# A Note on Self-dual Cones in Hilbert Spaces

SACHINDRANATH JAYARAMAN

*School of Mathematics*
*Indian Institute of Science Education and Research Thiruvananthapuram*
*CET Campus, Engineering College P.O.*
*Thiruvananthapuram– 695 016, Kerala, India*
*sachindranathj@gmail.com, sachindranathj@iisertvm.ac.in*

*Abstract*: A result of Tam says that if a nonnegative matrix $A$ has a nonnegative generalized inverse $X$ (that is, $X$ satisfies the equation $AXA = A$) then, $A(\mathbb{R}^n_+) = R(A) \cap \mathbb{R}^m_+$ and are simplicial (the image of the nonnegative orthant under an invertible linear map). Although in general, a simplicial cone need not be self-dual, there is another inner product with respect to which it is self-dual. The aim of this note to bring out an analoge of this in infinite dimensional separable Hilbert spaces, although there is no notion of simpliciality in such spaces.

*Key words*: Self-dual / regular cone, nonnegative reflexive generalized inverse, Riesz basis.

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## 1. INTRODUCTION

The geometry of a convex cone in a Banach space plays a crucial role in understanding a linear operator that leaves it invariant. For instance, an interesting result due to Lozanovsky says that if $X_1, X_2$ are partially ordered Banach spaces with closed cones and if the cone of $X_1$ is also generating, then every positive operator $A : X_1 \longrightarrow X_2$ is continuous [1, Corollary 2.5]. Another interesting result is that, if $H_1$ and $H_2$ are Hilbert spaces equipped with self-dual cones $P_1$ and $P_2$, respectively, and $A$ is a bounded linear operator such that $A(P_1) \subseteq P_2$, then $A^*(P_2) \subseteq P_1$ (See Lemma 3.8, Examples 3.10 and 3.11 in [9]). Thus, the effect of the topology of the underlying space on the cone structure plays a crucial role in the theory of positive operators. Duality of cones in Hilbert spaces, together with interesting examples can be found in Section 2 of the article by Borwein and Yost [4]. Self-dual cones in Hilbert spaces has also been studied by several authors; see for instance the articles by Barker and Foran [2] and Penney [14], respectively. For a good reference on various applications of positive operators in applied mathematics, one can

refer the monograph by Berman and Plemmons [3] or Krasnoselskii et al [10]. The question we would like to address in this article is the following :

QUESTION 1.1. Let $H_1$ and $H_2$ be Hilbert spaces equipped with pointed self-dual cones $P_1$ and $P_2$, respectively. Let $A$ be a bounded operator between $H_1$ and $H_2$ with closed range and let $T$ be a bounded reflexive generalized inverse of $A$ (that is, $T$ is a bounded operator that satisfies the equations $ATA = A, TAT = T$). Assume that $A(P_1) \subseteq P_2$ and $T(P_2) \subseteq P_1$. Does there exist an inner product on $R(A)$, different from the one induced by $H_2$, but equivalent to it and with respect to which the cone $A(P_1)$ is self-dual ?

A motivation for this problem arose from an interesting problem in matrix theory, namely, the nonnegative rank factorization problem and its infinite dimensional version. Berman and Plemmons proved that if an (entrywise) nonnegative $m \times n$ matrix $A$ has a nonnegative generalized inverse $X$ (that is, $A = AXA$), then there exist matrices $B$ and $C$ of sizes $m \times r$ and $r \times n$, respectively, such that $A = BC$, where $r = \text{rank}(A) = \text{rank}(B) = \text{rank}(C)$.

A notion of the above to infinite dimensional spaces was proposed by the author and Sivakumar in connection with nonnegativity of various generalized inverses (see Definitions 3.1 and 3.2 and Theorem 3.16 in [9] and Definitions 3.1 and 3.3, Theorems 3.20 and 3.21 in [8]) : A bounded operator $A$ between Hilbert spaces $H_1$ and $H_2$, equipped with self-dual cones $P_1$ and $P_2$, respectively, is said to admit a nonnegative rank factorization if there exists a Hilbert space $H$ and a self-dual cone $P$ in $H$, and bounded operators $B$ and $C$ from $H$ to $H_2$ and $H_1$ to $H$, respectively, such that $A = BC$ and such that $B(P) \subseteq P_2, C(P_1) \subseteq P$. We retain the word rank in the above definition, although it is a little out of place.

