EXTRACTA MATHEMATICAE Vol. **29**, Núm. 1 - 2, 97 – 115 (2014)

On Positive Definite Kernels, Related Problems and Applications

R. Bruzual, A. de la Barrera, M. Domínguez

Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela, Apartado Postal 47686*, Caracas* 1041-*A, Venezuela*
ramonbruzual.ucv@gmail.com , ramon.bruzual@ciens.ucv.ve

Departamento de Matemáticas, Universidad de Pamplona, Colombia
abarrera1994@gmail.com

Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela, Apartado Postal 47159*, Caracas* 1041-*A, Venezuela*
marisela.dominguez@ciens.ucv.ve , dominguez.math@gmail.com

Presented by Manuel González — Received August 1, 2014

Abstract: We give a representation result for regular positive definite Toeplitz kernels and, as a corollary, we obtain a representation result for equivalent kernels. We obtain a stability result which is used to show that, under certain conditions, a special perturbation of a positive definite Toeplitz kernel is equivalent to the perturbed kernel. Some applications to stochastic processes are given.

Key words: positive definite function, Toeplitz kernel, weakly stationary process.

AMS *Subject Class.* (2010): 42A82, 60G10.

1. Introduction

Positive definite kernels play a prominent role in many results of analysis and probability where they appear naturally. It is important to highlight that this notion allows to consider, in a unified way, problems of both areas. In this paper we will consider kernels that are equivalent to positive definite Toeplitz kernels.

The Wold decomposition appeared for the first time on [7] (see also [8]) and its harmonic analysis version for operators on Hilbert spaces was given on [6]. In this paper we will use the Wold decomposition of a translation operator to obtain a representation result for regular positive definite Toeplitz kernels (see Theorem 5). This theorem is analogous to a result for stochastic process

The first and the third authors were supported in part by the CDCH of the Universidad Central de Venezuela.

given in [2]. Considering Riesz basis we give a similar representation result for kernels equivalent to positive definite Toeplitz kernels (see Theorem 7).

We prove a stability result for positive definite kernels related with the Paley-Wiener theorem about stability of bases given in [4] (see Theorem 8). This stability result is used to show that, under certain conditions, a special perturbation of a positive definite Toeplitz kernel is equivalent to the perturbed kernel (see Theorem 13).

Some applications to stochastic process are given. Recall that a discrete stochastic process is a sequence of random variables on a probability space. A stochastic process is said to be stationary if its finite dimensional distributions are invariant under translations of time. A wider class of stochastic processes is given by the weakly stationary process in the definition of which one only imposed those conditions that are absolutely necessary for using Hilbert space methods and Fourier analysis methods. In this case the covariance kernel is a positive definite Toeplitz kernel.

Using our representation results for regular positive definite Toeplitz kernels we obtain a theorem of Gihman and Skorokhod for stochastic processes proved in [2] (see Theorem 15).

We consider a special class of stochastic processes, called approximately weakly stationary, that were introduced by Strandell. For this type of processes we prove a representation result already given in [5] (see Theorem 17) and a perturbation result similar to a theorem given in [5] (see Theorem 18).

2. Some properties of positive definite kernels

Let $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ be a kernel. It is said that K is *positive definite* if

$$\sum_{m,n \in \mathbb{Z}} K(n,m)\, a_m\, \overline{a_n} \geq 0$$

for every sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ with finite support.

Let $\mathcal{E}_o$ de the space of the sequences $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ with finite support.

Let $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ a positive definite kernel. For $a = \{a_n\}_{n\in\mathbb{Z}}$ and $b = \{b_n\}_{n\in\mathbb{Z}}$ in $\mathcal{E}_o$ define

$$\langle a, b \rangle = \sum_{m,n \in \mathbb{Z}} K(n,m)\, a_m\, \overline{b_n}\,.$$

It holds that $\langle \cdot\,, \cdot \rangle$ is a, possibly degenerated, positive definite sesquilinear form on $\mathcal{E}_o$.

Let $\mathcal{E}_{o,K}$ be the pre-Hilbert space obtained after the natural quotient on $\mathcal{E}_o$ and let $\mathcal{H}_K$ be the completion of $\mathcal{E}_{o,K}$.

The product and the norm on $\mathcal{H}_K$ will be denoted by $\langle\ ,\ \rangle_{\mathcal{H}_K}$ and $\|\ \|_{\mathcal{H}_K}$ respectively. This norm will be called the *norm induced by* K.

For $n \in \mathbb{Z}$ let $\delta^{(n)}$ be the element of $\mathcal{E}_o$ defined by

$$\delta^{(n)}_m = \begin{cases} 1 & \text{if } m = n\,, \\ 0 & \text{if } m \neq n\,. \end{cases}$$

Let $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ be a finite support sequence.

The equivalence class of the element $\sum\limits_{n\in\mathbb{Z}} a_n \delta^{(n)}$ will be denoted by

$$\left[\sum_{n\in\mathbb{Z}} a_n \delta^{(n)}\right]_K .$$

Observe that

$$\left\| \left[\sum_{n\in\mathbb{Z}} a_n \delta^{(n)}\right]_K \right\|^2_{\mathcal{H}_K} = \sum_{m,n\in\mathbb{Z}} K(n,m)\, a_m\, \overline{a_n}\,.$$

Definition 1. Let $K_1, K_2 : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be two positive definite kernels.

