

## Global Asymptotic Stability for Semilinear Equations via Thompson’s Metric

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*Abstract:* In ordered Banach spaces we prove the global asymptotic stability of the unique strictly positive equilibrium of the semilinear equation  $u' = Au + f(u)$ , if  $A$  is the generator of a positive and exponentially stable  $C_0$ -semigroup and  $f$  is a contraction with respect to Thompson’s metric. The given estimates show that convergence holds with a uniform exponential rate.

*Key words:* Ordered Banach spaces, Thompson metric, global stability, semilinear equations.

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### 1. INTRODUCTION

Let  $(E, \|\cdot\|)$  be a real Banach space ordered by a normal cone  $K$ , that is  $K$  is a closed convex subset of  $E$  such that  $\lambda K \subseteq K$  ( $\lambda \geq 0$ ),  $K \cap (-K) = \{0\}$ , inducing an ordering by  $x \leq y : \iff y - x \in K$ , and

$$\exists c \geq 1 : \quad 0 \leq x \leq y \quad \Rightarrow \quad \|x\| \leq c\|y\|.$$

Moreover we assume that  $K$  is solid, that is  $K$  has nonempty interior  $K^\circ$ , and we set  $x \ll y : \iff y - x \in K^\circ$ . In this situation  $K^\circ$  endowed with the Thompson metric [11]

$$d(x, y) := \log \left( \min\{\alpha \geq 1 : x \leq \alpha y, y \leq \alpha x\} \right)$$

is a complete metric space. A function  $f : K^\circ \rightarrow K^\circ$  is  $d$ -Lipschitz continuous with constant  $l \geq 0$  if and only if, for all  $x, y \in K^\circ$ ,

$$x \leq \alpha y \wedge y \leq \alpha x \quad \Rightarrow \quad f(x) \leq \alpha^l f(y) \wedge f(y) \leq \alpha^l f(x).$$

We study in this paper the dynamical system

$$(1) \quad \begin{aligned} u'(t) &= Au(t) + f(u(t)) & (t \geq 0), \\ u(0) &= x \in K^\circ, \end{aligned}$$

where  $A$  is the generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  of positive operators on  $E$ , i.e.,

$$x \geq 0 \quad \Rightarrow \quad T(t)x \geq 0 \quad \text{for all } t \geq 0.$$

In particular,  $A$  is quasimonotone increasing (cf. Section 2). We are interested in the semiflow given by mild solutions of (1), i.e., by solutions of the integral equation

$$(2) \quad u(t) = T(t)x + \int_0^t T(t-s)f(u(s)) \, ds \quad (t \in [0, t_{\max}(x)], x \in K^\circ).$$

If  $l < 1$  and the semigroup is exponentially stable, then we obtain a unique  $x_0 \in K^\circ \cap D(A)$  such that

$$Ax_0 + f(x_0) = 0$$

(cf. Proposition 6 below); i.e., the dynamical system given by (1) has a unique equilibrium  $x_0$  in  $K^\circ$ . In our main result Theorem 1 below we show that, under these assumptions, the equilibrium  $x_0$  is globally asymptotically stable in  $K^\circ$ , i.e., any mild solution  $u(t, x)$  of (1) converges to  $x_0$  as  $t \rightarrow \infty$ . Moreover, we establish an explicit estimate for  $d(u(t, x), x_0)$  which shows that convergence holds with a uniform exponential rate. Under the additional assumption that  $x_0$  is an eigenvalue of  $A$  we obtain in Theorem 2 an optimal estimate on  $d(u(t, x), x_0)$ .

As an introductory example consider  $E = l^\infty(\mathbb{Z})$  endowed with the supremum norm  $\|\cdot\|_\infty$  and ordered by the normal and solid cone

$$K := \{x \in l^\infty(\mathbb{Z}) : x_n \geq 0 \text{ for all } n \in \mathbb{Z}\}.$$

Then

$$K^\circ = \left\{x \in l^\infty(\mathbb{Z}) : \inf_{n \in \mathbb{Z}} x_n > 0\right\},$$

and for  $|\beta| < 2$  the function  $f : K^\circ \rightarrow K^\circ$  defined by

$$f(x) = \left(2 + \beta \sin(\log(x_{n-1}x_{n+1}))\right)_{n \in \mathbb{Z}}$$

is  $d$ -Lipschitz continuous with constant  $\frac{2|\beta|}{\sqrt{4-\beta^2}}$  (cf. Section 4). Thus our results yield that, for  $|\beta| < 2/\sqrt{5}$ , all solutions of the infinite system

$$\begin{aligned} u'_n(t) &= u_{n+1}(t) - 4u_n(t) + u_{n-1}(t) + 2 \\ &\quad + \beta \sin(\log(u_{n-1}(t)u_{n+1}(t))) \quad (n \in \mathbb{Z}), \end{aligned}$$

with initial values in  $K^\circ$  are exponentially decaying to the constant sequence  $(1)_{n \in \mathbb{Z}}$  in  $l^\infty(\mathbb{Z})$  as  $t \rightarrow \infty$ , with respect to Thompson's metric.

