

# Uniformly Bounded Superposition Operators in the Space of Functions of Bounded $n$ -Dimensional $\Phi$ -Variation

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*Abstract:* We prove that if a superposition operator maps a subset of the space of all functions of  $n$ -dimensional bounded  $\Phi$ -variation in the sense of Riesz, into another such space, and is uniformly bounded, then the non-linear generator  $h(x, y)$  of this operator must be of the form  $h(x, y) = A(x)y + B(x)$  where, for every  $x$ ,  $A(x)$  is a linear map.

*Key words:* Nemytskij operator,  $n$ -dimensional  $\Phi$ -variation,  $\varphi$ -function.

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## 1. INTRODUCTION

Given two (non-empty) sets  $A$  and an  $B$ , the notation  $B^A$  will stand for the set of all functions from  $A$  to  $B$ . As usual, if  $M$  is a normed space,  $\mathcal{L}(M)$  denotes the set of all bounded operators on  $M$ .

Let  $A$ ,  $B$  and  $C$  be non-empty sets. If  $h : A \times C \rightarrow B$  is a given function,  $X \subset C^A$  and  $Y \subset B^A$  are linear spaces then, the nonlinear superposition (Nemytskij) operator  $\mathbf{H} : X \rightarrow Y$ , generated by the function  $h$ , is defined as

$$(\mathbf{H}f)(\mathbf{t}) := h(\mathbf{t}, f(\mathbf{t})), \quad \mathbf{t} \in A.$$

This operator plays a central role in many mathematical fields (e.g. in the theory of nonlinear integral equations), by its applications to a variety of nonlinear problems, and has been studied thoroughly. Apart from conditions for the mere inclusion  $\mathbf{H}(X) \subset Y$ , the boundedness or the continuity of  $\mathbf{H}$  (cf. [2]), another important problem has been widely investigated:

to find conditions on the generating function  $h$  in the case in which  $(X, d_X), (Y, d_Y)$  are also metric spaces and the superposition operator  $\mathbf{H}$  is uniformly continuous or satisfies some global or local Lipschitz condition of the form

$$d_Y(\mathbf{H}(f_1), \mathbf{H}(f_2)) \leq \alpha d_X(f_1, f_2), \quad f_1, f_2 \in X,$$

cf. e.g., [10, 11, 5].

In this paper we will investigate a problem related to this last situation.

Throughout this paper the letter  $n$  denotes a positive integer. Let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  and  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  be points in  $\mathbb{R}^n$ . We will use the notation  $\mathbf{a} < \mathbf{b}$  to mean that  $a_i < b_i$  for each  $i = 1, \dots, n$  and accordingly we define  $\mathbf{a} = \mathbf{b}$ ,  $\mathbf{a} \leq \mathbf{b}$ ,  $\mathbf{a} \geq \mathbf{b}$  and  $\mathbf{a} > \mathbf{b}$ . If  $\mathbf{a} < \mathbf{b}$ , the set  $\mathbf{J} := [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^n [a_i, b_i]$  will be called an  $n$ -dimensional closed interval.

Given an  $n$ -dimensional closed interval  $\mathbf{J}$ , a metric vector space  $M$  and  $\Phi$  a  $\varphi$ -function, the space of all functions, defined on  $\mathbf{J}$  of  $n$ -dimensional bounded  $\Phi$ -variation will be denoted by  $BRV_{\Phi}^n(\mathbf{J}; M)$ . Suppose that  $N$  is another vector metric space,  $\mathcal{C}$  is a convex subset of  $M$ ,  $\Psi$  is another  $\varphi$ -function and  $h : \mathbf{J} \times \mathcal{C} \rightarrow N$  is a given function. In this paper we prove that if the superposition operator  $\mathbf{H}$ , generated by  $h$ , maps the set  $\{f \in BRV_{\Phi}^n(\mathbf{J}; M) : f(\mathbf{J}) \subset \mathcal{C}\}$  into  $BRV_{\Psi}^n(\mathbf{J}; N)$  and is *uniformly bounded*, in the sense introduced in [12], then there is a linear operator  $A \in L(M, N)$  and a function  $B \in N^{\mathbf{J}}$  such that

$$h(\mathbf{x}, y) = A(\mathbf{x})y + B(\mathbf{x}), \quad \mathbf{x} \in \mathbf{J}, \quad y \in \mathcal{C}.$$

This is a counterpart of a result of Matkowski proved in [12] for the space of Lipschitz continuous functions.

## 2. FUNCTIONS OF BOUNDED $n$ -DIMENSIONAL $\Phi$ -VARIATION

In this section we present the definition and main basic aspects of the notion of  $n$ -dimensional  $\Phi$ -variation for functions defined on  $n$ -dimensional closed intervals of  $\mathbb{R}^n$ , that take values on a *metric semigroup*, as introduced in [3]. This generalization of the notion of bounded variation for functions of several variables is inspired in the works of Chistyakov and Talalyan [6, 13]. Here, we also combine the notions of variations given by Vitali ([14]) and later generalized by Hardy and Krause (cf. [4, 7]).

**DEFINITION 2.1.** A metric semigroup is a structure  $(M, d, +)$  where  $(M, +)$  is an abelian semigroup and  $d$  is a translation invariant metric on  $M$ .