It is said that K_1 and K_2 are *equivalent* if the corresponding induced pre-Hilbert norms, $\|\ \|_{\mathcal{H}_{K_1}}$ and $\|\ \|_{\mathcal{H}_{K_2}}$, on the space $\mathcal{E}_o$ are equivalent.

Proposition 2. *Let $K_1, K_2 : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be two positive definite kernels. Then the following conditions are equivalent:*

(i) *The kernels K_1 and K_2 are equivalent.*

(ii) *There exists a bounded bijective linear map, with bounded inverse,*

$$\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$$

such that

$$\Phi\big[\delta^{(n)}\big]_{K_1} = \big[\delta^{(n)}\big]_{K_2} .$$

(iii) *There exist two constants A, B with $0 < A \le B$ such that*

$$\begin{aligned} A \sum_{m,n\in\mathbb{Z}} K_1(n,m)\, a_m\, \overline{a_n} &\le \sum_{m,n\in\mathbb{Z}} K_2(n,m)\, a_m\, \overline{a_n} \\ &\le B \sum_{m,n\in\mathbb{Z}} K_1(n,m)\, a_m\, \overline{a_n} \end{aligned}$$

for every finite support sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$.

Proof. Note that condition (iii) means that

$$A\big\|[h]_{K_1}\big\|^2_{\mathcal{H}_{K_1}} \leq \big\|[h]_{K_2}\big\|^2_{\mathcal{H}_{K_2}} \leq B\big\|[h]_{K_1}\big\|^2_{\mathcal{H}_{K_1}}$$

for $h \in \mathcal{E}_o$.

Thus condition (ii) implies condition (iii).

By definition conditions (i) and (iii) are equivalent. So it is enough to show that condition (iii) implies condition (ii).

Suppose that condition (iii) holds. Then the map

$$\Phi_o : \mathcal{E}_{o,K_1} \to \mathcal{E}_{o,K_2}$$

defined by

$$\Phi_o\left(\left[\sum_{n\in\mathbb{Z}} a_n\delta^{(n)}\right]_{K_1}\right) = \left[\sum_{n\in\mathbb{Z}} a_n\delta^{(n)}\right]_{K_2}$$

is well defined and it is linear and continuous.

Let $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$ be the continuous linear extension of Φ_o.

We have that $\{[h_m]_{K_2}\}$ is a Cauchy sequence in $\mathcal{E}_{o,K_2}$ if and only if $\{[h_m]_{K_1}\}$ is a Cauchy sequence in $\mathcal{E}_{o,K_1}$. From this fact it follows that Φ is onto with continuous inverse. ∎

Recall that $K : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ is a *Toeplitz kernel* if there exists a sequence $\tau : \mathbb{Z} \to \mathbb{C}$ such that

$$K(n,m) = \tau(n-m) \qquad \text{for all } n,m \in \mathbb{Z}\,.$$

The sequence τ is *positive definite* if the corresponding Toeplitz kernel is positive definite.

If K is a positive definite Toeplitz kernel then the operator $T : \mathcal{E}_{o,K} \to \mathcal{E}_{o,K}$ defined by

$$T\left(\left[\sum_{n\in\mathbb{Z}} a_n\delta^{(n)}\right]_{K}\right) = \left[\sum_{n\in\mathbb{Z}} a_n\delta^{(n-1)}\right]_{K}$$

gives raise to a unitary operator on $\mathcal{H}_K$, that will be denoted by T also. As usual it will called the *translation operator*.

3. Regular positive definite Toeplitz kernels

If $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is a positive definite kernel, for $j \in \mathbb{Z}$ let $\mathcal{H}_K^j$ be the closed subspace of $\mathcal{H}_K$ generated by the elements of the form $\left[\delta^{(n)}\right]_K$ with $n \leq j$, that is

$$\mathcal{H}_K^j = \overline{\operatorname{span}}\left\{\left[\delta^{(n)}\right]_K \,:\, n \leq j\right\}.$$

Note that if K is a Toeplitz kernel, then $\mathcal{H}_K^j$ is invariant by the translation operator T.

Definition 3. Let K be a positive definite kernel, K is said to be that *regular* if

$$\bigcap_{j\in\mathbb{Z}} \mathcal{H}_K^j = \{0\}.$$

We will avoid the trivial case $K \equiv 0$.

Proposition 4. *Let K be a positive definite Toeplitz kernel with translation operator T. If K is regular then* $\dim\left(\mathcal{H}_{\mathrm{K}}^0 \ominus \mathrm{T}\mathcal{H}_{\mathrm{K}}^0\right) = 1$.

Proof. It holds that $T\mathcal{H}_K^0 + \operatorname{span}\left\{\left[\delta^{(0)}\right]_{\mathrm{K}}\right\} = \mathcal{H}_{\mathrm{K}}^0$, then $\dim\left(\mathcal{H}_{\mathrm{K}}^0 \ominus \mathrm{T}\mathcal{H}_{\mathrm{K}}^0\right)$ is 0 or 1.

If this dimension is equal to 0, then $\mathcal{H}_K^j = T^j\mathcal{H}_K^0 = \mathcal{H}_K^0$, thus

$$\bigcap_{j\in\mathbb{Z}} \mathcal{H}_K^j = \mathcal{H}_K^0 \neq \{0\}.$$

∎

As usual $\ell^2(\mathbb{N})$ will denote the space of square summable sequences $a = \{a_n\}_{n\in\mathbb{N}}$ with norm $\|a\|_2 = \left(\sum_{n\in\mathbb{N}} |a_n|^2\right)^{1/2}$, analogously for $\ell^2(\mathbb{Z})$.