Concerning our assumptions we remark that, for  $l > 1$ , global existence for solutions of (1) fails in general, as one can already see in the one-dimensional case  $E = \mathbb{R}$  for  $Ax = -x$  and  $f(x) = x^l$ . The case  $l = 1$  is special in the sense that it includes linear functions. For  $E = \mathbb{R}$ ,  $Ax = -x$  and  $f(x) = 2x$  in (1) existence of equilibria in  $K^\circ$  fails, and all solutions in  $K^\circ$  grow exponentially.

Motivated by the analysis of generalized Riccati equations and other non-linear matrix equations, the contraction rate of flows with respect to Thompson's metric has been investigated in [7], [10]. In particular, Gaubert and Qu [3] studied the time-dependent case for order-preserving flows. In our case the semiflow generated by (1) is not order-preserving in general, since  $f$  need not to be quasimonotone increasing in the sense of Volkmann [12].

The paper is organized as follows: In the preliminary Section 2 we state some properties of Thompson's metric in our situation and of positive semigroups. Moreover, we show existence of global solutions to (1) and nonexpansiveness of the semiflow with respect to Thompson's metric under the assumption that  $l < 1$  and that  $A$  generates a semigroup of positive operators. In Section 3 we state and prove our main results on global asymptotic stability. Section 4 contains some examples.

## 2. PRELIMINARIES

The interrelation between the topologies generated by  $d$  and  $\|\cdot\|$  on  $K^\circ$  are characterized by the following Proposition 1, see [5]. Let  $p \in K^\circ$  be fixed and let  $\|\cdot\|_p$  denote the Minkowski functional

$$\|x\|_p = \min \{ \alpha \geq 0 : -\alpha p \leq x \leq \alpha p \} \quad (x \in E).$$

Then  $\|\cdot\|_p$  is an equivalent norm on  $E$  [2, Prop.19.9], and we write

$$\text{dist}_p(x, \partial K) := \inf \{ \|x - y\|_p : y \in \partial K \}$$

for the distance to the boundary of  $K$  with respect to this norm.

PROPOSITION 1. For all  $x, y \in K^\circ$ :

- (i)  $\|x - y\|_p \leq (\exp(d(x, y)) - 1) \exp(\max\{d(x, p), d(y, p)\})$ ;
- (ii)  $d(x, y) \leq \|x - y\|_p / \min \{ \text{dist}_p(x, \partial K), \text{dist}_p(y, \partial K) \}$ ;
- (iii)  $-\log(\text{dist}_p(x, \partial K)) \leq d(x, p)$ .

In particular, if  $f$  is  $d$ -Lipschitz continuous then  $f$  is locally Lipschitz continuous in norm, and thus (2) has unique local solutions, i.e., initial value problems for (1) have unique mild solutions on maximal time intervals. In case that  $f$  is a  $d$ -contraction, we will see that (1) generates a nonexpansive semiflow on  $[0, \infty)$ .

We assume from now on that  $A$  is the generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  of positive operators, i.e.,  $T(t)(K) \subseteq K$  for each  $t \geq 0$ . Then  $A$  is quasimonotone increasing, i.e., for all  $x \in D(A) \cap K$  and each

$$\phi \in K^* := \{\psi \in E^* : \psi \geq 0 \text{ on } K\}$$

such that  $\phi(x) = 0$  we have  $\phi(Ax) \geq 0$  (cf. [1], [6]). We recall the following result from [4], for which we give a new proof here. We also recall the directional derivative  $m_+$  from [8], which we use for  $\|\cdot\|_p$  here,

$$m_+[x, y] := \lim_{h \rightarrow 0^+} \frac{\|x + hy\|_p - \|x\|_p}{h} \quad (x, y \in E),$$

and the property

$$\left(\frac{d}{dt}\right)_+ \|T(t)x\|_p = m_+[T(t)x, AT(t)x] \quad (x \in D(A)),$$

where  $\left(\frac{d}{dt}\right)_+$  denotes the right side derivative.

**PROPOSITION 2.** *If  $p \in D(A) \cap K^\circ$  and  $\omega := \min\{\alpha \in \mathbb{R} : Ap \leq \alpha p\}$  then  $m_+[p, Ap] = \omega$  and  $m_+[x, Ax] \leq \omega\|x\|_p$  for all  $x \in D(A)$ . Moreover,  $\|T(t)x\|_p \leq e^{\omega t}\|x\|_p$  for all  $t \geq 0$  and  $x \in E$ , in particular  $T(t)p \leq e^{\omega t}p$  for all  $t \geq 0$ .*

*Proof.* Replacing  $A$  by  $A - \omega I$  and  $T(t)$  by  $e^{-\omega t}T(t)$  we may assume  $\omega = 0$ . Since  $p \in K^\circ$  we have  $p + hAp \in K^\circ$  for small  $h > 0$ . For these  $h$  we have

$$\begin{aligned} \|p + hAp\|_p &= \min \{ \alpha \in \mathbb{R} : p + hAp \leq \alpha p \} \\ &= \min \left\{ \alpha \in \mathbb{R} : Ap \leq \frac{\alpha - 1}{h} p \right\} = 1, \end{aligned}$$

which implies  $m_+[p, Ap] = 0$ . If  $x \in D(A)$  and  $\alpha = \|x\|_p$  then  $-\alpha p \leq x \leq \alpha p$ , which implies

$$-\alpha T(t)p \leq T(t)x \leq \alpha T(t)p \quad \text{and} \quad \|T(t)x\|_p \leq \alpha \|T(t)p\|_p.$$