In particular, the triangle inequality implies that, for all  $u, v, p, q \in M$ ,

$$\begin{aligned} d(u, v) &\leq d(p, q) + d(u + p, v + q), \quad \text{and} \\ d(u + p, v + q) &\leq d(u, v) + d(p, q). \end{aligned} \quad (2.1)$$

In this paper we will use the following standard notation:  $\mathbb{N}$  (resp.  $\mathbb{N}_0$ ) denotes the set of all positive integers (resp. non-negative integers) and a typical point of  $\mathbb{R}^n$  is denoted as  $\mathbf{x} = (x_1, x_2, \dots, x_n) := (x_i)_{i=1}^n$ ; but, the canonical unit vectors of  $\mathbb{R}^n$  are denoted by  $\mathbf{e}_j$  ( $j = 1, 2, \dots, n$ ); that is,

$$\mathbf{e}_j := (e_r^{(j)})_{r=1}^n \text{ where, } e_r^{(j)} := \begin{cases} 0 & \text{if } r \neq j \\ 1 & \text{if } r = j \end{cases}.$$

The zero  $n$ -tuple  $(0, 0, \dots, 0)$  will be denoted by  $\mathbf{0}$ , and by  $\mathbf{1}$  we will denote the  $n$ -tuple  $\mathbf{1} = (1, 1, \dots, 1)$ .

If  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , with  $\alpha_j \in \mathbb{N}_0$ , is a  $n$ -tuple of non-negative integers then we call  $\alpha$  a *multi-index* ([1]).

If  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  we use the notation  $\mathbf{a} < \mathbf{b}$  to mean that  $x_i < y_i$  for each  $i = 1, \dots, n$  and similarly are defined  $\mathbf{a} = \mathbf{b}$ ,  $\mathbf{a} \leq \mathbf{b}$ ,  $\mathbf{a} \geq \mathbf{b}$  and  $\mathbf{a} > \mathbf{b}$ . If  $\mathbf{a} < \mathbf{b}$ , the set  $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^n [a_i, b_i]$  will be called a  *$n$ -dimensional closed interval*. The euclidean volume of an  $n$ -dimensional closed interval will be denoted by  $\text{Vol} [\mathbf{a}, \mathbf{b}]$ ; that is,  $\text{Vol} [\mathbf{a}, \mathbf{b}] = \prod_{i=1}^n (b_i - a_i)$ .

In addition, for  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$  and  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  we will use the notations

$$|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n \quad \text{and} \quad \alpha \mathbf{x} := (\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_n x_n).$$

We will denote by  $\mathcal{N}$  the set of all strictly increasing continuous convex functions  $\Phi : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\Phi(t) = 0$  if and only if  $t = 0$  and  $\lim_{t \rightarrow \infty} \Phi(t) = +\infty$ .

$\mathcal{N}_\infty$  the set of all functions  $\Phi \in \mathcal{N}$ , for which the Orlicz condition (also called  $\infty_1$  condition) holds:  $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = +\infty$ . Functions from  $\mathcal{N}$  are often called  $\varphi$ -functions.

One says that a function  $\Phi \in \mathcal{N}$  satisfies a condition  $\Delta_2$ , and writes  $\Phi \in \Delta_2$ , if there are constants  $K > 2$  and  $t_0 > 0$  such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for all } t \geq t_0. \quad (2.2)$$

Now we define two important sets.

$$\begin{aligned}\mathcal{E}(n) &:= \{ \theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is even} \} \\ \mathcal{O}(n) &:= \{ \theta \in \mathbb{N}_0^n : \theta \leq \mathbf{1} \text{ and } |\theta| \text{ is odd} \}.\end{aligned}$$

Notice that these sets are related in a one to one fashion; indeed, if  $\theta = (\theta_1, \dots, \theta_n) \in \mathcal{E}(n)$  then we can define  $\tilde{\theta} := (1 - \theta_1, \theta_2, \dots, \theta_n) \in \mathcal{O}(n)$ , and this operation is clearly invertible.

In what follows  $M$  is supposed to be a metric semigroup and  $[\mathbf{a}, \mathbf{b}]$  an  $n$ -dimensional closed interval.

DEFINITION 2.2. ([7, 6, 14]) Given a function  $f : [\mathbf{a}, \mathbf{b}] \rightarrow M$ , we define the  $n$ -dimensional Vitali difference of  $f$  over an  $n$ -dimensional closed interval  $[\mathbf{x}, \mathbf{y}] \subseteq [\mathbf{a}, \mathbf{b}]$ , by

$$\Delta_n(f, [\mathbf{x}, \mathbf{y}]) := d\left( \sum_{\theta \in \mathcal{E}(n)} f(\theta \mathbf{x} + (1 - \theta)\mathbf{y}), \sum_{\theta \in \mathcal{O}(n)} f(\theta \mathbf{x} + (1 - \theta)\mathbf{y}) \right). \quad (2.3)$$

This difference is also called *mixed difference* and it is usually associated to the names of Vitali, Lebesgue, Hardy, Krause, Fréchet and De la Vallée Poussin ([7, 6, 8]).

Now, in order to define the  $\Phi$ -variation of a function  $f : [\mathbf{a}, \mathbf{b}] \rightarrow M$ , we consider *net* partitions of  $[\mathbf{a}, \mathbf{b}]$ ; that is, partitions of the kind

$$\xi = \xi_1 \times \xi_2 \times \dots \times \xi_n \quad \text{with} \quad \xi_i := \{t_j^{(i)}\}_{j=0}^{k_i}, \quad i = 1, \dots, n. \quad (2.4)$$

where,  $\{k_i\}_{i=1}^n \subset \mathbb{N}$  and for each  $i$ ,  $\xi_i$  is a partition of  $[a_i, b_i]$ . The set of all net partitions of an  $n$ -dimensional closed interval  $[\mathbf{a}, \mathbf{b}]$  will be denoted by  $\pi([\mathbf{a}, \mathbf{b}])$ .

