Notes for a Licenciatura

Grothendieck at the Undergraduate Level

Based on Lectures of J. Sancho

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Preface

During the seventies, the *Licenciatura* (the undergraduate studies) in Mathematics at the University of Salamanca had a highly coherent structure devised by late Prof. Juan B. Sancho Guimerá, and he taught us at how the main ideas of Grothendieck pervade mathematics:

How separable extensions of a field k may be understood as coverings of a one point space $\operatorname{Spec} k$, the localization process being the change of the base field and the globalization process encoding Galois theory.

How any morphism $X \to T$ may be understood as a family of spaces parameterized by T, so that absolute statements about a space X really refer to the projection $X \to p$ onto the one point space, and how absolute statements always have relative versions about families $X \to T$, which greatly simplify the theory because of the freedom they provide.

How any morphism $T \to X$ may be understood as a point of X (parameterized by T) and that obviously such functor of points of X fully determines the whole structure of X, so reducing definitions of morphisms to the silly case of sets –provided that they are natural definitions– and constructions to the problem of representing a functor.

How sheaf cohomology is well-suited for Algebraic Geometry, Analysis and Topology.

How Grothendieck's representability theorem is very simple in the case of categories of abelian sheaves, and that it directly provides the existence of dualizing sheaves and dualizing complexes in the case of smooth curves (representing the functor $H^1(C,-)^*$, Riemann-Roch's theorem), smooth varieties (Serre's duality), topological manifolds (Poincaré's duality) and proper morphisms (Grothendieck's duality).

How toposes may unify Algebraic Geometry, Arithmetic and Topology.

And above all, how deeply rooted we have the wish of simple and natural definitions, statements and proofs, and that such yearning may be always overwhelmingly accomplished.

The aim of these student notes based on the undergraduate lectures of Sancho and his collaborators, updating them and redacting any topic as best as I know today, is to achieve a unified presentation of a significant part of the university teaching in Mathematics. And also to develop the courses with complete proofs, in a concise and coordinated way; mainly focusing on the logical and meaning interdependencies of the involved topics, devoid of anything that would put the central points in the shade. Hence the pages will be filled with definitions, statements and proofs, while with scarce examples, applications and motivations. And the aim of this preface is to bring these dry pages into life, to invite the reader to place them into the perspective of the much broader Grothendieck's oeuvre and life.

Fortunately Grothendieck has given us a very inspired and appealing description of its own mathematical works in the *Promenade à travers une oeuvre – ou l'enfant et la Mère*[1] at the very beginning of *Récoltes et Semailles*[2], a work where (following the path of Descartes, Pascal, Leibnitz,...) he has contributed to embed Mathematics as a significant part of a much broader spiritual adventure: man's self-conscience unfolding. In this *Promenade* he presents the relationship of any man with the spiritual goods under a double aspect. On the one hand, the luminous aspect (the passion for knowledge) represented by the figure of a child. On the other hand, the obscure aspect of *la peur et de ses antidotes vaniteux, et les insidieux blocages de la*

[1] Promenade through an oeuvre – or the child and the Mother.

[2] *Reapings and Sowings*, first published in french in 2022 by Gallimard.

creativité qui en derivent[3]. A deep and valuable lesson of the main body of *Récoltes et Semailles* is summarized at the end of the Introduction:

> *Si quelque chose pourtant est saccagé et mutilé, et desamorcé de sa force originelle, c'est en ceux qui oublient la force qui repose en eux-mêmes et qui s'imaginent saccager une chose à leur merci, alors qu'ils se coupent seulement de la vertue créatrice de ce qui est à leur disposition comme elle est à la disposition de tous, mais nullement à leur merci ni au pouvoir de personne.*[4]

This *Promenade* touched me deeply. First by the innocence of the child making mathematics, of our children untiringly asking *why?* and *what is it?* instead of the weary and clumsy *what is it for?* of adults. And above all by the silently, quite, attentive, feminine attitude of listening our whispering interior voice and readiness to accept it, resounding me Mary's fiat: *be it unto me according to your word.* But the exigence of solitude in any creative work shocked me, and when I wrote Grothendieck in 1987 that I missed it, he said me in a letter:

> *Vous soulignez, à juste titre, la difficulté psychique de la création solitaire... C'est là la situation qui a été la mienne plus ou moins pendant toute ma vie, depuis mon enfance, tant sur le plan de la création mathématique, que dans mon itinéraire spirituel... Et sans cela, rien de grand ne s'accomplit, ni dans l'aventure individuelle, ni dans l'aventure collective – que ce soit au plan intellectuel, ou au plan spirituel... Au niveau spirituel, la plus grande ouevre (à mes yeux) qu'un homme ait accomplie, était la Passion du Christ et sa mort sur la croix... Cette oeuvre était et ne pouvait être que solitaire. Et même c'était la solitude suprême, car Dieu Lui-même s'est retiré, pour que l'Oeuvre s'accomplisse sans le secours d'une consolation.*[5]

In fact any great creative labor usually has a harsh period of solitude, and Grothendieck and his parents had a very tough life, which is a paradigmatic and astonishing incarnation of the whole XXth century history. A life that we may foresee in the autobiographical chapters III, VI and VII of *La Clef des Songes*[6], a partially published work where he embeds mathematics into man's religious adventure. He says:

> *Les lois mathématiques peuvent être découvertes par l'homme, mais elles ne sont créés ni par l'homme ni même par Dieu. Que deux plus deux égale quatre n'est pas un décret de Dieu, qu'Il aurait été libre de changer en deux plus deux égale trois, ou cinq. Je sens les lois mathématiques comme faisant partie de la nature même de Dieu – une partie infime, certes, la plus superficielle en quelque sorte, et la seule qui*

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[3]fear and its vain antidotes, and the insidious blocks to creativity that derive from them (*Avant-propos*, p. A3).

[4]If anything, however, is despoiled and mutilated, defused of its original force, it is in those unaware of the force resting within themselves and who imagine they are despoiling something that is at their mercy, while they are only cutting themselves off from the creative virtue of what is at their disposal, just as it is at the disposal of everyone else, but in no way at their mercy or under the power of anyone (Introduction II.10, p. xxii).

[5]You remark, deservedly, the psychological difficulty of any lonely creation... That's the situation I have been in more or less all my life, since I was a child, both in terms of mathematical creation, and in my spiritual itinerary... And without that, nothing great is achieved, either in the individual adventure, or in the collective adventure – whether on the intellectual level, or on the spiritual level... On the spiritual level, the greatest oeuvre (in my eyes) that a man has accomplished was the Passion of the Christ and his death on the cross... This oeuvre was and could only be solitary. In fact, it was even the supreme solitude, since God Himself withdrew, so that the Oeuvre could be accomplished without the help of any consolation.

[6]*The Key of Dreams*, first published in french (without the Notes including "The Mutants") in 2024 by Éditions du Sandre.

soit accesible à la seule raison.[7]

And without this skin, without that flesh and blood, and without that fire resting inside us, the following hundreds of tight pages will only be a nude and dead skeleton.

Juan Antonio Navarro González

[7] Mathematical laws may be discovered by man, but they are not created by man, nor even by God. That two plus two equals four is not a decree of God that He is free to change into two plus two equals three, or five. I sense the mathematical laws as being part of the very nature of God – a tiny part, certainly, the most superficial in some sense, and the only part accessible to reason alone (III 31, *Les retrouvailles perdues...*, p. 100).

Contents

Introduction

On the occasion of the Symposium in memory of late Prof. J. Sancho Guimerá held at Salamanca in 2014, I decided to write some notes based on Sancho's lectures, with the same initial aim as Bourbaki at the 40's: to undertake the undergraduate mathematical teaching from the outset, with precise definitions, clear statements and complete proofs.