Theorem 5. *Let K be a positive definite kernel. Then the following conditions are equivalent*

(i) *K is a regular and Toeplitz.*

(ii) *There exists a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ and an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}_K$ such that*

$$\mathcal{H}_K^j = \overline{\operatorname{span}}\{e_n \,:\, n \leq j\} \qquad \textit{for all } j \in \mathbb{Z}$$

and

$$\left[\delta^{(n)}\right]_K = \sum_{j=0}^{+\infty} a_j e_{n-j}.$$

Proof. Suppose (i) holds.

We will use the Wold decomposition of the translation operator $T : \mathcal{H}_K \to \mathcal{H}_K$ (for details about the Wold decomposition see [6, Theorem 1.1, p. 3]).

Since $T^p \mathcal{H}_K^q = \mathcal{H}_K^{q-p}$, if $p \geq 0$ it holds that

$$\bigcap_{j=0}^{+\infty} T^j \mathcal{H}_K^0 = \bigcap_{j=0}^{+\infty} \mathcal{H}_K^{-j} = \bigcap_{j=-\infty}^{0} \mathcal{H}_K^j .$$

Since $\mathcal{H}_K^0 \subset \bigcap_{j=1}^{+\infty} \mathcal{H}_K^j$, we have that

$$\bigcap_{j=0}^{+\infty} T^j \mathcal{H}_K^0 \subset \bigcap_{j=-\infty}^{+\infty} \mathcal{H}_K^j = \{0\} ,$$

so the operator T is a unilateral shift.

From Proposition 4 it follows that $\dim \left(\mathcal{H}_\mathrm{K}^0 \ominus \mathrm{T}\mathcal{H}_\mathrm{K}^0\right) = 1$, thus the multiplicity of T is 1.

Let e_0 be a unitary vector in $\mathcal{H}_K^0 \ominus T\mathcal{H}_K^0$.

From the Wold decomposition it follows that $\{T^p e_0\}_{p\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}_K$ and $\{T^p e_0\}_{p=p_o}^{+\infty}$ is an orthonormal basis of $\mathcal{H}_K^{-p_o}$.

For $p \in \mathbb{Z}$ let

$$e_p = T^{-p} e_0 .$$

Since $\left[\delta^{(0)}\right]_K \in \mathcal{H}_K^0$ there exists a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ such that

$$\left[\delta^{(0)}\right]_K = \sum_{j=0}^{+\infty} a_j e_{-j} .$$

Thus

$$\left[\delta^{(n)}\right]_K = T^{-n} \left[\delta^{(0)}\right]_K = \sum_{j=0}^{+\infty} a_j e_{n-j} .$$

Suppose (ii) holds.

Then, for $n, m \in \mathbb{Z}$ it holds that

$$K(n,m) = \left\langle \left[\delta^{(m)}\right]_K, \left[\delta^{(n)}\right]_K \right\rangle_{\mathcal{H}_K} = \sum_{j=0}^{+\infty} a_j \overline{a}_{n-m+j} .$$

So K is a Toeplitz kernel.

Since $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}_K$ and $\mathcal{H}_K^j = \overline{\operatorname{span}}\{e_n : n \le j\}$ for all $j \in \mathbb{Z}$ it holds that

$$\bigcap_{j\in\mathbb{Z}} \mathcal{H}_K^j = \{0\},$$

so K is regular. ∎

PROPOSITION 6. *Let K_1 and K_2 be two equivalent positive definite kernels. Then K_1 is regular if and only if K_2 is regular.*

Proof. With the same notation of Proposition 2 we have that $\Phi(\mathcal{H}_{K_1}^j) = \mathcal{H}_{K_2}^j$ for $j \in \mathbb{Z}$. So the result follows. ∎

Recall that if $\mathcal{H}$ is a separable Hilbert space a sequence $\{v_n\}_{n\in\mathbb{Z}}$ is called a *Riesz basis* if there exist a bounded linear operator with bounded inverse $V : \mathcal{H} \to \mathcal{H}$ and an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}$ such that $v_n = Ve_n$ for $n \in \mathbb{Z}$.

For more details about bases in Banach spaces see [1, 3].

THEOREM 7. *Let K be a positive definite kernel. Then the following conditions are equivalent:*

(i) *K is regular and equivalent to a positive definite Toeplitz kernel.*

(ii) *There exists a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ and a Riesz basis $\{v_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}_K$ such that*

$$\left[\delta^{(n)}\right]_K = \sum_{j=0}^{+\infty} a_j v_{n-j}$$

and

$$\mathcal{H}_K^j = \overline{\operatorname{span}}\{v_n : n \le j\} \qquad \text{for all } j \in \mathbb{Z}.$$

Proof. Suppose that (i) holds. Let K_1 be a positive definite Toeplitz kernel equivalent to K. From Proposition 6 it follows that K_1 is regular, so from Theorem 5 it follows that there exist a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ and a orthonormal base $\{e_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}_{K_1}$ such that

$$\left[\delta^{(n)}\right]_{K_1} = \sum_{j=0}^{+\infty} a_j e_{n-j}$$

and

$$\mathcal{H}_{K_1}^j = \overline{\operatorname{span}}\{e_n : n \le j\} \qquad \text{for every } j \in \mathbb{Z}.$$

Since K_1 and K are equivalent, the function $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_K$ defined by

$$\Phi\big[\delta^{(n)}\big]_{K_1} = \big[\delta^{(n)}\big]_K$$

is well defined and it is bounded with bounded inverse.

