EXTRACTA MATHEMATICAE Vol. **31**, Núm. 2, 145 – 167 (2016)

Ascent and Essential Ascent Spectrum of Linear Relations

Ezzeddine Chafai, Maher Mnif

Département de Mathématiques, Faculté des Sciences de Sfax, Université de Sfax, Route de Soukra Km 3.5*, B. P.* 1171*,* 3000*, Sfax, Tunisia*

ezzeddine.chafai@ipeis.rnu.tn *maher.mnif@gmail.com*

Presented by Manuel González Received February 28, 2016

Abstract: In the present paper, we study the ascent of a linear relation everywhere defined on a Banach space X and the related essential ascent spectrum. Some properties and characterization of such spectra are given. In particular, we show that a Banach space X is finite dimensional if and only if the ascent and the essential ascent of every closed linear relation in X is finite. As an application, we focus on the stability of the ascent and the essential ascent spectrum under perturbations. We prove that an operator F in X has some finite rank power, if and only if, $\sigma^e_{asc}(T+F) = \sigma^e_{asc}(T)$, for every closed linear relation T commuting with F.

Key words: Ascent, essential ascent, perturbation, spectrum, linear relations.

AMS *Subject Class.* (2010): 47A06, 47A53, 47A10.

1. Introduction

Let X denote a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multivalued linear operator in X or simply a *linear relation* in X is a mapping from a subspace $D(T) \subset X$, called the *domain* of T, into the collection of nonempty subsets of X such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T x_1 + \alpha_2 T x_2$, for all nonzero scalars $\alpha_1, \alpha_2 \in \mathbb{K}$ and $x_1, x_2 \in D(T)$. We use the convention that the domain of T is $D(T) := \{x \in X : Tx \neq \emptyset\}$. Then we have $Tx = \emptyset$, for all $x \in X \backslash D(T)$. The class of such linear relations T is denoted by $LR(X)$. The subspace $T(0)$ is called the *multivalued part* of T, and we say that T is a *single valued* linear operator or simply an operator if $T(0) = \{0\}$, that is equivalent to T maps the points of its domain to singletons. A linear relation T in X is uniquely determined by its *graph*, $G(T)$, which is defined by $G(T) := \{(x, y) \in X \times X : x \in D(T),\ y \in Tx\}$, so that we can identify T with $G(T)$. We say that $T \in LR(X)$ is *closed* if its graph $G(T)$ is a closed subspace of $X \times X$. We designate by $CR(X)$ the class of all closed linear relations in X.

Given a subset $A \subset X$, the *image* of A is defined by $T(A) := \cup\{Ta : a \in A \cap D(T)\}$, while $R(T) := T(D(T))$ is called the *range* of T. The linear relation T is said to be *surjective* whenever its range $R(T)$ coincides with X. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y,x) : (x,y) \in G(T)\}$. Let $\emptyset \neq B \subset X$, then the *inverse image* of B under T is defined to be the set $T^{-1}(B) := \{x \in D(T) : B \cap Tx \neq \emptyset\}$. The *kernel* of T is the subspace $N(T) := T^{-1}(0) = \{x \in D(T) : 0 \in Tx\}$, and T is called *injective* if $N(T) = \{0\}$. When T is injective and surjective we say that T is *bijective*. The quantities $\alpha(T) := \dim N(T)$ and $\beta(T) := \dim X/R(T)$ are called the *nullity* and the *conullity* of T, respectively, and the *index* of T is defined by $\mathrm{ind}(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite. If $\alpha(T)$ and $\beta(T)$ are both infinite, then T is said to have no index.

Let M be a subspace of X such that $M \cap D(T) \neq \emptyset$. Then the *restriction* of T to M, denoted by $T_{/M}$, is given by $G(T_{/M}) := \{(m,y) \in G(T) : m \in M \cap D(T)\}$. For linear relations S and T such that $D(T) \cap D(S) \neq \emptyset$ and $\lambda \in \mathbb{C}$, the linear relations $S+T$ and λT are given by $G(S+T) := \{(x, y+z) : (x,y) \in G(S),\ (x,z) \in G(T)\}$ and $G(\lambda T) := \{(x, \lambda y) : (x,y) \in G(T)\}$. $T - \lambda$ stands for $T - \lambda I$, where I is the identity operator on X. Let $S, T \in LR(X)$ such that $R(T) \cap D(S) \neq \emptyset$. The *product* ST is defined as the relation $G(ST) := \{(x,z) : (x,y) \in G(T),\ (y,z) \in G(S) \text{ for some } y \in X\}$. The product of linear relations is clearly associative. Hence for $T \in LR(X)$ and $n \in \mathbb{Z}$, T^n is defined as usual with $T^0 = I$ and $T^1 = T$. It is easily seen that $(T^{-1})^n = (T^n)^{-1}, n \in \mathbb{Z}$. The singular chain manifold $R_c(T)$ of T is defined by

$$R_c(T) := \Big(\bigcup_{n=1}^{+\infty} T^n(0) \Big) \cap \Big(\bigcup_{n=1}^{+\infty} N(T^n) \Big)$$

and we say that the linear space $R_c(T)$ is trivial if $R_c(T) = \{0\}$.

For a given closed subspace M of a normed space X, let Q_M denote the natural quotient map from X onto X/M. If $T \in LR(X)$, then we shall denote $Q_{\overline{T(0)}}$ by Q_T. Clearly $Q_T T$ is single valued, and T is closed, if and only if, $Q_T T$ and $T(0)$ are both closed (see [12, II.5.3]). For $x \in X$, we define $\|Tx\| := \|Q_T Tx\|$ and thus $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\} = \|Q_T T\|$. The quantity $\|T\|$ is referred as the norm of T , though we note that it is in fact a pseudonorm, since $\|T\| = 0$ does not imply $T = 0$. A linear relation T is said to be *continuous* if for each open subset V in $R(T)$, $T^{-1}(V)$ is an open subset in $D(T)$, equivalently $\|T\| < \infty$, *open* if its inverse T^{-1} is continuous and *bounded below* if T is injective and open. Continuous everywhere defined linear relations are referred to as *bounded* relations.

It is very well known (see [22, Lemmas 3.4 and 3.5]) that $(N(T^n))_{n\in\mathbb{N}}$ is an increasing sequence and if $N(T^m) = N(T^{m+1})$, for some nonnegative integer

m, then $N(T^m) = N(T^n)$, for all $n \geq m$. Similarly, $(R(T^n))_{n\in\mathbb{N}}$ is a decreasing sequence and if $R(T^m) = R(T^{m+1})$, for some $m \in \mathbb{N}$, then $R(T^m) = R(T^n)$ for all $n \geq m$. These statements lead to the introduction of the *ascent* and the *descent* of a linear relation T in X by

$$a(T) := \min\{r \in \mathbb{N} : N(T^r) = N(T^{r+1})\};$$

$$d(T) := \min\{s \in \mathbb{N} : R(T^s) = R(T^{s+1})\};$$

respectively, whenever these minima exist. If no such numbers exist the ascent and the descent of T are defined to be ∞.

Likewise, the statements of Lemma 2.5 below lead to the introduction of the *essential ascent* of a linear relation T, which are due to Chafai and Mnif [7], by

$$a_e(T) := \min\{n \in \mathbb{N} : \ \alpha_n := \dim N(T^{n+1})/N(T^n) < \infty\},$$

where the minimum over the empty set is taken to be infinite. If $a_e(T)$ is finite, we denote

$$p(T) := \min\{p \in \mathbb{N} : \ \alpha_n(T) = \alpha_p(T), \ \forall n \geq p\}.$$

In the sequel, X will be a complex Banach space and $T \in CR(X)$. We say that T is *upper semi-Fredholm*, usually denoted $T \in \Phi_+(X)$, if $R(T)$ is closed and $\alpha(T)$ is finite. Clearly, every upper semi-Fredholm linear relation has a finite essential ascent precisely we have $a_e(T) = 0$. The such class of linear relations contains every linear relation with finite ascent.

