The bounded arbitrage for the multi-period model of a financial market with discrete time

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Abstract

In a general discrete time market model with portfolio constraints, an ε-arbitrage opportunity is an opportunity to make a riskless profit of amount at least ε while trading a limited amount of assets. In a one-period framework with contingent initial data and in a multi-period framework we prove no-ε-arbitrage criteria similar to the classical fundamental theorem of asset pricing. More precisely, we prove that there is no ε-arbitrage iff there exists such an equivalent measure with a bounded density that the discounted price process is close to a martingale w.r.t. this measure.

1. Introduction

An extensively used approach of a financial market modeling is the following one. An investor works on a market, where stock price increments are random, and transactions are allowed only in some finite set of time instances. At each of these instances, the investor, who has no knowledge of the future price dynamics, can reorganize his portfolio, but no cash outflow or inflow is allowed.

First question about every model of a financial market is whether an arbitrage opportunity, or an opportunity to make a riskless profit, exists. It is well-known that for discrete-time models the absence of an arbitrage opportunity is equivalent to the existence of a martingale measure for the discounted price process. This fact is called the fundamental theorem of asset pricing. It is explained in details for instance in [2] and [6]. Among many other works about the classical arbitrage notion for discrete and continuous time markets we mention [3], [7], [10], see also references therein.

For large financial markets it is usually assumed that an arbitrary amount of any asset can be traded. In reality, there are some portfolio constraints. Thus investor can use the existing arbitrage opportunities in a limited way. We are interested when an investor, using a constrained strategy, can guarantee himself a riskless profit of amount ε. We call such situations ε-arbitrage. Investigating ε-arbitrage absence requires more complicated mathematical technique than in the classical case.

The notion of ε-arbitrage was introduced in [9], where an analogue of fundamental theorem was proved for a one-period market model with non-random initial data. In this paper we construct an ε-arbitrage theory similar to the classical one. We generalize the classical fundamental theorem of asset pricing to a model with contingent initial data and to a multi-period financial market model.

2. The main theorem for one period model with contingent initial data

First we introduce some notations. Let $(\Omega, \mathbb{F}, P)$ be a complete probability space equipped with filtration $(\mathbb{F}_t)_{t=\overline{0,T}}, \mathbb{F}_T = \mathbb{F}$. Also $S = (S_t), t = \overline{0,T}$ is $\mathbb{F}_t$-adapted $(d+1)$-dimensional process (vector of prices). We assume that S^0 is a riskless asset: $S_0^0 = 1$, $S_t^0 = (1+r)^t$, where $r > 0$ is the interest rate.

We will call a $(d+1)$-dimensional process $\xi = (\xi^0, \xi) = (\xi_t^0, \xi_t^1, ..., \xi_t^d)$, $t = \overline{1,T}$ a *strategy* if it is $\mathbb{F}$-predictable, i.e., if ξ_t is $\mathbb{F}_{t-1}$-measurable for all $t = \overline{1,T}$.

We will call a strategy $\xi = (\xi_t), t = \overline{1,T}$ *self-financing* if following holds:

$$\xi_t \cdot S_t = \xi_{t+1} \cdot S_t, \forall t = \overline{1, T-1}.$$

This condition amounts to absence of exogenous cash flows and outflows.

The *discounted price process* is $X_t := (X_t^0, X_t) = (X_t^0, X_t^1, \dots, X_t^d)$, where

$$X_t^i := \frac{S_t^i}{S_t^0}, \quad t = 0, \dots, T; \quad i = 0, \dots, d.$$

Define the discounted price increments as $\triangle X_t := X_t - X_{t-1}$, $\Delta X := \Delta X_1$ and the *discounted capital process* associated with ξ as

$$V_0 := \xi_1 \cdot X_0; \quad V_t := \xi_t \cdot X_t,\ t = \overline{1,T}.$$

Now we give a definition of an ε-arbitrage opportunity for the new model.