Self-duality of the underlying cones are crucial in the above definition, for, if $A$ has a nonnegative rank factorization $A = BC$, then $A^* = C^*B^*$ gives a nonnegative rank factorization of $A^*$. Self-duality of the cones cannot be dispensed with in deducing nonnegativity of the adjoint. Unlike Berman and Plemmons's result, where the cones are fixed (the nonnegative orthants), the cone $P$ in $H$ in the above definition is arbitrary. Therefore, strictly speaking the above definition is not a generalization of the usual definition of nonnegative rank factorization for nonnegative matrices, but still is a worthy extension to infinite dimensions and also in finite dimensional spaces, where the cones are non-polyhedral. It is easy to see that if $A$ is a bounded operator with closed range, then $A$ can always be factorized as $A = BC$, where $C : H_1 \longrightarrow R(A)$ is defined by $Cx = Ax$ and $B$ is the inclusion operator from $R(A)$ to $H_2$.

Moreover, any rank factorization is necessarily of this form. (We retain the word rank, although it is a little out of place). Therefore, the most obvious choice of the Hilbert space $H$ is the range of $A$ and the cones being $A(P_1)$ and $R(A) \cap P_2$. There are now three possibilities.

(1) Embed $A(P_1)$ in some self-dual cone $P$ in $H = R(A)$.

(2) Prove self-duality of $A(P_1)$ directly.

(3) See if there is some other complete inner product on $R(A)$ with respect to which the cone $A(P_1)$ is self-dual.

If option (1) were true, then since any factorization is of the above form, a simple duality argument will yield that $P = A(P_1)$. It was proved recently by the author that if $P_1$ and $P_2$ are self-dual cones and if there exists another bounded operator $T : H_2 \to H_1$ such that $ATA = A, TAT = T$ (such a $T$ is called a reflexive generalized inverse) with $T(P_2) \subseteq P_1$ then, the cone $A(P_1)$ is self-dual in $R(A)$ [8, Theorem 3.11]. In this note we present an affirmative answer to Question 1.1, when the Hilbert spaces are separable. This will prove that option (3) above also holds.

The paper is organized as follows. The main results are presented in Section 2. The basic definitions in the theory of cones in Banach spaces, such as normality, nonoblateness, acuteness, regularity and self-duality are presented first. A result due to Khudalov [11, Theorem 1], that self-duality and regularity are equivalent for a pointed cone in a Hilbert space is highlighted. A result concerning Riesz bases is stated next (Theorem 2.1). This theorem says that a Riesz basis (an isomorphic image of an orthonormal basis) induces in a natural way a new inner product on the space that makes it into a Hilbert space in such a way that the two induced norms are equivalent. An affirmative answer to Question 1.1 is presented next (Theorem 2.6). This is then applied to the nonnegative rank factorization problem (Theorem 2.7).

## 2. Main Results

A common source for various definitions presented below is the monograph by Krasnoselskii et al [10]. However, individual references are also cited wherever necessary.

A real vector space $X$ is called a partially ordered vector space if there is a partial order $\leq$ defined on it satisfying the following : For $x, y \in X, x \leq y \implies x + u \leq y + u$ for all $u \in X$ and $\alpha x \leq \alpha y$ for all $\alpha \geq 0$. The subset