But, contrary to Bourbaki's book, without avoiding sheaves and categories.

Hence all theorems included in these notes have concise and complete proofs, but this is not a textbook for self-study of students in a standard sense:

1. It tries to reflect the terse style of the study notes of a student, always filled in with the explanations of a lecturer. And always assuming the atmosphere of each course, with some implicit assumptions and conventions, that surely will be clear to anyone who carefully reads any section from the beginning. However, the defined concepts are always written in boldface and they figure in the index of terms.

2. It focuses on the core of each course, the simple concepts and ideas that directly lead to the main results, leaving aside details, examples and applications that surely are required to assimilate the theory, but would put the central points in the shadow.

3. All the courses of a year must be studied simultaneously, as students actually do, and no logical order may be imposed upon them, each one massively using definitions, results and ideas from the companion chapters, sometimes placed at subsequent pages, due to the linear character of any book. It is non sense to read a chapter alone, and all the courses of a year must be studied simultaneously as a whole. So the course *Linear Algebra* uses some properties of polynomials, proved in the companion but posterior chapter *Algebra I*; the course *Analysis III* massively uses from the start smooth manifolds, vector fields and differential forms (and by the end metrics of constant curvature), studied in the companion but posterior chapter *Differential Geometry I*; and so on.

I do not pretend to develop each course in a self-contained way, without references to the previous or simultaneous courses. On the contrary, I pretend to enlighten the mutual connections, to show that it would be unnatural to parcel out mathematics in several unrelated fields (algebra, analysis, differential geometry, topology,...), that the main ideas are simple and everywhere fruitful, so reflecting the essential unity of Mathematics. Even if this choice force us to use concepts, results and ideas eventually placed in a posterior chapter, when all the courses of a year are written in a single book.

I have tried to be brief, since my aim is to exhibit many university courses, underlining the logical interdependencies and the central concepts and methods. To show that, when each course is developed with a regard to the resources and needs of the other parts of Mathematics, the main results have more natural and simple proofs. To show that, again and again, when a gentle hand provides the correct definitions, the adequate point of view, surprisingly the knots untie, the difficulties dissolve and the backbone of each theory may be exposed in a handful of pages.

I have been able to put black on white fifteen annual courses:

First Year

- **Analysis I:** Cardinal and ordinal numbers. Real and complex numbers. Metric spaces and topological spaces. Differential and integral calculus. Power series.
- **Linear Algebra:** Vector spaces and linear maps. The dual space. Euclidean vector spaces. Tensors and p-forms.
- **Algebra I:** The quotient ring. Principal ideal domains. Field extensions and roots. Rings of fractions. Elimination.

Second Year

- **Analysis II:** σ-compact spaces. The Lebesgue measure. Differential and integral calculus with several real variables. Convergence in rings of differentiable functions.
- **Algebra II:** Actions of a group. Modules. Categories. Spectrum of a ring. Module of differentials. Finite algebras over a field. Galois theory of fields.
- **Projective Geometry:** Projective and affine spaces. Classification of metrics, modules over a principal ideal domain, and pairs of metrics. Semisimple rings and the Brauer group.

Third Year

- **Commutative Algebra:** Noetherian rings. Primary decomposition. Completion. Dimension theory. Integral dependence. Dedekind domains. Galois theory of rings.
- **Topology:** Lattice semirings. Separation properties. Noetherian spaces. Compactifications. Dimension theory. Uniform spaces. Galois theory of coverings. The fundamental group.
- **Analysis III:** Rings of smooth functions. Ordinary differential equations. Pfaff systems. Integration of differential forms. De Rham cohomology. Riemann surfaces. Meromorphic functions. Riemann mapping theorem.
- **Differential Geometry I:** Smooth manifolds. Tensor fields and the exterior differential calculus. Linear connections. Riemannian metrics. Lie groups.

Fourth Year

- **Algebraic Geometry I:** Sheaf cohomology. Schemes and coherent sheaves. Algebraic curves and the Riemann-Roch theorem. The projective spectrum.
- **Algebraic Topology I:** The cohomology ring. Homological algebra. Local cohomology. Duality theorem. Characteristic classes. Spectral sequences.
- **Analysis IV:** Uniformization theorem. Fréchet spaces. Compact Riemann surfaces.
- **Differential Geometry II:** Valued differential calculus. Calculus of variations. Natural bundles. Chern classes and curvature.

Fifth Year

- **Algebraic Geometry II:** Local algebra. Quasi-coherent sheaves. K-theory and the Riemann-Roch-Grothendieck theorem. Duality theory. Grothendieck topologies and descent theory.

Many significant parts of the university teaching lack (functional analysis and partial differential equations, homology and homotopy theory, number theory, measure theory and statistics, mathematical physics...), but I think that these fifteen courses[8] suffice to show that many Grothendieck ideas may be introduced at the undergraduate level when the courses are developed in a coordinate way.

The first three years are mainly devoted to the concepts of *structure* and *morphism*, the central results being "Galois type" theorems stating the equivalence of two different structures (i.e. of two categories), while the last two years focus on the concepts of *sheaf* and *cohomology*, the central results being duality theorems. Moreover, representable functors and the concepts of *point* and *space* pervade the notes from start to finish, as well as the clarification of the adjectives *natural*, *intrinsic*, *canonical* and *universal* that we frequently use in a naive sense.

Here it is a brief description of the sense and content of each annual course:

Analysis I: We begin with the construction of integer, rational, real and complex numbers, the *genetic* method of reducing all mathematical concepts to the process of counting (sets and natural numbers); so initiating the accomplishment of the fantastic pythagorean vision[9]: *all is a music of numbers.* This genetic method is the backbone of Analysis and culminates in the theory of functions of several complex variables and the theory of partial differential equations.

We study cardinal numbers, and we put $|X| \leq |Y|$ when there is an injective map $X \to Y$. The main result is that any set of cardinal numbers has a first element. So, there is the first infinite cardinal $\aleph_0 = |\mathbb{N}|$, the first cardinal $\aleph_1$ bigger than $\aleph_0$,... and so we may parameterize the infinite cardinals by the ordinal numbers.

We construct the real numbers using Cauchy sequences of rational numbers, and generalize this method to complete any metric space. We also prove that the compact sets in $\mathbb{R}^n$ are just the bounded closed sets, a central result that readily proves that the image of any continuous function $f\colon [a,b] \to \mathbb{R}$ is a closed interval, $f([a,b]) = [m,M]$, a statement encoding the main results on continuous functions.

Now, since a real function f of a real variable is differentiable at a point $x = a$ when we have $f = f(a) + (x-a)u$ for some function u continuous at a, where $u(a) = f'(a)$, we show that the differential calculus readily follows from the properties of continuous functions. Then, after an exposition of the Riemann integral, we show that a bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable if and only if it is almost everywhere continuous. We end with an introduction to the power series expansions, and we use them to study the exponential and trigonometric functions.

Linear Algebra: In this chapter we take the path of the *axiomatic* method. Geometry bursts luminously in Greece when the Greek genius realized that properties of such subtle and fundamental concepts as point, line, right angle, etc., should not be reduced to properties of previous concepts, but that we should better put on show their basic mutual relations, stating them as axioms or postulates of the theory, and then derive the remaining properties. In modern terms, any question is submerged in an implicit structure, and a breaking point is to put it into light. Besides, in the 19th century the German world discovered that it is crucial to clarify what structures are considered to be equal (the isomorphisms) and it placed as the central problem of any theory the classification of all possible structures up to isomorphisms[10]. As a first example of

[8]Essentially reflecting courses that I took in the 70s at the University of Salamanca, except for *Analysis* I, II and IV. These three courses are by far the weakest and most questionable part of the present notes, but I have include them so that all the theorems of these notes have complete proofs.