Definition 1 *The multi-period financial market has ε-*arbitrage *opportunity if there exists a self-financing strategy $\xi = (\xi_t)$ bounded in the sense*

$$\|\xi_t\|_1 := \left\| \sum_{i=1}^{d} |\xi_t^i| \right\|_{L_\infty(\Omega)} \le 1, \quad \forall t = \overline{0, T-1}, \tag{1}$$

whose associated capital process satisfies

$$V_0 \le -\varepsilon, \quad P\{V_T \ge 0\} = 1, \quad P\{V_T > 0\} > 0.$$

Lemma 1 ([9]) *The following statements are equivalent:*

1. *Financial market has ε-arbitrage opportunity;*
2. *There exists $\mathbb{F}_0$-measurable vector ξ such that $\left\| \sum_{i=1}^{d} |\xi^i| \right\|_{L_\infty(\Omega)} \leq 1$ and $\xi \cdot \triangle X \geq \varepsilon$, P-a.s., and $P\{\xi \cdot \triangle X > \varepsilon\} > 0$.*

We are going to transfer Theorem **??** to a model with contingent initial data. An absence of ε-arbitrage follows from the existence of measure in a rather straightforward manner and one needs no density boundedness assumption for that; so we formulate this as a separate lemma.

Lemma 2 *If there exists a measure P^* equivalent to P such that*

$$E_{P^*}\left[\ \|E_{P^*}[\triangle X/\mathbb{F}_0]\|_\infty\right] \leq \varepsilon, \tag{2}$$

then the financial market has no ε-arbitrage opportunity

It is much harder to show how the existence of measure follows from the absence of ε-arbitrage. We adopt the main ideas and the scheme of proof of the classical Dalang–Morton–Willinger theorem [2]. First, assuming no ε-arbitrage for the model with non-random initial data, we will prove the existence of an equivalent measure with continuous density. Let $\overline{\mathbb{R}}^d$ denote the one-point compactification of $\mathbb{R}^d$. For a vector $\alpha \in \mathbb{R}^d$ we define $H^\varepsilon_{\alpha\varpi} = \{x : \alpha \cdot x \,\varpi\, \varepsilon\}$, $\varpi \in \{\leq, <, =, >, \geq\}$.

Lemma 3 *Assume that for an integrable random vector $Y \in \mathbb{R}^d$ with distribution function $P(dx)$ there is no ε-arbitrage opportunity, that is, $P(H^\varepsilon_{\alpha\geq}) = 1$ for $\alpha \in \mathbb{R}^d$ with $\|\alpha\|_1 \leq 1$ implies $P(H^\varepsilon_{\alpha>}) = 0$. Then there exists a positive function $g \in C(\overline{\mathbb{R}}^d)$ such that $\mathsf{E}[g(Y)] = 1$, $\|\mathsf{E}[Y g(Y)]\|_\infty \leq \varepsilon$.*

The abovementioned lemma allows us to show the existence of an equivalent "ε-martingale" measure for a model with contingent initial data with an approach similar to the one used in [2]. Thus we will only pinpoint the major milestones of the proof, without going much into details, which one can find in the paper cited.

Theorem 1 *For our market the following conditions are equivalent:*

1. *It has no ε-arbitrage opportunities;*
2. *There exists P^* equivalent to P, such that $\frac{dP^*}{dP} \leq C$ for some $C > 0$ and $E_{P^*}\left[\ \|E_{P^*}[\triangle X/\mathbb{F}_0]\|_\infty\ \right] \leq \varepsilon$.*

Proof [$2 \Rightarrow 1$] It is an immediate consequence of Lemma 2.
[$1 \Rightarrow 2$] First of all, without loss of generality we assume that

$$E[X_t^i] < \infty; i = \overline{1,d}, t = 0,1. \qquad (3)$$

Indeed, we can define the probability measure $\tilde{P}$ as

$$\frac{d\tilde{P}}{dP} := c\left(1 + \sum_{i=1}^{d}\sum_{t=0}^{1} X_t^i\right)^{-1},$$

where c is such that integral on the right side equals 1. It is clear that (3) holds for $\tilde{P}$. The first statement of our theorem holds for P iff it holds for the equivalent measure $\tilde{P}$. At last, if the density $\frac{dP^*}{d\tilde{P}}$ is bounded for P^*, then $\frac{dP^*}{dP} = \frac{dP^*}{d\tilde{P}}\frac{d\tilde{P}}{dP}$ is bounded as well. Therefore $1 \Rightarrow 2$ holds for P iff it holds for $\tilde{P}$.

