Approximations to Most Powerful Invariant Tests for Multinormality

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Abstract

We consider the problem of testing multinormality against alternatives invariant with respect to some subgroup of affine transformations. In [3], a general form of the most powerful invariant (MPI) test has been obtained. Unfortunately, applicability of the MPI test is rather limited, due to complicated, intractable integrals. With the aid of the Laplace method for integrals, we derive large sample approximations for the MPI tests. The cases of bivariate exponential and uniform alternatives are studied in details, whereas higher dimensional extensions are outlined. It is shown in the both bivariate cases, that a further approximation for the Laplace approximation can be given. This leads to the likelihood ratio (LR) test statistic. A final conclusion is that the likelihood ratio test statistic can be seen as a formal expansion of the MPI test statistic, with a known upper bound for the relative error of the approximation. The Monte Carlo simulation study shows, that powers of both, the Laplace approximation, as well as the LR test are very close to the power of the most powerful invariant test even in small sample sizes.

1. Introduction

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be p-dimensional *iid* observations from a probability distribution absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^p$, and let $\mathbf{X}$ denote the (p, n) random matrix of observations. Let G^* denote the group of affine transformations of the form

$$\mathbb{R}^{pn} \ni \mathbf{X} \to \mathbf{A} \cdot \mathbf{X} + \mathbf{b}\mathbf{1}_n^T \in \mathbb{R}^{pn},$$

with $\mathbf{A}$ being a (p, p) upper triangular matrix with positive diagonal, $\mathbf{b} \in \mathbb{R}^p$ and $\mathbf{1}_n^T = (1, \ldots, 1)^T \in \mathbb{R}^n$. In what follows, we deal with the problem of testing:

$$H_0 : \ \mathbf{X}_i \sim \mathcal{N}_p(\mathbf{m}, \Sigma) \quad \text{vs} \quad H_1 : \ \mathbf{X}_i \sim \mathcal{F}\big[\mathbf{U}(\mathbf{Y} - \mathbf{m})\big], \tag{1}$$

where $\mathbf{m} \in \mathbb{R}^p$, $\mathcal{F}$ is a distribution function of a random vector $\mathbf{Y} \in \mathbb{R}^p$ and $\mathbf{U}$ is an upper triangular (p, p) matrix with positive diagonal. The problem (1) is G^*-invariant, and it is natural to require that any „reasonable" test should posses the G^*-invariant property. Fortunately, the most powerful G^*-invariant (MPI) test exists and is given by (see [3]):

$$\varphi(\mathbf{X}) = \begin{cases} 1 & \text{if} \quad F_0^*(\mathbf{X})/F_1^*(\mathbf{X}) \leq c_{cr} \\ 0 & \quad \text{otherwise} \end{cases}, \tag{2}$$

where F_0 and F_1 are given by multidimensional integrals. $F_0(\mathbf{X})$ - corresponds to the p-variate normal distribution and has a simple analytical form, depending on the sample covariance matrix only. Thus, the test can be applied to all these alternatives, for which one can compute the $F_1(\mathbf{X})$. This, however, even for $p = 2$, turns out to be a difficult task, due to complicated, intractable integrals.

2. Testing binormality against bivariate uniformity

Consider the testing problem (2) with $\mathcal{F}$ being a distribution function of a bivariate random vector uniformly distributed over $[0, 1] \times [0, 1]$. Let $R(x_1, \ldots, x_n) := x_{n:n} - x_{1:n}$ be the range of the vector $(x_1, \ldots, x_n)$ and let $\mathbf{A}^{(i)}$ denote the i-th row of the matrix $\mathbf{A}$. The rejection region of the MPI test, given in (2) may be written as $T_U^*(\mathbf{X}) \leq c_{cr}$, with suitably selected c_{cr}, where

$$T_U^*(\mathbf{X}) = \frac{R\Big(\mathbf{X}^{(2)}\Big) \left[\int\limits_{-\infty}^{\infty} \Big\{ R\Big(\mathbf{X}^{(1)} + s\mathbf{X}^{(2)}\Big) \Big\}^{-(n-1)} ds \right]^{-1/(n-1)}}{\sqrt[n-1]{\hat{\sigma}_2^2} \Big\{ \hat{\sigma}_1 \hat{\sigma}_2 \sqrt{1 - r_{12}^2} \Big\}^{\frac{n-2}{n-1}}}, \tag{3}$$

and where $\hat{\sigma}_1^2, \hat{\sigma}_2^2$ and r_{12} denote respectively the sample variance for the first and second row and the sample correlation coefficient To complete a derivation one needs to compute the integral in a nominator in (3). This is possible, it however yields to a complicated algorithm and numerically unstable formulas. Thus an approximation for this integral is desired. It can be shown, that the base of the integrand is almost surely a convex and unimodal function of $s \in \mathbb{R}$. As a consequence, the Laplace expansion for integrals can be used to obtain a large sample approximation for the integral under study. A simple example of the Laplace expansion can be, roughly speaking, described in a following way. Let $f : \mathbb{R} \to \mathbb{R}$ be a function attaining a unique minimum at $a \in \mathbb{R}$ and let $f'(a_+) > 0$. If regularity conditions are fulfilled, then:

$$\int_a^{\infty} e^{-nf(t)} dt = \frac{e^{-nf(a)}}{nf'(a_+)} \left\{ 1 + O\big(n^{-1}\big) \right\}. \tag{4}$$