[9]Aristotle (*Metaphysics*, Book I, Part 5, 986a): *τα των αριθμων στωιχεια των οντων στωιχεια παντων υπελαβον ειναι, και τον ολον ουρανον αρμονιαν ειναι και αριθμον*. ([The pythagoreans supposed] the numbers to be the elements of all things, and the whole heaven to be a musical scale and a number).

[10]One of the first to glimpse such classification problems was Goethe, as he wrote in his diary entries (Naples,

the great insight of the German point of view, just compare the genetic definition of the addition of integer numbers given in p. 15 with the definition steaming from the classification of cyclic groups (p. 46): it is the unique possible operation defining a structure of infinite cyclic group (so recovering our infantile vision: *what* is added does not matter, only *how* to do the addition).

After a very concise introduction to the structures of group and ring, the course focuses on the structure of vector space (so that we may exhibit for the first time the structure of the Euclidean Geometry[11], at least with a fixed point and a fixed length unit) and we prove that vector spaces are classified by the dimension, whenever it is finite.

A big emphasis is put on universal properties (our first contact with representable functors) which simplify the theory in many points. For example in the intrinsic definition of the contraction of indices, otherwise cumbersome and in need of some checking, and in the proof of the natural isomorphism $(\Lambda^p E)^* = \Lambda^p E^*$. The systematic use of linear maps and exact sequences from the very beginning of the course also simplifies the theory.

Algebra I: This chapter is devoted to the study of the structure of ring. We understand it as a generalized arithmetic: numbers are ideals and the divisibility relation is the inclusion relation, so that prime numbers are maximal ideals, the quotient ring encoding the gaussian theory of congruences. The main result is Kronecker's formula for a root of any polynomial equation $p(x) = 0$, discovered when he realized that the extraction of a root involves two quite different steps. The first one, subtle, humble and usually of long gestation, is the highlight of a structure of field (radical expressions, real or complex numbers, etc.); while the second one is to

May 17, 1787): *Die Urpflanze wird das wunderlichste Geschöpf von der Welt, um welches mich die Natur selbst beneiden soll. Mit diesem Modell und dem Schlüssel dazu kann man alsdann noch Pflanzen ins Unendliche erfinden, die konsequent sein müssen, das heißt, die, wenn sie auch nicht existieren, doch existieren könnten und nicht etwa malerische oder dichterische Schatten und Scheine sind, sondern eine innerliche Wahrheit und Notwendigkeit haben.* (The *Urpflanze* is going to be the most wonderful creation of the world, which Nature herself shall envy me. With this model and the key to it, one can then go on inventing plants ad infinitum, which must be consistent; that is, even if they do not exist, they could exist and not as picturesque or poetic shadows and illusions, but with inner truth and necessity). In his quiet walks along the Italian botanical gardens, he dreamed about the plant structure and its possible realizations, so that he could know all the plants that Linnaeus had catalogued, the unknown plants of Africa and America that brave and courageous explorers were looking for, the extinct plants, the future ones and those that will remain in God's mind forever. Even if he could not achieve this wonderful dream, it is not a mere coincidence that the periodic table of elements and the first classifications of mathematical structures and elementary particles were so close to Goethe, in time, space and spirit.

[11]Stricto senso, the higher dimensional real Euclidean geometries studied in this course were the first "Non Euclidean" geometries, glimpsed in 1747 by Immanuel Kant (*Gedanken von der wahren Schätzung der lebendigen Kräfte* §10–11): *Diesem zu folge halte ich dafür: daß die Substanzen in der existirenden Welt, wovon wir ein Theil sind, wesentliche Kräfte von der Art haben, daß sie in Vereinigung miteinander nach dem doppelten umgekehrten Verhältniß der Weiten ihre Wirkungen von sich ausbreiten; zweitens, daß das Ganze, was daher entspringt, vermöge dieses Gesetzes die Eigenschaft der dreifachen Dimension habe; drittens, daß dieses Gesetz willkürlich sei, und da Gott dafür ein anderes, zum Exempel des umgekehrten dreifachen Verhältnisses, hätte wählen können; daß endlich viertens aus einem andern Gesetze auch eine Ausdehnung von andern Eigenschaften und Abmessungen geflossen wäre. Eine Wissenschaft von allen diesen möglichen Raumesarten wäre unfehlbar die höchste Geometrie, die ein endlicher Verstand unternehmen könnte... Wenn es möglich ist, daß es Ausdehnungen von andern Abmessungen gebe, so ist es auch sehr wahrscheinlich, daß sie Gott wirklich irgendwo angebracht hat. Denn seine Werke haben alle die Größe und Mannigfaltigkeit, ist, daß es die sie nur fassen können.* (Thoughts on the True Estimation of Living Forces §10–11: Accordingly, I am of the opinion that substances in the existing world, of which we are a part, have essential forces of such a kind that they propagate their effects in union with each other according to the inverse square of the distances; secondly, that the whole to which this gives rise has, by virtue of this law, the property of being three dimensional; thirdly, that this law is arbitrary, and that God could have chosen another, e.g., the inverse-cube, relation; fourthly, and finally, that an extension with different properties and dimensions would also have resulted from a different law. A science of all these possible kinds of space would undoubtedly be the highest geometry that a finite understanding could undertake... If it is possible that there are extensions of different dimensions, then it is also very probable that God has really produced them somewhere. For His works have all the greatness and diversity that they can possibly contain).

express a root inside such structure. But men were looking for a root without questioning the convenience of the structure where they limited the search, enclosed in an invisible iron circle[12] that Kronecker innocently crossed when he discovered that, if $p(x)$ is irreducible, an obvious root is just the class $[x]$ in the field of residue classes modulo $p(x)$, so reducing the extraction of roots to the problem of decomposing polynomials into irreducible factors.

At first glance Kronecker's astonishing answer may seem a *mere tautology* or a *sterile formality.* Certainly it is a tautology presented with all the formal rigor of which we are able, as any other theorem; but, after applying it to prove D'Alembert's and Hamilton-Cayley theorems (p. 80 and p. 70), and to solve the millennial questions on ruler-and-compass constructions (p. 82), we leave to the reader the consideration that the adjectives *mere* and *sterile* deserve.

Analysis II: After a brief study of topological spaces, we study the differential calculus with functions of several real variables. The main tool is a key lemma in the calculus with infinitesimals: *If f is a function of class $\mathcal{C}^m$ on an open ball U around the origin $0 \in \mathbb{R}^n$ and $f(0) = 0$, then $f = \sum_i f_i x_i$ for some functions f_i of class $\mathcal{C}^{m-1}$.* This key lemma is proved using Barrow's rule and the differentiation rule under the integral sign, and it readily gives Schwarz's theorem, $\partial_i\partial_j f = \partial_j\partial_i f$, Taylor's expansions and also the inverse mapping theorem, using the relative version along the diagonal: *any function $f \in \mathcal{C}^m(U \times U)$, vanishing on the diagonal, is $f = \sum_i f_i \cdot (y_i - x_i)$ for some functions $f_i \in \mathcal{C}^{m-1}(U \times U)$.*

Then we study the Lebesgue measure and the corresponding integration of functions, the main results being the change of variables formula and Sard's theorem. We prove both using the key lemma along the diagonal, that shows that in a neighborhood of a point p, the image of any cube is very close to the image by the tangent linear map at p.