Denote $Y = \triangle X$. Let $P(\omega, dx)$ be the regular conditional distribution of Y under condition $\mathbb{F}_0$. We will understand $C(\overline{\mathbb{R}}^d)$ as a metric space with the norm $\|g\|_{\sup} = \max_{\overline{\mathbb{R}}^d} |g|$. Define $c(\omega) = \sup_{\|y\|_1=1} \operatorname{ess\,inf}_{x\in\mathbb{R}^d} x \cdot y$, where the essential infimum is taken with respect to $P(\omega, \cdot)$. It is clear that $c(\omega) < \infty$ and the function c is $\mathbb{F}_0$-measurable. Moreover, it is clear that $c(\omega) \leq \operatorname{ess\,inf}_\Omega \|\mathsf{E}[Y(\omega)/\mathbb{F}_0]\|_\infty \leq d\,\mathsf{E}[\|Y\|/\mathbb{F}_0]$, therefore $\mathsf{E}[c(\omega)] < \infty$.

Consider two cases.

1. $P(c(\omega) < \varepsilon) = 0$. Note that this implies $c(\omega) = \varepsilon$ a.s., because there is no ε-arbitrage.

Define the set

$$\mathfrak{H} := \left\{(\omega, g) \in \Omega \times C(\overline{\mathbb{R}}^d) : g \geq 0, \int_{\mathbb{R}^d} g(x)P(\omega, dx) = 1, \int_{\mathbb{R}^d} xg(x)P(\omega, dx) \in [-\varepsilon, \varepsilon]^d\right\}.$$

It is proved the same way as in [2] that $\mathfrak{H} \in \mathbb{F}_0 \times \mathcal{B}(C(\overline{\mathbb{R}}^d))$. The absence of ε-arbitrage implies that for almost all ω ε-arbitrage is also absent for a random vector with distribution $P(\omega, dx)$. Because of Lemma 3 we can argue that for almost all ω there exists $g_\omega \in C(\overline{\mathbb{R}}^d)$ such that $(\omega, g_\omega) \in \mathfrak{H}$, or, in other words, the projection of $\mathfrak{H}$ on Ω has probability 1. The space $C(\overline{\mathbb{R}}^d)$ is a complete separable space, so we can apply the measurable selection theorem [4] and state that there exists an $\mathbb{F}_0$-measurable function $G : \Omega \to C(\overline{\mathbb{R}}^d)$ such that $(\omega, G(\omega)) \in \mathfrak{H}$ a.s. The map $(\omega, x) \to G(\omega, x) := G(\omega)(x)$ is $\mathbb{F}_0 \times \mathcal{B}(\mathbb{R}^d)$-measurable as a composition of measurables. Therefore $D(\omega) := M(\omega)G(\omega, Y(\omega))$ with $M(\omega) = 1/(1 + K(\omega))$, $K(\omega) = \|G(\omega, \cdot)\|_{\sup}$, is $\mathbb{F}_1$-measurable.

