Repairable systems with general repair

Jaroslav Ševčík
Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic. Pod Vodárenskou věží 4, Praha 8, 182 08, Czech Republic.
& Dep. of Probability and Math. Statistics, Fac. of Mathematics and Physics, Charles University Prague. Sokolovská 83, Praha 8, 186 75, Czech Rep.

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Abstract

The contribution deals with general repair models for repairable system. Various ways of modeling the impact of repairs on a system condition are examined. The most common is to assume the repairs impact the failure intensity following a virtual age process proposed by Kijima. Another option considers repairs performing the time-dependent scale transformation of the governing distribution function. Comparison of these two concepts is given.

1. Repairable system

Consider a sequence of positive random variables $\{T_i\},\ i = 1, 2, \dots$ which is interpreted as the sequence of operation times of repairable system (machine) with instantaneous repair upon failure. Lets call this sequence the repair process. We assume that all random elements are defined over the underlying probability space $(\Omega, \mathcal{A}, P)$. Let $F_i(t), t \in [0, \infty)$ denote the distribution function (DF) of $T_i,\ i = 1, 2, \dots$ and assume that it is absolutely continuous. Given that we can define the point process of instants of failures $\{N(t),\ t \geq 0\}$ like $N(t) = \sup\{n \geq 0 : S_n \leq t\}$, where $S_n = \sum_{i=1}^{n} T_i,\ n \geq 1, S_0 = 0$ is the partial sum process which represents the time elapsed since the system has been put in operation. Except the DF of T_i we will often characterize i^{th} operation time T_i (time to failure of the system after $(i-1)^{\text{th}}$ repair) also by hazard rate (failure intensity, risk function)

$$h_i(t) = \lim_{\epsilon \to 0_+} \frac{P(t \leq T_i \leq t + \epsilon | T_i \geq t)}{\epsilon} = \frac{f_i(t)}{1 - F_i(t)}, \quad t \geq T_{i-1}.$$

The convenient mathematical description of repair process uses also a concept of failure intensity $h(t)$ which is defined considering the foregoing notation as follows

$$h(t) = \sum_{i=1}^{\infty} h_i(t) I(T_{i-1} \leq t < T_i).$$

We can assume that random variables $\{T_i\},\ i = 1, 2, \dots$ are independent as in renewal processes or there is some stochastic dependency between operation times. In engineering applications it means that the degree of repair on a current cycle and its duration are dependent on the history of the repair process. This contribution deals with this case, which is also more general. Finally, we focus on the repair which is not perfect so the "quality" of the repaired system is worse than the system's "quality" at the beginning of the previous cycle (*degrading system*). It is reasonable to assume for probabilistic modeling of this situation that T_i are stochastically decreasing with i:

$$T_{i+1} \leq_{st} T_i \Leftrightarrow \bar{F}_{i+1}(t) \leq \bar{F}_i(t),\ i = 1, 2, \ldots, \forall t \in [0, \infty),$$

where $\bar{F}_i(t) \equiv 1 - F_i(t)$ denotes the survival probability of i^{th} operation time (lifetime of the system after $(i-1)^{\text{th}}$ repair).

2. Lifetime aging property of a system

Consider a standard renewall process. Assume that the generic DF $F(t)$ is IFR (increasing failure rate), which means that the corresponding failure rate $h(t)$ is not decreasing. Therefore $F(t)$ is an aging distribution. What can be said about the aging properties of the renewal process? It is reasonable to conclude, that as the repair is perfect, there is no aging in this process, as after each perfect repair the age of the system is 0. Thus, the perfect repair does not lead to accumulation of damage in the described sense[1]. But this is not so when the repair is not perfect, which is definitely the case in most technical systems.

Aging property of a system is usually modeled by considering some additional information about the state of system's wear (damage). For instance, deterioration of a system can be modeled by a single, predictible, increasing stochastic process with independent increments $W_t,\ t \geq 0$, e.g. gamma process. Using this additional information about deterioration we can define the lifetime of the system as $T = \inf\{t \in \mathcal{R}_+\ :\ W_t \geq S\}$, i.e., as the first time the damage hits a given level S. Here S can be a constant or, more general, a random variable independent of the damage process (*damage threshold models*).