Finally, we introduce topological vector spaces and we prove that $\mathcal{C}^m(U)$ is complete with the topology of the compact convergence of the functions and the iterated partial derivatives.

Algebra II: This is a crucial course with an explosion of many central ideas and constructions that will pervade the remaining chapters.

1. A big step is given in the clarification of the concepts of *point* and *function.* Any commutative ring A defines a topological space $\operatorname{Spec} A$ whose points correspond to the prime ideals of A, the elements of A are viewed as functions on $\operatorname{Spec} A$, and the ideals of A as equations of geometric loci. Integers, gaussian integers, etc., may be understood as functions, and we may apply them our geometric intuitions and resources, so starting the unification of Arithmetic and Geometry dreamed by Kronecker.

2. Localization of modules, proving that most properties of modules are local[13], in the sense that they only have to be checked on the local rings of the points of $\operatorname{Spec} A$.

3. Tensor products. Hence the base change of algebras and modules, so that we may introduce the concepts of *geometric* property (stable under changes $k \to K$ of the base field) and *local* property (valid whenever it holds after some base change $k \to K$).

4. Categories and functors, aiming to give a first rigorous approximation to the fundamental concepts of *structure, natural* and *canonical.*

[12]Our questions always have an implicit horizon embracing all the possible answers. To make it clear, and to extend it if necessary, are the strong moments of man's self-conscience unfolding.

[13]A basic fact that we constantly use throughout all the notes, even without further mention. Surprisingly, any other property is easily reduced to the critical property of being 0. Here, as in any other case, nothingness, as soon as it is senseful, becomes a driving force.

5. Points of Spec A with coordinates in a field K are viewed as morphisms Spec $K \to$ Spec A and, in general, Grothendieck's representability theorem gives necessary and sufficient conditions for a functor to be a functor of points.

6. The module of differentials, giving an algebraic treatment of the differential calculus.

7. Injective and projective modules, a basic tool of the forthcoming Homological Algebra.

The central result is Galois theorem, stated as an antiequivalence of the category[14] of k-algebras trivial on a given Galois extension $k \to L$ of Galois group G with the category of finite G-sets. The proof[15] directly follows from the fundamental exact sequence $k \to A \rightrightarrows A \otimes_k A$ and the stability of the algebra of invariants A^G under base changes, $(A^G) \otimes_k L = (A \otimes_k L)^G$.

We also introduce the Frobënius automorphism of a polynomial $q(x) \in \mathbb{Z}[x]$ at a prime p, so relating the global properties of $q(x)$ to the behavior of its reductions $\bar{q}(x) \in \mathbb{F}_p[x]$.

Finally, we prove that the maximal separable subalgebra is stable under base changes $k \to K$, and the elementary theory of inseparable k-algebras readily follows.

Projective Geometry: We introduce the projective space $\mathbb{P}(E)$ attached to a vector space E, but only as a mere set, without elucidating the underlying structure. To figure out the implicit structure grounding projective geometry has been a major topic in the history of Geometry, and so it will be in these notes. Nevertheless, in this course at least we know the group of automorphisms of such elusive structure: an isomorphism of a k-vector space E into a k'-vector space E' clearly should be defined as a group isomorphism $f\colon E \to E'$ such that $f(\lambda e) = \sigma(\lambda) f(e)$ for some ring isomorphism $\sigma\colon k \to k'$; hence the group of automorphisms of $\mathbb{P}(E)$ is the group $PSl(E)$ of all bijections $\mathbb{P}(E) \to \mathbb{P}(E)$ represented by a semilinear automorphism $E \to E$.

Klein, in his foundational *Erlangen Programme*, states that the structure of the projective space is given by the action of the group $PGl(E)$ of all projectivities[16] on $\mathbb{P}(E)$, and that in general any action of a group G on a set defines a geometry: concepts of the geometry being just G-invariant concepts, statements being relations between concepts, and theorems being true relations. Considering different subgroups of $PGl(E)$, in this course we also study affine geometry, Euclidean geometry and Non-Euclidean geometries. In this course we show that the structure of the projective spaces of dimension ≥ 2 is captured by the lattice of linear subvarieties[17]; but to figure out the structure of the real projective line as a ringed space (so grounding projective geometry on non-algebraically closed fields) was a first success of the theory of schemes.

On the other hand, any concept of a geometry gives rise to a classification problem, considered as the determination of the set of orbits of the induced G-action. In this course we study the projective and affine classifications of quadrics (reduced to the linear classification of symmetric metrics), the projective classification of projectivities (reduced to the linear classification of endomorphims) and the projective classification of pencils of quadrics (reduced to the classification of metrics on modules over the finite k-algebras $k[x]/(p(x))$ considered in the companion course *Algebra II*). In this sense, this is a second course in Linear Algebra.

[14]In fact the category of finite direct sums of intermediate fields, so that the theorem not only determines the intermediate fields but also their k-morphisms.

[15]Two alternative proofs are sketched in exercises 112b and 112c, p. 508. The first one reduces the proof to the simple case $k = L$, once the exactness properties of the involved functors are checked; while the second one uses Grothendieck's characterization of the forgetful functor on the category of finite G-sets.

[16]so that the group of automorphisms of $\mathbb{P}(E)$ is just the normalizer of $PGl(E)$ in the group of all bijections $\mathbb{P}(E) \to \mathbb{P}(E)$, which may be shown (ex. 7, p. 513) to be $PSl(E)$.

[17]The so called fundamental theorem of projective geometry (p. 165); but today such fundamental theorem, illuminating the structure of projective spaces, is the universal property of $\mathbb{P}(E)$ given in p. 335.

We classify endomorphisms, and finitely generated modules over a principal ideal domain A, using the crucial fact that $B = A/p^nA$ is an injective B-module when p is irreducible, a result that follows directly from the Ideal Criterion studied in *Algebra II.*

It is worthy of attention that, in the former course *Linear Algebra*, the characteristic polynomial of an endomorphism was introduced in a non-intrinsic way, defining it in a base and checking that it does not depend on the base. Now we introduce the K-group to give Grothendieck's astonishing definition of the characteristic polynomial: it is the universal additive function on the finitely generated torsion $k[x]$-modules.

Finally, we study semisimple rings and the Brauer group, obtaining that the quaternions are the unique non-commutative finite extension of the real numbers.

Commutative Algebra: First we introduce the "structural sheaf" of the spectrum of a ring, so that we may present all the topics with its full geometric comprehension and flavor:

1. Primary decomposition of ideals and completion.

2. Krull dimension theory and regular rings.

3. Finite morphisms and birrational finite morphisms, obtaining the desingularization of curves, and the calculation of intersection numbers, by quadratic transformations.

Finally we show that the theory of unramified finite flat ring morphisms $A \to B$ is totally analogous to the Galois theory of separable finite k-algebras and that, when considering the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, it is totally analogous to the theory of coverings of a topological space, so obtaining the definition of the fundamental group $\pi_1(\operatorname{Spec} A, p)$ of $\operatorname{Spec} A$ at a geometric point $p\colon \operatorname{Spec} \bar{k} \to \operatorname{Spec} A$.

In these notes the Galois theory of fields, of noetherian rings and of coverings of a topological space, have a unified presentation, with equal statements and proofs.