Adapting the expansion to our context, one obtains an approximate critical region of the MPI test. The null hypothesis is rejected, if

$$T_U^{\dagger}(\mathbf{X}) := \frac{R\Big(\mathbf{X}^{(2)}\Big)\left\{\min_{s\in\mathbb{R}} R\Big(\mathbf{X}^{(1)}+s\mathbf{X}^{(2)}\Big)\right\}^{\frac{n-2}{n-1}}}{\sqrt[n-1]{\hat{b}^{-1}-\hat{a}^{-1}}\cdot\sqrt[n-1]{\hat{\sigma}_2^2}\left\{\hat{\sigma}_1\hat{\sigma}_2\sqrt{1-r_{12}^2}\right\}^{\frac{n-2}{n-1}}} \leq c_{cr}, \tag{5}$$

where $\hat{a}$ and $\hat{b}$ are (random) slopes in the vicinity of a mode of the integrand. Moreover, $T_U^*(\mathbf{X}) = T_U^{\dagger}(\mathbf{X})\cdot(1+O(n^{-2}))$. It can be shown, that $T_U^{\dagger}$ is G^*-invariant. The approximation $T_U^{\dagger}$ yields a considerable simplification of the MPI test statistic T_U^*. Nevertheless, it requires an evaluation of the derivatives $\hat{a}$ and $\hat{b}$. This may be a difficult task. Fortunately, we have the following

Lemma 1 *Under both, the null and the alternative hypothesis, we have*

$$\frac{1}{\sqrt[n-1]{\hat{b}^{-1}-\hat{a}^{-1}}} = 1 + O_P\left(\frac{\log n}{n}\right). \tag{6}$$

And consequently, after some algebra

Theorem 1 *The test statistic T_U^*, can be expressed as*

$$T_U^*(\boldsymbol{X}) = T_U^{\ddagger}(\boldsymbol{X})\left\{1 + O_P\left(\frac{\log n}{n}\right)\right\}, \tag{7}$$

where

$$T_U^{\ddagger}(\boldsymbol{X}) := \left\{R\Big(\boldsymbol{Y}^{(2)}\Big)\min_{s\in\mathbb{R}} R\Big(\boldsymbol{Y}^{(1)}+s\,\boldsymbol{Y}^{(2)}\Big)\right\}^{\frac{n-2}{n-1}}. \tag{8}$$

Moreover, $T_U^{\ddagger}$ is equivalent to the likelihood ratio test.

The power of the obtained approximations are given in Section 4.

3. The double exponential alternative and multivariate extensions

The analogous approximations for the double exponential alternative can be obtained in a similar way. They are given by the same formulas as in the previous section, but with the range R replaced by $\bar{R}(x_1,\ldots,x_n) = \bar{x} - x_{1:n}$.

The extensions of the approximations to higher dimensions ($p \geq 3$), are not straightforward. The main idea remains however unchanged. The Laplace method provides an expansion of the integral around the maximum of the integrand. Then, if an appropriate version of Lemma 1 holds, the likelihood ratio test statistic can be seen as an approximated test statistic of the MPI test.

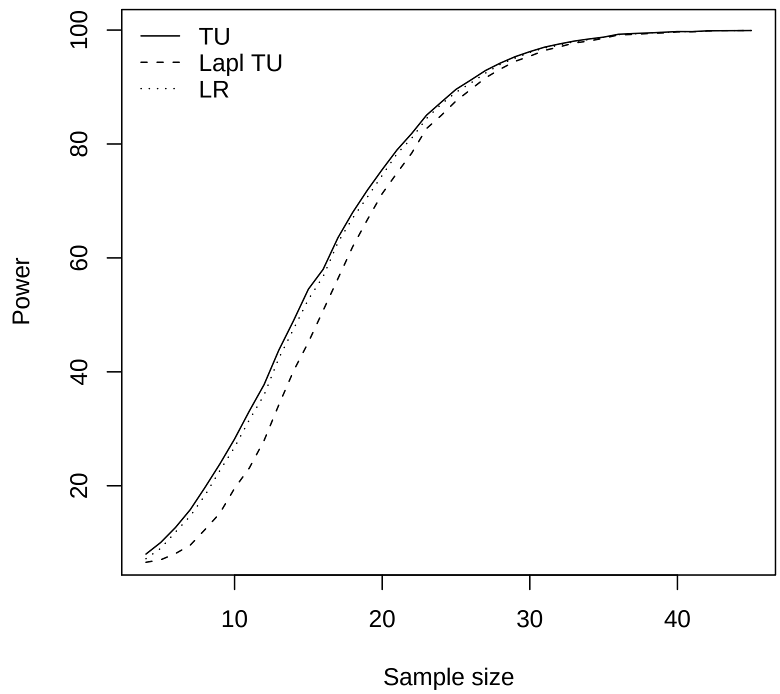

Figure 1: Variations in n of the power function of the test T_U^* and its approximations for $\alpha = 0,05$. (Replications: $H_0 = 50.000$, $H_1 = 50.000$).

4. Simulations

In this section, as an illustration, we present results of a little study of powers of the MPI test statistic and its approximations via the Monte Carlo simulations. These powers are approximated upper bounds for the power of any G^*-invariant test in problems considered in Section 2.

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5. Bibliography

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