$P := \{x \in X : x \geq 0\}$ is called the positive cone of $X$. It is easy to verify that the set $P$ satisfies the usual definition of a cone : $\alpha P \subseteq P \ \forall \ \alpha \geq 0$ and $P + P \subseteq P$. We shall simply call the above set as a cone. Note that $P$ is a pointed cone i.e., $P \cap -P = \{0\}$. $P$ is said to be generating or reproducing if $X = P - P$. It is well known that in finite dimensional spaces, a cone is generating if and only if it has non-empty interior, which is not true in infinite dimensional spaces. A Banach space $X$ which is partially ordered is said to be a partially ordered Banach space. If $X$ is also a Hilbert space, then it is called a partially ordered Hilbert space. If $P$ is a pointed cone in a Banach space $(X, ||.||)$, then the norm $||.||$ is called monotone if $0 \leq x \leq y \Rightarrow ||x|| \leq ||y||$ and semi-monotone if $0 \leq x \leq y \Rightarrow ||x|| \leq b||y||$ for some universal constant $b$. A cone $P$ is called acute if the norm is monotone and normal if it is semi-monotone. A cone $P$ is said to be 1-normal if for any two elements $x, y \in X, \pm x \leq y$ implies $||x|| \leq ||y||$ (See [12, Definition 1(1)]). It can be shown that every normal cone is acute with respect to some equivalent norm (Refer Theorem 4.4, [10]). A cone $P$ is said to be 1-nonoblate if for any $x \in P$, there exists a unique $y \geq x$ such that $||x|| = ||y||$. $P$ is said to be regular if it is both 1-normal and 1-nonoblate. Regularity can also be defined in terms of convergence of a non-decreasing sequence from $P$. A cone $P$ in a Banach space $X$ is called regular if every sequence $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$, which is bounded from above, converges in norm. $P$ is said to be completely regular if every sequence $0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$, that is norm bounded converges. Every regular cone is normal [10, Theorem 5.1] and every completely regular cone is regular [10, Theorem 5.2]. For a cone $P$ in a Hilbert space $H$, the dual cone of $P$ will be denoted by $P^*$. A cone $P$ in a Hilbert space is acute if and only if it is 1-normal; equivalently, $P$ is acute if and only if $P \subseteq P^*$ (Refer Assertions 1 and 2, [12]). A cone $P$ in a Hilbert space is said to be obtuse if $P^* \cap \{\overline{span(P)}\}$ is acute [6]. A cone $P$ in a Hilbert space $H$ is 1-nonoblate if and only if for every $x \in H$, $\exists \ u, v \in P$ such that $x = u - v$ and $u \perp v$. Moreover, if $P$ is 1-nonoblate, then $P^* \subseteq P$ (Refer Assertions 3 and 4, [12]). Incidentally, Novikoff had also defined obtuseness of a cone in a Hilbert space to be one for which $P^* \subseteq P$ [13]. A cone is self-dual if it is both acute and obtuse. It was proved by Khudalov [11, Theorem 1] that for a cone $H_+$ in a Hilbert space $H$, regularity is equivalent to self-duality. It is also known and easy to prove that a normal cone $P$ in a weakly sequentially complete Banach space is completely regular, and hence regular. The following theorem due to Holub proves quite useful.

THEOREM 2.1. [7, Theorem 2.1] *Let $\{x_i, f_i\}$ be a normalized Riesz basis*

*for a Hilbert space $H$. Then there is an equivalent inner product on $H$ in which $\{x_i\}$ is an orthonormal basis for $H$ under the norm induced by this inner product.*

We now prove a result that will be used later on to answer the question posed in the introduction.

THEOREM 2.2. *Let $P$ be a pointed self-dual cone in a separable Hilbert space $H$. Then there is an inner product $\langle .,. \rangle_1$ on $H$ that makes it into a Hilbert space and with respect to this new inner product, $P$ is self-dual.*

*Proof.* The existence of a new inner product $\langle .,. \rangle_1$, the norm induced (denoted by $||.||_1$) by which is equivalent to the one induced by the inner product $\langle .,. \rangle$ (denoted by $||.||$), follows from the proof of Theorem 2.1. Thus, there exist positive constants $\alpha$ and $\beta$ such that $||.||_1 \leq \alpha ||.|| \leq \beta ||.|_1$. Therefore for $0 \leq x \leq y$, we have $||x||_1 \leq \alpha ||x|| \leq \alpha ||y|| \leq \alpha \beta ||y||_1$, which implies that the norm $||.||_1$ is semi-monotone for $P$. The conclusion now follows from the discussion preceeding Theorem 2.1.

We now apply the above result to answer Question 1.1. We shall denote the set of all bounded operators between Hilbert spaces $H_1$ and $H_2$ by $\mathcal{B}(H_1, H_2)$. Note that $A(P_1)$ is a pointed cone if $P_1$ is pointed. For the image cone $A(P)$, obtuseness is defined to be acuteness of $(A(P))^* \cap R(A)$ [6]. A linear operator $T : H_2 \rightarrow H_1$ is called a generalized inverse (or a $\{1\}$-inverse) of $A$ if it satisfies the equation $ATA = A$. $T$ is called a reflexive generalized inverse (or a $\{1,2\}$-inverse) of $A$ if it satisfies the equations $ATA = A$ and $TAT = T$. Such a transformation exists as a bounded operator if and only if $R(A)$ is closed [5]. The terminology $\{1\}$ and $\{1,2\}$ come from the first two Penrose equations in the definition of the (unique) Moore-Penrose inverse : $ATA = A, TAT = T, (AT)^* = AT, (TA)^* = TA$. The following characterization of nonnegativity of generalized inverses is due to Tam. Other similar results concerning nonnegative generalized inverses can be found in [16] (For instance, Lemma 2.7 and the discussion following it, Proposition 2.8, Theorems 4.1 and 4.2 and the discussions following them).

