Statistical problems in a discrete time random field HJM type interest rate model

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Abstract

We consider a discrete time Heath-Jarrow-Morton (HJM) type forward interest rate model, where the interest rate curves are driven by a geometric spatial autoregression field. Our aim is to test the autoregression parameter ϱ. We study stable ($|\varrho| < 1$) and unstable ($|\varrho| = 1$) cases. We study strong consistency of the maximum likelihood estimator of ϱ, and local asymptotic normality (LAN) of the sequence of the related statistical experiments. The main gain of the second result is that we obtain at once asymptotically optimal tests.

1. Introduction

Our aim is to consider some statistical questions arising in a HJM type interest rate model. Such models were proposed by Gáll, Pap and Zuijlen [3]. We will study a no-arbitrage discrete time forward interest rate curve model $\{f_{k,\ell}^{(\varrho)} : k, \ell \geq 0\}$ given for each parameter $\varrho \in \mathbb{R}$ by

$$\begin{cases} f_{k,\ell}^{(\varrho)} - f_{k-1,\ell+1}^{(\varrho)} - \varrho(f_{k,\ell-1}^{(\varrho)} - f_{k-1,\ell}^{(\varrho)}) = \eta_{k,\ell} + G_\ell(\varrho), & k, \ell \geq 1, \\ f_{k,0}^{(\varrho)} - f_{k-1,1}^{(\varrho)} = \eta_{k,0} + G_0(\varrho), & k \geq 1, \end{cases}$$

where $\{\eta_{k,\ell} : k \geq 1,\ \ell \geq 0\}$ are independent standard normal random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, $G_\ell(\varrho) := \frac{1}{2}\sum_{j=0}^{2\ell} \varrho^j$ for $\varrho \in \mathbb{R}$, and the initial values $\{f_{0,\ell}^{(\varrho)} : \ell \geq 0\}$ are given real numbers. Here $f_{k,\ell}^{(\varrho)}$ means the forward interest rate at time $k \geq 0$ with time to maturity date $\ell \geq 0$, i.e., it is supposed to hold for the time period $[k + \ell, k + \ell + 1)$.

The process $\{f_{k,\ell}^{(\varrho)} : k, \ell \geq 0\}$ is a spatial autoregressive process. It is called *stable, unstable*, or *explosive*, if $|\varrho| < 1$, $|\varrho| = 1$, or $|\varrho| > 1$, respectively.

The main goal of this paper is to test the autoregression parameter ϱ based on a sample $\mathbf{f}^{(\varrho)} := (f_{k,\ell}^{(\varrho)})_{1 \leq k \leq K,\, 0 \leq \ell \leq L}$. Of course, the difficulty is that the sample consists of nonindependent random variables. Moreover, no explicit formula is available for the maximum likelihood estimators of ϱ.

2. Strong consistency

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, $(d_n)_{n\in\mathbb{N}}$ a sequence of positive integers, and for each $n \in \mathbb{N}$ and $\theta \in \Theta$, let $\xi_n^{(\theta)} : \Omega \to \mathbb{R}^{d_n}$ be a random vector such that its distribution $\mathsf{P}_{n,\theta}$ on $\big(\mathbb{R}^{d_n}, \mathcal{B}(\mathbb{R}^{d_n})\big)$ is absolutely continuous with a density function $x \mapsto L_n(x;\theta)$ from $\mathbb{R}^{d_n}$ into $[0,\infty)$. Then $\Lambda_n(x;\theta) := \log L_n(x;\theta) \in [-\infty,\infty)$, $x \in \mathbb{R}^{d_n}$, is the loglikelihood function, where we put $\log 0 := -\infty$.

Assume that the parameter set $\Theta \subset \mathbb{R}^p$ is compact, and for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^{d_n}$, the likelihood function $\theta \mapsto L_n(x;\theta)$ is continuous on Θ. Then for each $n \in \mathbb{N}$, there exists a measurable maximum likelihood estimator $\widehat{\theta}_n : \mathbb{R}^{d_n} \to \Theta$ for the parameter, i.e. there exist a measurable function $\widehat{\theta}_n$ such that $L(x;\widehat{\theta}(x)) = \sup_{\theta\in\Theta} L(x;\theta)$ for all $x \in \mathbb{R}^{d_n}$.

