Resampling methods for the change analysis of dependent data

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Abstract

We study an AMOC-model with an abrupt change in the mean and dependent errors that form a linear process. Approximations of the critical values for change-point tests are obtained through permutation methods. The theoretical results show that the original test statistics and their corresponding block permutation counterparts follow the same distributional asymptotics. Some simulation studies illustrate that the permutation tests usually behave better than the original tests if performance is measured by α- and β-errors.

1. Change-point analysis

In our daily lives we encounter changes – or the possibility thereof – frequently and everywhere in such diverse fields as economics, finance, medicine, geology, climate, physics and so on. Therefore the detection, location and investigation of changes is of particular interest.

Change-point analysis provides the statistical tools to decide whether a given (ordered) data set remains stable over time or whether it follows a certain model up to an unknown time-point and a different model afterwards. Usually that means that we observe a stochastic process with certain parameters that change somewhere during the observational period.

The questions arising in that context are testing, estimation of the change-point and confidence intervals for it. Moreover one distinguishes between off-line procedures, where one already has observed a complete sequence, and on-line or sequential procedures, where new data arrives steadily.

In this talk we will focus on the off-line testing problem for the at-most-one-change (AMOC) location model, i.e. a model with one or no change in the mean, and dependent errors. Critical values for such testing procedures are usually obtained by distributional asymptotics. These critical values, however, do not sufficiently reflect dependency. Moreover it is a well-known fact that convergence rates especially for extreme-value statistics are very slow. Using resampling methods we obtain better approximations, which take possible dependency structures more efficiently into account.

We consider the following AMOC location model

$$X_i = \mu + d\,1_{\{i>m^*\}} + e(i), \quad 1 \leqslant i \leqslant n, \tag{1}$$

where the errors $\{e(i), 1 \leqslant i \leqslant n\}$ are given by the linear process

$$e(i) = \sum_{j \geqslant 0} w_j \, \epsilon(i-j)$$

and $m^* = [\vartheta n]$, $d = d(n)$ may depend on n; m^* is called the change-point. We are interested in testing the null hypothesis of "no change"

$$H_0: \quad m^* = n$$

against the alternative of a change in the mean

$$H_1: \quad 1 \leqslant m^* < n \text{ and } d \neq 0.$$

Moreover we assume that the innovations $\{\epsilon(i) : -\infty < i < \infty\}$ are i.i.d. random variables with

$$\mathrm{E}\,\epsilon(i) = 0, \quad 0 < \sigma^2 = \mathrm{E}\,\epsilon(i)^2 < \infty, \quad \mathrm{E}\,|\epsilon(i)|^\nu < \infty \text{ for some } \nu > 2. \qquad (2)$$

We suppose that the weights $\{w_j : j \geqslant 0\}$ satisfy

$$\sum_{j \geqslant 0} w_j \neq 0, \quad \sum_{j \geqslant 0} \sqrt{j}\,|w_j| < \infty. \qquad (3)$$

The theory for the above model with i.i.d. errors has been well developed. In that situation test statistics are usually based on the partial sums $S_m = \sum_{i=1}^m (X_i - \overline{X}_n)$ where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. It turns out that these statistics work also very well in the present setup (confer e.g. Antoch et al. [2]).

In this talk we focus on the classical CUSUM (cumulative sum) statistic

$$T_n(X_1, \ldots, X_n) = \max_{1 \leqslant m < n} \left(\frac{1}{\sqrt{n}} \, |S_m| \right),$$

yet the assertions remain true for a much larger class of statistics, for details we refer to Kirch [3, 4].

To give an idea on how the statistic works, Figure 1 gives a time series (i.i.d. errors) with a small change in the mean at time 70 and the corresponding cumulative sums. The dotted line gives the asymptotic critical value for the level 0.05.

Note that the time at which the maximum is obtained is very close to the actual change-point - and in fact one can prove that this is a good estimator for the change-point.

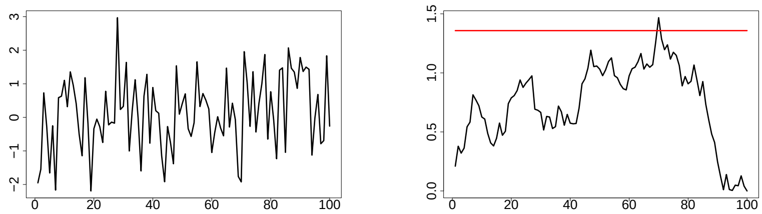


Figure 1: I.i.d. sequence with a change in the mean at 70 and its cumulative sums

2. Asymptotic results and bootstrap

The following theorem gives the asymptotic distribution of the CUSUM statistic under the null hypothesis, which yields asymptotic critical values.

Theorem 1 *Assume that (1) - (3) and H_0 holds. Then*

$$\frac{1}{\widehat{\tau}} T_n(X_1, \ldots, X_n) \xrightarrow{\mathcal{D}} \sup_{0<t<1} |B(t)| \qquad \text{as } n \to \infty,$$

where $\{B(t) : 0 \leqslant t \leqslant 1\}$ *is a Brownian bridge,* $\widehat{\tau} - \tau = o_P(1)$ *and* $\tau^2 = \sigma^2 \left(\sum_{s \geqslant 0} w_s\right)^2$.

Proof. Confer Theorem 2.1 in Antoch et al. [2]. ■

The convergence of the asymptotic critical values is known to be very slow (especially for extreme value statistics such as the weighted CUSUM statistic). Moreover for some other statistics the asymptotic distribution depends on unknown parameters. Finally some simulations suggest that the asymptotic critical values do not sufficiently take the dependency structure into account.

