Quickest detection of intensity change for Poisson process in generalized Bayesian setting

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Abstract

We observe a stochastic process $X_t = \int_0^t I(s<\theta)dN_s^{\lambda_0} + \int_0^t I(s \geq \theta)dN_{s-\theta}^{\lambda_1}$, $X_0 = 0$, where $N_t^{\lambda_0}$ and $N_t^{\lambda_1}$ are Poisson processes with known intensities λ_0 and λ_1, $\lambda_0 > \lambda_1$, $I(\cdot)$ is an indicator function. Nonrandom unknown parameter $\theta \in [0,\infty]$ is a time when the "disorder" appears. Let $\tau = \tau(\omega)$ be a finite stopping time with respect to the filtration generated by the process X_t. We interpret τ as the decision that the disorder has happened at the time $\tau(\omega)$. Let P_t be the distribution of the process X_t under the assumption that the disorder happened at the time $\theta = t$. For every $T > 0$ we denote by $\mathcal{M}_T$ the set of stopping times with the mean time $T = \mathrm{E}_\infty \tau$ until the false alarm. For $\tau \in \mathcal{M}_T$ introduce the risk $B(T;\tau) = \frac{1}{T}\int_0^\infty \mathrm{E}_\theta(\tau-\theta)^+ d\theta$. The stopping time $\tau_T^* \in \mathcal{M}_T$ is called optimal if $B(T) = B(T;\tau_T^*) = \inf_{\tau \in \mathcal{M}_T} B(T;\tau)$. Let ψ_t be the Shiryaev's process, $\psi_t = \int_0^t \frac{L_t}{L_s} ds$, where $L_t = \exp\left(\log\left(\frac{\lambda_1}{\lambda_0}\right) X_t - (\lambda_1 - \lambda_0)t\right)$ is the likelihood process. Denote $\rho = \frac{1}{2}\left(\frac{\lambda_1-\lambda_0}{\sqrt{\lambda_0}}\right)^2$, $\beta = \frac{1}{6}\left(\frac{\lambda_1-\lambda_0}{\sqrt{\lambda_0}}\right)^3$, $(-Ei(-x)) = \int_x^{+\infty} \frac{e^{-t}}{t} dt$ $(x > 0)$ – integral exponential function.

Theorem. *For any $T > 0$ an optimal stopping time τ_T^* exists in $\mathcal{M}_T$ such that $\tau_T^* = \inf\{t \geq 0 : \psi_t \geq T\}$. Moreover, the generalized Bayesian risk $B(T) = \frac{V(0;T)}{T}$, where $V(x;T)$ solves differential-difference equation $(1-(\lambda_1-\lambda_0)x)V'(x;T) + \lambda_0(V(x\frac{\lambda_1}{\lambda_0};T) - V(x;T)) = -x$, $V(T;T) = 0$. If $\lambda_0 \to \infty$ and $\lambda_1 \to \infty$ such that $\lambda_1 = \lambda_0 + C\sqrt{\lambda_0}$ for some fixed constant $C < 0$, then $B(T) = B_0(T) + \varepsilon B_1(T) + \dots$, where $\varepsilon = 1/\sqrt{\lambda_0}$, $B_0(T) = \frac{1}{\rho}\left[e^{\frac{1}{\rho T}}\left(-Ei\left(-\frac{1}{\rho T}\right)\right) - \left(1 - \frac{1}{\rho T}\int_0^\infty \frac{e^{-t}\log(1+\rho T\cdot t)}{t} dt\right)\right]$, $B_1(T) = \frac{\beta}{\rho^2}\left[\frac{3}{\rho T}\int_0^\infty \frac{e^{-t}\log(1+\rho T\cdot t)}{t}dt + e^{\frac{1}{\rho T}} Ei\left(-\frac{1}{\rho T}\right)\cdot\left(\frac{1}{2\rho T} - 1\right) - \frac{5}{2}\right]$.*

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