But then

$$\begin{aligned} m_+[x, Ax] &= \lim_{t \rightarrow 0^+} \frac{\|T(t)x\|_p - \alpha}{t} \\ &\leq \alpha \lim_{t \rightarrow 0^+} \frac{\|T(t)p\|_p - \|p\|_p}{t} = \alpha m_+[p, Ap] = 0. \end{aligned}$$

Hence

$$\left(\frac{d}{dt}\right)_+ \|T(t)x\|_p = m_+[T(t)x, AT(t)x] \leq 0$$

for all  $t \geq 0$  and  $x \in D(A)$ , which implies  $\|T(t)x\|_p \leq \|x\|_p$  for all  $t \geq 0$  and  $x \in E$ . ■

Here we remark that if  $p \in D(A) \cap K^\circ$  and  $-Ap \in K^\circ$  then  $\omega < 0$  and the semigroup is exponentially stable. On the other hand, exponential stability of the semigroup implies that  $A : D(A) \rightarrow E$  is bijective and  $-A^{-1}$  is positive as can be seen from

$$(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x dt \quad (x \in E)$$

for  $\lambda \geq 0$ .

**PROPOSITION 3.** *For each  $t \geq 0$  we have  $T(t)(K^\circ) \subseteq K^\circ$ . If the semigroup is exponentially stable then we have  $-A^{-1}(K^\circ) \subseteq K^\circ$ .*

*Proof.* Choose  $p \gg 0$ . By strong continuity, we find  $\delta > 0$  such that  $T(t)p \gg 0$  for  $t \in [0, \delta]$ . If  $x \gg 0$  we find  $\alpha > 0$  with  $p \leq \alpha x$  which implies  $T(t)p \leq \alpha T(t)x$  for any  $t \geq 0$ . In particular,  $T(t)x \gg 0$  for all  $t \in [0, \delta]$ . We have shown  $T(t)(K^\circ) \subseteq K^\circ$  for  $t \in [0, \delta]$ . By the semigroup property, this holds for all  $t \geq 0$ .

Let the semigroup be exponentially stable and  $x \gg 0$ . Since  $-A^{-1}$  is positive, we have  $y := -A^{-1}x \geq 0$ , and if  $y \notin K^\circ$  we find  $\phi \in K^* \setminus \{0\}$  such that  $\phi(y) = 0$ . But then  $\phi(Ay) \geq 0$  in contradiction to  $Ay = -x \in -K^\circ$ . ■

For  $T > 0$  let  $C([0, T], E)$  be endowed with the maximum norm  $\|\cdot\|_\infty$  and ordered by the normal and solid cone

$$K_T := \{u \in C([0, T], E) : u(t) \geq 0 \text{ for all } t \in [0, T]\}.$$

Note that (since  $[0, T]$  is compact)

$$K_T^\circ = \{u \in C([0, T], E) : u(t) \gg 0 \text{ for all } t \in [0, T]\}.$$

The Thompson metric with respect to  $K_T$  is denoted by  $d_T$ . We will see that  $d$ -contractivity of  $f$  yields existence of global mild solutions for (1):

**PROPOSITION 4.** *Let  $f : K^\circ \rightarrow K^\circ$  be  $d$ -Lipschitz continuous with constant  $l < 1$  and let  $A$  be the generator of a positive semigroup. Then each initial value problem*

$$u'(t) = Au(t) + f(u(t)), \quad u(0) = x \gg 0,$$

*has a unique mild solution on  $[0, \infty)$ . In particular,  $u(t) \gg 0$  for each  $t \geq 0$ .*

*Proof.* Fix  $T > 0$  and consider the integral operator  $S : K_T^\circ \rightarrow K_T^\circ$  defined as

$$(Sv)(t) = T(t)x + \int_0^t T(t-s)f(v(s)) \, ds.$$

Let  $v, w \in K_T^\circ$ . We have  $v \leq \exp(d_T(v, w))w$  and  $w \leq \exp(d_T(v, w))v$ , that is to say

$$v(t) \leq \exp(d_T(v, w))w(t) \quad \text{and} \quad w(t) \leq \exp(d_T(v, w))v(t) \quad \text{for all } t \in [0, T].$$

Thus

$$\begin{aligned} (Sv)(t) &\leq T(t)x + \int_0^t T(t-s) \exp(ld_T(v, w))f(w(s)) \, ds \\ &\leq \exp(ld_T(v, w))S(w)(t) \quad (t \in [0, T]), \end{aligned}$$

and therefore  $Sv \leq \exp(ld_T(v, w))Sw$ ; analogously  $Sw \leq \exp(ld_T(v, w))Sv$ . Thus

$$d_T(Sv, Sw) \leq ld_T(v, w) \quad (v, w \in K_T^\circ),$$

and therefore  $S$  has a unique fixed point  $u \in K_T^\circ$  which is a mild solution of (1) on  $[0, T]$ . Since  $T > 0$  was arbitrary, the assertion follows. ■

In the sequel, let  $u(\cdot, x) : [0, \infty) \rightarrow K^\circ$  denote the mild solution of (1) with initial value  $u(0) = x \gg 0$ .