Other option is to consider the system a subject to shocks that occur from time to time and add a random amount to the damage. The successive times of occurrence of shocks, S_n, are given by an increasing sequence $0 < S_1 \leq S_2 \leq ...$ of random variables. Each time point S_n is associated with a real-valued random mark V_n, which describe the additional damage caused by the n^{th} shock. From this marked point process $(S, V) = (S_n, V_n),\ n \in \mathcal{N}$

[1] We should be careful and not to confuse the accumulated hazard function (CHR) $H(t) = \int_0^t h(s)ds$ of repairable system by its cumulated damage (CD). The CHR is an increasing function reflecting the expectation of system's failures up to specific time (in the case of homog. Poisson process: $h(t) = \lambda,\ t \in [0, \infty)$ and $H(t) = \lambda t,\ t \in [0, \infty)$), in contrast of CD which could also decrease (typically after a perfect repair).

the corresponding compound point process X with $X_t = \sum_{n=1}^{\infty} I(S_n \leq t)V_n$ is derived, which describes the accumulated damage up to time t.

3. Modeling the impact of repairs

3.1. Virtual age concept

Suppose that the system has the virtual age $V_{n-1} = y$ immediately after the $(n-1)^{\text{th}}$ repair, then we can model the survival function of the n^{th} failure-time T_n by survival function $\bar{F} = 1 - F$ of a new system as follows

$$\bar{F}_n(t) \equiv \bar{F}(t|V_{n-1} = y) = \frac{\bar{F}(t+y)}{\bar{F}(y)}.$$

The DF of the n^{th} failure-time T_n then has the form

$$F_n(t) \equiv P(T_n \leq t|V_{n-1} = y) = \frac{F(t+y) - F(y)}{1 - F(y)}.$$

A general repair is represented as a sequence of random variables ξ_n taking values between 0 and 1, which denotes the degree of repairs. The case $\xi_n = 1$ denotes minimal repair and $\xi_n = 0$ means a perfect repair. There are two models for virtual age process V_n (the age of the system at time S_n which correspond to its actual wear) depending on how the repair affects the state of a system [1]: $V_n = V_{n-1} + \xi_n T_n$ (MI), $V_n = \xi_n(V_{n-1} + T_n)$ (MII).

Suppose that our system starts working with an initial presribed failure rate $h_1(t) = h(t)$. Then using the concept of virtual age the failure intensity during the $(n+1)^{\text{th}}$ sojourn is determined by

$$h_{n+1}(t) = h(t - S_n + V_n), \quad S_n \leq t < S_{n+1}, n \geq 0.$$

It is easy to see that in both models $N(t)$ (or $\{S_n\}_0^\infty$ [2]) is a non-homogeneous Poisson process when $\xi_n = 1$ for all $n \geq 1$ and is a renewal process when $\xi_n = 0$ for all $n \geq 1$. It is also worth noting that minimal repair does not reduce wear ($h_n(t) = h(t)$ for all $n \geq 1$). Situation when $\xi_n = \xi$ $(0 \leq \xi \leq 1)$ for all $n \geq 1$ in model MI is examined in [2]. Other than these, the process $\{S_n\}$ is not Markov and its analytical evaluation for a general repair is therefore very difficult ([1]).

3.2. Time transformation concept

In [3] author uses the concept of AML to define the non-ideal repair of the repair process as the repair that changes the DF of the forthcoming cycle according to the following model. Let $F_0(t)$ be some reference DF. Then

[2]This process may be called the real age process, as S_n is the elapsed time till the n^{th} failure since the system was put in operation

sequence $\{F_i(t)\},\ i = 1, 2, \ldots$ is defined as

$$
\begin{aligned}
F_1(t) &= F_0(W_0(t)), \\
F_2(t) &= F_1(W_1(t)) = F_0(W_0(W_1(t))), \\
&\ldots \\
F_i(t) &= F_{i-1}(W_{i-1}(t)) = \ldots = F_0(W_0(W_1(\ldots(W_{i-1}(t))\ldots))),
\end{aligned}
$$

where $W_i(t)$ is monotonically increasing, $W_i(t) \to \infty$ as $t \to \infty$ and moreover continuous and differentiable in $[0, \infty)$: $W_i(t) = \int_0^t w_i(u)du;\ w_i(t) > 0;\ t \in [0, \infty)$. ($W_i(t)$ can be interpreted as a measure of relative degradation and $w_i(t)$ as speed of this deterioration).