The *resolvent* set of T is the set

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is bijective } \};$$

and the *spectrum* of T is defined as the set $\sigma(T) : \mathbb{C}\backslash\rho(T)$. It is shown (see [12, VI.1.3]), that $\rho(T)$ is an open set and hence $\sigma(T)$ is closed. The *descent resolvent*, the *ascent resolvent* and the *essential ascent resolvent* sets of T are defined by

$$\rho_{des}(T) := \{\lambda \in \mathbb{C} : d(T - \lambda) < \infty\};$$
$$\rho_{asc}(T) := \{\lambda \in \mathbb{C} : a(T - \lambda) < \infty \text{ and } R((T - \lambda)^{a(T-\lambda)+1}) \text{ is closed}\};$$
$$\rho^e_{asc}(T) := \{\lambda \in \mathbb{C} : a_e(T - \lambda) < \infty \text{ and } R((T - \lambda)^{a_e(T-\lambda)+1}) \text{ is closed}\};$$

respectively. The *descent spectrum* $\sigma_{des}(T)$, the *ascent spectrum* $\sigma_{asc}(T)$ and the *essential ascent spectrum* $\sigma^{e}_{asc}(T)$ are defined by

$$\sigma_{des}(T) := \mathbb{C}\backslash\rho_{des}(T); \qquad \sigma_{asc}(T) := \mathbb{C}\backslash\rho_{asc}(T); \qquad \sigma^{e}_{asc}(T) := \mathbb{C}\backslash\rho^{e}_{asc}(T);$$

respectively.

Linear relations were introduced into Functional Analysis by J. von Neumann [24], motivated by the need to consider adjoints of non-densely defined linear differential operators which are considered by Coddington [9], Coddington and Dijksma [10], Dikjsma, Sabbah and De Snoo [13], among others. One main reason why linear relations are more convenient than operators is that one can define the inverse, the closure and the completion for a linear relation. Interesting works on multivalued linear operators include the treatise on partial differential relations by Gromov [18], the application of multivalued methods to solution of differential equations by Favini and Yagi [15], the development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and also to many problems of fuzzy theory (see, for example [1]) and several papers on semi-Fredholm linear relations and other classes related to them (see, for examples [5] and [4]).

For an operator in a linear space, the notion of ascent and essential ascent was studied in several articles, for instance, we cite [6], [8], [16], [17], [19], [20] and [23]. Later, these concepts are extended to the multivalued case. In particular, some well known results concerning the ascent and the essential ascent for the case of linear operators remain valid in the context of linear relations. Sometimes, an additional condition is needed which is the linear relation having a trivial singular chain manifold. In [6], the authors study the ascent and the essential ascent spectrum of an operator acting on a Banach space. They show that a Banach space X has a finite dimension, if and only if, the essential ascent of every operator on X is finite. The aim of this paper is to find conditions under which results of the type mentioned above will still be true in the most general setting of multivalued linear operators between Banach spaces.

The structure of this work is as follows. Throughout Section 2, we give some auxiliary results, sometimes purely algebraic, which are used to prove the main results. Section 3 is devoted to the study of the ascent spectrum and the essential ascent spectrum of a closed linear relation acting on a Banach space. We show that they are closed subsets of the spectrum, and that $\sigma^{e}_{asc}(T)$ is empty precisely when $\sigma_{asc}(T)$ is empty. We shall also prove that X has a finite dimension, if and only if, the essential ascent of every closed linear relation is finite. Finally, in Section 4, we are concerned with the stability of the essential ascent spectrum under finite rank perturbations. We prove that F^k has a finite dimensional range, for some $k \in \mathbb{N}$, if and only if, $\sigma^e_{asc}(T+F) = \sigma^e_{asc}(T)$ (equivalently, $\sigma_{asc}(T+F) = \sigma_{asc}(T)$) for every closed linear relation T in the commutant of F.

2. Preliminary and auxiliary results

In this section we collect some algebraic results of the theory of multivalued linear operators which will be needed in the following sections. Firstly, we recall the next elementary lemma.

Lemma 2.1. ([12, I.3.1]) *Let X be a linear space and $T \in LR(X)$. Then*

(i) $TT^{-1}(M) = M \cap R(T) + T(0)$, *for all $M \subset X$.*

(ii) $T^{-1}T(M) = M \cap D(T) + N(T)$, *for all $M \subset X$.*

(iii) $T(M+N) = T(M) + T(N)$, *for all $M \subset X$ and $N \subset D(T)$.*

A proof of the next lemma can be found in [22].

Lemma 2.2. ([22, Lemmas 4.1 and 4.4]) *Let T be a linear relation in a linear space X and let $n, m \in \mathbb{N}$. Then*

(i) $D(T^m)/(R(T^n) + N(T^m)) \cap D(T^m) \simeq R(T^m)/R(T^{m+n})$.

(ii) *If, moreover,* $R_c(T) = \{0\}$, *then* $N(T^{m+n})/N(T^n) \simeq N(T^m) \cap R(T^n)$.

As an immediate consequence of Lemma 2.2, we mention the following useful result which is valid for every linear relation everywhere defined in a linear space and having a trivial singular chain manifold.

$$N(T) \cap R(T^p) = N(T) \cap R(T^{p+n}), \quad \text{for all} \quad n \in \mathbb{N}, \text{ where } p := p(T).$$

The inverse image of a closed linear space M of a Banach space X under a linear relation T in X is not, in general, a closed subspace of X. In the following lemma, we give conditions for which $T^{-1}(M)$ remains closed.

Lemma 2.3. *Let T be an everywhere defined linear relation in a Banach space X and let M be a closed subspace of X such that $T(0) \subset M$. Then $T^{-1}(M)$ is closed.*

Proof. According to [12, III.4.2] and, since T is closed and everywhere defined, one can deduce that T is bounded. So that, $Q_T T$ is a bounded operator. On the other hand, since M and $T(0)$ are both closed, it follows, from [3, Lemma 13]), that $Q_T(M) = (M + T(0))/T(0) = M/T(0)$ is also closed. Therefore $(Q_T T)^{-1} Q_T(M)$ is closed. But clearly, $(Q_T T)^{-1} Q_T(M) = T^{-1}(M + N(Q_T)) = T^{-1}(M + T(0)) = T^{-1}(M)$. ∎

The next lemma is used to prove Lemma 3.2 below.

LEMMA 2.4. *Let X be a Banach space, $T \in CR(X)$ be everywhere defined and let M be a closed subspace of X such that $T(0) \cap M = \{0\}$ or $T(0) \subset M$. Suppose that $M + R(T)$ and $M \cap R(T)$ are closed. If either $T(0)$ or $M \cap R(T)$ has a finite dimension then, $R(T)$ is closed.*

Proof. Write for short $N = (M + T(0)) \cap R(T) = M \cap R(T) + T(0)$ if $T(0) \cap M = \{0\}$ and $N = M \cap R(T)$ if $T(0) \subset M$. Clearly N is a closed subspace of X and hence, using Lemma 2.3, it follows that $T^{-1}(N)$ is also closed . Now, let us consider the linear relation

$$\widehat{T} : (X/T^{-1}(N)) \oplus (M/N) \to (R(T) + M)/N$$

defined canonically by

$$\widehat{T}(\overline{x} + \overline{m}) := \{\overline{y} + \overline{m} : y \in Tx\}.$$