Topology: When studying continuous functions, the particular value of a function f at a point x usually does not matter, only wether $f(x) = 0$ or $f(x) \neq 0$. So it is natural to identify all the non-zero real numbers, and so we obtain $\mathbb{K} = \{0, g\}$, with a closed point 0 and a dense point g (the generic real number). We have a natural structure of semifield[18] with unity $g = 1$ on $\mathbb{K}$,

$$\begin{array}{cccccc} 0+0=0 & , & 0+1=1+0=1 & , & 1+1=1 & , \\ 0\cdot 0=0 & , & 0\cdot 1=1\cdot 0=0 & , & 1\cdot 1=1 & . \end{array}$$

In any topological space X, we may consider the "ring" $A(X)$ of all continuous functions $X \to \mathbb{K}$, which is not a true ring, since the existence of opposite element fails, but a semiring with the extra properties $1 + a = 1$ and $a^2 = a$. We name them *lattice semirings.* In fact, since any open set U in X is the cozero-set of a unique continuous function $\chi_U\colon X \to \mathbb{K}$, we see that $A(X)$ is canonically isomorphic to the lattice of all open sets in X, the addition corresponding to the union and the product to the intersection.

Now, many operations introduced in the elementary theory of rings (quotients, localizations, tensor products, spectra, etc.) remain meaningful in the framework of lattice semirings, and they preserve their geometric meanings. Moreover, most statements and proofs remain valid, and the purpose of this chapter is to show that this semiring of continuous functions $A(X)$ provides a handy tool in some topics:

1. The functor Spec clarifies the theory of compactifications.

[18] In a suggestive way, Connes uses to say that semirings where $1 + 1 = 1$ have characteristic 1, so that $\mathbb{K}$ is a semifield of characteristic 1.

2. Finite topological spaces are dual to finite lattice semirings, and essentially encode the theory of polyhedra.

3. Typically the Krull dimension of $A(X)$ is infinite, but Sancho realized that the minimal Krull dimension of a base[19] $B \subseteq A(X)$ provides a good definition for the dimension of a topological space X, since it coincides with Grothendieck's dimension (the maximal length of a chain of irreducible closed sets) when X is a noetherian space, and with the inductive dimension and Lebesgue cover dimension when X is a separable metric space.

Finally we study uniform spaces, and the theory of coverings $X \to S$ is developed parallel to the Galois theory of fields and rings, taking advantage of the existence of a universal covering with Galois group $\pi_1(S, p)$.

Analysis III: First we study partitions of unity and we show how a smooth manifold X may be reconstructed from the ring of smooth functions $\mathcal{C}^\infty(X)$, the basic facts being the natural homeomorphism $X = \operatorname{Spec}_{\mathbb{R}} \mathcal{C}^\infty(X)$ and the coincidence of $\mathcal{C}^\infty(U)$ with the ring of fractions of $\mathcal{C}^\infty(X)$ with respect to the multiplicative system of all smooth functions without zeros in the open subset U.

In the theory of ordinary differential equations, we show that the argument of the contractive map, classically used to prove the existence and uniqueness of a solution, also proves the continuous and differentiable dependence on the initial conditions when considering adequate functional spaces.

Then we study Pfaff systems, the central result being a geometrically obvious proof of a projection theorem given necessary and sufficient condition for a Pfaff system $\mathcal{P}$ on a smooth manifold X to be projectable by a regular projection $\pi\colon X \to Y$, in the sense that $\mathcal{P} = \pi^*\mathcal{Q}$ for some Pfaff system $\mathcal{Q}$ on Y. As a direct consequence we obtain Frobënius theorem (involutive distributions are integrable) and Darboux's classification of germs of 1-forms.

We also study the integration of differential forms on smooth manifolds, obtaining the fundamental Stoke's theorem. We apply it to harmonic functions, De Rham cohomology and the theory of complex analytic functions.

Finally, we introduce Riemann surfaces as ringed spaces. It is easy to prove that locally any non constant analytic morphism is $u = z^n$ for some exponent $n \geq 1$ and, as a direct consequence of this local classification of morphisms, most elementary properties of Riemann surfaces follow. We prove the Riemann mapping theorem with a systematic use of the hyperbolic plane geometry.

Differential Geometry I: We introduce smooth manifolds as ringed spaces X locally isomorphic to open sets in Euclidean spaces with the sheaf of smooth functions (so eliminating the cumbersome atlases) and submanifolds as topological subspaces $Y \hookrightarrow X$ such that $(Y, \mathcal{C}^\infty_Y)$ is a smooth manifold, where $\mathcal{C}^\infty_Y$ is the sheaf of restrictions of smooth functions on X. Then we develop the tensor calculus and the exterior differential calculus on smooth manifolds.

We study linear connections, the main result being that any torsion free linear connection of null curvature is locally Euclidean, and riemannian manifolds, proving that any simply connected and complete riemannian surface of constant curvature is isometric to the Euclidean plane, the hyperbolic plane or the sphere. Then we present the theory of riemannian embeddings.

Finally we present a bit of the theory of Lie groups, up to the point of obtaining the classification of abelian Lie groups.

Algebraic Geometry I: This chapter may be considered the hearth of these notes. We introduce sheaves and we define the cohomology groups $H^p(X, \mathcal{F})$ using Godement's resolution.

[19] a subring whose cozero-sets form a base of open sets of X in the usual sense.

Then we prove a cohomological bound theorem

$$H^p(X, \mathcal{F}) = 0, \ p > \dim X,$$

when X is a noetherian space, and the acyclicity theorem for arbitrary rings,

$$H^p(\operatorname{Spec} A, \widetilde{M}\,) = 0, \ p > 0.$$

Now, after a massive use of the spectrum of a ring in the courses *Algebra II, Topology* and *Commutative Algebra*, and the comprehension of smooth manifolds and Riemann surfaces as ringed spaces in *Differential Geometry I* and *Analysis III*, we may naturally introduce schemes as ringed spaces locally isomorphic to $(\operatorname{Spec} A, \widetilde{A}\,)$. The major example of scheme in this course will be the Riemann variety of a finite extension of $k(t)$, the Riemann variety of $k(t)$ itself being the projective line $\mathbb{P}_1$ over the field k.

First we calculate the cohomology groups of the structural sheaf of $\mathbb{P}_1$, projecting it onto a finite space with two closed points and a dense point, and it readily follows the determination of the cohomology groups of the line sheaves $\mathcal{O}_{\mathbb{P}_1}(n)$. Then we prove the fundamental finiteness theorem for coherent sheaves, $\dim H^p(X, \mathcal{M}) < \infty$, projecting the curve X onto $\mathbb{P}_1$, since the direct image preserves cohomology.

Now the weak Riemann-Roch theorem for line sheaves L_D is immediate,

$$\dim H^0(X, L_D) - \dim H^1(X, L_D) = 1 - g + \deg D.$$

Since, $H^0(X, L_{K-D}) = \operatorname{Hom}_{\mathcal{O}_X}(L_D, L_K)$, to prove the strong Riemann-Roch theorem we only have to see that

$$\dim H^1(X, L_D) = \dim \operatorname{Hom}_{\mathcal{O}_X}(L_D, L_K),$$

where K stands for a canonical divisor of the curve X. Since $H^2(X, \mathcal{M}) = 0$, Grothendieck's representability theorem directly gives the existence of a dualizing sheaf:

$$H^1(X, \mathcal{M})^* = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{D}_X)$$

for some quasi-coherent sheaf $\mathcal{D}_X$, and the problem is to determine $\mathcal{D}_X$.

It is easy to show that $\mathcal{D}_X$ is a line sheaf, but it is hard to prove that it is the sheaf of differentials. We present two proofs of this fundamental result, the first one based on a laborious local calculation of the conductor of a projection $X \to \mathbb{P}_1$, while the more natural second proof uses the diagonal embedding $X \to X \times_k X$ and the stability of the cohomology of coherent sheaves under base changes (so pointing that the theory of curves is naturally entangled with the theory of higher dimensional varieties).