There are two ways how we can model stochastic decreasing of T_i (deterioration on each stage) in this model:

1. $T_{i+1} \leq_{st} T_i \Leftrightarrow W_i(t) \geq t,\ i = 1, 2, \ldots,\ \forall t \in [0, \infty)$
2. $\bar{F}_0(\tilde{W}_i(t)) \leq \bar{F}_0(\tilde{W}_{i-1}(t)),\ i = 1, 2, \ldots,\ \forall t \in [0, \infty) \Rightarrow \tilde{W}_i(t) \geq \tilde{W}_{i-1}(t),\ i = 1, 2, \ldots,\ \forall t \in [0, \infty)$, where $\tilde{W}_{i-1}(t) = W_0(W_1(\ldots(W_{i-1}(t))\ldots))$

New relative characteristics, which show more specifically (than the general condition $W_i(t) \geq t$) the pattern of the corresponding deterioration, were also derived ([3]).

3.3. Comparison

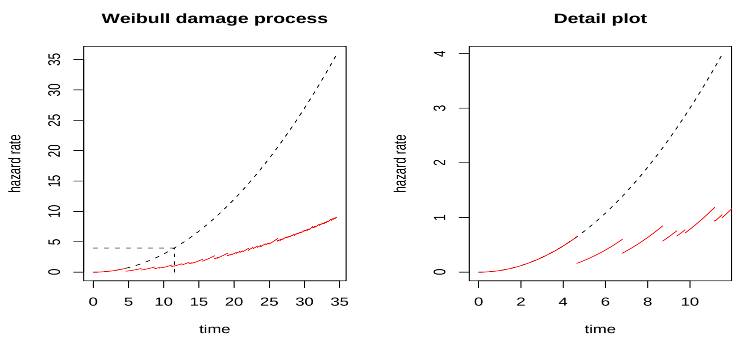

Figure 1: Hazard rate (solid discont. line) of Weibull damage process (WDP) with constant repair $\xi = 0.5$ simulated according to Kijima virtual age model of type I. Number of simulations is $n = 100$, weibull shape parameter $\alpha = 3$ and weibull scale parameter $\lambda = 0.01$. Dashed line denotes hazard rate of WDP with the same parameters but with minimal repair, which is represented by hazard rate of simple weibull distribution, in this case $h(t) = 0.03t^2$.

The main advantage of virtual age concept proposed by Kijima is his intuitiveness, straightforward interpretation of the impact of various repair's degrees on a system and simple form of each cycle's hazard rate. It is easy to see that hazard rate of any operation time is hazard rate of a new system simply shifted according to corresponding virtual age (Fig. 1). DF of each operation time can be derived using the DF of a new system and virtual age process up to last failure (repair). Also, another asset of this concept is its utility for modeling preventive maintenance actions (preventive repair reduces system's virtual age and consecutively also system's hazard rate).

Multhiphase ALM is usually used for describing the non-ideal repair model with deteriorating cycles. This model is useful especially for modeling the case of aging accelerated after each failure's repair. Hazard rates of operation times have more complicated form than hazard rates in the previous case ($h_i(t) = w_{i-1}(t)h_{i-1}(W_{i-1}(t))$), but they are more flexible. DF of each operation time can be derived using the DF of a new system and the sequence $\{W_i(t)\}$ of relative degradation measures. Preventive repairs are not commonly modeled by this concept. It is worth to mention that under some additional assumptions (special case) one can create link between multhiphase AML and the virtual age concept [3].

4. Conclusion

The main purpose of the paper was to describe and to discuss two main approaches of modeling the impact of a general repair on a repairable system. So-called general repair denotes a repair which lies between two well-known boundary cases, minimal repair and perfect repair. Analyzing this situation can help us to control (to slower) the growth of the system's damage and to prolong the lifetime of the device in a general case.

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