A point in a net partition  $\xi$  is called a *node* ([13]) and it is of the form

$$\mathbf{t}_\alpha := (t_{\alpha_1}^{(1)}, t_{\alpha_2}^{(2)}, t_{\alpha_3}^{(3)}, \dots, t_{\alpha_n}^{(n)})$$

where  $\mathbf{0} \leq \alpha = (\alpha_i)_{i=1}^n \leq \kappa$ , with  $\kappa := (k_i)_{i=1}^n$ .

For the sake of simplicity in notation, we will simply write  $\xi = \{\mathbf{t}_\alpha\}$ , to refer to all the nodes that form a given partition  $\xi$ .

A cell of an  $n$ -dimensional closed interval  $[\mathbf{a}, \mathbf{b}]$  is an  $n$ -dimensional closed subinterval of the form  $[\mathbf{t}_{\alpha-1}, \mathbf{t}_\alpha]$ , for  $\mathbf{0} < \alpha \leq \kappa$ .

Note that

$$\begin{aligned}\mathbf{t}_\mathbf{0} &= (t_0^{(1)}, t_0^{(2)}, \dots, t_0^{(n)}) = (a_1, a_2, \dots, a_n) \text{ and} \\ \mathbf{t}_\kappa &= (t_{k_1}^{(1)}, t_{k_2}^{(2)}, \dots, t_{k_n}^{(n)}) = (b_1, b_2, \dots, b_n).\end{aligned}$$

DEFINITION 2.3. Let  $f : [\mathbf{a}, \mathbf{b}] \rightarrow M$  and  $\Phi \in \mathcal{N}$ . The  $\Phi$ -variation, in the sense of Vitali-Riesz of  $f$  is defined as

$$\rho_{\Phi}^n(f, [\mathbf{a}, \mathbf{b}]) := \sup_{\xi \in \pi[\mathbf{a}, \mathbf{b}]} \rho_{\Phi}^n(f, [\mathbf{a}, \mathbf{b}], \xi). \quad (2.5)$$

where

$$\rho_{\Phi}^n(f, [\mathbf{a}, \mathbf{b}], \xi) := \sum_{1 \leq \alpha \leq \kappa} \Phi \left( \frac{\Delta_n(f, [\mathbf{t}_{\alpha-1}, \mathbf{t}_{\alpha}])}{\text{Vol}[\mathbf{t}_{\alpha-1}, \mathbf{t}_{\alpha}]} \right) \text{Vol}[\mathbf{t}_{\alpha-1}, \mathbf{t}_{\alpha}].$$

We now need to define *the truncation* of a point, of an  $n$ -dimensional closed interval and of a function, by a given multi-index  $\mathbf{0} < \eta \leq \mathbf{1}$ . Notice that in this case, the entries of a such  $\eta$  are either 0 or 1.

- The truncation of a point  $\mathbf{x} \in \mathbb{R}^n$  by a multi-index  $\mathbf{0} < \eta \leq \mathbf{1}$ , which is denoted by  $\mathbf{x}[\eta]$ , is defined as the  $|\eta|$ -tuple that is obtained if we suppress from  $\mathbf{x}$  the entries for which the corresponding entries of  $\eta$  are equal to 0. That is,  $\mathbf{x}[\eta] = (x_i : i \in \{1, 2, \dots, n\}, \eta_i = 1)$ . For instance, if  $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$  and  $\eta = (0, 1, 1, 0, 1)$  then  $\mathbf{x}[\eta] = (x_2, x_3, x_5)$ .
- The truncation of an  $n$ -dimensional closed interval  $[\mathbf{a}, \mathbf{b}]$  by a multi-index  $\mathbf{0} < \eta \leq \mathbf{1}$ , is defined as  $[\mathbf{a}, \mathbf{b}][\eta] := [\mathbf{a}[\eta], \mathbf{b}[\eta]]$ .
- Given a function  $f : [\mathbf{a}, \mathbf{b}] \rightarrow M$ , a multi-index  $\mathbf{0} < \eta \leq \mathbf{1}$  and a point  $\mathbf{z} \in [\mathbf{a}, \mathbf{b}]$ , we define  $f_{\eta}^{\mathbf{z}} : [\mathbf{a}, \mathbf{b}][\eta] \rightarrow M$ , the truncation of  $f$  by la  $\eta$ , by the formula

$$f_{\eta}^{\mathbf{z}}(\mathbf{x}[\eta]) := f(\eta\mathbf{x} + (\mathbf{1} - \eta)\mathbf{z}), \quad \mathbf{x} \in [\mathbf{a}, \mathbf{b}].$$

Note that the function  $f_{\eta}^{\mathbf{z}}$  depends only on the  $|\eta|$  variables  $x_i$  for which  $\eta_i = 1$ .

