Corrigendum to
"Moore-Penrose Inverse and Operator Inequalities"
Extracta Mathematicae 30 (2015), 29 – 39

Ameur Seddik

Department of Mathematics, Faculty of Mathematics and Computer Science,
University of Batna 2, Batna, Algeria
seddikameur@hotmail.com

Abstract: We correct a mistake which affect our main results, namely the proof of Lema 1. The main results of the article remain unchanged.

Key words: Closed range operator, Moore-Penrose inverse, normal operator, operator inequality.

AMS *Subject Class.* (2010): 47A30, 47A03, 47B15.

The paper mentioned in the title includes the following result as Lemma 1:

LEMMA 1. *Let $S \in \mathfrak{B}(H)$. If S is surjective or injective with closed range and satisfies the following inequality*

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2 X \right\| + \left\| X S^2 \right\| \geq 2 \left\| SXS \right\|, \quad (*)$$

then S is normal.

In the proof of this lemma, the matrix representation of the operator R^2 was computed incorrectly obtaining $\begin{bmatrix} S_1^* S_1 & 0 \\ 0 & S_2^* S_2 \end{bmatrix}$, while the correct form of this matrix is $R^2 = \begin{bmatrix} S_1^* S_1 & 0 \\ 0 & (S_1 S_2)^* (S_1 S_2) \end{bmatrix}$. Since all the results of the paper are based on this lemma, we shall give here a correct proof of it.

The original proof is given in two cases. The second case follows immediately from the first one. The proof of the first case is divided in six steps. The mistake is in the fourth step.

For $S \in \mathfrak{B}(H)$ with closed range, S^+ denotes the Moore-Penrose inverse of S.

Proof of Lemma 1. Assume that $S \neq 0$ and that all 2×2 matrices used in this proof are given with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$. Then $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$.

We put $P = |S|$, $Q = |S^*|$, $P_1 = |S_1|$, $P_2 = |S_2|$, $Q_1 = (S_1 S_1^* + S_2 S_2^*)^{\frac{1}{2}}$. So we have $S^* S = P^2 = \begin{bmatrix} P_1^2 & S_1^* S_2 \\ S_2^* S_1 & P_2^2 \end{bmatrix}$, $SS^* = Q^2 = \begin{bmatrix} Q_1^2 & 0 \\ 0 & 0 \end{bmatrix}$. It is clear that Q_1 is invertible and $Q^+ = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

Case 1. Assume that S is injective with closed range and satisfies $(*)$. Then $S^+ S = I$, $\ker P = \ker S = \{0\}$, and $R(P) = R(S^*S)$ is closed (since $R(S^*)$ is also closed). Thus $\ker P = \{0\}$ and $R(P) = (\ker P)^\perp = H$. So, P is invertible.

Note that inequality $(*)$ implies the following inequality:

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2 S^+ X S^+ \right\| + \left\| S^+ X S \right\| \geq 2 \left\| SS^+ X \right\|. \tag{1}$$

The proof is given in four steps.

Step 1. $\left(S^2 \right)^+ S = S^+$. See Step 2 of the original proof.

Step 2. $\left(S^2 \right)^+ = (S^+)^2$. See Step 3 of the original proof.

Step 3. $\ker S^* = \{0\}$. Since S is injective, then $\ker S^* = \{0\}$ if and only if $S_2 = 0$. Assume that $S_2 \neq 0$.

Since $\left(S^2 \right)^+ = (S^+)^2$, then the two operators S^*S and SS^+ commute (see [1, 2]). Thus $P^2 = \begin{bmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{bmatrix}$, hence $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$.

Since $\ker S^* \neq \{0\}$, then $\sigma(Q^2) = \sigma(Q_1^2) \cup \{0\}$. From the fact that $\sigma(P^2) = \sigma(Q^2) - \{0\}$, we have $\sigma(P^2) = \sigma(Q_1^2)$. Then $\sigma(P_1^2) \cup \sigma(P_2^2) = \sigma(Q_1^2)$. Hence $\sigma(P_1^2) \subset \sigma(Q_1^2)$. Thus $\sigma(P_1) \subset \sigma(Q_1)$.

