Gaussian Process Simulation with application of the Theory of Square-Gaussian Processes

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Abstract

In the paper the simulation of Gaussian stochastic processes is considered. For this purpose the estimation for distribution of supremum of square-Gaussian processes is found. This result is used for model construction of Gaussian stochastic process, taking into account the derivative of the process, with given reliability and accuracy in Banach space.

1. Introduction

We'll consider some properties of square-Gaussian stochastic processes. A theorem about large deviation of supremum of the process was proved in [1]. This result was used for simulation of Gaussian stochastic process, that is entered the system (filter) as input process, taking into account output process, with given reliability and accuracy in Banach space $C([0,T]^d)$. In this paper the particular case of [1] is considered. Namely, when the output process is equal to the derivative of input process.

The paper consists of three parts and references. The first part is introduction, in the second one the main definitions and properties of square-Gaussian process are given. In last part we construct the model for Gaussian stochastic process which is entered the system and give conditions under which this model approximates the process, taking into account derivative of input process, with give accuracy and reliability in Banach space.

2. Square-Gaussian processes

Let $(\Omega, \mathcal{F}, P)$ be probability space and $(\mathbf{T}, \rho)$ be a compact metric space with metric ρ.

Definition 1. Let $\Xi = \{\xi_t,\, t \in \mathbf{T}\}$ be a family of jointed Gaussian random variables, $\mathbf{E}\xi_t = 0$ (for example, let ξ_t, $t \in \mathbf{T}$, be a Gaussian random process).

A space $SG_\Xi(\Omega)$ is called the space of square-Gaussian random variables if any $\eta \in SG_\Xi(\Omega)$ can be represented in such way

$$\eta = \bar{\xi}^T A \bar{\xi} - \mathbf{E}\bar{\xi}^T A \bar{\xi}, \tag{1}$$

where $\bar{\xi}^T = (\xi_1, \xi_2, \ldots, \xi_n)$, $\xi_k \in \Xi$, $k = 1, \ldots, n$, A is real-valued matrix,

or $\eta \in SG_\Xi(\Omega)$ can be represented as a limit in mean square of the sequence of random variables from (1)

$$\eta = l.i.m._{n\to\infty}(\bar{\xi}_n^T A\bar{\xi}_n - \mathbf{E}\bar{\xi}_n^T A\bar{\xi}_n).$$

Definition 2. A stochastic process $X = \{X(t),\, t \in \mathbf{T}\}$ is called square-Gaussian if for any $t \in \mathbf{T}$ a random variable $X(t)$ belongs to the space $SG_\Xi(\Omega)$.

Example 1. Consider a family of Gaussian centered stochastic processes $\xi_1(t),\, \xi_2(t), \ldots, \xi_n(t),\, t \in \mathbf{T}$. Let a matrix $A(t)$ be symmetric. Then

$$X(t) = \bar{\xi}^T(t)A(t)\bar{\xi}(t) - \mathbf{E}\bar{\xi}^T(t)A(t)\bar{\xi}(t),$$

where $\bar{\xi}^T(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_n(t))$, is square-Gaussian stochastic process.

Some properties of square-Gaussian stochastic processes can be found in papers [3, 4].

We consider the space $\mathbf{T} = [0, T]^d$, $d \geq 1$, with respect to metric $\rho(t, s) = \max\limits_{1\leq i\leq d} |t_i - s_i|$. Let $X = \{X(t),\, t \in \mathbf{T}\}$ be square-Gaussian stochastic process.

In the paper [1] the following theorem was proved.

Theorem 1. *Let $X(t), t \in [0, T]^d$, be separable square-Gaussian stochastic process and*

$$\sup_{\rho(t,s)\leq h} (\mathbf{Var}\,(X(t) - X(s)))^{\frac{1}{2}} \leq \sigma(h) = C \cdot h^\alpha, \qquad \alpha \in (0, 1], C > 0.$$

If for integer $M > 1$, $x > 0$

$$x > \frac{\sqrt{2}\gamma_0 Md}{\alpha} \max\Big\{1; \Big((\frac{T}{2})^\alpha C \frac{1}{\gamma_0} \Big)^{\frac{1}{M-1}} \Big\}, \tag{2}$$

then the next estimation holds

$$\begin{aligned} \P\Big\{\sup_{t\in \boldsymbol{T}} |X(t)| > x\Big\} \quad &\leq \quad 2^{1+d} e^{\frac{(M+1)d}{\alpha}} \exp\Big\{-\frac{x}{\sqrt{2}\gamma_0}\Big\} \\ &\times \quad \Big(\frac{\alpha x}{\sqrt{2}\gamma_0 Md}\Big)^{Md/\alpha} \Big(1 + \frac{2x}{\sqrt{2}\gamma_0}\Big)^{1/2}, \end{aligned} \tag{3}$$

where $\gamma_0 = \sup\limits_{t\in\mathbf{T}} (\mathbf{Var}\, X(t))^{\frac{1}{2}}$.