Defining v_n by

$$v_n = \Phi e_n$$

the first part of the result is obtained.

For the last equality it is enough to observe that

$$\Phi(\mathcal{H}^j_{K_1}) = \mathcal{H}^j_K\,.$$

Suppose that (ii) holds. Then there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}_K$ and a bounded linear operator with bounded inverse $V : \mathcal{H}_K \to \mathcal{H}_K$ such that $v_n = Ve_n$ for $n \in \mathbb{Z}$.

We have that

$$V^{-1}\big[\delta^{(n)}\big]_K = \sum_{j=0}^{+\infty} a_j e_{n-j}\,.$$

Let $K_1 : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be the kernel defined by

$$K_1(n,m) = \Big\langle V^{-1}\big[\delta^{(m)}\big]_K, V^{-1}\big[\delta^{(n)}\big]_K\Big\rangle_{\mathcal{H}_K}\,.$$

It holds that $K_1(n,m) = \sum_{j=0}^{+\infty} a_j\overline{a}_{n-m+j}$, so K_1 is a Toeplitz kernel.

Note that, for a finite support sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$

$$\sum_{m,n\in\mathbb{Z}} K_1(n,m)\, a_m\, \overline{a_n} = \left\| V^{-1}\left(\sum_{n\in\mathbb{Z}} a_n \big[\delta^{(n)}\big]_K\right)\right\|^2_{\mathcal{H}_K}\,.$$

Since V is bounded with bounded inverse, by Proposition 2 it holds that K and K_1 are equivalent.

Finally the regularity of K follows as in Theorem 5. ∎

4. Perturbations of Toeplitz kernels

The following result is related with a Paley-Wiener theorem about stability of bases [4] (see also [9, Theorem 10, p. 38]). A similar result for stochastic process was given in [5, Theorem 2].

THEOREM 8. *Let K be a positive definite kernel.*
If $\{g_n\}_{n=-\infty}^{\infty} \subset \mathcal{H}_K$ satisfies

$$\left\| \sum_{n\in\mathbb{Z}} a_n \left(\left[\delta^{(n)}\right]_K - g_n \right) \right\|_{\mathcal{H}_K} \leq \lambda \left\| \sum_{n\in\mathbb{Z}} a_n \left[\delta^{(n)}\right]_K \right\|_{\mathcal{H}_K}$$

for every finite support sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$, where $\lambda \in (0,1)$, then the kernel K_1 defined by

$$K_1(n,m) = \langle g_n, g_m \rangle_{\mathcal{H}_K}$$

is equivalent to K.

Proof. From the hypothesis it follows that there exists a bounded linear operator $J : \mathcal{H}_K \to \mathcal{H}_K$ such that $\|J\| \leq \lambda$ and

$$J\left(\left[\sum_{n\in\mathbb{Z}} a_n \delta^{(n)} \right]_K \right) = \sum_{n\in\mathbb{Z}} a_n \left(\left[\delta^{(n)}\right]_K - g_n \right)$$

for each finite support sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$.

Let $f \in \mathcal{H}_K$ be given by

$$f = \sum_{n\in\mathbb{Z}} a_n \left[\delta^{(n)}\right]_K ,$$

where $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ is a finite support sequence.

We have that

$$(I - J)(f) = \sum_{n\in\mathbb{Z}} a_n \, g_n$$

and

$$(1-\lambda)\|f\|_{\mathcal{H}_K} \leq \|(I-J)f\|_{\mathcal{H}_K} \leq 2\|f\|_{\mathcal{H}_K} .$$

Then

$$\begin{aligned}(1-\lambda)^2 \sum_{m,n\in\mathbb{Z}} K(n,m)\, a_m\, \overline{a_n} &\leq \sum_{m,n\in\mathbb{Z}} K_1(n,m)\, a_m\, \overline{a_n} \\ &\leq 4 \sum_{m,n\in\mathbb{Z}} K(n,m)\, a_m\, \overline{a_n} .\end{aligned}$$

∎

The following result follows from the Riesz representation theorem, for more details see [9, Theorem 2, p. 151].

Lemma 9. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in a Hilbert space $\mathcal{H}$, let $\{c_n\}_{n\in\mathbb{Z}}$ be a sequence of scalars and let $M > 0$. Then the following conditions are equivalent:*

(i) *There exists $f \in \mathcal{H}$ such that $\|f\|^2_{\mathcal{H}} \leq M$ and $\langle f, f_n\rangle_{\mathcal{H}} = c_n$ for $n \in \mathbb{Z}$.*

(ii) *For every finite support sequence of scalars $\{a_n\}_{n\in\mathbb{Z}}$ it holds that*

$$\left|\sum_{n\in\mathbb{Z}} a_n \overline{c_n}\right|^2 \leq M \left\|\sum_{n\in\mathbb{Z}} a_n f_n\right\|^2_{\mathcal{H}}.$$

The following result can be obtained from a proof of [9, Proposition 2, p. 154]. Since in [9] the proof is left as an exercise, we include a proof of our lemma here.