It is easy to check that $\widehat{T}$ is correctly defined. Moreover, for $y \in T(0)$, we have $\overline{y} = \overline{0}$ (as $T(0) \subset N$), so that $\widehat{T}(\overline{0}) = \overline{0}$. Which implies that $\widehat{T}$ is an operator. On the other hand, if $x \in X$ and $m \in M$ such that $\widehat{T}(\overline{x} + \overline{m}) = \overline{0}$, then $Tx + m \subset N$ and hence $Tx \subset (M + N) \cap R(T) = N$. It follows that $m \in N$ and $x \in T^{-1}(N)$, so that $\overline{x} + \overline{m} = \overline{0}$. This implies that $\widehat{T}$ is injective and, obviously, $\widehat{T}$ is surjective. Thus $\widehat{T}$ is bijective. Now, since $TT^{-1}(N) = N$, then for all $x \in X, m \in M$ and $y \in Tx$ we have

$$\begin{aligned}
\|\widehat{T}(\overline{x} + \overline{m})\| &= \|\overline{y} + \overline{m}\| = d(y + m, N) \\
&\leq d(y, N) + d(m, N) \\
&= d(y, TT^{-1}(N)) + d(m, N) \\
&= \inf_{x' \in T^{-1}(N)} d(y, Tx') + d(m, N).
\end{aligned}$$

Which implies that

$$\begin{aligned}\|\widehat{T}(\overline{x}+\overline{m}) &\leq \inf_{x'\in T^{-1}(N)} d(Tx,Tx')+d(m,N)\\ &= \inf_{x'\in T^{-1}(N)} \|Tx-Tx'\|+d(m,N)\\ &\leq \|T\| \inf_{x'\in T^{-1}(N)} \|x-x'\|+d(m,N)\\ &= \|T\|d(x,T^{-1}(N))+d(m,N)\\ &\leq (1+\|T\|)[d(x,T^{-1}(N))+d(m,N)]\\ &= (1+\|T\|)\|\overline{x}+\overline{m}\|.\end{aligned}$$

Thus $\widehat{T}$ is bounded and, since $\widehat{T}$ is bijective, then, by the open mapping theorem of linear operators, $\widehat{T}(X/(T^{-1}(N)))$ is closed. Now let us consider $P : R(T)+M \to (R(T)+M)/N$ the canonical projection. Then $R(T) = P^{-1}(R(T)/N) = P^{-1}(\widehat{T}(X/T^{-1}(N)))$. Consequently, $R(T)$ is closed. ∎

In order to introduce the ascent and the essential ascent of linear relations we will need the next result.

Lemma 2.5. *Let T be a linear relation in a linear space X such that $R_c(T) = \{0\}$. Then, for $n \geq 1$,*

(i) $\dim N(T^{n+1})/N(T^n) \leq \dim N(T^n)/N(T^{n-1})$.

(ii) *If there exists $n \in \mathbb{N}$ such that* $\dim N(T^{n+1})/N(T^n)$ *is finite then* $\dim N(T^{m+1})/N(T^m)$ *is finite, for all $m \geq n$.*

(iii) $\dim N(T^{n+1})/N(T^n) < \infty$ *if and only if* $\dim N(T^{n+k})/N(T^n) < \infty$, *for all* $k \geq 1$.

Proof. (i) We note, by Lemma 2.2, that

$$N(T^{n+1})/N(T^n) \simeq R(T^n)\cap N(T) \subset R(T^{n-1})\cap N(T) \simeq N(T^n)/N(T^{n-1}).$$

Thus, obviously, $\dim N(T^{n+1})/N(T^n) \leq \dim N(T^n)/N(T^{n-1})$.

(ii) Follows immediately from the part (i).

(iii) Suppose that $\dim N(T^{n+1})/N(T^n) < \infty$ then, for any $k \geq 1$,

$$\dim N(T^{n+k})/N(T^n) = \sum_{i=0}^{k-1} \dim N(T^{n+i+1})/N(T^{n+i}) < \infty.$$

The reverse implication is trivial. ∎

We close this section with the next lemma which is sometimes useful.

Lemma 2.6. *Let T be a linear relation in a Banach space X. Then*

(i) $R_c(T) = \{0\}$ *if and only if* $R_c(T - \lambda) = \{0\}$ *for every* $\lambda \in \mathbb{C}$.

(ii) *If* $\rho(T) \neq \emptyset$ *then* $R_c(T) = \{0\}$.

Proof. (i) See [22, Lemma 7.1].

(ii) Follows immediately from [21, Lemma 6.1]. ∎

3. Ascent and essential ascent spectrum of linear relations

Throughout this section, X will denote a complex Banach space. The *regular* spectrum (see Definition 3.1 below) of T is defined as those complex numbers λ for which $T - \lambda$ is not regular. In this section, our interest concentrates on proving that, if $0 \notin \sigma^e_{asc}(T)$, then either T is regular or 0 is an isolated point of its regular spectrum. This extend the result of Theorem 2.3, described in [6], to the multivalued case. The proof requires the following technical lemmas.

Lemma 3.1. *Let $T \in LR(X)$ be everywhere defined. Then*

(i) $d(T)$ *is finite if and only if* $R(T) + N(T^d) = X$ *for some* $d \in \mathbb{N}$.
If, moreover, $R_c(T) = \{0\}$, *then*

(ii) $a(T)$ *is finite if and only if* $N(T) \cap R(T^p) = \{0\}$ *for some* $p \in \mathbb{N}$.

(iii) $a_e(T)$ *is finite if and only if* $N(T) \cap R(T^p)$ *has a finite dimension in* X *for some* $p \in \mathbb{N}$.

Proof. Follows immediately from Lemma 2.2. ∎

Lemma 3.2. *Let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$ and $a_e(T)$ is finite. If $R(T^n)$ is closed, for some $n > a_e(T)$, then $R(T^n)$ is closed, for all $n \geq a_e(T)$.*

Proof. The use of [14, Lemma 3.1] proves that T^n is closed and everywhere defined, for all $n \in \mathbb{N}$. Moreover, from Lemma 2.6, we get $R_c(T) = \{0\}$. Now, suppose that $R(T^n)$ is closed, for some $n > a_e(T)$. We need only to show that $R(T^{n-1})$ and $R(T^{n+1})$ are both closed. Since T is continuous and $T(0) \subset R(T^n)$, then, by Lemma 2.3, one can deduce that $T^{-1}(R(T^n)) =$

$R(T^{n-1})+N(T)$ is closed. Now, using Lemma 3.1, we get $R(T^{n-1})\cap N(T)$ is finite dimensional and hence it is closed. According to Lemma 2.4, it follows that $R(T^{n-1})$ is closed. Let T_0 be the restriction of T to the Banach space $R(T^{n-1})$ onto the Banach space $R(T^n)$. Evidently, T_0 is surjective and closed, which implies that T_0 is open. Write for short $M := R(T^n) + N(T_0) = R(T^n) + N(T) \cap R(T^{n-1})$. Clearly, since M and T_0 are both closed, $T_1 := T_0/M$ is also closed . According to [12, II.6.1] and the fact that T_0 is open, it follows that T_1 is open. Consequently, from [12, II.5.3], we deduce that $R(T_1) = T(M) = R(T^{n+1}) + T(N(T) \cap R(T^{n-1}) = R(T^{n+1})$ is closed. This completes the proof. ∎

Remark 3.1. Since $a_e(T) \leq a(T)$, we obviously see, by Lemma 3.2, for $T \in CR(X)$ everywhere defined such that $R_c(T) = \{0\}$, that

$$\sigma^e_{asc}(T) \subseteq \sigma_{asc}(T) \subseteq \sigma(T).$$

The following purely algebraic lemma helps to read Definition 3.1 below. There exhibits some useful connections between the kernels and the ranges of the iterates T^n of a linear relation T in X.

Lemma 3.3. ([2, Lemma 3.7]) *Let $T \in LR(X)$. Then the following statements are equivalent.*

(i) *$N(T) \subset R(T^n)$ for each $n \in \mathbb{N}$.*

(ii) *$N(T^m) \subset R(T)$ for each $m \in \mathbb{N}$.*

(iii) *$N(T^m) \subset R(T^n)$ for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$.*

Definition 3.1. We say that a linear relation $T \in LR(X)$ is *regular* if $R(T)$ is closed and T verifies one of the equivalent conditions of Lemma 3.3.

Trivial examples of regular linear relations are surjective multivalued operators as well as injective multivalued operators with closed range. The next perturbation results are shown in [2] and used in the sequel.