*Remark 2.4.* Given a function  $f : [\mathbf{a}, \mathbf{b}] \rightarrow M$  and a multi-index  $\eta \neq \mathbf{0}$ , then the  $|\eta|$ -dimensional Vitali difference for  $f_{\eta}^{\mathbf{a}}$  (cf. (2.3)), is given by

$$\begin{aligned} & \Delta_{|\eta|}(f_{\eta}^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}]) \\ & := d \left( \sum_{\substack{\theta \in \mathcal{E}(n) \\ \theta \leq \eta}} f(\eta(\theta\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + (\mathbf{1} - \eta)\mathbf{a}), \sum_{\substack{\theta \in \mathcal{O}(n) \\ \theta \leq \eta}} f(\eta(\theta\mathbf{x} + (\mathbf{1} - \theta)\mathbf{y}) + (\mathbf{1} - \eta)\mathbf{a}) \right). \end{aligned}$$

DEFINITION 2.5. Let  $\Phi \in \mathcal{N}$  and let  $(M, d, +, \cdot)$  be a metric semigroup. A function  $f : [\mathbf{a}, \mathbf{b}] \rightarrow M$  is said to be of *bounded  $\Phi$ -variation*, in the sense of Riesz, if the total  $\Phi$ -variation

$$TRV_{\Phi}(f, [\mathbf{a}, \mathbf{b}]) := \sum_{\mathbf{0} \neq \eta \leq \mathbf{1}} \rho_{\Phi}^{|\eta|}(f_{\eta}^{\mathbf{a}}, [\mathbf{a}, \mathbf{b}]|\eta), \quad (2.6)$$

is finite. The set of all functions  $f$  that satisfy  $TRV_{\Phi}(f, [\mathbf{a}, \mathbf{b}]) < +\infty$  will be denoted by  $RV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; M)$ .

### 3. THE NORMED SPACE $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; M)$

So far, our consideration of metric semigroups as targets sets suffices adequately to define a notion of  $n$ -dimensional variation; however, as we need to study a *superposition operator problem* between linear spaces in which the presence of this notion is desired, it will be necessary to ask for additional structure on the target set  $M$ . The one that we will considerate is that of *vectorial metric space*.

DEFINITION 3.1. By a metric vector space (MVS) we will understand a topological vector space  $(\mathcal{M}, \tau)$  in which the topology  $\tau$  is induced by a metric  $d$  that satisfies the following conditions:

1.  $d$  is a translation invariant metric.
2.  $d(\alpha a, \alpha b) \leq |\alpha| d(a, b)$  for any  $\alpha \in \mathbb{R}$  and  $a, b \in \mathcal{M}$ .

Note that, in particular, any MVS is a metric semigroup. In what follows  $\mathcal{M}$  is supposed to be a MVS and  $[\mathbf{a}, \mathbf{b}]$  an  $n$ -dimensional closed interval.

*Remark 3.2.* It readily follows from 2.1 that given two functions  $f, g : [\mathbf{a}, \mathbf{b}] \rightarrow \mathcal{M}$ , a multi-index  $\eta \neq \mathbf{0}$  and an  $n$ -dimensional closed interval  $[\mathbf{x}, \mathbf{y}] \subset [\mathbf{a}, \mathbf{b}]$ , then the  $|\eta|$ -dimensional Vitali difference (c.f. (2.3)) of the truncation  $(f + g)_{\eta}^{\mathbf{a}}$  satisfies the inequality

$$\Delta_{|\eta|} \left( f_{\eta}^{\mathbf{a}} + g_{\eta}^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}] \right) \leq \Delta_{|\eta|} \left( f_{\eta}^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}] \right) + \Delta_{|\eta|} \left( g_{\eta}^{\mathbf{a}}, [\mathbf{x}, \mathbf{y}] \right). \quad (3.1)$$

LEMMA 3.3. *The functional  $TRV_{\Phi}(\cdot, [\mathbf{a}, \mathbf{b}])$  is convex.*

*Proof.* The lemma is consequence of (3.1) and of the fact that  $\Phi$  is a nondecreasing convex function. ■

THEOREM 3.4. *The class  $RV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$  is symmetric and convex.*

*Proof.* That  $RV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$  is symmetric is consequence of property (2) (since  $d(-a, -b) \leq d(a, b)$ ) of Definition 3.1 while convexity follows from Lemma 3.3. ■

As a consequence of Theorem 3.4, the *linear space* generated by the set  $RV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$  is

$$\{f : [\mathbf{a}, \mathbf{b}] \rightarrow \mathcal{M} : \lambda f \in RV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M}) \text{ for some } \lambda > 0\},$$

which, we will call, the *space of functions of bounded  $\Phi$ -variation in the sense of Vitali-Hardy-Riesz* and will denote as  $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$ .

LEMMA 3.5. *The set*

$$\Lambda := \{f \in BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M}) : TRV_{\Phi}(f, [\mathbf{a}, \mathbf{b}]) \leq 1\}$$

*is a convex, balanced and absorbent subset of  $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$ .*

*Proof.* To prove convexity suppose that  $f, g \in \Lambda$  and let  $\alpha, \beta$  be non-negative real numbers such that  $\alpha + \beta = 1$ . Then  $TRV_{\Phi}(f, [\mathbf{a}, \mathbf{b}]) \leq 1$ ,  $TRV_{\Phi}(g, [\mathbf{a}, \mathbf{b}]) \leq 1$  and by Lemma 3.3

$$\begin{aligned} TRV_{\Phi}(\alpha f + \beta g, [\mathbf{a}, \mathbf{b}]) &\leq \alpha TRV_{\Phi}(f, [\mathbf{a}, \mathbf{b}]) + \beta TRV_{\Phi}(g, [\mathbf{a}, \mathbf{b}]) \\ &\leq \alpha + \beta = 1. \end{aligned}$$

Hence  $\Lambda$  is convex.