Lemma 10. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in a Hilbert space $\mathcal{H}$, let $\mathcal{M} = \overline{\operatorname{span}}\{f_n\}_{n\in\mathbb{Z}}$ and let $\mathcal{L}$ be a closed subspace of $\ell^2(\mathbb{Z})$. Suppose that for each sequence $x = \{x_n\}_{n\in\mathbb{Z}} \in \mathcal{L}$ the problem*

$$\langle f, f_n\rangle_{\mathcal{H}} = x_n\,, \qquad n \in \mathbb{Z}$$

has a solution $f \in \mathcal{H}$.

Then, for each sequence $x = \{x_n\}_{n\in\mathbb{Z}} \in \mathcal{L}$, this problem has a unique solution $Tx \in \mathcal{M}$ and the function $x \mapsto Tx$, from $\mathcal{L}$ to $\mathcal{M}$, is linear and bounded.

Proof. Let $P^{\mathcal{H}}_{\mathcal{M}} : \mathcal{H} \to \mathcal{M}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.

Suppose that $x = \{x_n\}_{n\in\mathbb{Z}} \in \mathcal{L}$. If $f \in \mathcal{H}$ is a solution of $\langle f, f_n\rangle_{\mathcal{H}} = x_n$, $n \in \mathbb{Z}$, then $P^{\mathcal{H}}_{\mathcal{M}} f$ is the only solution of this problem in $\mathcal{M}$, so it is enough to take $Tx = P^{\mathcal{H}}_{\mathcal{M}} f$.

Clearly T is linear, so it is enough to show that T is closed.

Let $\{x^{(j)}\}_{j=1}^{+\infty} \subset \mathcal{L}$ be a sequence such that

$$x^{(j)} \to x \in \mathcal{L} \qquad \text{and} \qquad Tx^{(j)} \to y \in \mathcal{M} \qquad \text{as } j \to +\infty\,,$$

then, for $n \in \mathbb{Z}$,

$$x^{(j)}_n \to x_n \qquad \text{as } j \to +\infty\,.$$

Since $\big\langle Tx^{(j)}, f_n\big\rangle_{\mathcal{H}} = x^{(j)}_n$ for all j, taking limit we obtain

$$\langle y, f_n\rangle_{\mathcal{H}} = \lim_{j\to+\infty} \langle Tx^{(j)}, f_n\rangle_{\mathcal{H}} = x_n\,,$$

therefore

$$y = Tx\,.$$

∎

Remark 11. There exists a constant $A > 0$ such that, under the same hypothesis of this last lemma, for each sequence $x = \{x_n\}_{n\in\mathbb{Z}} \in \mathcal{L}$, the problem

$$\langle f, f_n\rangle_{\mathcal{H}} = x_n\,, \qquad n \in \mathbb{Z}$$

has a solution $f \in \mathcal{H}$ such that

$$\|f\|^2_{\mathcal{H}} \le A \sum_{n\in\mathbb{Z}} |x_n|^2\,.$$

The following result is related with [5, Lemma 3.3].

LEMMA 12. *Let $\{f_n\}_{n\in\mathbb{Z}}$ be a sequence in a Hilbert space $\mathcal{H}$ and let $\{b_n\}_{n\in\mathbb{Z}}$ be a sequence of numbers such that $b_n = 1$ or $b_n = 0$.*

Suppose that for each sequence $c = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ the problem

$$\langle f, f_n\rangle_{\mathcal{H}} = b_n c_n\,, \qquad n \in \mathbb{Z}$$

has a solution $f \in \mathcal{H}$.

Then there exists a constant $A > 0$ such that

$$\sum_{n\in\mathbb{Z}} |a_n b_n|^2 \le A \left\| \sum_{n\in\mathbb{Z}} a_n f_n \right\|^2_{\mathcal{H}}$$

for every finite support sequence of scalars $\{a_n\}_{n\in\mathbb{Z}}$.

Proof. Let $\mathcal{L} = \big\{\{b_n z_n\}_{n\in\mathbb{Z}} : \{z_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\big\}$, then $\mathcal{L}$ is a closed subspace of $\ell^2(\mathbb{Z})$ and it holds that, for each sequence $w = \{w_n\}_{n\in\mathbb{Z}} \in \mathcal{L}$, the problem

$$\langle f, f_n\rangle_{\mathcal{H}} = w_n \qquad \text{for } n \in \mathbb{Z}$$

has a solution $f \in \mathcal{H}$.

From Lemma 10 (see also Remark 11) it follows that there exists $A > 0$, not depending on w, such that this problem has a solution $f \in \mathcal{H}$ which satisfies

$$\|f\|^2_{\mathcal{H}} \le A \sum_{n\in\mathbb{Z}} |w_n|^2\,.$$

In particular, if $z = \{z_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $\|z\|_2 \leq 1$, then the problem

$$\langle f, f_n\rangle_{\mathcal{H}} = b_n z_n \qquad \text{for } n \in \mathbb{Z}$$

has a solution $f \in \mathcal{H}$ such that

$$\|f\|_{\mathcal{H}}^2 \leq A\,.$$

So, from Lemma 9 it follows that

$$\left|\sum_{n\in\mathbb{Z}} a_n b_n \overline{z_n}\right|^2 \leq A \left\|\sum_{n\in\mathbb{Z}} a_n f_n\right\|_{\mathcal{H}}^2,$$

for each finite support sequence of scalars $\{a_n\}_{n\in\mathbb{Z}}$ and each scalar sequence $z = \{z_n\}_{n\in\mathbb{Z}}$ such that $\|z\|_2 \leq 1$.