Theorem 2.1 *Let $\theta_0 \in \Theta$. If there exists a sequence $(k_n)_{n\in\mathbb{N}}$ of real numbers with $\liminf_{n\to\infty} k_n > 0$, and for every $\theta \in \Theta \setminus \{\theta_0\}$, there exist a neighborhood $N(\theta,\theta_0)$ of θ and a quantity $I(\theta,\theta_0)$ such that $\inf_{\phi \in N(\theta,\theta_0)} I(\phi,\theta_0) > 0$ and*

$$\lim_{n\to\infty} \sup_{\phi \in N(\theta,\theta_0)} \left| \frac{1}{k_n}\big(\Lambda_n(\xi_n^{(\theta_0)};\phi) - \Lambda_n(\xi_n^{(\theta_0)};\theta_0)\big) + I(\phi,\theta_0)\right| = 0 \qquad a.s., \tag{2.1}$$

then every sequence $(\widehat{\theta}_n)_{n\in\mathbb{N}}$ of measurable maximum likelihood estimators is strongly consistent estimator of the true value $\theta_0 \in \Theta$ of the parameter, i.e.

$$\widehat{\theta}_n(\xi_n^{(\theta_0)}) \to \theta_0 \qquad a.s. \qquad as\ \ n \to \infty.$$

Theorem 2.2 *Let $\{K_n, L_n : n \in \mathbb{N}\}$ be positive integers such that $K_n = nK + o(n)$ and $L_n = nL + o(n)$ as $n \to \infty$ with some $K > 0$ and $L > 0$. Let $a, b \in \mathbb{R}$ with $-1 < a < b < +1$ and $\varrho_0 \in \Theta$ where*

$$\Theta = \begin{cases} [-1;b], & \text{if } \ \varrho_0 = -1, \\ [a;b], & \text{if } \ |\varrho_0| < 1, \\ [a;+1], & \text{if } \ \varrho_0 = +1, \end{cases}$$

Then

$$\sup_{\varrho\in[a,b]} \left| r_n \left(\Lambda_{K_n,L_n}\left(\mathbf{f}_n^{(\varrho_0)};\varrho\right) - \Lambda_{K_n,L_n}\left(\mathbf{f}_n^{(\varrho_0)};\varrho_0\right)\right) + I(\varrho,\varrho_0)\right| \to 0 \qquad a.s.$$

as $n \to \infty$, where

$$r_n(\varrho_0) = \begin{cases} n^{-3}, & \text{if } \ \varrho_0 = -1, \\ n^{-2}, & \text{if } \ |\varrho_0| < 1, \\ n^{-6}, & \text{if } \ \varrho_0 = +1, \end{cases}$$

$$I(\varrho,\varrho_0) := \begin{cases} \frac{(1+\varrho)^2}{2} \int\limits_0^K \left(\int\limits_0^L t \,\mathrm{d}t \right) \mathrm{d}s & \textit{if} \;\; \varrho_0 = -1, \\ \frac{KL(\varrho_0-\varrho)^2}{2(1-\varrho_0^2)} + \frac{K(K+2L)(\varrho_0-\varrho)^2(2-\varrho_0-\varrho)^2}{16(1-\varrho_0)^4(1-\varrho)^2} & \textit{if} \;\; |\varrho_0| < 1, \\ \frac{(1-\varrho)^2}{8} \left[\int\limits_0^K \left(\int\limits_0^L t^4 \,\mathrm{d}t \right) \mathrm{d}s + \int\limits_0^K \frac{1}{s} \left(\int\limits_L^{L+s} t^2 \,\mathrm{d}t \right)^2 \mathrm{d}s \right] & \textit{if} \;\; \varrho_0 = +1, \end{cases}$$

Moreover, for every $\varrho \in [a,b] \setminus \{\varrho_0\}$ *and for every neighborhood* N *of* ϱ *with* $\varrho_0 \notin \overline{N}$ *(where* $\overline{N}$ *denotes the closure of* N*) we have* $\inf_{\phi \in N} I(\phi,\varrho_0) > 0$.