3. Simulation of Gaussian stochastic processes

Consider the same space $\mathbf{T} = [0,T]^d$, $d > 1$ with metric $\rho(t,s) = \max\limits_{1\leq i\leq d} |t_i - s_i|$, where t, s are vectors from $\mathbf{T}$. Let $\xi = \{\xi(t), t \in \mathbf{T}\}$ be centered Gaussian stochastic process and

$$\xi(t) = \sum_{n=0}^{\infty} \xi_n f_n(t), \tag{4}$$

where the functions $f_n(t)$, $n \geq 0$, are continuous and such that for all $t \in \mathbf{T}$

$$\sum_{n=0}^{\infty} f_n^2(t) < \infty,$$

ξ_n, $n = 0, 1, 2, \dots$, are independent Gaussian random variables, $\mathbf{E}\xi_n = 0$, $E\xi_n^2 = 1$. Since

$$\mathbf{E}\xi^2(t) = \sum_{n=0}^{\infty} f_n^2(t) < \infty,$$

then $\sum\limits_{n=0}^{\infty} \xi_n f_n(t)$ converges with probability one (see, for example, [5]).

Consider such situation: Let Σ be some system(filter, device), which is intended for transformation of signals (functions) $f_n(t)$. The function which has to be transformed is called input function on system; the transformed function is called output function or reaction on input function. Under $g_n(t)$ we will define output function. More information about filter can be found in [9].

Remark 1. In particular case $g_n(t) = z_n \cdot f_n(t)$. It means that transformation doesn't change the shape of signal.

In particular case can be also considered when $g_n(t) = f_n'(t)$.

If input process on the system Σ is $\xi(t) = \sum\limits_{n=0}^{\infty} \xi_n f_n(t)$, then output process is $\eta(t) = \sum\limits_{n=0}^{\infty} \xi_n g_n(t)$. Suppose that for all $t \in \mathbf{T}$ the series $\sum\limits_{n=0}^{\infty} g_n^2(t)$ converges. It's sufficient condition for convergence with probability one of the series $\eta(t) = \sum\limits_{n=0}^{\infty} \xi_n g_n(t)$.

Definition 3. The process $\tilde{\xi}_N(t)$ is called the model of $\xi(t)$, $t \in \mathbf{T}$ if

$$\tilde{\xi}_N(t) = \sum_{k=0}^{N} \xi_k f_k(t), \quad t \in \mathbf{T}.$$

Let's define the difference between the process and the model under

$$\xi_N(t) = \xi(t) - \tilde{\xi}_N(t) = \sum_{k=N+1}^{\infty} \xi_k f_k(t), \quad t \in \mathbf{T}.$$

In the same way $\eta_N(t)$ can be defined:

$$\eta_N(t) = \sum_{k=N+1}^{\infty} \xi_k g_k(t), \quad t \in \mathbf{T}.$$

We'll investigate conditions under which the model $\tilde{\xi}_N(t)$ approximates $\xi(t)$ with given accuracy and reliability in Banach space $C([0,T]^d)$ taking into account the process $\eta(t)$. For this purpose the relationship $\xi^2(t) + \eta^2(t)$ can be analyzed. If generalize this case we can consider a semi-positive quadratic form

$$X(x,y) = a \cdot x^2 + 2c \cdot x \cdot y + b \cdot y^2,$$

where a, b, c are such that $a > 0$, $ab - c^2 > 0$.

For convenience,under $X_N(t)$ we'll define a quadratic form which is defined on the processes $\xi_N(t)$, $\eta_N(t)$:

$$X_N(t) = X(\xi_N(t), \eta_N(t)) = a \cdot (\xi_N(t))^2 + 2c \cdot \xi_N(t) \cdot \eta_N(t) + b \cdot (\eta_N(t))^2.$$

Stochastic process $X_N(t)$ is equal to

$$X_N(t) = \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} \xi_k \xi_n \phi_{kn}(t), \tag{5}$$

where

$$\phi_{kn}(t) = a f_k(t) f_n(t) + c(f_k(t) g_n(t) + g_k(t) f_n(t)) + b g_k(t) g_n(t). \tag{6}$$

$\phi_{kn}(t)$ is symmetric with respect to k and n. Hence, $\phi_{kn}(t) = \phi_{nk}(t)$.

Denote

$$\bar{X}_N(t) = X_N(t) - \mathbf{E} X_N(t).$$

Definition 4. The model $\tilde{\xi}_N(t)$ approximates stochastic process $\xi(t)$ on input of the system, taking into account output process, with given reliability $1 - \nu$, $\nu \in (0,1)$ and accuracy $\delta > 0$ in Banach space $C([0,T]^d)$, if

$$\P \left\{ \sup_{t \in \mathbf{T}} |\bar{X}_N(t)| > \delta \right\} \le \nu.$$

Notice that $X_N(t) - \mathbf{E} X_N(t) = \bar{X}_N(t)$, $t \in [0,T]^d$, is square-Gaussian stochastic process.

The following theorem holds true.