Finally, if for a finite support sequence $\{a_n\}_{n\in\mathbb{Z}}$ we take

$$z_n = \begin{cases} 0 & \text{if } \|\{a_n b_n\}\|_2 = 0\,, \\ \dfrac{a_n b_n}{\|\{a_n b_n\}\|_2} & \text{in other case}\,, \end{cases}$$

we obtain

$$\sum_{n\in\mathbb{Z}} |a_n b_n|^2 \leq A \left\|\sum_{n\in\mathbb{Z}} a_n f_n\right\|_{\mathcal{H}}^2.$$

∎

Recall that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ on a Hilbert space is called *minimal* if, for each $p \in \mathbb{Z}$,

$$x_p \notin \overline{\operatorname{span}}\,\{x_n \,:\, n \in \mathbb{Z}\,,\ n \neq p\}\,.$$

If $\{x_n\}_{n\in\mathbb{Z}}$ is a minimal sequence on the Hilbert space $\mathcal{H}$, then there exists a sequence $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$ which is *biorthogonal* to $\{x_n\}_{n\in\mathbb{Z}}$, that is

$$\langle x_n, h_m\rangle_{\mathcal{H}} = \delta_{nm}$$

(more details can be found in [9, p. 28]).

THEOREM 13. *Let $K : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be a positive definite Toeplitz kernel such that the sequence $\left\{\left[\delta^{(n)}\right]_K\right\}_{n\in\mathbb{Z}} \subset \mathcal{H}_K$ is minimal.*

Let $I \subset \mathbb{Z}$ be a finite subset and let $\{e_n\}_{n\in I} \subset \mathcal{H}_K$ be an orthonormal system.

Then there exists a constant $B > 0$ such that for any sequence of numbers $\{c_n\}_{n\in I}$ that satisfies $0 \le c_n < B$ the kernel $K+K_1$ is equivalent to the kernel K, where

$$K_1(n,m) = \begin{cases} \rho(n,m) + \rho(m,n) + c_n\delta_{nm} & \text{if } m,n \in I\,, \\ \rho(n,m) & \text{if } m \in I, n \notin I\,, \\ \rho(m,n) & \text{if } m \notin I, n \in I\,, \\ 0 & \text{if } m \notin I, n \notin I \end{cases}$$

and

$$\rho(n,m) = \sqrt{c_m}\left\langle e_m, \left[\delta^{(n)}\right]_K \right\rangle_{\mathcal{H}_K}.$$

Proof. Let $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}_K$ be a sequence biorthogonal to $\left\{\left[\delta^{(n)}\right]_K\right\}_{n\in\mathbb{Z}}$.

Let $\{b_n\}_{n\in\mathbb{Z}}$ be the sequence defined by $b_n = 1$ if $n \in I$ and $b_n = 0$ in other case.

If $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then the problem

$$\left\langle f, \left[\delta^{(n)}\right]_K \right\rangle_{\mathcal{H}_K} = b_n c_n \qquad \text{for } n \in \mathbb{Z}$$

has the solution

$$f = \sum_{n\in I} c_n h_n\,.$$

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a finite support sequence of scalars.

From Lemma 12 it follows that there exists a constant $A > 0$ such that

$$\sum_{n\in\mathbb{Z}} |a_n b_n|^2 \le A \left\| \sum_{n\in\mathbb{Z}} a_n \left[\delta^{(n)}\right]_K \right\|^2_{\mathcal{H}_K}.$$

Let $B > 0$ such that $A\,B < 1$. Suppose that $\{c_n\}_{n\in\mathbb{Z}}$ also satisfies $0 \le c_n < B$.

If $\{g_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}_K$ is defined by

$$g_n = \begin{cases} \left[\delta^{(n)}\right]_K + \sqrt{c_n}e_n & \text{if } n \in I\,, \\ \left[\delta^{(n)}\right]_K & \text{other case}\,, \end{cases}$$

then

$$
\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}} a_n\left(\left[\delta^{(n)}\right]_K - g_n\right)\right\|^2_{\mathcal{H}_K} &= \left\|\sum_{n\in I} a_n\sqrt{c_n}\, e_n\right\|^2_{\mathcal{H}_K} \\
&\le B\left\|\sum_{n\in I} a_n\, e_n\right\|^2_{\mathcal{H}_K} = B\sum_{n\in\mathbb{Z}}|a_n b_n|^2 \\
&\le A\,B\left\|\sum_{n\in\mathbb{Z}} a_n\left[\delta^{(n)}\right]_K\right\|^2_{\mathcal{H}_K}.
\end{aligned}
$$

Since

$$K(n,m) + K_1(n,m) = \langle g_n, g_m\rangle_{\mathcal{H}_K}$$

the result follows from Theorem 8. ∎

5. Applications to stochastic processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The well known Hilbert space $L^2(\Omega, \mathcal{F}, P)$ will be denoted by $L^2(P)$.

Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stochastic process. We will always suppose that $X_n \in L^2(P)$ and $\mathbb{E}(X_n) = 0$ for every $n \in \mathbb{Z}$. Let

$$\mathcal{H}(X) = \overline{\operatorname{span}}\,\{X_n \,:\, n\in\mathbb{Z}\} \subset L^2(P)$$

and

$$\mathcal{H}^j(X) = \overline{\operatorname{span}}\,\{X_n \,:\, n\le j\} \subset L^2(P) \qquad \text{for all } j\in\mathbb{Z}\,.$$

The process is said to be *regular* if

$$\bigcap_{j\in\mathbb{Z}} \mathcal{H}^j(X) = \{0\}\,.$$

The kernel associated to the process is the *covariance kernel* K defined by

$$K(n,m) = \operatorname{cov}(X_m, X_n) = \mathbb{E}(X_m\overline{X_n}) = \langle X_m, X_n\rangle_{L^2(P)}\,.$$

We have that K is a positive definite kernel.

Remark 14. Define $\Psi : \mathcal{H}_K \to \mathcal{H}(X)$ by

$$\Psi\left(\left[\delta^{(n)}\right]_K\right) = X_n\,.$$

Then

$$\begin{aligned}\left\langle \left[\delta^{(m)}\right]_K, \left[\delta^{(n)}\right]_K \right\rangle_{\mathcal{H}_K} &= K(n,m) = \langle X_m, X_n\rangle_{L^2(P)} \\ &= \left\langle \Psi(\left[\delta^{(m)}\right]_K), \Psi(\left[\delta^{(n)}\right]_K)\right\rangle_{L^2(P)},\end{aligned}$$

therefore Ψ is a unitary operator such that $\Psi(\mathcal{H}_K^j) = \mathcal{H}^j(X)$ for all $j \in \mathbb{Z}$.

Note that the kernel K is regular if and only if the process X is regular.

The process $\{X_n\}_{n\in\mathbb{Z}}$ is said to be *weakly stationary* if

$$\mathbb{E}(X_m\overline{X_n}) = \tau(m-n) \qquad \text{for all } n,m \in \mathbb{Z}$$

for a sequence $\tau : \mathbb{Z} \to \mathbb{C}$, that is, the kernel associated to the process is Toeplitz.

As applications we give proofs of [2, Theorem 2, p. 292] and [5, Theorem 6]. We also obtain a result similar to [5, Theorem 3].

Theorem 15. ([2, Theorem 2, p. 292]) *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stochastic process. Then the following conditions are equivalent:*

(i) *$X = \{X_n\}_{n\in\mathbb{Z}}$ is regular and weakly stationary.*

(ii) *There exists a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ and a orthonormal basis $\{\xi_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}(X)$ such that*

$$X_n = \sum_{j=0}^{+\infty} a_j\, \xi_{n-j}$$

and

$$\mathcal{H}^j(X) = \overline{\operatorname{span}}\,\{\xi_n \,:\, n \le j\} \qquad \textit{for all } j \in \mathbb{Z}\,.$$

Proof. Let K be the covariance kernel of the process X.

Suppose (i) holds. Then K is regular and Toeplitz.

By Theorem 5 there exists a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ and a orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}_K$ such that

$$\left[\delta^{(n)}\right]_K = \sum_{j=0}^{+\infty} a_j\, e_{n-j}$$

and

$$\mathcal{H}_K^j = \overline{\operatorname{span}}\,\{e_n \,:\, n \leq j\} \qquad \text{for } j \in \mathbb{Z}\,.$$

Consider the unitary operator $\Psi : \mathcal{H}_K \to \mathcal{H}(X)$ defined by

$$\Psi\left(\left[\delta^{(n)}\right]_K\right) = X_n \qquad \text{for } n \in \mathbb{Z}\,.$$

Then

$$X_n = \Psi\left(\left[\delta^{(n)}\right]_K\right) = \sum_{j=0}^{+\infty} a_j\, \Psi(e_{n-j})\,.$$

For $n \in \mathbb{Z}$, take

$$\xi_n = \Psi(e_n)\,.$$

Then $\{\xi_n\}_{n\in\mathbb{Z}}$ is a orthonormal basis of $\mathcal{H}(X)$.

And for $j \in \mathbb{Z}$ we have that

$$\mathcal{H}^j(X) = \Psi(\mathcal{H}_K^j) = \overline{\operatorname{span}}\,\{\Psi(e_n) \,:\, n \leq j\} = \overline{\operatorname{span}}\,\{\xi_n \,:\, n \leq j\}\,.$$

The converse follows using again Theorem 5. ∎

According to the definition given in [5, p. 17] a stochastic process $\{X_n\}_{n\in\mathbb{Z}}$ on $L^2(P)$ is *approximately weakly stationary* if there exists a positive definite sequence $\tau : \mathbb{Z} \to \mathbb{C}$ such that

$$\begin{aligned} A \sum_{m,n\in\mathbb{Z}} \tau(n-m)\, a_m\, \overline{a_n} &\leq \left\| \sum_{n\in\mathbb{Z}} a_n\, X_n \right\|_{L^2(P)}^2 \\ &\leq B \sum_{m,n\in\mathbb{Z}} \tau(n-m)\, a_m\, \overline{a_n} \end{aligned}$$

for every finite support sequence $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$.