**PROPOSITION 5.** *Under the assumptions of Proposition 4, for all  $x, y \gg 0$*

$$t \mapsto d(u(t, x), u(t, y)) \text{ is monotone decreasing on } [0, \infty),$$

*i.e., the semiflow of (1) is non-expansive with respect to Thompson's metric.*

*Proof.* For  $x, y \in K^\circ$  and  $v := u(\cdot, x)$ ,  $w := u(\cdot, y)$  we have

$$\begin{aligned} v(t) &= T(t)x + \int_0^t T(t-s)f(v(s)) \, ds \quad (t \geq 0), \\ w(t) &= T(t)y + \int_0^t T(t-s)f(w(s)) \, ds \quad (t \geq 0), \end{aligned}$$

thus

$$\begin{aligned} v(\tau) &\leq \max \{ \exp(d(x, y)), \exp(ld_t(v, w)) \} w(\tau) \quad (\tau \in [0, t]), \\ w(\tau) &\leq \max \{ \exp(d(x, y)), \exp(ld_t(v, w)) \} v(\tau) \quad (\tau \in [0, t]), \end{aligned}$$

which implies

$$d_t(v, w) \leq \max \{ d(x, y), ld_t(v, w) \} \quad (t \geq 0).$$

Since  $l < 1$  and  $d(v(t), w(t)) \leq d_t(v, w)$  ( $t \geq 0$ ) we get

$$d(v(t), w(t)) \leq d(x, y) \quad (t \geq 0),$$

and the assertion follows by translation of  $v$  and  $w$ . ■

### 3. GLOBAL ASYMPTOTIC STABILITY

**PROPOSITION 6.** *Let  $f : K^\circ \rightarrow K^\circ$  be  $d$ -Lipschitz continuous with constant  $l < 1$  and let  $A$  be the generator of a positive semigroup which is exponentially stable. Then there is a unique solution  $x_0 \in K^\circ \cap D(A)$  of the equation  $Ax_0 + f(x_0) = 0$ .*

*Remark.* Observe that existence of  $x_0 \in K^\circ \cap D(A)$  with  $Ax_0 + f(x_0) = 0$  implies via  $-Ax_0 \in K^\circ$  exponential stability of the semigroup (cf. Section 2), which thus is a necessary assumption.

*Proof.* Proposition 3 yields  $-A^{-1}(f(K^\circ)) \subseteq K^\circ$ . Since  $-A^{-1} : E \rightarrow E$  is linear and order-preserving, the function  $-A^{-1} \circ f : K^\circ \rightarrow K^\circ$  is again  $d$ -Lipschitz continuous with constant  $l$ . Thus  $-A^{-1} \circ f$  has a unique fixed point  $x_0$  in  $K^\circ$  by Banach's Fixed Point Theorem. Since  $A^{-1}(E) = D(A)$ , we have  $x_0 \in D(A)$ , and the assertion follows. ■

The following is our main result on global asymptotic stability.

**THEOREM 1.** *Let  $f : K^\circ \rightarrow K^\circ$  be  $d$ -Lipschitz continuous with constant  $l < 1$  and let  $A$  be the generator of a positive semigroup which is exponentially stable. Let  $x_0 \in K^\circ \cap D(A)$  be the unique solution of the equation  $Ax_0 + f(x_0) = 0$ . Then for every  $x \in K^\circ$  we have  $u(t, x) \rightarrow x_0$  as  $t \rightarrow \infty$  and*

$$d(u(t, x), x_0) \leq \frac{e^{d(x, x_0)} - 1}{\mu} e^{-\gamma t} \quad (t \geq 0),$$

where

$$\begin{aligned} \delta &:= -\min \{ \omega \in \mathbb{R} : Ax_0 \leq \omega x_0 \} > 0, \\ \mu &:= l(1 - e^{-\delta/2}) + e^{-\delta/2} \in (0, 1), \\ \gamma &:= -\log \mu > 0. \end{aligned}$$

*Remark.* We recall from Proposition 2 that, with  $p := x_0$ , we have in Theorem 1

$$-\delta = m_+[x_0, Ax_0] = \lim_{h \rightarrow 0^+} \frac{\|x_0 + hAx_0\|_p - \|x_0\|_p}{h}.$$

*Proof.* Let  $x \in K^\circ$  and  $u(t) := u(t, x)$ ,  $h(t) := e^{d(u(t), x_0)}$  ( $t \geq 0$ ). Recall that  $h$  is decreasing by Proposition 5 and  $\geq 1$ , and observe  $-Ax_0 = f(x_0) \in K^\circ$ . By Proposition 2 we have  $T(t)x_0 \leq e^{-\delta t}x_0$  where  $\delta > 0$  here. Hence

$$T(t+s)x_0 = T(t)T(s)x_0 \leq T(t) \left( e^{-\delta s} x_0 \right) \leq T(t)x_0 \quad (s, t \geq 0),$$

which means that  $t \mapsto T(t)x_0$  is decreasing on  $[0, \infty)$ . From

$$u(t) = T(t)x + \int_0^t T(t-s)f(u(s)) \, ds \quad (t \geq 0)$$

we obtain by splitting the integral as  $\int_0^{t/2} + \int_{t/2}^t$ :



