambda)^{n-k} \mu^k U^{*k} \phi \\ &= \sum_{k=0}^{N} \left(\sum_{n=k}^{N} a_n d_{n,k} (-\lambda)^{n-k} \mu^k \right) U^{*k} \phi \\ &= \sum_{k=0}^{N} d_k U^{*k} \phi\end{aligned}$$

where $d_k = \sum_{n=k}^{N} a_n d_{n,k}(-\lambda)^{n-k}\mu^k$. Thus we have $d_0 = d_1 = d_2 = \cdots = d_N = 0$ if and only if $a_0 = a_1 = a_2 = \cdots = a_N = 0$.

As a consequence, we get that the vectors $\{U^{*k}\phi\}_{k=0}^{N}$ are linearly dependent if and only if the vectors $\{(\mu U^* - \lambda)^k\phi\}_{k=0}^{N}$ are so. But, the matrix whose columns are formed by the vectors $\{U^{*k}\phi\}$ is a Hankel operator with symbol ϕ and hence the Hankel operator with symbol ϕ is of finite rank if and only if X is of finite rank. Now the result follows by applying the classical theorem of Kronecker for Hankel operators. ∎

In the Theorem 3.2, boundedness of X is not discussed so it is quite appealing to know $\phi = Xe_0$ so that X (densely defined on polynomials) satisfying $\mu U^*X - XU = \lambda X$ is bounded.

4. Connection between (λ, μ)-Hankel and λ-Hankel operators

In this section, we begin with the results that form bridges between the notions of (λ, μ)-Hankel and λ-Hankel operators. Most of the results obtained in this section are communicated to us by the referee. If $\mu \in \mathbb{C}$ then D_μ given by $D_\mu e_n = \mu^n e_n$, satisfies

$$\begin{aligned} D_\mu U^* e_n &= \begin{cases} 0 & \text{if } n = 0 \\ \mu^{n-1} e_{n-1} & \text{if } n \geq 1 \end{cases} \\ &= \frac{1}{\mu} U^* D_\mu e_n \end{aligned}$$

and

$$U D_\mu e_n = \mu^n e_{n+1} = \frac{1}{\mu} D_\mu U e_n.$$

For the boundedness of D_μ, we need $|\mu| \leq 1$. Now we have the following.

Theorem 4.1. *Let $\mu \in \mathbb{C}$ and $0 < |\mu| \leq 1$.*

1) *If X is a (λ, μ)-Hankel operator then $D_\mu X$ is a λ-Hankel operator.*
2) *If Y is a $\frac{\lambda}{\mu}$-Hankel operator then $Y D_\mu$ is a (λ, μ)-Hankel operator.*

Proof. We use the properties of D_μ, in the equations obtained by multiplying the equations $\mu U^*X - XU = \lambda X$ and $U^*Y - YU = \frac{\lambda}{\mu} Y$ respectively on left and right by D_μ. It provide the proof of (1) and (2). ∎

Analogously, it can be shown that

THEOREM 4.2. *Let* $\mu \in \mathbb{C}$ *and* $|\mu| \geq 1$.

1) *If* X *is a* (λ, μ)*-Hankel operator then* $XD_{\frac{1}{\mu}}$ *is a* $\frac{\lambda}{\mu}$*-Hankel operator.*
2) *If* Y *is a* λ*-Hankel operator then* $D_{\frac{1}{\mu}}Y$ *is a* (λ, μ)*-Hankel operator.*

We use these theorems to see the existence of (λ, μ)-Hankel operators already discussed in Theorem 3.1(2),(3).

COROLLARY 4.3. *There exist non-zero* (λ, μ)*-Hankel operators in each of the situations* $0 < |\mu| \leq 1$, $|\lambda| < 2|\mu|$ *and* $|\mu| > 1$, $|\lambda| < 2$.

Proof. In one situation $|\frac{\lambda}{\mu}| < 2$ and in other $|\lambda| < 2$. Existence of $\frac{\lambda}{\mu}$-Hankel operators and λ-Hankel operators follow by applying [1, Theorem 4.3]. Now results follow on utilizing Theorem 4.1(2) and Theorem 4.2(2) in respective situations. ∎

For a fixed λ, a λ-Hankel operator can be generated from a (λ, μ)-Hankel operator by taking $\mu = 1$, but it can be seen that there is a large choice of values for μ to get a λ-Hankel operator from the (λ, μ)-Hankel operators.

COROLLARY 4.4. ([1]) *There exist infinite* λ*-Hankel operators for* $|\lambda| < 2$.

Proof. It is evident that for a fixed λ with $|\lambda| < 2$, we can find μ in abundance satisfying $|\lambda| < 2|\mu|$, $|\mu| \leq 1$. Now by Theorem 3.1(2), we find a (λ, μ)-Hankel operator X_μ corresponding to each μ and we apply Theorem 4.1(1) to find λ-Hankel operators in abundance as each $D_\mu X_\mu$ is a λ-Hankel operator. ∎

Further applications of these theorems, provide an alternative, in fact, a shorter proof of Theorem 3.3.

Alternate proof of Theorem 3.3*:* We prove that a (λ, μ)-Hankel operator is of finite rank if and only if Xe_0 is a rational function. We divide the proof in two cases.

Case 1: Suppose that $0 < |\mu| \leq 1$. If X is of finite rank, then $D_\mu X$, which is a λ-Hankel operator, is also of finite rank. Now by applying [1, Theorem 5.3], $D_\mu Xe_0$ is a rational function and hence Xe_0 is a rational function.

Conversely, assume that Xe_0 is a rational function. Then $D_\mu Xe_0$ is a rational function and hence the λ-Hankel operator $D_\mu X$ has finite rank. Since D_μ has dense range, we have X has finite rank.

Case 2: Suppose $|\mu| > 1$. Now Xe_0 is rational if and only if $XD_{\frac{1}{\mu}}e_0 = Xe_0$ is rational. Also, X is of finite rank if and only if $XD_{\frac{1}{\mu}}$ is of finite rank. The result follows using the fact that D_μ has dense range. This completes the proof. ∎

Theorem 3.1 and Theorem 3.2 do not discuss the existence of (λ, μ)-Hankel operators once $|\lambda| = 1 + |\mu|$. In this case if $|\mu| = 1$ then $|\lambda| = 2$ and as the only λ-Hankel operator for $|\lambda| = 2$ is the zero operator [2], so in the light of Theorem 4.1(1) we get that there are only zero (λ, μ)-Hankel operators once. But it is yet not known whether there exist non-zero (λ, μ)-Hankel operators or not if $|\lambda| = 1 + |\mu|$ and $|\mu| \neq 1$.

As the solutions that are found in each of the cases discussed in Theorem 3.1, are compact operators so it is interesting to know whether non-compact solutions exist or not, particularly, in cases (2) and (3) of the Theorem 3.1. If $\mu = 1$ then the case shifts to find the λ-Hankel operator and it is seen in [1] that if λ with $|\lambda| < 2$ is a purely imaginary complex number then we have non-compact solutions. Further, if $|\mu| = 1$ and λ with $|\lambda| < 2$ is a purely imaginary complex number then existence of non-compact (λ, μ)-Hankel operators can be seen by using Theorem 4.2(2).

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