Finally we introduce the projective spectrum of a graded ring, and determine the cohomology groups of the line sheaves $\mathcal{O}_{\mathbb{P}_d}(n)$ over the projective space $\mathbb{P}_d = \operatorname{Proj} k[x_0, \dots, x_d]$.

Algebraic Topology I: This chapter is devoted to the cohomology of sheaves over σ-compact spaces. After introducing the fundamental exact sequences (Mayer-Vietoris, closed subspace and local cohomology exact sequences) we determine the cohomology groups $H^p([0,1]^n, \mathbb{Z})$ of cubes, hence of spheres and many classical spaces.

Then we introduce the inverse image and the cup product, and we prove some fundamental results: calculation of the cohomology of the fibres, theorem of base change, finiteness theorem, universal coefficients formula, Künneth theorem, and the classification of line sheaves. We also use local cohomology groups to define the cohomology class of a closed submanifold and to develop a topological intersection theory.

Now we get down to the duality theorem. So as to include non locally Euclidean spaces, we do not consider the dual of the last cohomology group $H^n_c(X, \mathcal{F})$, but of a complex determining all the cohomology groups with compact supports. Again the representability theorem gives the existence of a dualizing complex $\mathcal{D}^\bullet_X$ such that

$$\mathbf{R}\mathrm{Hom}(\mathbf{R}\Gamma_c(\mathcal{F}), \mathbb{Z}) = \mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathcal{D}^\bullet_X)$$

and in this case it is not hard to determine it when X is a topological manifold of dimension n: the dualizing complex $\mathcal{D}^\bullet_X$ has a unique non-zero cohomology sheaf[20] $\mathbb{T}_X$, locally constant and placed at degree $-n$, so that $\mathrm{Hom}_{\mathbb{Z}}(H^n_c(X, \mathcal{F}), \mathbb{Z}) = \mathrm{Hom}(\mathcal{F}, \mathbb{T}_X)$.

Then we determine the cohomology groups of a projective bundle, a result grounding the theory of characteristic classes for vector bundles.

Finally we introduce some useful spectral sequences.

Analysis IV: After using Perron's method to solve the Dirichlet problem, we use it and sheaf cohomology to prove the uniformization theorem: any simply connected Riemann surface is the complex plane, the complex projective line or the unit disk.

Then we present a brief introduction to Fréchet spaces and the duality theory of locally convex spaces, up to the point of obtaining Schwartz theorem, the crucial technical tool to prove (using Cech cohomology) that locally free sheaves on Riemann compact surfaces always have finite dimensional cohomology groups.

This finiteness theorem readily gives the existence of non constant meromorphic functions, and the equivalence of the category of compact Riemann surfaces to the category of complete and non singular algebraic curves over the field $\mathbb{C}$, studied in the companion course *Algebraic Geometry I*. Hence, most of the results there obtained for complete non singular curves over arbitrary fields may be extended to compact Riemann surfaces, in particular the fundamental Riemann-Roch theorem. However, we show that now the residue theorem provides a very simple and natural proof of the coincidence of the dualizing sheaf with the sheaf of analytic differentials, the fundamental result that was so hard to prove for algebraic curves.

Differential Geometry II: Many mathematical and physical concepts may be viewed as differential forms with values in a vector bundle (or a locally free sheaf, the point of view adopted in this chapter) and the differential calculus with such forms requires a linear connection on the vector bundle. We develop this fruitful differential calculus up to the point of obtaining Bianchi's identities and Cartan's structure equations. We also determine the Chern classes of a complex vector bundle in terms of the curvature 2-form of any linear connection.

We present the variational calculus, obtaining the fundamental Poincaré-Cartan form and Noether's theorem on the infinitesimal symmetries of a variational problem.

Finally, since most bundles used in Geometry and Physics are natural bundles, we include the Galois theory of natural bundles (of a given finite order). The statements are parallel to those of the Galois theory of fields, rings and coverings, but unfortunately proofs differ.

Algebraic Geometry II: We begin with a study of Local Algebra, obtaining Serre's theorem on regular rings and some characterizations of Cohen-Macaulay rings. Then, after a brief study of quasi-coherent sheaves (including Deligne's formula and the base change theorem) we get down to the central topics of this course: the Riemann-Roch and duality theorems.

[20]that would be obtained in case of dualizing the last cohomology group, as in *Algebraic Geometry I*.

1. First we study the K-theory of coherent sheaves up to the point of obtaining Chern classes of vector bundles with values in the graded K-theory $GK^\bullet(X)$. Then we prove the fundamental result: the K-theory is the universal multiplicative[21] cohomology theory.

   Now, if a cohomology theory $A(X)$ follows the additive law $c_1(L\otimes L') = c_1(L) + c_1(L')$, then we may modify the direct image in $A(X)\otimes\mathbb{Q}$ with an exponential, so that it follows the multiplicative law and we have a functorial ring morphism $K(X)\to A(X)\otimes\mathbb{Q}$ preserving the new direct image. This is just the Riemann-Roch-Grothendieck theorem.

   A direct consequence, in the complex case, is the existence of a natural ring morphism $GK^\bullet(X)\otimes\mathbb{Q}\to H^{2\bullet}(X_{\rm an},\mathbb{Q})$ preserving inverse and direct images, hence Chern classes. Since $GK^0(\operatorname{Spec}\mathbb{C})\otimes\mathbb{Q} = H^0(\mathrm{pt},\mathbb{Q}) = \mathbb{Q}$, the algebraic and topological definitions of any numerical cohomological invariant coincide. For example, if X is a projective smooth variety, the topological and algebraic self-intersection numbers of the diagonal coincide,

   $$\textstyle\sum_i(-1)^i\dim_{\mathbb{Q}}H^i(X_{\rm an},\mathbb{Q}) = \sum_{p,q}(-1)^{p+q}\dim_{\mathbb{C}}H^p(X,\Omega^q_X).$$

2. Given a projective morphism $f\colon X\to S$, the representability theorem gives the existence of a dualizing complex $\mathcal{D}_{X/S}$ such that for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$,

   $$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_S}(\mathbf{R}f_*(\mathcal{M}),\mathcal{O}_S) = \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{D}_{X/S})$$

   and again the crux is to calculate it in the case of a smooth morphism; to prove that it is the highest exterior power of the sheaf of differentials, $\mathcal{D}_{X/S}\simeq\Omega^d_{X/S}$.

   Using the Koszul complex, we may see that the dualizing sheaf of a closed regular embedding $Y\to X$ of codimension d is just $\Lambda^d N_{Y/X}$, where $N_{Y/X}$ stands for the normal bundle. Now the calculation of the dualizing sheaf of a smooth projective morphism $X\to S$ is a simple question. Just consider the composition

   $$X \xrightarrow{\ \Delta\ } X\times_S X \xrightarrow{\ \pi_1\ } X$$

   of the diagonal embedding with the first projection (obviously it is the identity), and the normal bundle N to the diagonal embedding (which is dual to the sheaf of differentials $\Omega_{X/S}$ by the very definition). We have

   $$\mathcal{O}_X = \mathcal{D}_{X/X} = \mathcal{D}_{X/X\times X}\otimes\Delta^*\mathcal{D}_{X\times X/X} = \Lambda^d N\otimes\Delta^*(\pi_1^*\mathcal{D}_{X/S}) = \Lambda^d N\otimes\mathcal{D}_{X/S}$$

   and we are done: $\mathcal{D}_{X/S} = (\Lambda^d N)^* = \Omega^d_{X/S}$. Once the question is placed in the relative and general setting (morphisms and arbitrary dimension instead of the absolute case of curves) the obvious compatibility properties of the theory dissolve the question.