Note that

$$\mathsf{E}[G(\omega, Y(\omega))] = \mathsf{E}[\mathsf{E}[G(\omega, Y(\omega))/\mathbb{F}_0]] = \mathsf{E}\left[\int_{\mathbb{R}^d} g(\omega, x)dP(\omega, x)\right] = 1$$

and similarly

$$\mathsf{E}[D(\omega)] = \mathsf{E}[M(\omega)].$$

Define $dP^*(\omega) = MD(\omega)dP(\omega)$ now. Here $M = 1/\,\mathsf{E}[D(\omega)]$. So, we have:

$$\begin{aligned}\mathsf{E}_{P^*}[\|\mathsf{E}_{P^*}[Y/\mathbb{F}_0]\|_\infty] &\le M\,\mathsf{E}[M(\omega)Y(\omega)G(\omega, Y(\omega))] \\ &= M\,\mathsf{E}[M(\omega)\,\mathsf{E}[Y(\omega)G(\omega, Y(\omega))/\mathbb{F}_0]] = \varepsilon.\end{aligned}$$

2. $P(c(\omega) < \varepsilon) > 0$. Then for some $\delta > 0$ we have $P(A(\delta)) > 0$, where $A(\delta) = \{c(\omega) + \delta < \varepsilon\}$. On the other hand, for every ω $(c(\omega) + \delta)$-arbitrage is absent for the distribution $P(\omega, dx)$, which follows from the definition of $c(\omega)$. Hence, as above, we can prove an existence of a measurable function $G : \Omega \to C(\overline{\mathbb{R}}^d)$ such that $\mathsf{E}[G(\omega, Y(\omega))/\mathbb{F}_0] = 1$ and $\|\mathsf{E}[Y(\omega)G(\omega, Y(\omega))/\mathbb{F}_0]\|_\infty \le c(\omega)+\delta$ hold. Let $M(\omega)$ be as above, $M_1 = 1/\,\mathsf{E}[M(\omega)\mathbb{1}_{A(\delta)}]$, $M_2 = 1/\,\mathsf{E}[M(\omega)\mathbb{1}_{A^c(\delta)}]$. We define

$$\rho_\theta(\omega) = M(\omega)\big(\theta M_1 \mathbb{1}_{A(\delta)} + (1-\theta)M_2\mathbb{1}_{A^c(\delta)}\big)G(\omega, Y(\omega)), \quad \theta \in (0,1).$$

The function ρ_θ is positive and bounded. $A(\delta) \in \mathbb{F}_0$ implies that after going to conditional expectations one gets $\mathsf{E}[\rho_\theta] = 1$ and

$$\mathsf{E}[\rho_\theta(c(\omega)+\delta)] = \theta\,\mathsf{E}[(c(\omega)+\delta)\mathbb{1}_{A(\delta)}] + (1-\theta)\,\mathsf{E}[(c(\omega)+\delta)\mathbb{1}_{A^c(\delta)}].$$

(the expectation is finite because ρ is bounded and c is integrable). The last expression tends to $\mathsf{E}[(c(\omega)+\delta)\mathbb{1}_{A(\delta)}] < \varepsilon$ as $\delta \to 0$. Therefore there exists $\theta_0 > 0$ such that $\mathsf{E}[\rho_\theta(c(\omega)+\delta)] < \varepsilon$. We now define

$$dP^* = \rho_{\theta_0}(\omega)dP$$

And we can deduce the following:

$$\mathsf{E}_{P^*}[\|\mathsf{E}_{P^*}[Y/\mathbb{F}_0]\|_\infty] = \mathsf{E}_{P^*}[\|\mathsf{E}[Y(\omega)G(\omega, Y(\omega))/\mathbb{F}_0]\|_\infty] \le \mathsf{E}[\rho_{\theta_0}(c(\omega)+\delta)] < \varepsilon.$$

□

Further in this section we formulate the results, which in a sense generalize Theorem 1 to an arbitrary norm on $\mathbb{R}^d$. They are not only interesting by themselves, but also crucial for proving the existence of an "ε-martingale" measure in a multi-period model. Let $\|\|_\sim$ be a norm on $\mathbb{R}^d$, and $\|\|_\sim^*$ be the conjugate norm. The following auxiliary statement is similar to Lemma 3.