Lemma 3.4. ([2, Theorems 23, 25 and 27]) *Let $T \in CR(X)$.*

(i) *If T is regular then there exists $\gamma > 0$ such that $T - \lambda$ is regular for all $|\lambda| < \gamma$.*

(ii) *If $T \in \Phi_+(X)$ then there exists $\gamma > 0$ such that $T - \lambda \in \Phi_+(X)$ and $\alpha(T - \lambda)$ is constant in the annulus $0 < |\lambda| < \gamma$. Moreover T is regular if and only if $\alpha(T - \lambda) = \alpha(T)$ for all $0 < |\lambda| < \gamma$.*

We shall make frequent use of the following result which is the multivalued version of the corresponding result for operators.

LEMMA 3.5. *Let $T \in LR(X)$ be regular with finite-dimensional kernel and such that $R_c(T) = \{0\}$. Then*

$$\alpha(T^n) = n\ \alpha(T).$$

Proof. Since T is regular then $N(T^{n-1}) \subset R(T)$. This implies, by Lemma 2.2, that $\dim N(T^{n-1}) = \dim(N(T^{n-1}) \cap R(T)) = \dim(N(T^n)/N(T))$. Thus $\dim N(T^n) = \dim N(T^{n-1}) + \dim N(T)$. By induction, we get $\dim N(T^n) = n\ \dim N(T)$ for all $n \in \mathbb{N}$. ∎

Remark 3.2. As a consequence of [11, Theorem 3.1], if T is an everywhere defined linear relation in X with an index and such that $R_c(T) = \{0\}$, then $\mathrm{ind}(T^n) = n\ \mathrm{ind}(T)$.

Now, we are ready to give our first main result of this section.

THEOREM 3.1. *Let $T \in CR(X)$ be everywhere defined and such that $\rho(T) \neq \emptyset$, $a_e(T) < \infty$ and $R(T^{a_e(T)+1})$ is closed. Let $p := p(T)$. Then there exists $\gamma > 0$ such that for each $0 < |\lambda| < \gamma$ the following assertions hold:*

(i) *$T - \lambda$ is regular.*

(ii) *$\dim(N(T-\lambda)^n) = n\ \dim(N(T^{p+1})/N(T^p))$ for each $n \in \mathbb{N}$.*

(iii) *$\mathrm{codim}\, R(T-\lambda)^n = n\ \dim(R(T^p)/R(T^{p+1}))$ for each $n \in \mathbb{N}$.*

Proof. According to [12, III.4.2(a)] and, since T is closed and everywhere defined, we deduce that T is bounded and $T(0)$ is closed. On the other hand, from Lemma 3.2, we infer that $R(T^{p+1})$ and $R(T^p)$ are closed. Let $T_0 := T_{/R(T^p)}$, be the restriction of T to $R(T^p)$, then T_0 is closed (as T and $R(T^p)$ are both closed). However, $N(T_0) = N(T) \cap R(T^p) = N(T) \cap R(T^{p+n})$, for all $n \in \mathbb{N}$. It follows, from Lemma 3.1(iii), that $N(T_0)$ is finite dimensional and that $N(T_0) \subset R(T^{p+n}) = R(T_0^n)$. Therefore T_0 is both regular and upper semi-Fredholm. According to Lemma 3.4, there exists $\gamma > 0$ such that $T_0 - \lambda$ is both regular and upper semi-Fredholm with $\alpha(T_0) = \alpha(T_0 - \lambda)$, for each $0 < |\lambda| < \gamma$. Furthermore, $R_c(T) = \{0\}$ (since $\rho(T) \neq \emptyset$) and hence

$R_c(T_0) \subset R_c(T) = \{0\}$. Which implies that $R_c(T_0 - \lambda) = \{0\}$ (by Lemma 2.6). It follows that

$$\begin{aligned}
\dim(N(T-\lambda)^n) &= \dim(N(T-\lambda)^n \cap R(T^p)) && \text{([22, Lemma 7.2])}\\
&= \dim(N(T_0-\lambda)^n) = n\ \dim N(T_0-\lambda) && \text{(Lemma 3.5)}\\
&= n\ \dim N(T_0) && \text{(Lemma 3.4)}\\
&= n\ \dim(N(T) \cap R(T^p))\\
&= \dim(N(T^{p+1})/N(T^p)) && \text{(Lemma 2.2).}
\end{aligned}$$

Now, for $n \geq 1$ and $\lambda \neq 0$, we let us consider the polynomials P and Q defined by $P(z) = (z-\lambda)^n$ and $Q(z) = z^p$, for all $z \in \mathbb{C}$. Clearly P and Q have no common divisors. Then there exist two polynomials u and v such that $1 = P(z)u(z)+Q(z)v(z)$ for all $z \in \mathbb{C}$. It follows that $X = R(T-\lambda)^n + R(T^p)$, and therefore

$$\begin{aligned}
\operatorname{codim} R(T-\lambda)^n &= \dim X/R(T-\lambda)^n\\
&= \dim[(R(T^p) + R(T-\lambda)^n)/R(T-\lambda)^n]\\
&= \dim[R(T^p)/R(T^p) \cap R(T-\lambda)^n] && \text{([22, Lemma 2.3])}\\
&= \operatorname{codim} R(T_0-\lambda)^n\\
&= \dim N(T_0-\lambda)^n - \operatorname{ind}(T_0-\lambda)^n\\
&= n[\dim N(T_0-\lambda) - \operatorname{ind}(T_0-\lambda)] && \text{(Lemma 3.5 and}\\
&= n[\dim N(T_0) - \operatorname{ind}(T_0)] && \text{Remark 3.2)}\\
&= n\ \operatorname{codim} R(T_0) = n\ \dim R(T^p)/R(T^{p+1}).
\end{aligned}$$

Finally, $N(T-\lambda) = N(T-\lambda) \cap R(T^p) = N(T_0-\lambda) \subseteq R(T_0-\lambda)^n \subseteq R(T-\lambda)^n$. Which means that $T-\lambda$ is regular. ∎

The next corollary is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let $T \in CR(X)$ be everywhere defined such that $a(T) < \infty$, $\rho(T) \neq \emptyset$ and $R(T^{a(T)+1})$ is closed. Then there exists $\gamma > 0$ such that, for each $0 < |\lambda| < \gamma$, the following assertions holds.*

(i) *$T-\lambda$ is regular.*

(ii) *$T-\lambda$ is bounded below.*

(iii) $\operatorname{codim} R(T-\lambda)^n = n\ \dim(R(T^{a(T)})/R(T^{a(T)+1}))$.

Corollary 3.2. *Let* $T \in CR(X)$ *be everywhere defined such that* $\rho(T) \neq \emptyset$. *Then* $\sigma_{asc}(T)$ *and* $\sigma^e_{asc}(T)$ *are two closed subsets of* $\sigma(T)$. *Moreover* $\sigma_{asc}(T)\backslash\sigma^e_{asc}(T)$ *is an open set.*

Proof. The closedness of $\sigma^e_{asc}(T)$ and $\sigma_{asc}(T)$ are two immediate consequences of Theorem 3.1 and Corollary 3.1, respectively. For the last assertion, let $\lambda \in \sigma_{asc}(T)\backslash\sigma^e_{asc}(T)$ and let $p := p(T-\lambda)$. Then by Theorem 3.1 there exists a deleted open neighborhood U of λ such that $U \cap \sigma^e_{asc}(T) = \emptyset$ and, for all $\alpha \in U$ and $n \in \mathbb{N}$,

$$\dim N(T-\alpha)^n \geq n \; \dim(N(T-\lambda)^{p+1}/N(T-\lambda)^p).$$

But, since $T-\lambda$ has an infinite ascent, $\dim(N(T-\lambda)^{p+1}/N(T-\lambda)^p)$ is nonzero, and consequently the sequence $(\dim N(T-\alpha)^n)_n$ is strictly increasing, for each $\alpha \in U$. Thus $U \subset \sigma_{asc}(T)$, which completes the proof. ∎

In the following we denote $E(T) := \rho_{asc}(T) \cap \rho_{des}(T) \cap \sigma(T)$.