On the other hand, from Definition 2.3 it readily follows that if  $f_0 \equiv 0$  then  $TRV_{\Phi}(f_0, [\mathbf{a}, \mathbf{b}]) = 0$ , thus  $f_0 \in \Lambda$  and therefore, by virtue of the convexity property of  $\Lambda$  just proved,  $\Lambda$  is balanced. Finally, the fact that  $\Lambda$  is absorbent follows from property (2) of Definition 3.1 and the convexity of  $\Phi$ . ■

By virtue of Lemma 3.5, the *Minkowski Functional* of  $\Lambda$

$$p_{\Lambda}(f) := \inf \left\{ t > 0 : TRV_{\Phi}\left(\frac{f}{t}, [\mathbf{a}, \mathbf{b}]\right) \leq 1 \right\},$$

defines a seminorm on  $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$ , and therefore

$$\|f\| := \|f\|_{BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})} := d(f(\mathbf{a}), 0) + p_{\Lambda}(f) \quad (3.2)$$

defines a norm on  $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$ .

LEMMA 3.6. Let  $f \in BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$ ,

(i) If  $\|f\| \neq 0$  then  $TRV_{\Phi}(f/\|f\|, [\mathbf{a}, \mathbf{b}]) \leq 1$ ;

(ii) if  $0 \neq \|f\| \leq 1$  then  $TRV_{\Phi}(f, [\mathbf{a}, \mathbf{b}]) \leq \|f\|$ .

*Proof.* (i) From definition 3.2  $p_{\Lambda}(f) \leq \|f\|$ .

If  $p_{\Lambda}(f) < \|f\|$  then there is  $\xi \in \Lambda$  such that  $p_{\Lambda}(f) < \xi \leq \|f\|$  and  $TRV_{\Phi}(\frac{f}{\xi}, [\mathbf{a}, \mathbf{b}]) \leq 1$ . So, since  $\Lambda$  is absorbent,  $\frac{f}{\|f\|} \in \Lambda$ .

If  $p_{\Lambda}(f) = \|f\|$ , then there is a sequence  $t_n \in \Lambda$  such that

$$t_n \rightarrow \|f\| \quad \text{and} \quad TRV_{\Phi}\left(\frac{f}{t_n}, [\mathbf{a}, \mathbf{b}]\right) \leq 1.$$

It follows, by the continuity of the functional  $TRV_{\Phi}(\cdot, [\mathbf{a}, \mathbf{b}])$ , that

$$TRV_{\Phi}\left(\frac{f}{\|f\|}, [\mathbf{a}, \mathbf{b}]\right) \leq 1.$$

(ii) follows from (i) and the convexity of  $TRV_{\Phi}(\cdot, [\mathbf{a}, \mathbf{b}])$ . ■

*Remark 3.7.* It follows from Lemma 3.6 (i) that if  $p_{\Lambda}(f) \neq 0$  and  $t > \|f\|$  then  $t \in \Lambda$ .

#### 4. COMPOSITION OPERATOR IN THE SPACE $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$

In this section we state and prove the main results of this paper concerning the action of a superposition operator between spaces of functions of bounded  $n$ -dimensional  $\Phi$ -variation. For the sake of clarity of exposition, we will denote the norm of  $BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M})$  by  $\|\cdot\|_{(\Phi, \mathcal{M})}$ .

**THEOREM 4.1.** *Suppose that  $[\mathbf{a}, \mathbf{b}] \subseteq \mathbb{R}^n$  is an  $n$ -dimensional closed interval, and that  $\Phi, \Psi$  are  $\varphi$ -functions. Let  $\mathcal{M}$  and  $\mathcal{N}$  be linear metric spaces,  $\mathcal{C} \subseteq \mathcal{M}$  a convex and closed set with non empty interior and let  $h : [\mathbf{a}, \mathbf{b}] \times \mathcal{C} \rightarrow \mathcal{N}$  be a continuous function. If the Nemytskij operator  $H$ , generated by the function  $h$ , applies the set  $\mathbf{K} = \{f \in BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M}) : f([\mathbf{a}, \mathbf{b}]) \subset \mathcal{C}\}$  into  $BRV_{\Psi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{N})$  and satisfies the inequality*

$$\|H(f_1) - H(f_2)\|_{(\Psi, \mathcal{N})} \leq \gamma \left( \|f_1 - f_2\|_{(\Phi, \mathcal{M})} \right) \quad f_1, f_2 \in \mathbf{K}, \quad (4.1)$$



for some function  $\gamma : [0, \infty) \rightarrow [0, \infty)$ , then there are functions  $A : [\mathbf{a}, \mathbf{b}] \rightarrow \mathcal{L}(\mathcal{M}, \mathcal{N})$  and  $B \in \mathcal{N}^{[\mathbf{a}, \mathbf{b}]}$  such that

$$h(\mathbf{x}, u) = A(\mathbf{x})u + B(\mathbf{x}), \quad \mathbf{x} \in [\mathbf{a}, \mathbf{b}], \quad u \in \mathcal{C}.$$

If  $0 \in \mathcal{C}$ , then  $B \in BRV_{\Psi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{N})$ .

*Proof.* We will show that  $h$  satisfies the Jensen equation in the second variable.

