Loss Rate Asymptotics

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Abstract

We consider a Lévy process $\{S_t\}$ which is reflected at 0 and $K > 0$. The reflected process $\{V_t^K\}$ is given as a solution to a Skorokhod problem, which implies a representation $V_t^K = V_0 + S_t + L_t^0 - L_t^K$, where $\{L_t^0\}$ and $\{L_t^K\}$ are the local times at 0 at K, respectively. The regenerative structure of $\{V_t^K\}$ yields a stationary distribution denoted π_K and the loss rate is defined as the mean of L_1^K in the stationary situation. The loss rate was studied in Asmussen & Pihlsgård [1], where it was expressed in terms of the characteristic triplet of $\{S_t\}$ and π_K, and asymptotics of the loss rate as $K \to \infty$ was derived in the case of negative drift and light tails. Asymptotics for positive drift is straightforward by reversing the role of the barriers 0 and K and using a conservation law.

We use the expression for the loss rate from [1] to derive asymptotics in the case of negative drift and heavy tails, as well as in the case of zero drift. In the zero drift case, functional limit theorems (with a Brownian or stable limit) play an important role and are based on continuity properties of the loss rate.

1. Introduction

In this paper, we consider a Lévy process $\{S_t\}$ reflected at 0 and at $K > 0$. The reflected process $\{V_t^K\}$ along with the local times at the boundaries $\{L_t^0\}$ and $\{L_t^K\}$ are given as a solution of a Skorokhod problem, and we obtain a representation

$$V_t^K = V_0 + S_t + L_t^0 - L_t^K.$$

We may view $\{V_t^K\}$ as a regenerative process, by defining cycles starting from 0, going to K and then returning to 0, and this regenerative structure yields a stationary distribution π_K.

The loss rate is defined as

$$\ell^K = \mathbb{E}_{\pi_K} L_1^K \tag{1}$$

where $\mathbb{E}_{\pi_K}$ refers to the stationary situation. The loss rate can be interpreted as the overflow rate in a dam model, or the bit loss rate in a finite data buffer. Our primary concern is asymptotic expressions for ℓ^K, as K tends to infinity. Such expressions have already been developed in several cases. In the case with positive drift in [1]. When the drift is negative, the form of the asymptotic expressions of ℓ^K depends on whether the right tail of S_1 is light- or heavy-tailed. In [1], an expression for ℓ^K in terms of π_K and the characteristic triplet of $\{S_t\}$ and an asymptotic expression for the light-tailed case is derived . We use this expression in to derive an asymptotic expression for ℓ^K in the heavy-tailed case. Furthermore we prove a continuity result, and use this to derive the asymptotics in case of zero drift. It turns out that the existence of a second moment determines the asymptotics in this case.

2. Preliminaries

In this section we present some standard results on Lévy processes and heavy tails, as well as the two-sided Skorokhod problem. We use the notation $a_K \sim b_K$ $K \to \infty$ for $\lim_{K\to\infty} a_K/b_K = 1$

2.1. Lévy processes

A *Lévy process* $\{S_t\}$ (w.r.t. a filtration $\mathcal{F}_t$) is an $\mathcal{F}_t$-adapted stochastic process on $\mathbb{R}^d$ with stationary $\mathcal{F}_t$-independent increments which is continuous in probability and such that $X_0 = 0$ $P - a.s.$ Throughout the paper, $d = 1$. We use the cadlag version of $\{S_t\}$ which exists as a consequence of the continuity in probability. Furthermore, we define $\Theta := \{\alpha \in \mathbb{C} \mid \mathbb{E}e^{\Re(\alpha)S_1} < \infty\}$ and let $\kappa(\cdot)$ denote the Lévy exponent, that is, the unique function $\kappa : \Theta \to \mathbb{C}$ such that $\kappa(0) = 0$ and $\mathbb{E}\mathrm{e}^{\alpha X_t} = \mathrm{e}^{t\kappa(\alpha)}$ $\alpha \in \Theta$. By the Lévy -Khintchine representation there exists $\theta \in \mathbb{R}$, $\sigma \geq 0$ and a measure ν (the Lévy measure) on $\mathbb{R} - \{0\}$ with $\int_{-\infty}^{\infty}(1 \wedge y^2)\nu(\mathrm{d}y) < \infty$ such that:

$$\kappa(\alpha) = \theta\alpha + \frac{\sigma^2\alpha^2}{2} + \int_{-\infty}^{\infty} \left[\mathrm{e}^{\alpha x} - 1 - \alpha x I(|x| \leq 1)\right] \nu(\mathrm{d}x),$$

and we refer to (θ, σ, ν) as the characteristic triplet. We assume throughout $\mathbb{E}|S_t| < \infty$.

2.2. Heavy Tails

The class of subexponential distributions $\mathcal{S}$ is defined by the requirement, that for B, a distribution on $[0, \infty)$, we have $B \in \mathcal{S}$ iff

$$\lim_{x\to\infty} \frac{\overline{B^{*n}}(x)}{\overline{B}(x)} = n \qquad n = 2, 3, \cdots .$$

The notion of heavy-tailedness carries over to Lévy processes through Theorem 1 in [3], which states that if we assume ν is tail equivalent to a subexponential distribution, that is $\overline{\nu}(x) := \int_x^{\infty} \nu(\mathrm{d}y) \sim \overline{B}(x)$ for $B \in \mathcal{S}$. Then setting $\overline{F}(x) := \mathbb{P}(S_1 > x)$, this assumption yields

$$\overline{F}(x) \sim \overline{\nu}(x) \tag{2}$$

An important subclass introduced in [5] of $\mathcal{S}$ is $\mathcal{S}^*$, where we require that the mean μ of B is finite and

$$\lim_{x\to\infty}\int_0^x \frac{\bar{B}(x-y)}{\bar{B}(x)}\bar{B}(y)\mathrm{d}y = 2\mu$$

2.3. The Skorokhod Problem

The following proposition characterizes the reflected process and the local times at the boundaries as a solution to a so-called Skorokhod problem. It may be be proved in the same way as Prop. 2.2. p. 251 in [2].