THEOREM 2.3. [15, Theorem 3.1] *Let $A \in \mathbb{R}^{m \times n}$ be nonnegative with respect to generating cones $P_1$ and $P_2$, respectively. Then, a necessary and sufficient condition for the existence of a nonnegative generalized inverse (or a $\{1\}$-inverse) is : There exists a subspace $H$ of $\mathbb{R}^n$ such that $span(H \cap P_1) = H$, $A$ takes $H \cap P_1$ isomorphically onto $R(A) \cap P_2$, and there is a projection $P$*

*nonnegative with respect to $P_2$ such that $R(P) = R(A)$. When this condition is satisfied, there is also a projection $Q$ that is nonnegative with respect to $P_1$, such that $R(Q) = H, A = AQ$.*

The subspace $H$ in Theorem 2.3 can actually be taken to be $R(T)$, where $T$ is a reflexive generalized inverse of $A$. It is also clear that one can work with the cones $T(P_2)$ and $A(P_1)$ in $R(T)$ and $R(A)$, respectively, in the place of $H \cap P_1$ and $R(A) \cap P_2$, respectively. The following theorem is a consequence of these.

THEOREM 2.4. *Let $A \in \mathcal{B}(H_1, H_2)$ have a reflexive generalized inverse $T \in \mathcal{B}(H_2, H_1)$. Further, assume that $A$ and $T$ are $(P_1, P_2)$- and $(P_2, P_1)$-nonnegative, respectively. Then the operator $M := A|_{R(T)}$ is a surjective isomorphism from $R(T)$ onto $R(A)$ with inverse $M^{-1} := T|_{R(A)}$. Moreover, the cones $T(P_2)$ and $A(P_1)$ are mapped one-to-one onto each other by $M$ and $M^{-1}$, respectively; that is, $T(P_2)$ and $A(P_1)$ are linearly isomorphic.*

*Proof.* The fact that $M := A|_{R(T)} \longrightarrow R(A)$ is invertible needs no proof. Let $y = Tx, x \in P_2$. Then, $My = Ay = ATx \in A(P_1)$, as $T(P_2) \subseteq P_1$. Thus, $M(T(P_2)) \subseteq A(P_1)$. On the other hand, if $y = Ax, x \in P_1$, then $y \in P_2$. Also, $y = ATy = Av, v = Ty \in T(P_2) \subseteq P_1$. Thus, $A(P_1) \subseteq M(T(P_2))$. Consequently, $M(T(P_2)) = A(P_1)$.

The following result, due to the author, says that when $A$ has a nonnegative reflexive generalized inverse (nonnegative with respect to two self-dual cones $P_2$ and $P_1$), the image cone $A(P_1)$ is obtuse (and hence self-dual).

THEOREM 2.5. [8, Theorem 3.11] *Let $A \in \mathcal{B}(H_1, H_2)$ have closed range. Let $P_1, P_2$ be self-dual cones in $H_1, H_2$, respectively. Further, assume that $T(P_2) \subseteq P_1$, where $T$ is a reflexive generalized inverse of $A$. Then $A(P_1)$ is obtuse.*

The following theorem gives an answer to Question 1.1.

THEOREM 2.6. *Let $H_1$ and $H_2$ be real separable Hilbert spaces equipped with pointed self-dual cones $P_1$ and $P_2$, respectively. Let $A \in \mathcal{B}(H_1, H_2)$ be $(P_1, P_2)$-nonnegative with a $(P_2, P_1)$-nonnegative reflexive generalized inverse $T \in \mathcal{B}(H_2, H_1)$. Then, there is a complete inner product $\langle , . , \rangle_3$ on $R(A)$ such that the norm induced $||.||_3$ is equivalent to the one induced by the inner product $\langle , . , \rangle_2$ and such that $A(P_1)$ is self-dual with respect to $\langle , . , \rangle_3$.*