Indeed, let  $\mathbf{t}_1 = (t_1^{(i)})_{i=1}^n$  and  $\mathbf{t}_2 = (t_2^{(i)})_{i=1}^n \in [\mathbf{a}, \mathbf{b}]$ , suppose further that  $\mathbf{t}_1 \leq \mathbf{t}_2$ , and define the functions

$$\eta_i(t) := \begin{cases} 0 & \text{if } a_i \leq t \leq t_1^{(i)} \\ \frac{t - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} & \text{if } t_1^{(i)} \leq t \leq t_2^{(i)} \\ 1 & \text{if } t_2^{(i)} \leq t \leq b_i. \end{cases}$$

Next, consider  $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{C}$ ,  $\mathbf{y}_1 \neq \mathbf{y}_2$  and define

$$f_j(\mathbf{x}) := \frac{1}{2} \left[ \prod_{i=1}^n \eta_i(x_i) (\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_j + \mathbf{y}_2 \right], \quad (4.2)$$

for  $j = 1, 2$ , where  $\mathbf{x} := (x_1, x_2, \dots, x_n)$ .

Notice that

$$\begin{aligned} f_1(\mathbf{x}) - f_2(\mathbf{x}) &= \frac{1}{2} \left[ \prod_{i=1}^n \eta_i(x_i) (\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_1 + \mathbf{y}_2 - \prod_{i=1}^n \eta_i(x_i) (\mathbf{y}_1 - \mathbf{y}_2) - \mathbf{y}_2 - \mathbf{y}_2 \right] \\ &= \frac{\mathbf{y}_1 - \mathbf{y}_2}{2}. \end{aligned}$$

Hence  $f_1 - f_2$  has zero  $\Phi$ -variation and

$$\begin{aligned} \|f_1 - f_2\|_{(\Phi, \mathcal{M})} &= d((f_1 - f_2)(\mathbf{a}), 0) + p_{\varphi}(f_1 - f_2) \\ &= d((f_1 - f_2)(\mathbf{a}), 0) = d\left(\frac{\mathbf{y}_1 - \mathbf{y}_2}{2}, 0\right) \\ &= d\left(\frac{\mathbf{y}_1}{2}, \frac{\mathbf{y}_2}{2}\right) > 0. \end{aligned}$$

Notice further that

- If  $\mathbf{x} = \mathbf{t}_\alpha$  where  $\alpha_i = 2$  for  $i = 1, 2, \dots, n$  then

$$\prod_{i=1}^n \eta(t_{\alpha_i}^{(i)}) = \prod_{i=1}^n \frac{t_{\alpha_i}^{(i)} - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} = 1.$$

- If  $\mathbf{x} = \mathbf{t}_\alpha$  with  $\alpha_i \neq 2$  for some  $1 \leq i \leq n$  then

$$\prod_{i=1}^n \eta(t_{\alpha_i}^{(i)}) = \prod_{i=1}^n \frac{t_{\alpha_i}^{(i)} - t_1^{(i)}}{t_2^{(i)} - t_1^{(i)}} = 0.$$

Thus, by (4.2)

- If  $\alpha_i = 2$  for  $i = 1, 2, \dots, n$  then

$$f_1(\mathbf{t}_\alpha) := \frac{1}{2} \left[ \prod_{i=1}^n \eta(t_{\alpha_i}^{(i)}) (\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_1 + \mathbf{y}_2 \right] = \mathbf{y}_1,$$

and

$$f_2(\mathbf{t}_\alpha) := \frac{1}{2} \left[ \prod_{i=1}^n \eta(t_{\alpha_i}^{(i)}) (\mathbf{y}_1 - \mathbf{y}_2) + \mathbf{y}_2 + \mathbf{y}_2 \right] = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}.$$

- If  $\alpha_k \neq 2$  for some  $1 \leq k \leq n$  then

$$\begin{aligned} f_1(\mathbf{t}_\alpha) &:= \frac{1}{2} [\mathbf{y}_1 + \mathbf{y}_2] = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}, \\ f_2(\mathbf{t}_\alpha) &:= \mathbf{y}_2. \end{aligned}$$

Thus, by the definition of  $\mathbf{H}$ , we have

$$\begin{aligned} \mathbf{H}f_1(\mathbf{t}_2) &= h(\mathbf{t}_2, f_1(\mathbf{t}_2)) = h(\mathbf{t}_2, \mathbf{y}_1) \\ \mathbf{H}f_2(\mathbf{t}_2) &= h(\mathbf{t}_2, f_2(\mathbf{t}_2)) = h\left(\mathbf{t}_2, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) \\ \mathbf{H}f_1(\mathbf{t}_1) &= h(\mathbf{t}_1, f_1(\mathbf{t}_1)) = h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) \\ \mathbf{H}f_2(\mathbf{t}_1) &= h(\mathbf{t}_1, f_2(\mathbf{t}_1)) = h(\mathbf{t}_1, \mathbf{y}_2), \end{aligned}$$

and, if  $\theta$  is a non-zero multi-index different from  $\mathbf{1}$

$$\begin{aligned} \mathbf{H}f_1(\theta \mathbf{t}_1 + (1 - \theta)\mathbf{t}_2) &= h\left(\theta \mathbf{t}_1 + (1 - \theta)\mathbf{t}_2, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) \\ \mathbf{H}f_2(\theta \mathbf{t}_1 + (1 - \theta)\mathbf{t}_2) &= h(\theta \mathbf{t}_1 + (1 - \theta)\mathbf{t}_2, \mathbf{y}_2). \end{aligned}$$