Lemma 4 *Assume that $\forall\delta > 0$ for a random vector $Y \in \mathbb{R}^d$ with the distribution $P(dx)$ the following holds: $P(H^{\varepsilon+\delta}_{\alpha\ge}) = 1$ for $\alpha \in \mathbb{R}^d$, $\|\alpha\|_\sim \le 1$ implies $P(H^{\varepsilon+\delta}_{\alpha>}) = 0$. Then $\forall\delta > 0$ there exists a positive function $g \in C(\overline{\mathbb{R}}^d)$ such that $\mathsf{E}[g(Y)] = 1$, $\|\mathsf{E}[Yg(Y)]\|_\sim^* \le \varepsilon + \delta$.*

Proof As in Lemma 3, assuming the contrary to the statement, one shows that there exists α with $\|\alpha\|_\sim \leq 1$ and $P(H^{\varepsilon+\delta}_{\alpha=}) = 1$. This immediately implies $P(H^{\varepsilon+\delta/2}_{\alpha>}) = 1$, which is impossible by the assumption. □

For a random vector $X \in \mathbb{R}^d$ in a one-period model we will call $\varepsilon\sim$arbitrage such an $\mathbb{F}_0$-measurable strategy $\xi \in \mathbb{R}^d$, that $\|\xi\|_\sim \leq 1$, $\xi \cdot X \geq \varepsilon$ a.s. and $P(\xi \cdot X > \varepsilon) > 0$. Define

$$B = \{ \ \mathbb{F}_0\text{-measurable } b : \exists \mathbb{F}_0\text{-measurable } \xi, \|\xi\|_\sim \leq 1, \xi \cdot X \geq b \text{ a.s.}\}$$

The set B is not empty, since $0 \in B$ clearly. Define

$$l = \operatorname{Ess\,sup} B,$$

where Ess sup denotes an essential supremum of a set of random variables, see [5].

Theorem 2 *For all $\delta > 0$ there exists an equivalent measure P^* with a bounded density, such that almost surely*

$$\|\mathsf{E}_{P^*}[X/\mathbb{F}_0]\|^*_\sim \leq l + \delta.$$

3. The main theorem for the multi-period model

Denote $R^\varepsilon_T := \{V_T : V_T = \sum_{t=1}^{T} \xi_t \cdot \triangle X_t - \varepsilon\}$, where ξ is an arbitrary self-financing strategy, bounded in the sense (1). In other words, R^ε_T is the set of all possible final outcomes for the initial capital $-\varepsilon$.

We set $A^\varepsilon_T := R^\varepsilon_T - L^0_+ = \{V - V^+ : V \in R^\varepsilon_T, V^+ \in L^0_+\}$.

In what follows for a set X of random variable we denote $\bar{X}$ the closure of X w.r.t. the almost sure convergence.

Theorem 3 *The following statements are equivalent:*

1. $\forall \delta > 0$ $R^{\varepsilon+\delta}_T \cap L^0_+ \subset \{0\}$ *(absence of $(\varepsilon+\delta)$-arbitrage)*
2. $\forall \delta > 0$ $A^{\varepsilon+\delta}_T \cap L^0_+ \subset \{0\}$
3. $\forall \delta > 0$ $A^{\varepsilon+\delta}_T \cap L^0_+ \subset \{0\}$ *and* $A^{\varepsilon+\delta}_T = \overline{A^{\varepsilon+\delta}_T}$
4. $\forall \delta > 0$ $\overline{A^{\varepsilon+\delta}_T} \cap L^0_+ \subset \{0\}$
5. $\forall \delta > 0$ $\exists P^* \sim P$ *such that* $E_{P^*}\left[\sum_{t=1}^{T} \| \ E_{P^*}[\triangle X_t/\mathbb{F}_{t-1}] \ \|_\infty\right] \leq \varepsilon + \delta$, *and* $\frac{dP^*}{dP} \leq C$ *for some* $C > 0$.

Remark 1 *In the classical theorem about conditions equivalent to no-arbitrage all intersections are* $\{0\}$*, for instance, the analogue of condition 1 in the classical theorem is* $R^0_T \cap L^0_+ = \{0\}$ *(see [8, 7, 11]). The reason why the zero does not in general enter these sets now is that the zero strategy does not give zero final outcome (recall that the initial capital is* $-\varepsilon$*).*

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