$$\begin{aligned}
u(t) &\leq h(0)T(t)x_0 + \int_0^t h(s)^l T(t-s)(-Ax_0) \, ds \\
&\leq h(0)T(t)x_0 + h(0)^l \int_0^{t/2} T(t-s)(-Ax_0) \, ds \\
&\quad + h(t/2)^l \int_{t/2}^t T(t-s)(-Ax_0) \, ds \\
&= h(0)T(t)x_0 + h(0)^l \left( T(t/2)x_0 - T(t)x_0 \right) + h(t/2)^l \left( x_0 - T(t/2)x_0 \right) \\
&\leq h(0)T(t/2)x_0 + h(t/2)^l \left( x_0 - T(t/2)x_0 \right) \\
&= h(t/2)^l x_0 + \left( h(0) - h(t/2)^l \right) T(t/2)x_0 \\
&\leq h(t/2)^l x_0 + \left( h(0) - h(t/2)^l \right) e^{-\delta t/2} x_0 \\
&= \left[ h(t/2)^l \left( 1 - e^{-\delta t/2} \right) + h(0)e^{-\delta t/2} \right] x_0
\end{aligned}$$

and

$$\begin{aligned}
u(t) &\geq h(0)^{-1}T(t)x_0 + \int_0^t h(s)^{-l}T(t-s)(-Ax_0) \, ds \\
&\geq h(0)^{-1}T(t)x_0 + h(0)^{-l} \int_0^{t/2} T(t-s)(-Ax_0) \, ds \\
&\quad + h(t/2)^{-l} \int_{t/2}^t T(t-s)(-Ax_0) \, ds \\
&= \frac{1}{h(0)}T(t)x_0 + \frac{1}{h(0)^l} \left( T(t/2)x_0 - T(t)x_0 \right) + \frac{1}{h(t/2)^l} \left( x_0 - T(t/2)x_0 \right) \\
&\geq h(0)^{-1}T(t/2)x_0 + h(t/2)^{-l} \left( x_0 - T(t/2)x_0 \right) \\
&= h(t/2)^{-l} x_0 + \left( h(0)^{-1} - h(t/2)^{-l} \right) T(t/2)x_0 \\
&\geq h(t/2)^{-l} x_0 + \left( h(0)^{-1} - h(t/2)^{-l} \right) e^{-\delta t/2} x_0 \\
&= \left[ h(t/2)^{-l} \left( 1 - e^{-\delta t/2} \right) + h(0)^{-1}e^{-\delta t/2} \right] x_0.
\end{aligned}$$

We thus have shown

$$h(t) \leq \max \left\{ \left( \frac{1 - e^{-\delta t/2}}{h(t/2)^l} + \frac{e^{-\delta t/2}}{h(0)} \right)^{-1}, h(t/2)^l(1 - e^{-\delta t/2}) + h(0)e^{-\delta t/2} \right\}.$$

Now let  $\alpha = h(0)$ ,  $\beta = h(t/2)^l \leq \alpha$ ,  $\lambda = e^{-\delta t/2} \in (0, 1)$  and observe

$$\begin{aligned} 1 &= (1 - \lambda)^2 + 2\lambda(1 - \lambda) + \lambda^2 \\ &\leq (1 - \lambda)^2 + \lambda(1 - \lambda)(\alpha/\beta + \beta/\alpha) + \lambda^2 \\ &= ((1 - \lambda)\beta + \lambda\alpha)((1 - \lambda)\beta^{-1} + \lambda\alpha^{-1}), \end{aligned}$$

which means  $((1 - \lambda)\beta^{-1} + \lambda\alpha^{-1})^{-1} \leq ((1 - \lambda)\beta + \lambda\alpha)$ . This yields

$$h(t) \leq h(t/2)^l(1 - e^{-\delta t/2}) + h(0)e^{-\delta t/2} \quad (t \geq 0).$$

Since the problem is autonomous we also have

$$h(s + t) \leq h(s + t/2)^l(1 - e^{-\delta t/2}) + h(s)e^{-\delta t/2} \quad (s, t \geq 0).$$

We know that  $h$  is decreasing, so  $h(t)$  tends to some  $c \geq 1$  as  $t \rightarrow \infty$ . The first inequality gives us  $c \leq c^l$ , and  $l < 1$  implies  $c = 1$ .

Rate of convergence: We let  $t = 1$ ,  $s = n \in \mathbb{N}_0$  in the second inequality. Then we have

$$h(n + 1) \leq h(n + 1/2)^l(1 - e^{-\delta/2}) + h(n)e^{-\delta/2} \quad (n \in \mathbb{N}_0),$$

and

$$h(n + 1) - 1 \leq (h(n + 1/2)^l - 1)(1 - e^{-\delta/2}) + (h(n) - 1)e^{-\delta/2} \quad (n \in \mathbb{N}_0).$$

By the mean value theorem and monotonicity of  $h$  we have

$$h(n + 1/2)^l - 1 \leq l(h(n + 1/2) - 1) \leq l(h(n) - 1),$$

and thus

$$h(n + 1) - 1 \leq \mu(h(n) - 1) \quad (n \in \mathbb{N}_0),$$

where  $\mu = l(1 - e^{-\delta/2}) + e^{-\delta/2} < 1$ . This yields

$$h(n) - 1 \leq \mu^n(h(0) - 1) \quad (n \in \mathbb{N}_0).$$

For  $t > 0$ , we choose  $n \in \mathbb{N}$  such that  $n - 1 < t \leq n$  and obtain

$$h(t) - 1 \leq h(n - 1) - 1 \leq \mu^{n-1}(h(0) - 1) \leq \mu^{t-1}(h(0) - 1),$$

and

$$d(u(t), x_0) = \log h(t) = \log(1 + h(t) - 1) \leq h(t) - 1 \leq \mu^{t-1}(h(0) - 1),$$

which is the claim. ■

Observe that we have in the proof  $Ax_0 \leq -\delta x_0$ . In case of equality  $Ax_0 = -\delta x_0$ , i.e., if  $x_0$  is an eigenvalue of  $A$ , we give another estimate which turns out to be sharp.