Then we prove the local duality theorem. Again, dualizing the local cohomology groups at a point x, the representability theorem directly gives the existence of a local dualizing complex, and the central point is to show that, in the case of a projective variety, it is just the stalk at x of the global dualizing complex. In the crucial case of a smooth variety X, we prove it determining the local cohomology groups $H^p_x(X,\mathcal{O}_X)$.

Finally, we introduce Grothendieck topologies, proving that quasi-coherent $\mathcal{O}_S$-modules and S-schemes define sheaves on the category of S-schemes with the fpqc topology, and that locally quasi-coherent sheaves are quasi-coherent (descent of quasi-coherent sheaves).

[21] in the sense that Chern classes of line sheaves follow the law $c_1(L\otimes L') = c_1(L)+c_1(L') - c_1(L)c_1(L')$ of the multiplicative group. Remark that $(1-x)(1-y) = 1-(x+y-xy)$.

Requirements: From a logical point of view, only the elementary properties of natural numbers and the intuitive set theory (including Zorn's lemma) are required, and further foundational questions are avoided. In fact, I feel that Zermelo-Fraenkel's axioms do not ground properly set theory because they assume more structure than required: they include unions of arbitrary sets and assume that the elements of a set are always sets, while in mathematics, only unions of subsets of a given set are sensible, and it is non-sense to consider the elements of the elements of any given set[22].

However, along the five years, an increasing maturity level is assumed.

Notations: Given a map $f\colon X \to Y$, the image of an element $x \in X$ is denoted by $f(x)$, and the composition with a map $g\colon Y \to Z$ is denoted by $gf = g \circ f$, so that $(gf)(x) = g(f(x))$.

Unless otherwise stated, any action of a group on a set, of a ring on an abelian group, etc. is assumed to be a left action, so that $\rho_{ab} = \rho_a \circ \rho_b$ (where ρ_a denotes the action of a).

The empty set is denoted by $\emptyset$ and the one point set by $*$.

When we have a binary operation with a neutral element e, such operation is assumed to be e in the case of an empty family of elements:

$\sum_{n\in\emptyset} a_n := 0 \quad , \quad \prod_{n\in\emptyset} a_n := 1.$

$\bigcup_{i\in\emptyset} X_i := \emptyset \quad , \quad \bigcap_{i\in\emptyset} X_i := X,$ when $\{X_i\}_{i\in I}$ is a family of subsets of a set X.

$\coprod_{i\in\emptyset} X_i := \emptyset \quad , \quad \prod_{i\in\emptyset} X_i := *.$

$\bigoplus_{i\in\emptyset} E_i = 0 \quad , \quad E^0 = 0.$

Moreover, we assume that there is a unique map $\emptyset \to X$ for any set X, and no map $X \to \emptyset$, except for $X = \emptyset$, the unique map $\emptyset \to \emptyset$ being the identity. In particular $0^0 := 1$.

Exceptionally we use some non-standard notations. The covariant derivative of a tensor field T in the direction of a vector field D is denoted $D^\nabla T$, instead of the usual $\nabla_D T$; and the Lie derivative $D^L T$, instead of the usual $L_D T$.

Sometimes T_p^q denotes a tensor of type (p, q); so $R_{2,2}$ denotes the Riemann-Christoffel tensor, R_2 denotes the Ricci tensor, a p-form is Ω_p, a metric is S_2, and so on.

When a fixed map $f\colon X \to Y$ is unambiguously understood, and $Z \subseteq Y$, we put

$$X \cap Z := f^{-1}(Z) = \{x \in X : f(x) \in Z\},$$

so that $X \cap Z$ is a subset of X but not a subset of Z.

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Also my gratitude to so many from whom I have taken ideas, examples and problems that are everywhere in these notes without further mention.

[22]So, even if the spectrum $\operatorname{Spec} A$ of a ring A is defined to be the set of all prime ideals of A, the statements $0 \in x \in \operatorname{Spec} A$ and $1 \notin x \in \operatorname{Spec} A$ are non-sense: points of $\operatorname{Spec} A$ are not prime ideals of A, but naturally *correspond* to prime ideals. Analogously the elements of a quotient set are not equivalence classes, but naturally correspond to equivalence classes...

Part I

First Year

Chapter 1

Analysis I

1.1 Integer and Rational Numbers

Definitions: A relation $\equiv$ on a set X is an **equivalence relation** if it is

1. *Reflexive:* $x \equiv x$, $\forall x \in X$.
2. *Symmetric:* $x, y \in X$, $x \equiv y \;\Rightarrow\; y \equiv x$.
3. *Transitive:* $x, y, z \in X$, $x \equiv y$, $y \equiv z \;\Rightarrow\; x \equiv z$.

The **equivalence class** of $x \in X$ is defined to be $\bar{x} = [x] = \{y \in X \colon x \equiv y\}$.
A subset $C \subseteq X$ is an equivalence class when $C = [x]$ for some $x \in X$.
The **quotient set** $X/\!\equiv$ is the set of all equivalence classes (or even better, it is a set whose elements correspond to the equivalence classes).
The surjective map $\pi \colon X \to X/\!\equiv$, $\pi(x) = [x]$, is the **canonical projection**.

Theorem: *In $X/\!\equiv$, only equivalent elements are identified;* $[x] = [y] \Leftrightarrow x \equiv y$.

Proof: If $[x] = [y]$, then $y \in [y] = [x]$, and $x \equiv y$.
Conversely, if $x \equiv y$, since $\equiv$ is reflexive, it is enough to show that $[y] \subseteq [x]$.
Now, if $z \in [y]$, then $y \equiv z$; hence $x \equiv z$ and $z \in [x]$.

Corollary: *Each element of X is in a unique equivalence class.*

Proof: If $x \in [y]$, then $y \equiv x$; hence $[y] = [x]$.

Construction of $\mathbb{Z}$: Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of **natural** numbers. The set of **integer** numbers $\mathbb{Z}$ is defined to be the quotient set of $\mathbb{N} \times \mathbb{N}$ by the equivalence relation

$$(m, n) \equiv (m', n') \quad \text{when} \quad m + n' = m' + n,$$

and $m - n$ denotes the class of (m, n). Any natural number n defines an integer number $n - 0$, and so we may identify $\mathbb{N}$ with a subset of $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
The **sum** and the **product** of $a = m - n$, $b = r - s$, are defined to be

$$a + b = (m + r) - (n + s) \quad , \quad a \cdot b = (mr + ns) - (nr + ms)$$

and both are well-defined. In fact, if $a = m' - n'$, then $m + n' = m' + n$, and

$$m + n' + r + s = m' + n + r + s$$
$$(m + n')r + (m' + n)s = (m' + n)r + (m + n')s$$

and we also have $a+b=m'+r-(n'+s)$, $ab=m'r+n's-(n'r+m's)$.

So we reduce statements on $\mathbb{Z}$ to statements on $\mathbb{N}$. The theory of integer numbers is just a part of the theory of natural numbers: *If the natural numbers are free from contradiction, so are the integer numbers.*

Example: Let us fix a natural number $n \geq 2$. The **congruence** relation, $a \equiv b$ (mod. n) when $b-a \in n\mathbb{Z}$, is an equivalence relation on $\mathbb{Z}$, and $[a]=a+n\mathbb{Z}=\{a+cn\colon c\in\mathbb{Z}\}$.