Corollary 3.3. *Let* $T \in CR(X)$ *be everywhere defined such that* $\rho(T) \neq \emptyset$. *If* $\lambda \in E(T)$ *then* λ *is an isolated point of the boundary of* $\sigma(T)$.

Proof. Let $\lambda \in E(T)$. Then $T-\lambda$ has a finite descent and ascent, moreover $R((T-\lambda)^{a(T-\lambda)+1})$ is closed. On the other hand, since T is closed and $R_c(T) = \{0\}$, one can deduce, by virtue of Lemma 2.6 and [12, II.5.16], that $T-\lambda$ is closed and $R_c(T-\lambda) = \{0\}$. Furthermore, applying Corollary 3.1 to $T-\lambda$ instead to T, we conclude that there exists $\gamma > 0$ such that $T-\alpha$ is injective and surjective, for all $0 < |\alpha-\lambda| < \gamma$. This implies that $D(\lambda,\gamma)\backslash\{\lambda\} \subset \rho(T)$. Thus λ is isolated and it is in the boundary of the spectrum of T. ∎

The ascent and the essential ascent spectrum of a linear relation T can be empty. As a consequence of the following theorem, we show that this occurs precisely when the boundary of the spectrum of T is a subset of the essential ascent resolvent.

Theorem 3.2. *Let* $T \in CR(X)$ *be everywhere defined such that* $\rho(T) \neq \emptyset$. *Then*

$$\rho^e_{asc}(T) \cap \partial\sigma(T) = \rho_{asc}(T) \cap \partial\sigma(T) = E(T). \tag{3.1}$$

Proof. From Corollary 3.3, we have the obviously inclusions $E(T) \subset \rho_{asc}(T) \cap \partial\sigma(T) \subset \rho^e_{asc}(T) \cap \partial\sigma(T)$. For the reverse inclusions, it suffices to show that $\rho^e_{asc}(T) \cap \partial\sigma(T) \subset E(T)$. Let λ be an element of the boundary of $\sigma(T)$ such that $T-\lambda$ has a finite essential ascent and $R(T^{a_e(T-\lambda)+1})$ is closed. Then $R(T^{a(T-\lambda)+1})$ is closed (see Lemma 3.2), $T-\lambda$ is closed (by [3, Lemma 14]) and $R_c(T-\lambda) = \{0\}$ (by [22, Lemma 7.1]). Moreover according to Theorem 3.1, there exists a punctured neighborhood V of λ such that $\dim N(T-\alpha) = \dim(N(T-\lambda)^p)/(N(T-\lambda)^p)$ and $\operatorname{codim} R(T-\alpha) = \dim(R(T-\lambda)^p/R(T-\lambda)^{p+1})$, for some $p \in \mathbb{N}$ and all $\alpha \in V$. Since $T-\alpha$ is closed, for all α (see [3, Lemma 14]) and $\lambda \in \partial\sigma(T)$, then there exists $\alpha_0 \in V \backslash \sigma(T) \neq \emptyset$. Hence

$$\begin{aligned} 0 &= \dim N(T-\alpha_0) = \operatorname{codim} R(T-\alpha_0) \\ &= \dim(N(T-\lambda)^{p+1})/N((T-\lambda)^p) \\ &= \dim(R((T-\lambda)^p)/R((T-\lambda)^{p+1})). \end{aligned}$$

It follows that $T-\lambda$ is of finite ascent and descent and $R(T^{a(T-\lambda)+1})$ is closed. This means that $\lambda \in E(T)$. ∎

Corollary 3.4. *Let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$. Then the following assertions are equivalent.*

(i) $\sigma_{asc}(T) = \emptyset$.

(ii) $\sigma^e_{asc}(T) = \emptyset$.

(iii) $\partial\sigma(T) \subseteq \rho_{asc}(T)$.

(iv) $\partial\sigma(T) \subseteq \rho^e_{asc}(T)$.

Proof. All the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow immediately from (3.1). For the implication (iv) $\Rightarrow$ (i), assume that $\partial\sigma(T) \subseteq \rho^e_{asc}(T)$ then $\partial\sigma(T) = E(T)$. According to Corollary 3.3, it follows that all points of $\partial\sigma(T)$ are isolated and hence $\partial\sigma(T) = \sigma(T)$. Now we get, by (3.1), $\sigma(T) \subset \rho_{des}(T) \cap \rho_{asc}(T)$. Which means that $\mathbb{C} = \sigma(T) \cup \rho(T) \subset \rho_{des}(T) \cap \rho_{asc}(T)$. Hence $\rho_{asc}(T) = \mathbb{C}$, and consequently $\sigma_{asc}(T) = \emptyset$. ∎

Corollary 3.5. *Let $K(X) := \{T \in CR(X) : D(T) = X$ and $\rho(T) \neq \emptyset\}$. The following assertions are equivalent.*

(i) *X has a finite dimension.*

(ii) *Every $T \in K(X)$ has a finite ascent and $R(T^{a(T)+1})$ is closed.*

(iii) *Every $T \in K(X)$ has a finite essential ascent and $R(T^{a_e(T)+1})$ is closed.*

(iv) $\sigma_{asc}(T)$ *is empty, for every* $T \in K(X)$.

(v) $\sigma^e_{asc}(T)$ *is empty, for every* $T \in K(X)$.

Proof. First observe that $T \in K(X)$, if and only if, $T - \lambda \in K(X)$, for every $\lambda \in \mathbb{C}$. However, all the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are obvious. From Corollary 3.4, it follows immediately that (iii) $\Rightarrow$ (iv). Now, suppose that $\sigma^e_{asc}(T)$ is empty for every $T \in K(X)$, then $\sigma^e_{asc}(T)$ is empty, for every bounded linear operator on X. It follows, from [6, Corollary 2.8], that X has a finite dimension. Thus (v) $\Rightarrow$ (i). ∎

Theorem 3.3. *Let* $T \in CR(X)$ *be everywhere defined such that* $\rho(T) \neq \emptyset$. *If* Ω *be a connected component of* $\rho^e_{asc}(T)$ *then*

$$\Omega \subset \sigma(T) \quad or \quad \Omega \backslash E(T) \subset \rho(T).$$

Proof. Let $\Omega_r := \{\lambda \in \Omega : T - \lambda$ is both regular and upper semi-Fredholm$\}$. From Theorem 3.2, clearly that $\Omega_1 := \Omega \backslash \Omega_r$ is at most countable and therefore Ω_r is connected. Suppose that $\Omega \cap \rho(T)$ is non-empty, so that is $\Omega_r \cap \rho(T)$. Let $\lambda \in \Omega \cap \rho(T)$. The use of Lemma 3.4 and [3, Corollary 17] leads to $\dim N(T - \lambda) = 0$ and, by continuity of the index (see [3, Theorem 15]), $\operatorname{codim} R(T - \lambda) = 0$. This implies that $\Omega_r \subset \rho(T)$. Thus $\Omega \backslash \Omega_r$ consists of isolated points of the spectrum with finite essential ascent, so that

$$\Omega \backslash \Omega_r \subset \rho^e_{asc}(T) \cap \partial\sigma(T) = E(T).$$

Consequently $\Omega \backslash E(T) \subset \Omega_r \subset \rho(T)$. ∎

Corollary 3.6. *Let* $T \in CR(X)$ *be everywhere defined such that* $\rho(T) \neq \emptyset$. *The following assertions are equivalent.*

(i) $\sigma(T)$ *is at most countable.*

(ii) $\sigma_{asc}(T)$ *is at most countable.*

(iii) $\sigma^e_{asc}(T)$ *is at most countable.*

In this case, $\sigma_{asc}(T) = \sigma^e_{asc}(T)$ *and* $\sigma(T) = \sigma_{asc}(T) \cup E(T)$.