**THEOREM 2.** *Let  $f : K^\circ \rightarrow K^\circ$  be  $d$ -Lipschitz continuous with constant  $l < 1$  and let  $A$  be the generator of a positive semigroup which is exponentially stable. Let  $x_0 \in K^\circ \cap D(A)$  be the unique solution of the equation  $Ax_0 + f(x_0) = 0$  and assume  $Ax_0 = -\delta x_0$ . Then, for all  $x \in K^\circ$  and  $t \geq 0$*

$$d(u(t, x), x_0) \leq \frac{1}{1-l} \log \left( 1 + (e^{(1-l)d(x, x_0)} - 1)e^{-\delta(1-l)t} \right).$$

*Remark.* For each  $a > 0$  the function  $t \mapsto \log(1 + ae^{-\delta(1-l)t})$  is strictly decreasing to 0 as  $t \rightarrow \infty$  and

$$\log(1 + ae^{-\delta(1-l)t}) \leq ae^{-\delta(1-l)t} \quad (t \geq 0).$$

Thus, also Theorem 2 gives an exponential decay estimate with respect to  $d$  for global asymptotic stability of the equilibrium  $x_0$  of (1). We shall see in the proof that the estimate is optimal.

*Proof of Theorem 2.* Let again  $u := u(\cdot, x)$  and  $h(t) := \exp(d(u(t), x_0))$  ( $t \geq 0$ ). We start the estimates as in the proof of Theorem 1. For  $t \geq 0$  we have

$$\begin{aligned} u(t) &\leq h(0)T(t)x_0 + \int_0^t h(s)^l T(t-s)(-Ax_0) ds \\ &= \left( h(0)e^{-\delta t} + \delta \int_0^t e^{-\delta(t-s)} h(s)^l ds \right) x_0 \end{aligned}$$

and

$$\begin{aligned} u(t) &\geq h(0)^{-1}T(t)x_0 + \int_0^t h(s)^{-l}T(t-s)(-Ax_0) \, ds \\ &= \left( h(0)^{-1}e^{-\delta t} + \delta \int_0^t e^{-\delta(t-s)}(h(s))^{-l} \, ds \right) x_0. \end{aligned}$$

Thus

$$h(t) \leq \max \left\{ \left( e^{-\delta t} \frac{1}{h(0)} + \int_0^t e^{-\delta(t-s)} \frac{1}{(h(s))^l} \, ds \right)^{-1}, \right. \\ \left. e^{-\delta t} h(0) + \int_0^t e^{-\delta(t-s)} (h(s))^l \, ds \right\}.$$

We next prove for any  $t \geq 0$ :

$$\left( e^{-\delta t} \frac{1}{h(0)} + \int_0^t e^{-\delta(t-s)} \frac{1}{(h(s))^l} \, ds \right)^{-1} \leq e^{-\delta t} h(0) + \int_0^t e^{-\delta(t-s)} (h(s))^l \, ds.$$

This inequality is equivalent to

$$\left( h(0) + \delta \int_0^t e^{\delta s} (h(s))^l \, ds \right) \left( \frac{1}{h(0)} + \delta \int_0^t e^{\delta s} \frac{1}{(h(s))^l} \, ds \right) \geq e^{2\delta t}.$$

According to the Cauchy-Schwarz inequality we have

$$\begin{aligned} e^{\delta t} - 1 &= \delta \int_0^t e^{\delta s} \, ds = \left( \int_0^t \sqrt{\delta} e^{\delta s/2} (h(s))^{l/2} \sqrt{\delta} e^{\delta s/2} \frac{1}{(h(s))^{l/2}} \, ds \right) \\ &\leq \left( \delta \int_0^t e^{\delta s} (h(s))^l \, ds \right)^{1/2} \left( \delta \int_0^t e^{\delta s} \frac{1}{(h(s))^l} \, ds \right)^{1/2}. \end{aligned}$$

Thus, from  $\sqrt{ab} + 1 \leq \sqrt{a+c} \sqrt{b+1/c}$  ( $a, b, c > 0$ ) we obtain

$$\begin{aligned} e^{\delta t} &\leq \left( \delta \int_0^t e^{\delta s} (h(s))^l \, ds \right)^{1/2} \left( \delta \int_0^t e^{\delta s} \frac{1}{(h(s))^l} \, ds \right)^{1/2} + 1 \\ &\leq \left( h(0) + \delta \int_0^t e^{\delta s} (h(s))^l \, ds \right)^{1/2} \left( \frac{1}{h(0)} + \delta \int_0^t e^{\delta s} \frac{1}{(h(s))^l} \, ds \right)^{1/2}. \end{aligned}$$