Consequently, the assumptions of Theorem 2.1 hold, hence both in stable and unstable cases, every sequence of measurable maximum likelihood estimators is strongly consistent.

3. Local asymptotic normality

If $(\Omega,\mathcal{A},\mathsf{P})$ is a probability space, $\xi_n : \Omega \to \mathbb{R}^{d_n}$, $n \in \mathbb{N}$, are random vectors and μ is a probability measure on $\big(\mathbb{R}^{d_n}, \mathcal{B}(\mathbb{R}^{d_n})\big)$, then we write $\xi_n \stackrel{\mathcal{L}(\mathsf{P})}{\longrightarrow} \mu$ if the distribution of ξ_n converges weakly to μ. The notation $o_{\mathsf{P}}(1)$ will be a short for a sequence of random vectors that converges to zero in probability.

If $\Theta \subset \mathbb{R}^p$ is an open set, $(\Omega,\mathcal{A},\mathsf{P})$ is a probability space, $\xi_n^{(\theta)} : \Omega \to \mathbb{R}^{d_n}$, $n \in \mathbb{N}$, $\theta \in \Theta$, are random vectors with distribution $\mathsf{P}_{n,\theta}$ on $\big(\mathbb{R}^{d_n}, \mathcal{B}(\mathbb{R}^{d_n})\big)$, then the sequence $\big(\mathbb{R}^{d_n}, \mathcal{B}(\mathbb{R}^{d_n}), \{\mathsf{P}_{n,\theta} : \theta \in \Theta\}\big)_{n\in\mathbb{N}}$ of statistical experiments is called *locally asymptotically normal at a point* $\theta \in \Theta$ if there exist (scaling) matrices $r_{n,\theta} \in \mathbb{R}^{p\times p}$, $n \in \mathbb{N}$, a nonzero, nonnegative definite symmetric (Fisher information) matrix $I_\theta \in \mathbb{R}^{p\times p}$, and measurable functions (statistics) $\Delta_{n,\theta} : \mathbb{R}^{d_n} \to \mathbb{R}^p$, $n \in \mathbb{N}$, such that $r_{n,\theta} \to 0$ as $n \to \infty$, and

$$\log \frac{\mathrm{d}\mathsf{P}_{n,\theta+r_{n,\theta}h_n}}{\mathrm{d}\mathsf{P}_{n,\theta}}(\xi_n^{(\theta)}) = h_n^\top \Delta_{n,\theta}(\xi_n^{(\theta)}) - \frac{1}{2} h_n^\top I_\theta \, h_n + o_{\mathsf{P}}(1) \qquad \text{as} \;\; n \to \infty,$$

$$\Delta_{n,\theta}(\xi_n^{(\theta)}) \stackrel{\mathcal{L}(\mathsf{P})}{\longrightarrow} \mathcal{N}(0, I_\theta) \qquad \text{as} \;\; n \to \infty,$$

whenever $h_n \to h$ with $h_n \in \mathbb{R}^p$, $n \in \mathbb{N}$, $h \in \mathbb{R}^p$.