On the other hand, for  $f_1, f_2 \in K$  we have

$$\|\mathbf{H}(f_1) - \mathbf{H}(f_2)\|_{(\Psi, \mathcal{N})} \leq \gamma \left( \|f_1 - f_2\|_{(\Phi, \mathcal{M})} \right),$$

thus

$$p_\Psi(\mathbf{H}(f_1) - \mathbf{H}(f_2)) \leq \|\mathbf{H}(f_1) - \mathbf{H}(f_2)\|_{(\Psi, \mathcal{N})} \leq \gamma \left( \|f_1 - f_2\|_{(\Phi, \mathcal{M})} \right).$$

Hence, by Remark 3.7 we have

$$\begin{aligned} \rho_\Phi^n \left( \frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})}, [\mathbf{a}, \mathbf{b}] \right) &\leq TRV_\Phi \left( \frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})}, [\mathbf{a}, \mathbf{b}] \right) \\ &\leq TRV_\Phi \left( \frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi, \mathcal{N})}} \frac{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi, \mathcal{N})}}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})}, [\mathbf{a}, \mathbf{b}] \right) \quad (4.3) \\ &\leq \frac{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi, \mathcal{N})}}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})} TRV_\Phi \left( \frac{\mathbf{H}f_1 - \mathbf{H}f_2}{\|\mathbf{H}f_1 - \mathbf{H}f_2\|_{(\Psi, \mathcal{N})}}, [\mathbf{a}, \mathbf{b}] \right) \leq 1. \end{aligned}$$

Thus

$$\begin{aligned} 1 &\geq \rho_\Phi^n \left( \frac{\mathbf{H}(f_1) - \mathbf{H}(f_2)}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})}, [\mathbf{a}, \mathbf{b}] \right) \\ &\geq \Phi \left( \frac{\Delta_n \left( \frac{\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1, \mathbf{t}_2]}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})} \right)}{\text{Vol}[\mathbf{t}_1, \mathbf{t}_2]} \right) \text{Vol}[\mathbf{t}_1, \mathbf{t}_2], \end{aligned}$$

which implies

$$\Phi^{-1} \left( \frac{1}{\text{Vol}[\mathbf{t}_1, \mathbf{t}_2]} \right) \text{Vol}[\mathbf{t}_1, \mathbf{t}_2] \geq \Delta_n \left( \frac{\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1, \mathbf{t}_2]}{\gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})})} \right)$$

and

$$\begin{aligned} \Delta_n(\mathbf{H}(f_1) - \mathbf{H}(f_2), [\mathbf{t}_1, \mathbf{t}_2]) \\ \leq \Phi^{-1} \left( \frac{1}{\text{Vol}[\mathbf{t}_1, \mathbf{t}_2]} \right) \text{Vol}[\mathbf{t}_1, \mathbf{t}_2] \gamma(\|f_1 - f_2\|_{(\Phi, \mathcal{M})}). \quad (4.4) \end{aligned}$$

Making  $\mathbf{t}_2 \rightarrow \mathbf{t}_1$  on the left hand side of (4.4) we get

$$\begin{aligned}
& \lim_{\mathbf{t}_2 \rightarrow \mathbf{t}_1} d\left(\sum_{\theta \leq 1} (-1)^{|\theta|} (\mathbf{H}(f_1) - \mathbf{H}(f_2))(\theta \mathbf{t}_1 + (\mathbf{1} - \theta)\mathbf{t}_2), 0\right) \\
&= d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)\right. \\
&\quad \left. + \lim_{\substack{\mathbf{t}_2 \rightarrow \mathbf{t}_1 \\ \theta \leq 1 \\ \theta \neq 1}} \sum_{\theta \leq 1} (-1)^{|\theta|} (\mathbf{H}(f_1) - \mathbf{H}(f_2))(\theta \mathbf{t}_1 + (\mathbf{1} - \theta)\mathbf{t}_2), 0\right) \\
&= d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)\right. \\
&\quad \left. + \lim_{\substack{\mathbf{t}_2 \rightarrow \mathbf{t}_1 \\ \theta \leq 1 \\ \theta \neq 1}} \sum_{\theta \leq 1} (-1)^{|\theta|} [h(\theta \mathbf{t}_1 + (\mathbf{1} - \theta)\mathbf{t}_2, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2})\right. \\
&\quad \left. - h(\theta \mathbf{t}_1 + (\mathbf{1} - \theta)\mathbf{t}_2, \mathbf{y}_2)], 0\right) \\
&= d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)\right. \\
&\quad \left. + \sum_{\substack{\theta \leq 1 \\ \theta \neq 1}} (-1)^{|\theta|} [h(\theta \mathbf{t}_1 + (\mathbf{1} - \theta)\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}) - h(\theta \mathbf{t}_1 + (\mathbf{1} - \theta)\mathbf{t}_1, \mathbf{y}_2)], 0\right) \\
&= d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)\right. \tag{4.5} \\
&\quad \left. + \sum_{\substack{\theta \leq 1 \\ \theta \neq 1}} (-1)^{|\theta|} [h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2)], 0\right).
\end{aligned}$$

Now, the number of  $n$ -tuples that contain  $k$  1s, with  $k > 0$ , is equal to