Summing up, we now have

$$e^{d(u(t), x_0)} = h(t) \leq e^{-\delta t} h(0) + \delta \int_0^t e^{-\delta(t-s)} (h(s))^l ds \quad (t \geq 0).$$

For  $T > 0$  consider  $C([0, T], \mathbb{R})$  ordered by the cone

$$K_r := \{ \xi \in C([0, T], \mathbb{R}) : \xi(t) \geq 0 \text{ for all } t \in [0, T] \},$$

with corresponding Thompson metric  $d_r$ , and  $S : K_r^\circ \rightarrow K_r^\circ$  defined by

$$(S\xi)(t) = e^{-\delta t} h(0) + \delta \int_0^t e^{-\delta(t-s)} (\xi(s))^l ds.$$

As in the proof of Proposition 4 we have  $d_r(S\xi, S\eta) \leq l d_r(\xi, \eta)$  ( $\xi, \eta \in K_r^\circ$ ), and moreover  $S$  is monotone increasing. Thus, there is a unique fixed point  $\xi_0$  of  $S$ , and  $h(t) \leq (Sh)(t)$  ( $t \in [0, T]$ ) implies  $h(t) \leq \xi_0(t)$  ( $t \in [0, T]$ ). Since  $T > 0$  was arbitrary, this proves that

$$h(t) \leq \xi_0(t) \quad (t \geq 0),$$

with  $\xi_0 : [0, \infty) \rightarrow (0, \infty)$  the solution of the initial value problem

$$(3) \quad \xi_0'(t) = \delta(\xi_0(t))^l - \delta\xi_0(t), \quad \xi_0(0) = h(0).$$

A simple calculation shows

$$\xi_0(t) = \left( 1 + e^{-\delta(1-l)t} (h(0)^{1-l} - 1) \right)^{1/(1-l)} \quad (t \geq 0).$$

Thus

$$\begin{aligned} d(u(t), x_0) &= \log(h(t)) \leq \log(\xi_0(t)) \\ &= \frac{1}{1-l} \log \left( 1 + e^{-\delta(1-l)t} (e^{(1-l)d(x, x_0)} - 1) \right) \quad (t \geq 0). \end{aligned}$$

■

*Remark.* As can be seen from the proof, the inequality in Theorem 2 is best possible since the equation (3) is of the form (1) where  $Ax = -\delta x$  and  $f(x) = \delta x^l$  is a  $d$ -contraction with constant  $l$ .

## 4. EXAMPLES

The following lemma gives a simple sufficient condition for a function  $f : (0, \infty) \rightarrow (0, \infty)$  to be a  $d$ -contraction.

LEMMA 1. *Let  $f : (0, \infty) \rightarrow (0, \infty)$  be a  $C^1$ -function such that*

$$l := \sup_{x>0} \frac{x|f'(x)|}{f(x)} < \infty.$$

*Then  $f$  is  $d$ -Lipschitz with constant  $l$ .*

*Proof.* First note that  $d(x, y) = |\log(y/x)| = |\log y - \log x|$  for  $x, y > 0$ . We thus have to show

$$|\log f(y) - \log f(x)| \leq l |\log y - \log x| \quad (x, y > 0).$$

Writing  $\log f(z) = \log(f(e^{\log z}))$  this follows from the mean value theorem applied to  $g(t) := \log(f(e^t))$ , since  $g'(t) = e^t f'(e^t)/f(e^t)$ . ■

As the example  $f(x) = x^l$  shows, the estimate in Lemma 1 is sharp.

EXAMPLES 1. (i) Consider  $f(x) = \log(\beta + x)$  with  $\beta > 1$ . Then we have for  $x > 0$  by the mean value theorem

$$\begin{aligned} \frac{xf'(x)}{f(x)} &= \frac{x}{(\beta + x)\log(\beta + x)} \\ &\leq \frac{x}{(\beta + x)\log(\beta + x) - \beta \log \beta} \leq \frac{1}{1 + \log \beta}. \end{aligned}$$

By Lemma 1,  $f$  is a  $d$ -contraction on  $(0, \infty)$  with constant  $l = \frac{1}{1 + \log \beta} < 1$ .

(ii) Consider  $f(x) = \arctan(\beta + x)$  where  $\beta > 0$ . Then we have for  $x > 0$  by the mean value theorem

$$\begin{aligned} \frac{xf'(x)}{f(x)} &= \frac{x}{(1 + (\beta + x)^2) \arctan(\beta + x)} \\ &\leq \frac{x}{(1 + (\beta + x)^2) \arctan(\beta + x) - (1 + \beta^2) \arctan \beta} \\ &\leq \frac{1}{1 + 2\beta \arctan \beta}. \end{aligned}$$

By Lemma 1,  $f$  is a  $d$ -contraction on  $(0, \infty)$  with constant  $l = \frac{1}{1+2\beta \arctan \beta} < 1$ .