*Proof.* Let us denote the inner product on $R(A)$ induced by $H_2$ by $\langle , . , \rangle_2$. From Theorem 2.4, we infer that the operator $N := (MM^*)^{-1}$ is a bounded operator on $R(A)$, which is also a linear isomorphism. Let $\{u_n\}$ be an orthonormal basis for $R(A)$ and let $\{w_n, z_n\}$ be the normalized Riesz basis obtained from $\{u_n\}$ and the isomorphism $N$. Although the sequence $\{w_n\}$ need not be an orthonormal basis for $R(A)$, it induces a complete inner product on it, say $\langle . , . \rangle_3$, such that the norm induced is equivalent to the one induced by $\langle . , . \rangle_2$ and such that $\{w_n\}$ will be an orthonormal basis for this new Hilbert space. Since $P_1$ is pointed, $A(P_1)$ is a pointed cone. By Theorem 2.5, we know that $A(P_1)$ is self-dual with respect to the inner product $\langle . , . \rangle_2$. The conclusion now follows from Theorem 2.2.

Theorem 2.6 can be thought of a possible analogue of Tam's result to infinite dimensions and also in finite dimensional spaces equipped with non-polyhedral cones. Let us indicate how we can obtain a different nonnegative rank factorization from a given one (Observe that any rank factorization of $A$ is necessarily of the form $A = BR^{-1}RC$ for some invertible operator $R$, where $A = BC$ is a rank factorization).

THEOREM 2.7. *Let $A \in \mathcal{B}(H_1, H_2)$ and $T \in \mathcal{B}(H_2, H_1)$ be nonnegative with respect to self-dual cones $P_1, P_2$ and $P_2, P_1$, respectively, where $T$ satisfies the equations $ATA = A, TAT = T$ (so that $A$ has a nonnegative rank factorization). Then, there exists a Hilbert space $H_3$ and a self-dual cone $P_3$ in $H_3$, a bounded operator $S : R(A) \longrightarrow H_3$ that is nonnegative with respect to $A(P_1)$ and $P_3$ so that $A$ has a nonnegative rank factorization $A = BS^{-1}SC$.*

*Proof.* The assumptions on $A$ and $T$ ensure that $A$ has a nonnegative rank factorization $A = BC$, where $B \in \mathcal{B}(H, H_2)$ and $C \in \mathcal{B}(H_1, H)$ are nonnegative with respect to the self-dual cones $A(P_1), P_2$ and $P_1, A(P_1)$, respectively, where $H = R(A)$. The existence of a Hilbert space $H_3$ and a self-dual cone in it are guaranteed by Theorem 2.6. It therefore suffices to prove the existence of an operator with the required properties. For the sake of clarity, let us denote by $K_1$ and $K_2$ the cone $A(P_1)$ in the Hilbert spaces $H$ and $H_3 (= H)$. Define $S : H \longrightarrow H_3$ by $u_n \mapsto Nu_n$, where $\{u_n\}$ is an orthonormal basis for $H$ and $N$ is as in Theorem 2.6. $S$ can be extended to a well defined bounded linear operator from $H$ into $H_3$, which we again denote by $S$. Suppose $y = \sum_{n=1}^{\infty} \langle y, u_n \rangle_2 u_n$ is such that $Sy = \sum_{n=1}^{\infty} \langle y, u_n \rangle_2 Nu_n = 0$. Then, $||Sy||_3^2 = 0$ and hence,

$\sum_{n=1}^{\infty} |\langle y, u_n \rangle_2|^2 = 0$ from which it follows that $y = 0$. Thus, the operator $S$ is injective. Similarly, the operator $S^*$ is also injective. Consequently, $S$ is an invertible bounded linear operator from $H$ onto $H_3$. Let $\tilde{B} := BS^{-1}$ and $\tilde{C} := SC$. It is then easy to check that $A = \tilde{B}\tilde{C}$ is a rank factorization of $A$. It only remains to prove that $S(K_1) \subseteq K_2$ and $S^{-1}(K_2) \subseteq K_1$. Let $x \in K_1$. For any $u \in K_2$, we have $\langle Sx, u \rangle_3 = (1/4)(||Sx+u||_3^2 + ||Sx-u||_3^2) \geq 0$. Therefore, $Sx \in K_2^*$ (the dual is in the Hilbert space $H_3$). Since $K_2$ is a self-dual cone in $H_3$, we see that $Sx \in K_2$, proving nonnegativity of $S$. A similar argument shows that $S^{-1}$ is also nonnegative. Therefore, $A = \tilde{B}\tilde{C}$ gives a nonnegative rank factorization of $A$.

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