Proof. All the implications are obvious except (iii) $\Rightarrow$ (i). To show this, assume that $\sigma^e_{asc}(T)$ is at most countable, then $\rho^e_{asc}(T)$ is connected. The use of Theorem 3.3 leads to $\rho^e_{asc}(T) \backslash E(T) \subset \rho(T)$. Consequently, $\sigma(T) = \sigma^e_{asc}(T) \cup E(T)$ is at most countable.

For the last assertion, suppose that $\sigma(T)$ is at most countable, then $\sigma_{asc}(T)\backslash\sigma^e_{asc}(T)$ is at most countable and open (see Corollary 3.2). Consequently it is empty, which means that $\sigma^e_{asc}(T) = \sigma_{asc}(T)$. ∎

4. Ascent, essential ascent spectrum and perturbations

In this section we are interested to investigate the stability of the ascent spectrum and the essential ascent spectrum of a linear relation, everywhere defined in a complex Banach space, under commuting finite rank perturbations. We start this section by some technical lemmas which are used in the sequel.

Lemma 4.1. *Let A and B be two linear relations in a linear space X and let C be an operator in X. Assume that $D(A) = D(C) = X$, $AB = BA$, $AC = CA$ and $A(0) \subset B(0)$. Then*

$$A^n(B + C) = (B + C)A^n = A^nB + A^nC, \quad \textit{for all } n \in \mathbb{N}. \tag{4.1}$$

Proof. First consider the case $n = 1$. From [12, I.4.2 (e)], we have

$$A^n(B + C) = A^nB + A^nC, \quad \text{for all } \in \mathbb{N}, \tag{4.2}$$

and, by using [12, I.4.3 (c)], it follows that

$$(B + C)A \subset A(B + C). \tag{4.3}$$

Now, let $y \in G(AB + AC)$. Then $y \in ABx + ACx = BAx + CAx$, for some $x \in X$. Which implies that $y \in By_1 + Cy_2$, for some $y_1, y_2 \in Ax$, so that $y \in By_1 + Cy_1 + C(y_2 - y_1) = By_1 + Cy_1 = (B + C)y_1 \subset (B + C)Ax$, because $C(y_2 - y_1) \in CA(0) = AC(0) = A(0) \subset B(0)$. Therefore

$$AB + AC \subset (B + C)A. \tag{4.4}$$

The use of (4.2), (4.3) and (4.4) leads to $A(B+C) = (B+C)A$ holds. Assume now that (4.1) holds, for some positive integer n. Then $A^{n+1}(B + C) = AA^n(B + C) = A(B + B)A^n = (B + C)AA^n = (B + C)A^{n+1}$. Thus (4.1) holds, for all $n \in \mathbb{N}$. ∎

Lemma 4.2. *Let T be an everywhere defined linear relation in a linear space X and let F be an operator in X such that $D(F) = X$ and $TF = FT$. Then,*

(i) $(T+F)^n = \sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^i$, *for all* $n \in \mathbb{N}$.

(ii) $T^n - F^n = \Big(\sum_{i=0}^{n-1} T^{n-1-i}F^i\Big)(T-F)$, *for all* $n \geq 1$.

Proof. (i) For $n = 0$ and $n = 1$ the result is trivial. Suppose that $(T+F)^n = \sum_{i=0}^{n} \binom{n}{i} T^{n-i}F^i$, for some $n \geq 1$. It follows that

$$\begin{aligned}
(T+F)^{n+1} &= (T+F)(T+F)^n \\
&= (T+F)\Big(\sum_{i=0}^{n} \tbinom{n}{i} T^{n-i}F^i\Big) \\
&= \sum_{i=0}^{n} \tbinom{n}{i}(T+F)T^{n-i}F^i && \text{([12, I.4.2 (e)])} \\
&= \sum_{i=0}^{n} [\tbinom{n}{i}T^{n-i}(T+F)F^i] && \text{(Lemma 4.1)} \\
&= \sum_{i=0}^{n} [\tbinom{n}{i}T^{n-i}F^i(T+F)] && \text{(Lemma 4.1)} \\
&= \sum_{i=0}^{n} \tbinom{n}{i}T^{n-i}F^iT + \sum_{i=0}^{n} \tbinom{n}{i}T^{n-i}F^iF && \text{([12, I.4.2 (e)]} \\
&= \sum_{i=0}^{n} \tbinom{n}{i}T^{n-i+1}F^i + \sum_{i=0}^{n} \tbinom{n}{i}T^{n-i}F^{i+1} && \text{(as } TF = FT) \\
&= \sum_{i=0}^{n} \tbinom{n}{i}T^{n-i+1}F^i + \sum_{i=1}^{n+1} \tbinom{n}{i-1}T^{n-i+1}F^i \\
&= T^{n+1} + \sum_{i=1}^{n} \tbinom{n+1}{i}T^{n-i+1}F^i + F^{n+1} \\
&= \sum_{i=0}^{n+1} \tbinom{n+1}{i}T^{n+1-i}F^i.
\end{aligned}$$

Therefore the required equality holds, for all $n \in \mathbb{N}$.

(ii) We can easily seen that $T^n + T^kF^j - T^kF^j = T^n$, for all $0 \leq k \leq n$ and

$j \in \mathbb{N}$. Applying this together with Lemma 4.1 and [12, I.4.2 (e)] we obtain

$$\Big(\sum_{i=0}^{n-1} T^{n-1-i}F^i\Big)(T-F)$$
$$= (T^{n-1} + T^{n-2}F + \cdots + F^{n-1})T - (T^{n-1} + T^{n-2}F + \cdots + F^{n-1})F.$$

We shall now to show, by induction on n, that

$$(T^{n-1} + T^{n-2}F + \cdots + F^{n-1})T = T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}). \quad (4.5)$$

The case $n = 1$ is evident. Assume that (4.5) holds for some positive integer n and take $A = T, B = T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1})$ and $C = F^n$. Then, $D(A) = D(C) = X$, $A(0) = T(0) \subset B(0) = T^n(0)$, $AC = TF^n = F^nT = CA$ (as $TF = FT$) and $AB = T(T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1})) = T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1})T = BA$. Now, this fact together with Lemma 4.1 and [12, I.4.2 (e)]), lead to

$$\begin{aligned}(T^n + T^{n-1}F + \cdots + F^n)T &= [T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}) + F^n]T \\ &= (B + C)A \\ &= A(B + C) \\ &= T[T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}) + F^n] \\ &= T(T^n + T^{n-2}F + \cdots + F^n).\end{aligned}$$

Hence (4.5) holds for all $n \geq 1$. We prove, arguing in a similar way as in preceding, that

$$(T^{n-1} + T^{n-2}F + \cdots + F^{n-1})F = F(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}). \quad (4.6)$$

Combining (4.5) and (4.6) we get

$$\begin{aligned}\Big(\sum_{i=0}^{n-1} T^{n-1-i}F^i\Big)(T-F) &= T(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}) \\ &\qquad - F(T^{n-1} + T^{n-2}F + \cdots + F^{n-1}) \\ &= (T^n + T^{n-1}F + \cdots + TF^{n-1}) \\ &\qquad - (T^{n-1}F + T^{n-2}F^2 + \cdots + F^n) \\ &= T^n + (T^{n-1}F - T^{n-1}F + \cdots \\ &\qquad + TF^{n-1} - TF^{n-1}) - F^n \\ &= T^n - F^n.\end{aligned}$$

∎

In the rest of this section, X, unless otherwise stated, will be a complex Banach space.