Theorem 2.[1] *Let $\xi(t), t \in [0,T]^d$, be separable Gaussian stochastic process for which*

$$\sup_{\rho(t,s)\le h} |\phi_{kn}(t) - \phi_{kn}(s)| \le d_{kn} h^{\alpha}, \qquad \alpha \in (0,1], \tag{7}$$

and

$$2 \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} d_{kn}^2 = C^2(N) < \infty,$$

where $\phi_{kn}(t)$ are from (6).

The model $\tilde{\xi}_N(t)$ approximates separable Gaussian process $\xi(t)$, taking into account output process, with given accuracy $\delta > 0$ and reliability $1-\nu$, $\nu \in (0,1)$, if for $N \ge 1$ the conditions are fulfilled

$$\delta > \frac{\sqrt{2}\gamma_0(N)Md}{\alpha} \max\Big\{1; \Big(\big(\frac{T}{2}\big)^{\alpha} \frac{C(N)}{\gamma_0(N)} \Big)^{\frac{1}{M-1}} \Big\},$$

$$2^{1+d} e^{\frac{(M+1)d}{\alpha}} \exp\Big\{-\frac{\delta}{\sqrt{2}\gamma_0(N)}\Big\} \Big(\frac{\alpha\delta}{\sqrt{2}\gamma_0(N)Md}\Big)^{Md/\alpha} \Big(1 + \frac{2\delta}{\sqrt{2}\gamma_0(N)}\Big)^{1/2} < \nu,$$

where $M > 1$ is arbitrary integer number, $\gamma_0(N) = \left(2 \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} s_{kn}^2\right)^{\frac{1}{2}}$.

Consider particular case. Let $\xi = \{\xi(t), t \in [0,T]\}$, be centered Gaussian process which can be represented in the form (4), where $f_n(t)$, $n \ge 0$, are continuously differentiable and for all $t \in [0,T]$ $\sum_{n=0}^{\infty} (f_n'(t))^2 < \infty$ and $\sum_{n=0}^{\infty} |f_n'(t)| < \infty$.

Consider the case when output process is equal to $\eta(t) = \xi'(t), t \in [0,T]$. There exists derivative of stochastic process $\xi'(t) = \sum_{n=0}^{\infty} f_n'(t)\xi_n$ in mean square.

The difference between the process and the model is

$$\xi_N(t) = \xi(t) - \tilde{\xi}_N(t) = \sum_{k=N+1}^{\infty} \xi_k f_k(t), \quad t \in \mathbf{T}$$

and the process $\eta_N(t)$ is equal to

$$\eta_N(t) = \sum_{k=N+1}^{\infty} \xi_k f_k'(t), \quad t \in \mathbf{T}.$$

Let's construct a semi-positive quadratic form $X_N(t)$, which is defined on the processes $\xi_N(t)$, $\eta_N(t)$

$$X_N(t) = X(\xi_N(t), \eta_N(t)) = a \cdot (\xi_N(t))^2 + 2c \cdot \xi_N(t) \cdot \eta_N(t) + b \cdot (\eta_N(t))^2.$$

The process $X_N(t)$ can be represented in the form

$$X_N(t) = \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} \xi_k \xi_n \phi_{kn}(t), \tag{8}$$

where

$$\phi_{kn}(t) = af_k(t)f_n(t) + c(f_k(t)f'_n(t) + f'_k(t)f_n(t)) + bf'_k(t)f'_n(t). \tag{9}$$

Then it can be used corollary 2 for stochastic process (8) which gives the conditions under which the model approximates separable Gaussian process, taking into account its derivative, with given accuracy and reliability. It's shown in the next theorem.

Theorem 3. *Let $\xi(t), t \in [0, T]$, be separable Gaussian stochastic process for which*

$$\sup_{|t-s| \leq h} |\phi_{kn}(t) - \phi_{kn}(s)| \leq d_{kn} h^{\alpha}, \qquad \alpha \in (0, 1], \tag{10}$$

and

$$2 \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} d_{kn}^2 = C^2(N) < \infty,$$

where $\phi_{kn}(t)$ are from (9).

The model $\tilde{\xi}_N(t)$ approximates separable Gaussian process $\xi(t)$, taking into account output process, with given accuracy $\delta > 0$ and reliability $1 - \nu$, $\nu \in (0, 1)$, if for N the next inequalities are satisfied

$$\delta > \frac{\sqrt{2}\gamma_0(N)M}{\alpha} \max\Big\{1; \left(\left(\frac{T}{2}\right)^{\alpha} \frac{C(N)}{\gamma_0(N)} \right)^{\frac{1}{M-1}} \Big\},$$

$$4e^{\frac{(M+1)}{\alpha}} \exp\Big\{-\frac{\delta}{\sqrt{2}\gamma_0(N)}\Big\} \Big(\frac{\alpha\delta}{\sqrt{2}\gamma_0(N)M}\Big)^{M/\alpha} \Big(1 + \frac{2\delta}{\sqrt{2}\gamma_0(N)}\Big)^{1/2} < \nu,$$

where $M > 1$is arbitrary integer number, $\gamma_0(N) = \left(2 \sum_{k=N+1}^{\infty} \sum_{n=N+1}^{\infty} s_{kn}^2 \right)^{\frac{1}{2}}.$

4. Bibliography

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