(iii) Consider  $f(x) = 2 + \beta \sin(\log x)$  with  $|\beta| < \sqrt{2}$ . Then we have for  $x > 0$

$$\frac{x|f'(x)|}{f(x)} = \frac{|\beta \cos(\log x)|}{2 + \beta \sin(\log x)} = \frac{|\beta| \sqrt{1 - y^2}}{2 + \beta y},$$

where  $y = \sin(\log x) \in [-1, 1]$ . A simple calculation shows that this is  $\leq \frac{|\beta|}{\sqrt{4-\beta^2}}$  with equality for  $y = -\frac{\beta}{2}$ . By Lemma 1,  $f$  is a  $d$ -contraction on  $(0, \infty)$  with constant  $l = \frac{|\beta|}{\sqrt{4-\beta^2}} < 1$ .

We give an application of Theorem 1.

EXAMPLE 2. Consider the problem

$$\begin{aligned} \partial_t u(t, s) &= \partial_s^2 u(t, s) + 2s \partial_s u(t, s) + \left( \frac{u(t, s)}{u(t, -s)} \right)^{1/4}, \\ u(0, s) &= u_0(s), \\ \partial_s u(t, 1) &= -u(t, 1), \\ \partial_s u(t, -1) &= u(t, -1), \end{aligned}$$

with  $s \in [-1, 1]$  and  $t \geq 0$ . We take  $E := C([-1, 1], \mathbb{R})$  ordered by the normal and solid cone

$$K := \{x \in E : x(s) \geq 0 \text{ for all } s \in [-1, 1]\}.$$

Clearly,  $x \mapsto f(x)(s) := \left( \frac{x(s)}{x(-s)} \right)^{1/4}$  ( $s \in [-1, 1]$ ) is  $d$ -Lipschitz with constant  $l = 1/2$ . We define

$$D(A) := \{x \in C^2([-1, 1], \mathbb{R}) : x'(1) = -x(1), x'(-1) = x(-1)\}$$

and  $Ax := x'' + 2sx'$  for  $x \in D(A)$ . Then  $A$  is a second order elliptic operator with real coefficients and Robin boundary conditions and thus generates a positive  $C_0$ -semigroup in  $E$  (cf., e.g., [9]). Letting

$$w(s) := -e^{-s^2} \int_0^s e^{\sigma^2} d\sigma \quad (s \in [-1, 1])$$

we have

$$w(-s) = -w(s) \quad \text{and} \quad w'(s) + 2sw(s) + 1 = 0 \quad \text{on } [-1, 1],$$

and  $c := w(-1) = \int_0^1 e^{\sigma^2-1} d\sigma \in [e^{-1}, 1]$ . Thus

$$x_0(s) := w(-1) + \int_{-1}^s w(\sigma) d\sigma \quad (s \in [-1, 1])$$

defines a function  $x_0 \in K^\circ \cap D(A)$ , satisfying

$$x_0''(s) + 2sx_0'(s) + 1 = 0 \quad \text{and} \quad x_0(-s) = x_0(s) \quad \text{on} \quad [-1, 1].$$

We conclude that  $x_0 \in K^\circ \cap D(A)$  satisfies  $Ax_0 + f(x_0) = 0$ . By the remark following Proposition 6 the semigroup generated by  $A$  is exponentially stable. Thus Theorem 1 is applicable. By

$$Ax_0 = -1 = -\frac{1}{c} \cdot c \leq -\frac{1}{c}x_0, \quad \min_{|s| \leq 1} x_0(s) = c,$$

we obtain

$$d(u(t, x), x_0) \leq \frac{e^{d(x, x_0)} - 1}{\mu} e^{-\gamma t} \quad (t \geq 0, x \gg 0),$$

$\delta = c^{-1} \in [1, e]$ ,  $\mu = (1 - e^{-1/(2c)})/2 + e^{-1/(2c)}$ . The numerical values are  $\delta \approx 1.85846$ ,  $\mu \approx 0.69743$ ,  $\gamma \approx 0.36036$ . We also see that Theorem 2 cannot be applied.

For an application of Theorem 2 we recall the example from the introduction.

EXAMPLE 3. Consider again  $E = l^\infty(\mathbb{Z})$  endowed with the supremum norm  $\|\cdot\|_\infty$  and ordered by the normal and solid cone

$$K := \{x \in l^\infty(\mathbb{Z}) : x_n \geq 0 \text{ for all } n \in \mathbb{Z}\}.$$

For  $|\beta| < \frac{2}{\sqrt{5}}$ , let  $f : K^\circ \rightarrow K^\circ$  be defined by

$$f(x) = \left(2 + \beta \sin(\log(x_{n-1}x_{n+1}))\right)_{n \in \mathbb{Z}}$$

and  $A : E \rightarrow E$  be defined by  $A(x_n) := (x_{n+1} - 4x_n + x_{n-1})_{n \in \mathbb{Z}}$ . Then  $A$  is quasimonotone increasing and generates a semigroup of positive operators. Letting  $p := (1)_{n \in \mathbb{Z}}$  we have  $Ap = -2p = -f(p)$ , and  $\delta = -2$ . For  $|\beta| = \frac{2}{\sqrt{17}}$ , the function  $f$  is a  $d$ -contraction with constant  $l = 1/2$ . By Theorem 2 we have the estimate

$$\begin{aligned} d(u(t, x), p) &\leq 2 \log(1 + (e^{d(x, p)/2} - 1) e^{-t}) \\ &\leq 2(e^{d(x, p)/2} - 1) e^{-t} \quad (t \geq 0, x \gg 0). \end{aligned}$$



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