Using the polar decomposition of S and S^* in inequality (1), we obtain the following inequality:

$$\forall X \in \mathfrak{B}(H), \ \left\| S^2 S^+ X P^{-1} \right\| + \left\| Q^+ X Q \right\| \geq 2 \left\| SS^+ X \right\|.$$

By taking $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $X = \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix}$), where $X_1 \in \mathfrak{B}(R(S))$ (resp. $X_2 \in \mathfrak{B}(\ker S^*, R(S))$) in the last inequality, and since $S^2 S^+ = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$, we deduce the two following inequalities

$$\forall X_1 \in \mathfrak{B}(R(S)), \ \left\| P_1 X_1 P_1^{-1} \right\| + \left\| Q_1^{-1} X_1 Q_1 \right\| \geq 2 \left\| X_1 \right\|, \tag{2}$$

$$\forall X_2 \in \mathfrak{B}(\ker S^*, R(S)), \; \left\| P_1 X_2 P_2^{-1} \right\| \geq 2 \left\| X_2 \right\|. \tag{3}$$

By taking $X_2 = x \otimes y$ (where $x \in (R(S))_1$, $y \in \ker S^*$) in (3), we obtain

$$\forall x \in (R(S))_1, \forall y \in \ker S^*, \; \left\| P_1 x \right\| \left\| P_2^{-1} y \right\| \geq 2 \left\| y \right\|.$$

So we have

$$\forall x \in (R(S))_1, \forall y \in (\ker S^*)_1, \; \left\| P_1 x \right\| \geq 2 \left\| P_2 y \right\|.$$

Thus $\|P_2 y\| \leq \frac{k}{2}$, for every $y \in (\ker S^*)_1$ (where $k = \inf_{\|x\|=1} \|P_1 x\| > 0$), and then $\left\langle P_2^2 y, y \right\rangle \leq k^2/4$, for every $y \in (\ker S^*)_1$. So we obtain $\sigma(P_2^2) \subset \left(0, \frac{k^2}{4}\right]$ and $\sigma(P_1^2) \subset [k^2, \infty)$.

Since $\sigma(P_1) \subset \sigma(Q_1)$, and P_1, Q_1 satisfy the inequality (2), then using a variation of [3, Theorem 3.6] (in that paper Theorem 3.6 is stated with equality between the spectra but the proof is the same for inclusion between the spectra), we obtain $P_1 = Q_1$. Hence $\sigma(Q_1^2) = \sigma(P_1^2) = \sigma(P_1^2) \cup \sigma(P_2^2)$. Then $\sigma(P_2^2) \subset \sigma(P_1^2)$, that is impossible since $\left(0, \frac{k^2}{4}\right] \cap [k^2, \infty) = \varnothing$. Therefore $\ker S^* = \{0\}$.

Step 4. S is normal. Since $\ker S^* = \{0\}$, we obtain $R(S) = H$. So that S is invertible and satisfies the inequality $(*)$. Hence S satisfies the following inequality

$$\forall X \in \mathfrak{B}(H), \; \left\| SXS^{-1} \right\| + \left\| S^{-1}XS \right\| \geq 2 \left\| X \right\|.$$

Therefore S is normal (using [4]).

Case 2. Assume that S is surjective and satisfies $(*)$. Then S^* is injective with closed range and satisfies inequality $(*)$. From Case 1, S^* is normal. Hence S is normal, and the proof is finished. ∎

REFERENCES

[1] R. BOULDIN, The pseudo-inverse of a product, *SIAM J. Appl. Math.* **24** (1973), 489–495.

[2] S. IZUMINO, The product of operators with closed range and an extension of the reverse order law, *Tôhoku Math. J. (2)* **34** (1) (1982), 43–52.

[3] A. SEDDIK, Some results related to the Corach-Porta-Recht inequality, *Proc. Amer. Math. Soc.* **129** (2001), 3009–3015.

[4] A. SEDDIK, On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities, *J. Math. Anal. Appl.* **351** (1) (2009), 277–284.