Theorem 3.1 *Let* K_n, L_n, $n \in \mathbb{N}$, *be positive integers such that* $K_n = nK + o(n)$ *and* $L_n = nL + o(n)$ *as* $n \to \infty$ *with some* $K > 0$ *and* $L > 0$. *For each* $n \in \mathbb{N}$ *and* $\varrho \in \mathbb{R}$, *let* $\mathsf{P}_{n,\varrho}$ *be the distribution of the sample* $\mathbf{f}_n^{(\varrho)} := (f_{k,\ell}^{(\varrho)})_{1\le k\le K_n,\, 0\le \ell \le L_n}$ *on* $\big(\mathbb{R}^{K_n(L_n+1)}, \mathcal{B}(\mathbb{R}^{K_n(L_n+1)})\big)$. *Then the sequence* $\big(\mathbb{R}^{K_n(L_n+1)}, \mathcal{B}(\mathbb{R}^{K_n(L_n+1)}), \{\mathsf{P}_{n,\varrho} : \varrho \in \mathbb{R}\}\big)_{n\in\mathbb{N}}$ *of statistical* *experiments is locally asymptotically normal at each point* $\varrho \in [-1, +1]$ *with*

$$r_{n,\varrho} = \begin{cases} n^{-2} & \text{if } \varrho = -1, \\ n^{-1} & \text{if } |\varrho| < 1, \\ n^{-3} & \text{if } \varrho = +1, \end{cases}$$

$$I_\varrho = \begin{cases} \frac{1}{4}\int_0^K \left(\int_0^L t^2 dt\right) \mathrm{d}s + \frac{1}{4}\int_0^K \frac{1}{s}\left(\int_L^{L+s} t dt\right)^2 \mathrm{d}s & \text{if } \varrho = -1, \\ KL\left(\frac{1}{(1-\varrho)^4} + \frac{1}{1-\varrho^2}\right) + \frac{K^2}{2(1-\varrho)^4} & \text{if } |\varrho| < 1, \\ \frac{9}{4}\int_0^K \left(\int_0^L t^4 dt\right) \mathrm{d}s + \frac{9}{4}\int_0^K \frac{1}{s}\left(\int_L^{L+s} t^2 dt\right)^2 \mathrm{d}s & \text{if } \varrho = +1, \end{cases}$$

and

$$\Delta_{n,\varrho}(\mathbf{x}_n) = r_{n,\varrho} \sum_{k=1}^{K_n} \sum_{\ell=1}^{L_n-1} \left(\Diamond_\varrho x_{k,\ell} - G_\ell(\varrho)\right)\left(G'_\ell(\varrho) + \sum_{i=0}^{\ell-1} \varrho^{\ell-i-1} \Diamond_\varrho x_{k,i}\right)$$
$$+ r_{n,\varrho} \sum_{k=1}^{K_n} \frac{1}{k}\left(\widetilde{\Diamond}_\varrho x_{k,L_n} - \sum_{\ell=L_n}^{L_n+k-1} G_\ell(\varrho)\right) \sum_{\ell=L_n}^{L_n+k-1}\left(G'_\ell(\varrho) + \sum_{i=0}^{\ell-1} \varrho^{\ell-i-1} G_i(\varrho)\right)$$

for $\mathbf{x}_n := (x_{k,\ell})_{1\le k\le K_n,\, 0\le \ell\le L_n} \in \mathbb{R}^{K_n(L_n+1)}$, *where* $x_{0,\ell} := f_{0,\ell}$ *for* $\ell \ge 1$ *and the modified difference operators* $\Diamond_\varrho$ *and* $\widetilde{\Diamond}_\varrho$ *are defined by*

$$\Diamond_\varrho x_{k,\ell} := \begin{cases} x_{k,\ell} - x_{k-1,\ell+1} - \varrho(x_{k,\ell-1} - x_{k-1,\ell}) & \text{for } k, \ell \ge 1, \\ x_{k,0} - x_{k-1,1} & \text{for } k \ge 1, \ \ell = 0, \end{cases}$$
$$\widetilde{\Diamond}_\varrho x_{k,\ell} := x_{k,\ell} - x_{0,k+\ell} - \varrho(x_{k,\ell-1} - x_{0,k+\ell-1}) \qquad \text{for } k, \ell \ge 1.$$