LEMMA 4.3. *Let $T \in CR(X)$.*

(i) *Let $F \in LR(X)$ be single valued with finite rank. If $T \in \Phi_+(X)$ then $T+F \in \Phi_+(X)$.*

(ii) *Let $S \in LR(X)$ be bounded. If $ST \in \Phi_+(X)$, then $T \in \Phi_+(X)$.*

Proof. Follows immediately from [12, V.3.2 and V.2.16]. ∎

LEMMA 4.4. *Let F be a bounded operator in X such that F^k is of finite rank, for some $k \geq 1$, and let $T \in CR(X)$ be everywhere defined. Suppose that $FT = TF$. If T is upper semi-Fredholm then $T+F$ is upper semi-Fredholm.*

Proof. According to Lemma 4.2, we can write $T^k - F^k = S(T-F)$ where $S := \sum_{i=0}^{k-1} T^{k-1-i}F^i$. Furthermore, since T is closed and upper semi-Fredholm, then so is T^k (see [2, Proposition 24]). From this together with Lemma 4.3, it follows that $T^k - F^k$ is also upper semi-Fredholm. On the other hand, since T is closed and everywhere defined and F is bounded, then S is bounded. This means, using Lemma 4.3, that $T-F$ is upper semi-Fredholm. Now, by interchanging F and $-F$, we obtain $T+F$ is upper semi-Fredholm. ∎

LEMMA 4.5. *Let X be a linear spaces and let $T \in LR(X)$ be injective. Then* $\dim D(T) \leq \dim(R(T))$.

Proof. We have, by [12, I.6.4], $\dim D(T) + \dim T(0) = \dim R(T) + \dim N(T)$. Therefore, since $N(T) = \{0\}$, we have obviously $\dim D(T) \leq \dim(R(T))$. ∎

Remark 4.1. As a direct consequence of the above lemma we have, for $T \in LR(X)$, $\dim D(T)/N(T) \leq \dim R(T)$.

LEMMA 4.6. *Let F be a bounded operator in X such that F^k is of finite rank, for some nonnegative integer k, and let $T \in CR(X)$ be everywhere defined. Assume that $FT = TF$. Then*

$$\dim \Big(N(T^n)/N((T+F)^{n+k-1}) \cap N(T^n) \Big) < \dim \big(R(F^k) \big) < \infty,$$

for all $n \geq 1$.

Proof. First, observe, since $TF = FT$, that $T^m(0) = (TF)^m(0) = (FT)^m(0) = F^m(T^m(0)) \subset R(F^m)$, for all $m \in \mathbb{N}$. Let $n \geq 1$ and let M be a subspace of $N(T^n)$ such that

$$N(T^n) = \Big[N((T+F)^{n+k-1}) \cap N(T^n)\Big] \oplus M.$$

It follows, by Lemma 4.2, that $(T+F)^{n+k-1}$ maps $N(T^n)$ into $T^{n+k-1}(0) + R(F^k) = R(F^k)$. Now, since $(T+F)^{n+k-1}$ is injective on M, and according to Lemma 4.5, we get $\dim(M) \leq \dim(T+F)^{n+k-1}(N(T^n)) \leq \dim(R(F^k)) < \infty$. This proves the Lemma. ∎

Now, we are in the position to give the main theorem of this section.

THEOREM 4.1. *Let F be a bounded operator on X such that F^k is of finite rank, for some nonnegative integer k, and let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$ and $\rho(T+F) \neq \emptyset$. Suppose that $TF = FT$. Then*

(i) *$a(T)$ is finite if and only if $a(T+F)$ is finite.*

(ii) *$a_e(T)$ is finite if and only if $a_e(T+F)$ is finite.*

In this case $R(T^{a_e(T)+1})$ is closed if and only if $R((T+F)^{a_e(T+F)+1})$ is closed.

Proof. Clearly, since F is a bounded operator and using Lemma 4.3, it suffices to show only one direction.
(i) Let $a := a(T)$ and, for $n \geq a$, we let us consider the sequences $(a_n(T))_{n\in\mathbb{N}}$ and $(b_n(T))_{n\in\mathbb{N}}$ defined as

$$\begin{aligned} a_n(T) &:= \dim\big(N(T^n)/N(T+F)^{n+k-1} \cap N(T^n)\big) \\ &= \dim\big(N(T^a)/N(T+F)^{n+k-1} \cap N(T^a)\big); \\ b_n(T) &:= \dim\big(N(T+F)^n/N(T+F)^n \cap N(T^{n+k-1})\big) \\ &= \dim\big(N(T+F)^n/N(T+F)^n \cap N(T^a)\big); \end{aligned}$$

respectively. Clearly $(a_n(T))_n$ is a decreasing sequence, which implies that there exists $p \geq a$ such that $a_n(T) = a_p(T)$, for all $n \geq p$. It follows that

$$N(T+F)^{n+k-1} \cap N(T^a) = N(T+F)^{p+k-1} \cap N(T^a), \text{ for all } n \geq p.$$

Furthermore $(b_n(T))_n$ is an increasing sequence and, by interchanging T and $T+F$ in Lemma 4.6, we may infer that $b_n(T) \leq \dim(R(F^k)) < \infty$. So, there exists $q \geq a$ such that $b_n(T) = b_q(T)$, for all $n \geq q$. Hence, for $n \geq q \geq p+k-1$,

$$\begin{aligned}\dim\big(N(T+F)^q/N(T+F)^q \cap N(T^a)\big)\\ &= \dim\big(N(T+F)^{n+k-1}/N(T+F)^{n+k-1} \cap N(T^a)\big)\\ &= \dim\big(N(T+F)^{n+k-1}/N(T+F)^{p+k-1} \cap N(T^a)\big)\\ &= \dim\big(N(T+F)^{n+k-1}/N(T+F)^q \cap N(T^a)\big).\end{aligned}$$

This implies that $N(T+F)^q = N(T+F)^{n+k-1}$, for all $n \geq q$. Thus $a(T+F) \leq q$.

(ii) Suppose that T has a finite essential ascent and let $p := p(T)$. Given $n \geq k+p$, it follows, by Lemma 4.6, that

$$\dim\big(N(T^n)/N((T+F)^{n+k-1}) \cap N(T^n)\big) < \infty. \tag{4.7}$$

By interchanging T and $T+F$ in (4.7), we obtain

$$\dim\big(N(T+F)^n/N(T^{n+k-1}) \cap N(T+F)^n\big) < \infty.$$

On the other hand, since $a_e(T) < \infty$, then $\dim N(T^{n+k-1})/N(T^p) < \infty$, which means that

$$\dim\big(N((T+F)^n)/N(T+F)^n \cap N(T^p))\big) < \infty. \tag{4.8}$$

Furthermore, $N(F^k) \cap N(T^p) \subset N(T+F)^n \cap N(T^p) \subset N(T^p)$ and, since F^k is finite dimensional range, it follows that

$$\dim\big(N(T^p)/(N(F^k) \cap N(T^p))\big) < \infty. \tag{4.9}$$

Hence

$$\dim\big(N(T^p)/(N(T+F)^n \cap N(T^p))\big) < \infty \tag{4.10}$$

Now, the use of (4.8) combined with (4.9), leads to

$$\begin{aligned}\dim\big(N(T+F)^n/N(F^k) \cap N(T^p)\big)\\ &= \dim\big(N(T+F)^n/N(T+F)^n \cap N(T^p)\big)\\ &\quad + \dim\big(N(T+F)^n \cap N(T^p)/N(F^k) \cap N(T^p)\big)\\ &\leq \dim\big(N(T+F)^n/N(T+F)^n \cap N(T^p)\big)\\ &\quad + \dim\big(N(T^p)/N(F^k) \cap N(T^p)\big)\\ &< \infty.\end{aligned}$$

Therefore $\dim(N(T+F)^{n+1}/N(T+F)^n) = \dim(N(T+F)^{n+1}/N(F^k) \cap N(T^p)) - \dim(N(T+F)^n/N(F^k) \cap N(T^p)) < \infty$.