This is why we are proposing resampling procedures to obtain critical values. We split the observation sequence of length n into L sequences of length K (i.e. $n = KL$). Then, we permute the blocks $X_{Kl+1}, \ldots, X_{K(l+1)}$, $l = 0, \ldots, L-1$, and compute the statistic using the permuted blocks (we keep the order of $X(\cdot)$ within the blocks). The idea is that the block contains enough information about the dependency structure so that the estimate is close to the null hypothesis.

We assume in the following that $L \to \infty$ and $K = K(L) \to \infty$, $n = n(L) = KL$, $K/L = O(1)$.

The distribution of the statistic of such a bootstrap sample given the observations can easily be computed using B independent permutations, B large. The next theorem shows that this conditional resampling statistic and the original statistic follow the same distributional asymptotics. This is true under the null hypothesis as well as alternatives, even though we did not alter the alternative observation sequence (by say using change-point estimators). Thus we can use the conditional distribution of the resampling statistic to obtain critical values for the test.

Theorem 2 *Assume that* $\{X_i : 1 \leqslant i \leqslant n\}$ *fulfills (1) - (3) with* $\nu > 4$. *Additionally, let the error sequence be sufficiently fast alpha-mixing - for details confer Kirch [3, 4]. Then, for all* $x \in \mathbb{R}$ *as* $L \to \infty$,

$$P\left(\frac{T_{L,K}(\boldsymbol{\pi}, X)}{\widehat{\tau}_{LK}} \leqslant x \middle| X_1, \ldots, X_n\right) \to P\left(\sup_{0<t<1} |B(t)| \leqslant x\right) \quad a.s.,$$

where $\{B(t) : 0 \leqslant t \leqslant 1\}$ *denotes a Brownian bridge,*

$$T_{L,K}(\boldsymbol{\pi}, X) = T_n(X_{K(\pi_1-1)+1}, \ldots, X_{K\pi_1}, X_{K(\pi_2-1)+1}, \ldots, X_{K\pi_L}),$$

$$\widehat{\tau}^2_{LK} = \frac{1}{KL} \sum_{l=0}^{L-1} \left[\sum_{k=1}^{K} (X_{Kl+k} - \overline{X}_n)\right]^2.$$

$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_L)$ *is a random permutation of* $(1, \ldots, L)$ *independent of* $\{X_i\}$.

Proof. Confer Theorem 3.8 in Kirch [3]. ■

In case of independent errors K can be chosen equal to 1 (cf. Antoch and Hušková [1]), in this case the permutation test is exact.

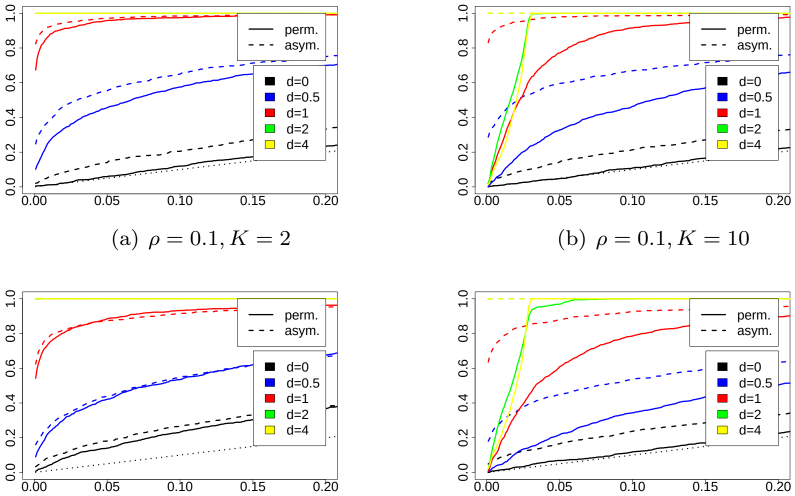
3. Selected simulations

In the previous section we have seen that the asymptotic as well as the resampling method yield asymptotically correct critical values. In this section we investigate their respective small sample behavior by some simulations.

The goodness of a test is essentially determined by its $\alpha-$ resp. $\beta-$ errors. To visualize this properties we create size-power curves, which plot the empirical distribution function of the respective p-values (thus the empirical size and power of the test). So the plot should be on the diagonal for the null hypothesis and as close to one as possible for alternatives.

Some results can be found in Figure 2, we use AR(1) errors ($e(i) = \rho e(i-1) + \epsilon(i)$) with standard normally distributed innovations $\{\epsilon(\cdot)\}$, the observation length is $n = 80$ with a change-point at $m = 40$. As variance estimator $\widehat{\tau}$ for the asymptotic test we use the Bartlett type estimator which has performed best in Antoch et al. [2].

The simulations show that the block permutation test behaves better than the asymptotic test for an appropriately chosen block-length K. For example, the actual level of the asymptotic test is almost 0.2 when we have chosen a nominal level of 0.1. The permutation level on the other hand equals



(a) $\rho = 0.1, K = 2$ (b) $\rho = 0.1, K = 10$

(c) $\rho = 0.3, K = 2$ (d) $\rho = 0.3, K = 10$

Figure 2: Size-power curves

the nominal level for appropriately chosen K. The β-errors of both tests are comparable if one takes the actual level into account.

Concerning the choice of the block-length we see that the block-length should be chosen longer the more dependent the data sequence is. On the other hand choosing it too large will not influence the level of the test, however slightly increase the β-errors. Hence, if we are uncertain about the degree of dependency in the sample we should rather choose the block-length somewhat larger.

4. Bibliography

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