Proposition 2.1
Let $\left\{L_t^{0,*}\right\}$ *and* $\left\{L_t^{K,*}\right\}$ *be any non-decreasing right-continuous processes such that the process* $\left\{V_t^*\right\}$ *given by* $\left\{V_0^*\right\} = x, V_t^* = S_t + L_t^{0,*} - L_t^{K,*}$ *satisfies* $0 \le V_t^* \le K$ *for all* t, $\int_0^T V_t^*\,\mathrm{d}L_t^{0,*} = 0\ \forall T$ *and* $\int_0^T (K - V_t^*)\,\mathrm{d}L_t^{K,*} = 0\ \forall T$ *then* $L_t^{0,*}(x) = L_t^0(x)$, $L_t^{K,*}(x) = L_t^K(x)$ *and* $V_t^* = V_t(x)$.

3. Loss Rate Asymptotics

Loss rate asymptotics in the cases $\mathbb{E}S_1 > 0$ and $\mathbb{E}S_1 < 0$ and light tails were derived in [1], thus we are left with the cases $\mathbb{E}S_1 < 0$ and heavy tails, and $\mathbb{E}S_1 = 0$. Both cases are treated using the following expression for the loss rate, which is derived in [1]:

$$\ell^K = \frac{\mathbb{E}S_1}{K}\int_0^K \overline{\pi}_K(x)\,\mathrm{d}x + \frac{\sigma^2}{2K} + \frac{1}{2K}\int_0^K \pi_K(\mathrm{d}x)\int_{-\infty}^{\infty}\varphi_K(x,y)\nu(\mathrm{d}y), \quad (3)$$

where

$$\varphi_K(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y \le -x \\ y^2 & \text{if } -x < y < K-x \\ 2y(K-x) - (K-x)^2 & \text{if } y \ge K-x \end{cases} \quad (4)$$

3.1. Negative drift and heavy tails

The loss rate defined above has an analogue in the discrete time case, where we consider an i.i.d. sequence (X_n) of random variables with distribution function F, and the associated random walk instead of a Lévy process. It was investigated in [4], and we have that if $\overline{F}(x) \sim \overline{B}(x)$ for some distribution $B \in \mathcal{S}^*$ and ℓ^K is the loss rate at K of the associated random walk, reflected in 0 and K as defined by (1) above, then

$$\ell^K \sim \int_K^\infty \overline{F}(y)\,\mathrm{d}y, \qquad K \to \infty. \quad (5)$$

In view of (5) and (2) it appears reasonable to investigate if perhaps

$$\ell^K \sim \int_K^\infty \overline{\nu}(y)\,\mathrm{d}y, \qquad K \to \infty,$$

And this is indeed the case if one of the following regularity conditions is fulfilled.

1. $\mathbb{E}S_1^2 < \infty$ and $\int_K^\infty \overline{\nu}_I(y)\,\mathrm{d}y / \overline{\nu}_I(K) \in O(K)$
2. $\overline{\nu}(K) \sim L(K)K^{-\alpha}$ where L is a locally bounded slowly varying function and $1 < \alpha < 2$

This assertion is proved using careful analysis of (5).
In order to derive the loss rate asymptotics in the case $\mathbb{E}S_1 = 0$, we need the following continuity, which is of independent interest.

3.2. Continuity of the loss rate

Let us consider a sequence of Lévy processes $\{S_t^{(n)}\}_{n\geq 1}$, such that $\{S_t^{(n)}\} \xrightarrow{w} \{S_t^{(0)}\}$, where $\xrightarrow{w}$ denotes weak convergence of processes in the Skorokhod topology. What, if any, regularity conditions are required to ensure weak convergence of the associated stationary distributions as well as $\ell^{K(n)} \to \ell^{K(0)}$? It turns out, that no additional regularity conditions are required for weak convergence of the associated stationary distributions, whereas uniform integrability of $\{S_1^{(n)}\}_{n\geq 1}$ ensures $\ell^{K(n)} \to \ell^{K(0)}$.

3.3. Zero drift

Using the continuity result mentioned above allows us to derive asymptotics in the case of zero drift. By proper scaling of the involved process, we have weak convergence towards a Brownian motion or a stable process, and since the loss rate of the latter processes is known, we get an asymptotic result. To be precise, if $\{S_t\}$ is square integrable, then

$$\ell^K \sim \frac{1}{2K}\int_{-\infty}^\infty y^2 \nu(dy) + \frac{\sigma^2}{2K} \qquad K \to \infty$$

whereas, if $\{S_t\}$ is a Lévy process with characteristic triplet (θ, σ, ν), and we assume that for some $1 < \alpha < 2$, there exists c and d such that

$$\lim_{x\to\infty} \overline{\nu}(x) \sim dx^{-\alpha} \qquad \lim_{x\to\infty} \nu(-x) \sim cx^{-\alpha}$$

Then, setting $\beta := (d-c)/(c+d)$ and $\rho = 1/2 + (\pi\alpha)^{-1}\arctan(\beta\tan(\pi\alpha/2))$ we have:

$$\ell^K \sim \frac{cB(2-\alpha\rho, \alpha\rho) + dB(2-\alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))(\alpha-1)(2-\alpha)} \frac{1}{K^{\alpha-1}} \qquad K \to \infty.$$

Where $B(\cdot,\cdot)$ is the Beta function.

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