4. Asymptotically optimal tests

If $\Theta \subset \mathbb{R}^p$ is an open set, $(\Omega, \mathcal{A}, \mathsf{P})$ is a probability space, $\xi^{(\theta)} : \Omega \to \mathbb{R}^d$, $\theta \in \Theta$, are random vectors with distribution P_θ on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$, then a *randomized test function* in the statistical experiment $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \{\mathsf{P}_\theta : \theta \in \Theta\}\big)$ is a measurable function $\phi : \Theta \to [0, 1]$. The interpretation is that if $\xi^{(\theta)}$ is observed, then a null hypothesis $H_0 \subset \Theta$ is rejected with probability $\phi(\xi^{(\theta)})$. The *power function* of ϕ is the function $\theta \mapsto \mathsf{E}\phi(\xi_n^{(\theta)})$. (This gives the probability that the null hypothesis H_0 is rejected.) Let $\alpha \in (0, 1)$. The test ϕ is of level α for testing a null hypothesis H_0 if

$$\sup\{\mathsf{E}\phi(\xi^{(\theta)}) : \theta \in H_0\} \le \alpha.$$

Combining [5, Theorem 15.4]) with Theorem 3.1, we obtain asymptotically optimal tests for our interest rate model.

Theorem 4.1 *Let K_n, L_n, $n \in \mathbb{N}$, be positive integers such that $K_n = nK + o(n)$ and $L_n = nL + o(n)$ as $n \to \infty$ with some $K > 0$ and $L > 0$. For each $n \in \mathbb{N}$ and $\varrho \in \mathbb{R}$, let $\mathsf{P}_{n,\varrho}$ be the distribution of the sample $\mathbf{f}_n^{(\varrho)} := (f_{k,\ell}^{(\varrho)})_{1\leq k\leq K_n,\, 0\leq \ell\leq L_n}$ on $\big(\mathbb{R}^{K_n(L_n+1)}, \mathcal{B}(\mathbb{R}^{K_n(L_n+1)})\big)$. Let $\varrho_0 \in [0,1]$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be differentiable at ϱ_0 with $\psi(\varrho_0) = 0$, $\psi'(\varrho_0) \neq 0$. Let $\alpha \in (0,1)$. For each $n \in \mathbb{N}$, let $\phi_n : \mathbb{R}^{K_n(L_n+1)} \to [0,1]$ be a test of level α for testing $H_0 : \psi(\varrho) \leq 0$ versus $H_1 : \psi(\varrho) > 0$, i.e., it is a measurable function, such that*

$$\sup\big\{\mathsf{E}\phi_n(\mathbf{f}_n^{(\varrho)}) : \varrho \in \mathbb{R},\, \psi(\varrho) \leq 0\big\} \leq \alpha.$$

Then for every $h \in \mathbb{R}$ with $\psi'(\theta_0)h > 0$,

$$\limsup_{n\to\infty} \mathsf{E}\phi_n\big(\mathbf{f}_n^{(\varrho_0 + r_{n,\varrho_0}h)}\big) \leq 1 - \Phi(z_\alpha - I_{\varrho_0}^{1/2}|h|),$$

where Φ denotes the standard normal distribution function, and z_α denotes the upper α-quantile of the standard normal distribution.

Moreover, considering the sequence

$$T_{n,\varrho_0} := \operatorname{sign}(\psi'(\varrho_0)) I_{\varrho_0}^{-1/2} \Delta_{n,\varrho_0}, \qquad n \in \mathbb{N}$$

of statistics, the sequence of tests that reject for values of T_{n,ϱ_0} exceeding z_α is asymptotically optimal for testing $H_0 : \psi(\varrho) \leq 0$ versus $H_1 : \psi(\varrho) > 0$ in the sense that for every $h \in \mathbb{R}$,

$$\mathsf{P}\big(T_{n,\varrho_0}(\mathbf{f}_n^{(\varrho_0 + r_{n,\varrho_0}h)}) \geq z_\alpha\big) \to 1 - \Phi\big(z_\alpha - \operatorname{sign}(\psi'(\varrho_0)) I_{\varrho_0}^{1/2} h\big) \qquad \text{as } n \to \infty.$$

5. Bibliography

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