Thus $a_e(T+F) \leq p+k$. Now, assume that $R(T^{a_e(T)+1})$ is closed and let $n \geq k+q$, where $q := \max(p(T), p(T+F))$. Denote by T_0 and F_0 the restrictions of T and F, respectively, to $R(T^q)$. Since $R(T^q)$ is closed (by Lemma 3.2) and T is closed, then T_0 is closed. On the other hand, $N(T_0) = N(T) \cap R(T^q)$ is finite dimensional (by Lemma 3.1), which means that T_0 is upper semi-Fredholm. Moreover $T_0 + F_0$ is closed (as T_0 is closed and F_0 is a bounded operator). Furthermore, clearly $T_0(0) = T(0) \subset N(F) \cap R(T^q) = N(F_0)$, and hence, by Lemma 4.4, $T_0 + F_0$ is upper semi Fredholm. It follows, from [2, Proposition 24], that $(T_0 + F_0)^n$ is also upper semi-Fredholm. This means that $T^q(R(T+F)^n) = R(T_0 - F_0)^n$ is closed. Therefore, since T^q is everywhere defined and closed and $T^q(0) \subset T^q(R(T+F)^n)$, we infer that $T^{-q}T^q(R(T+F)^n)$ is closed. Thus $R((T+F)^n) + N(T^q) = T^{-q}T^q(R(T+F)^n)$ is closed. On the other hand, using (4.10), we get

$$\dim\big(R(T+F)^n \cap N(T^q)/R(T+F)^n \cap N(T+F)^n \cap N(T^q)\big) < \infty.$$

However, since $a_e(T+F) < \infty$ and using Lemma 2.2, we deduce that $N(T+F)^n \cap R(T+F)^n$ has a finite dimension. This implies that $\dim R(T+F)^n \cap N(T^q) < \infty$, in particular $R(T+F)^n \cap N(T^q)$ is closed. By the hypothesis $R_c(T) = \{0\}$, we infer that $(T+F)^n(0) \cap N(T^q) = T^n(0) \cap N(T^q) = \{0\}$. Further, since $T+F$ is closed and $\rho(T+F) \neq \emptyset$, then $(T+F)^n$ is closed. The use of [14, Lemma 3.2] leads to $R(T+F)^n$ is closed. Hence, applying Lemma 3.2, it follows that $R(T+F)^{a_e(T+F)+1}$ is closed. For the reverse implication it suffices to interchange T and $T+F$. ∎

As applications of Theorem 4.1 we give the following corollaries.

Corollary 4.1. *Let F be a bounded operator in X and let*

$$K_F := \big\{T \in CR(X) : D(T) = X,\ TF = FT,\ \rho(T) \neq \emptyset \text{ and } \rho(T+F) \neq \emptyset\big\}.$$

Then the following assertions are equivalent.

(i) *F^k has a finite rank, for some $k \geq 1$.*

(ii) *$\sigma_{asc}(T+F) = \sigma_{asc}(T)$, for every $T \in K_F$.*

(iii) *$\sigma^e_{asc}(T+F) = \sigma^e_{asc}(T)$, for every $T \in K_F$.*

The proof of this Corollary requires the following lemma.

LEMMA 4.7. ([6, Theorem 3.2]) *Let F be a bounded operator in X. The following conditions are equivalent:*

(i) *There exists a positive integer n such that F^n has a finite rank.*

(ii) $\sigma^e_{asc}(T+F) = \sigma^e_{asc}(T)$, *for all bounded operator $T \in LR(X)$ commuting with F.*

(iii) $\sigma_{asc}(T+F) = \sigma_{asc}(T)$, *for all bounded operator $T \in LR(X)$ commuting with F.*

Proof of Corollary 4.1. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow immediately from Theorem 4.1. Now, since all bounded operators commuting with F on X belong to K_F and using Lemma 4.7, we conclude that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). ∎

COROLLARY 4.2. *Let $T \in CR(X)$ be everywhere defined such that $\rho(T) \neq \emptyset$. Then*

$$\sigma^e_{asc}(T) \subset \bigcap_{F \in \mathcal{F}_T(X)} \sigma_{asc}(T+F)$$

where $\mathcal{F}_T(X)$ denotes the set of all bounded finite-rank operators F on X commuting with T and such that $\rho(T+F) \neq \emptyset$.

REFERENCES

[1] R. P. AGARWAL, M. MEEHAN, D. O'REGAN, "Fixed Point Theory and Applications", Cambridge Tracts in Mathematics, 141, Cambridge University Press, Cambridge, 2001.

[2] T. ÁLVAREZ, On regular linear relations, *Acta Math. Sin.* (English Series) **28** (1) (2012), 183 – 194.

[3] T. ÁLVAREZ, Small perturbation of normally solvable relations, *Publ. Math. Debrecen* **80** (1-2) (2012), 155 – 168.

[4] T. ÁLVAREZ, Perturbation theorems for upper and lower semi-Fredholm linear relations, *Publ. Math. Debrecen* **65** (1-2), (2004), 179 – 191.

[5] T. ÁLVAREZ, R. CROSS, D. WILCOX, Quantities related to upper and lower semi-Fredholm type linear relations, *Bull. Austral. Math. Soc.* **66** (2) (2002), 275 – 289.

[6] O. BEL HADJ FREDJ, M. BURGOS, M. OUDGHIRI, Ascent and essential ascent spectrum, *Studia Math.* **187** (1) (2008), 59 – 73.

[7] E. CHAFAI, M. MNIF, Spectral mapping theorem for ascent, essential ascent, descent and essential descent spectrum of linear relations, *Acta Math. Sci.* **34 B** (4) (2014), 1212 – 1224.

[8] H. Chandra, P. Kumar, Ascent and descent of composition operators on l^p spaces, *Demonstratio Math.* **43** (1) (2010), 161 – 165.

[9] E. A. Coddington, Multi-valued operators and boundary vValue problems, in “Analytic theory of Differential Equations”, Lecture Notes in Math., 183, Springer-Verlag, Berlin, 1971, 1 – 8.

[10] E. A. Coddington, A. Dijksma, Self-adjoint subspaces and eigenfunction expansions for ordinary differential subspaces, *J. Differential Equations* **20** (2) (1976), 473 – 526.

[11] R. W. Cross, An index theorem for the product of linear relations, *Linear Algebra Appl.* **277** (1-3) (1998), 127 – 134.

[12] R. W. Cross, “Multivalued Linear Operators”, Monographs and Textbooks in Pure and Applied Mathematics, 213, Marcel Dekker, New York, 1998.

[13] A. Dijksma, A. EL Sabbah, H. S. V. de Snoo, Selfadjoint extensions of regular canonical systems with Stieltjes boundary conditions, *J. Math. Anal. Appl.* **152** (2), (1990), 546 – 583.

[14] F. Fakhfakh, M. Mnif, Perturbation theory of lower semi-Browder multivalued linear operators, *Publ. Math. Debrecen* **78** (3-4) (2011), 595 – 606.

[15] A. Favini, A. Yagi, Multivalued linear operators and degenerate evolution equations, *Ann. Mat. Pura. Appl. (4)* **163**, (1993), 353 – 384.

[16] S. Grabiner, Uniform ascent and descent of bounded operators, *J. Math. Soc. Japan* **34** (2) (1982), 317 – 337.

[17] S. Grabiner and J. Zemánek, Ascent, descent and ergodic properties of linear operators, *J. Operator Theory* **48** (1) (2002), 69 – 81.

[18] M. Gromov, “Partial Differential Relations”, Springer-Verlag, Berlin, 1986.

[19] M. A. Kaashoek, D. C. Lay, Ascent, descent and commuting perturbation, *Trans. Amer. Math. Soc.* **169** (1972), 35 – 47.

[20] M. Mbekhta, Ascente, descente et spectre essentiel quasi-Fredholm, *Rend. Circ. Mat. Palermo (2)* **46** (2) (1997), 175 – 196.

[21] A. Sandovici, H. de Snoo, An index formula for the product of linear relations, *Linear Algebra Appl.* **431** (11) (2009), 2160 – 2171.

[22] A. Sandovici, H. de Snoo, H. Winkler, Ascent, descent, nullity, defect, and related notions for linear relations in linear spaces, *Linear Algebra Appl.* **423** (2-3) (2007), 456 – 497.

[23] A. E. Taylor, Theorems on ascent, descent, nullity and defect of linear operators, *Math. Ann.* **163** (1966), 18 – 49.

[24] J. Von Neumann, Functional operators II, The geometry of orthogonal spaces, Annals of Math. Studies, 22, Princeton University Press, Princeton N. J., 1950.