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \text{ thus}$$

$$\begin{aligned}
& \sum_{\substack{\theta \leq 1 \\ \theta \neq 0}} (-1)^{|\theta|} \left[ h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2) \right] \\
&= \left[ h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2) \right] \sum_{k=1}^n (-1)^k \binom{n}{k} \\
&= \left[ h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2) \right] \left\{ \sum_{k=0}^n (-1)^k \binom{n}{k} - \binom{n}{0} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left[ h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2) \right] \left\{ (-1 + 1)^n - \binom{n}{0} \right\} \\
&= \left[ h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2) \right] \{-1\}.
\end{aligned}$$

Hence, substituting this last identity in (4.5) we get

$$\begin{aligned}
&\lim_{\mathbf{t}_2 \rightarrow \mathbf{t}_1} d\left(\sum_{\theta \leq 1} (-1)^{|\theta|} (\mathbf{H}(f_1) - \mathbf{H}(f_2))(\theta \mathbf{t}_1 + (1 - \theta)\mathbf{t}_2), 0\right) \\
&= d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)\right. \\
&\quad \left.+ \sum_{\substack{\theta \leq 1 \\ \theta \neq 0}} (-1)^{|\theta|} [h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h(\mathbf{t}_1, \mathbf{y}_2)], 0\right) \\
&= d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) + h(\mathbf{t}_1, \mathbf{y}_2), 0\right). \quad (4.6)
\end{aligned}$$

On the other hand, the limit as  $\mathbf{t}_2 \rightarrow \mathbf{t}_1$  on the right side of (4.4) is zero, therefore

$$d\left(h(\mathbf{t}_1, \mathbf{y}_1) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) - h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right) + h(\mathbf{t}_1, \mathbf{y}_2), 0\right) = 0$$

or equivalently

$$\frac{h(\mathbf{t}_1, \mathbf{y}_1) + h(\mathbf{t}_1, \mathbf{y}_2)}{2} = h\left(\mathbf{t}_1, \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right).$$

Thus  $h(\mathbf{t}_1, \cdot)$  is solution for the Jensen equation in  $\mathcal{C}$  for  $\mathbf{t}_1 \in [\mathbf{a}, \mathbf{b}]$ .

Adapting the classical standard argument (cf. Kuczma [9], see also [12]) we conclude that there exist  $A(\mathbf{t}_1) \in \mathcal{L}(\mathcal{M}, \mathcal{N})$  and  $B \in \mathcal{N}^{[\mathbf{a}, \mathbf{b}]}$  such that

$$h(\mathbf{t}_1, \mathbf{y}) = A(\mathbf{t}_1)\mathbf{y} + B(\mathbf{t}_1) \quad \mathbf{y} \in \mathcal{C}. \quad (4.7)$$

Finally, notice that if  $0 \in \mathcal{C}$ , then taking  $y = 0$  in (4.7), we have  $h(\mathbf{t}, \mathbf{0}) = B(\mathbf{t})$ , for  $\mathbf{t} \in [\mathbf{a}, \mathbf{b}]$ , which implies that  $B \in BRV_{\Psi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{N})$ . ■

Notice that condition (4.1) is a generalization of the classical Lipschitz condition; indeed, that is the case if, in particular, the function  $\gamma$  is an increasing linear function.

In [12] J. Matkowski gives the following definition

DEFINITION 4.2. Let  $Y$  and  $Z$  be two metric (or normed) spaces. We say that the map  $H : Y \rightarrow Z$  is uniformly bounded if, for all  $t > 0$  there exists a real number  $\gamma(t)$  such that for all non empty set  $B \subset Y$ :

$$\text{diam } B \leq t \implies \text{diam } H(B) \leq \gamma(t). \quad (4.8)$$

COROLLARY 4.3. Suppose that  $[\mathbf{a}, \mathbf{b}] \subseteq \mathbb{R}^n$  is an  $n$ -dimensional closed interval, and that  $\Phi, \Psi$  are  $\varphi$ -functions. Let  $\mathcal{M}$  and  $\mathcal{N}$  be linear metric spaces,  $\mathcal{C} \subseteq \mathcal{M}$  a convex and closed set with non empty interior and let  $h : [\mathbf{a}, \mathbf{b}] \times \mathcal{C} \rightarrow \mathcal{N}$  be a continuous function. If the Nemytskij operator  $H$ , generated by the function  $h$ , applies the set  $K = \{f \in BRV_{\Phi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{M}) : f([\mathbf{a}, \mathbf{b}]) \subset \mathcal{C}\}$  into  $BRV_{\Psi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{N})$  and is uniformly bounded then there are functions  $A : [\mathbf{a}, \mathbf{b}] \rightarrow \mathcal{L}(\mathcal{M}, \mathcal{N})$  and  $B \in \mathcal{N}^{[\mathbf{a}, \mathbf{b}]}$  such that

$$h(\mathbf{x}, u) = A(\mathbf{x})u + B(\mathbf{x}), \quad \mathbf{x} \in [\mathbf{a}, \mathbf{b}], \quad u \in \mathcal{C}.$$

If  $0 \in \mathcal{C}$ , then  $B \in BRV_{\Psi}^n([\mathbf{a}, \mathbf{b}]; \mathcal{N})$ .

*Proof.* If  $f_1, f_2 \in K$  then  $\text{diam}(\{f_1, f_2\}) = \|f_1 - f_2\|_{\Phi}$ . Since  $H$  is uniformly bounded we have

$$\text{diam } H(\{f_1, f_2\}) = \|H(\varphi) - H(\psi)\|_{\Psi} \leq \gamma(\|\varphi - \psi\|_{\Phi}),$$

and the result readily follows from Theorem 4.1. ■

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