LEMMA 16. *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stochastic process on $L^2(P)$. Then the following conditions are equivalent:*

(i) *The process is approximately weakly stationary.*

(ii) *The covariance kernel, K, of the process X is equivalent to a positive definite Toeplitz kernel.*

Proof. The result follows from the definitions and from the following equality

$$\left\| \sum_{n\in\mathbb{Z}} a_n\, X_n \right\|^2_{L^2(P)} = \sum_{m,n\in\mathbb{Z}} K(n,m)\, a_m\, \overline{a_n}\,. \qquad \blacksquare$$

THEOREM 17. ([5, THEOREM 6]) *Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a stochastic process on $L^2(P)$. Then the following conditions are equivalent:*

(i) *$X = \{X_n\}_{n\in\mathbb{Z}}$ is regular and approximately weakly stationary.*

(ii) *There exists a sequence $\{a_n\}_{n=0}^{+\infty} \in \ell^2(\mathbb{N})$ and a Riesz basis $\{\xi_n\}_{n\in\mathbb{Z}}$ of $\mathcal{H}(X)$ such that*

$$X_n = \sum_{j=0}^{+\infty} a_j\, \xi_{n-j}$$

and

$$\mathcal{H}^j(X) = \overline{\operatorname{span}}\,\{\xi_n \,:\, n \le j\} \qquad \text{for } j \in \mathbb{Z}\,.$$

Proof. Let K be the covariance kernel of the process X. Suppose $X = \{X_n\}_{n\in\mathbb{Z}}$ is regular and approximately weakly stationary. By Lemma 16 the kernel K is equivalent to a positive definite Toeplitz kernel K_1.

Using Theorem 7 the proof follows in a similar way to the proof of Theorem 15. ∎

The following result is similar to [5, Theorem 3].

THEOREM 18. *Let $S = \{S_n\}_{n\in\mathbb{Z}}$ be a weakly stationary process such that the sequence $\{S_n\}_{n\in\mathbb{Z}}$ is minimal. Then for any orthonormal process $\{e_n\}_{n\in I} \subset \mathcal{H}(S)$, where I is a finite subset of $\mathbb{Z}$, there exists a constant $B > 0$, such that for any sequence of numbers $\{c_n\}_{n\in I}$ such that $0 \le c_n < B$, the stochastic process X defined by*

$$X_n = \begin{cases} S_n + \sqrt{c_n}\, e_n & \text{if } n \in I\,, \\ S_n & \text{other case}\,, \end{cases}$$

is approximately weakly stationary.

Proof. Let K be the kernel given by $K(n,m) = \operatorname{cov}(S_m, S_n)$. Then K is a Toeplitz kernel.

Let $\Psi : \mathcal{H}_K \to \mathcal{H}(S)$ defined by

$$\Psi\left(\left[\delta^{(n)}\right]_K\right) = S_n$$

(see Remark 14).

Since $\{S_n\}_{n\in\mathbb{Z}}$ is minimal we have that $\{\left[\delta^{(n)}\right]_K\}_{n\in\mathbb{Z}}$ is minimal.

By Theorem 13 there exists a constant $B > 0$ such that for any sequence of numbers $\{c_n\}_{n\in I}$ that satisfies $0 \le c_n < B$ the kernel $K + K_1$ is equivalent to the kernel K, where

$$K_1(n,m) = \begin{cases} \rho(n,m) + \rho(m,n) + c_n\delta_{nm} & \text{if } m, n \in I\,, \\ \rho(n,m) & \text{if } m \in I, n \notin I\,, \\ \rho(m,n) & \text{if } m \notin I, n \in I\,, \\ 0 & \text{if } m \notin I, n \notin I \end{cases}$$

and

$$\rho(n,m) = \sqrt{c_m}\left\langle e_m, \left[\delta^{(n)}\right]_K\right\rangle_{\mathcal{H}_K}.$$

Since

$$\operatorname{cov}(X_m, X_n) = K(n,m) + K_1(n,m)\,,$$

it follows that the covariance kernel of $X = \{X_n\}_{n\in\mathbb{Z}}$ is equivalent to the Toeplitz kernel K.

So from Lemma 16 it follows that the process $S = \{S_n\}_{n\in\mathbb{Z}}$ is approximately weakly stationary. ∎

References

[1] N.L. Carothers, "A Short Course on Banach Space Theory", London Math. Soc. Student Texts 64, Cambridge University Press, Cambridge, 2005.

[2] I.I. Gihman, A.V. Skorokhod, "The Theory of Stochastic Processes I", Translated from the Russian by S. Kotz, Die Grundlehren der mathematischen Wissenschaften, Band 210, Springer-Verlag, New York-Heidelberg, 1974.

[3] J. Lindenstrauss, L. Tzafriri, "Classical Banach Spaces I", Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92, Springer-Verlag, Berlin-New York, 1977.

[4] R. Paley, N. Wiener, "Fourier Transforms in the Complex Domain", Amer. Math. Soc. Colloquium Publications 19, AMS, Providence, RI, 1934.

[5] G. Strandell, Approximately stationary stochastic processes, in "Stationary in Hilbert Spaces", U.U.D.M. Report 2001:31, Publications of the Department of Mathematics, Uppsala University, 2001, 13–32, http://www2.math.uu.se/research/pub/Strandell1.pdf.

[6] B. Sz.-Nagy, C. Foias, "Harmonic Analysis of Operators on Hilbert Space", North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest 1970.

[7] H.O.A. Wold, "A Study in the Analysis of Stationary Time Series, 2d ed. with an appendix by Peter Whittle", Almqvist and Wiksell, Stockholm, 1954.

[8] H.O.A. Wold, A large sample test for moving averages, *J. Roy. Statist. Soc. Ser. B* **11** (1949), 297–305.

[9] R.M. Young, "An Introduction to Nonharmonic Fourier Series", Pure and Applied Mathematics 93, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980; Academic Press, New York, 1980.