The quotient set $\mathbb{Z}/n\mathbb{Z}=\{[1],[2],\ldots,[n]=[0]\}$ has n elements.

Construction of $\mathbb{Q}$: The set of rational numbers $\mathbb{Q}$ is defined to be the quotient set of the direct product $\mathbb{Z}\times(\mathbb{Z}-\{0\})$ by the equivalence relation

$$(a,s)\equiv(b,t) \quad \text{when} \quad at=bs,$$

and $\frac{a}{s}$ denotes the class of (a,s). Each integer a defines a rational number $\frac{a}{1}$ and we may identify $\mathbb{Z}$ with a subset of $\mathbb{Q}$. The **sum** and the **product** of $q=\frac{a}{s}$ and $r=\frac{b}{t}$ are defined to be

$$q+r=\frac{at+bs}{st} \quad , \quad q\cdot r=\frac{ab}{st}$$

and both are well-defined. In fact, if $q=\frac{a'}{s'}$, then $as'=a's$, so that

$$\begin{aligned}(at+bs)s't &= a'tst+bss't=(a't+bs')st,\\ abs't &= a'bst,\end{aligned}$$

and we also have $q+r=\frac{a't+bs'}{s't}$, $qr=\frac{a'b}{s't}$.

We say that a rational number is ≥ 0 when it may be represented as a quotient $\frac{n}{m}$ of two natural numbers. If $q,r\in\mathbb{Q}$, we put $q\leq r$ when $r-q\geq 0$.

So we reduce statements on $\mathbb{Q}$ to statements on $\mathbb{Z}$. The theory of rational numbers is just a part of the theory of integer numbers: *If the natural numbers are free from contradiction, so are the rational numbers.*

1.2 Cardinal and Ordinal Numbers

Definitions: Let X, Y be two sets. A **map** $f\colon X\to Y$ assigns to any element $x\in X$ a unique element $f(x)\in Y$, and the **composition** with another map $g\colon Y\to Z$ is

$$g\circ f\colon X\to Z \ , \quad (g\circ f)(x)=g\big(f(x)\big) \ .$$

If $A\subseteq X$, we put $f(A)=\{f(a)\colon a\in A\}\subseteq Y$.

If $B\subseteq Y$, we put $B\cap X := f^{-1}(B)=\{x\in X\colon f(x)\in B\}\subseteq X$.

A map $f\colon X\to Y$ is **injective** if $f(x)=f(x') \ \Rightarrow \ x=x'$, **surjective** when $f(X)=Y$, and **bijective** if it is injective and surjective: any element $y\in Y$ is the image of a unique element $f^{-1}(y)\in X$, so that $f^{-1}\colon Y\to X$ also is a bijection and $f^{-1}\circ f=\mathrm{Id}_X$, $f\circ f^{-1}=\mathrm{Id}_Y$.

Compositions of injective (surjective, bijective) maps are injective (surjective, bijective).

Definition: Let X, Y be two sets. If there exists a bijection $f\colon X\to Y$, then we say that X and Y have the same **cardinal** and we put $|X|=|Y|$. If there exists an injection $f\colon X\to Y$, then we say that $|X|\leq|Y|$, and we put $|X|<|Y|$ if moreover $|X|\neq|Y|$.

Sets of cardinal $\leq \aleph_0 := |\mathbb{N}|$ are said to be **countable**.

Schröder-Bernstein Theorem: *If $|X|\leq|Y|$ and $|Y|\leq|X|$, then $|X|=|Y|$.*

Proof: Let $f\colon X \to Y$ and $g\colon Y \to X$ be injective maps, so that $h = gf\colon X_0 := X \to X_2 := h(X)$ is a bijection, and we have $X_2 = (gf)(X) \subseteq X_1 := g(Y)$, where $|Y| = |X_1|$.

Put $X_{n+2} := h(X_n)$, $X_\infty := \bigcap_n X_n$, and $Z_n := X_n - X_{n+1}$.

The map h defines bijections $Z_{2n} \to Z_{2n+2}$, $n \geq 0$, and the identity defines bijections $Z_{2n+1} \to Z_{2n+1}$, $n \geq 0$. We see that $|X| = |X_1| = |Y|$, since we have a bijection

$$X = Z_0 \cup Z_1 \cup Z_2 \cup \ldots \cup X_\infty \to Z_1 \cup Z_2 \cup Z_3 \cup \ldots \cup X_\infty = X_1 = g(Y).$$

Definitions: Given two cardinals $\mathfrak{a} = |X|$, $\mathfrak{b} = |Y|$, the **sum** $\mathfrak{a}+\mathfrak{b}$ is the cardinal of the disjoint union $X \amalg Y$, the **product** $\mathfrak{ab}$ is the cardinal of the cartesian product $X \times Y$, and the **power** $\mathfrak{a}^\mathfrak{b}$ is the cardinal of the set X^Y of all maps[1] $Y \to X$.

Any subset Y of a set X may be viewed as a map $I_Y\colon X \to \{0,1\}$, where $I_Y(Y) = 1$ and $I_Y(X - Y) = 0$; hence the cardinal of the set $\mathcal{P}(X)$ of all subsets of X is just $2^{|X|}$.

Cantor's Theorem: $|X| < 2^{|X|}$.

First proof: Obviously $|X| \leq |\mathcal{P}(X)|$.

If there exists a bijection $f\colon X \to \mathcal{P}(X)$, we consider the subset $Y := \{x \in X : x \notin f(x)\}$.

Since f is surjective, we have $Y = f(y)$ for some $y \in X$.

If $y \in Y$, then $y \notin f(y) = Y$. If $y \notin Y$, then $y \in f(y) = Y$. Absurd.

Hence such bijection does not exist, and $|X| \neq |\mathcal{P}(X)| = 2^{|X|}$.

Second proof: Let us give an alternative proof, using Cantor's *Diagonalverfahren.* Given a map $\phi\colon X \to \{0,1\}^X$, $x \mapsto \phi_x$, we may consider the map $\varphi\colon X \to \{0,1\}$ such that $\varphi(x) = 1$ when $\phi_x(x) = 0$, and $\varphi(x) = 0$ when $\phi_x(x) = 1$, so that $\varphi \neq \phi_x$, $\forall x \in X$, and we see that ϕ is not surjective. Hence $|X| \neq |\{0,1\}^X| = 2^{|X|}$.

Definitions: A relation $\leq$ on a set X is an **order relation** if it is

1. *Reflexive:* $x \leq x$, $\forall x \in X$.
2. *Antisymmetric:* $x, y \in X$, $x \leq y$, $y \leq x \;\Rightarrow\; x = y$.
3. *Transitive:* $x, y, z \in X$, $x \leq y$, $y \leq z \;\Rightarrow\; x \leq z$.

and we put $x < y$ when $x \leq y$ and $x \neq y$.

Any order on X clearly induces an order on any subset $Y \subseteq X$.

Let X, X' be two ordered sets. A map $f\colon X \to X'$ is a **morphism** of ordered sets when

$$x_1 \leq x_2 \;\Rightarrow\; f(x_1) \leq f(x_2).$$

It is an **isomorphism** of ordered sets when it is bijective and f^{-1} also is a morphism (it is bijective and $x_1 \leq x_2 \;\Leftrightarrow\; f(x_1) \leq f(x_2)$), and it is an **anti-isomorphism** when it is bijective and $x_1 \leq x_2 \;\Leftrightarrow\; f(x_1) \geq f(x_2)$.

Definitions: Let X be an order. We say that $x \in X$ is the **last** element when $y \leq x$, $\forall y \in X$ (if it exists, it is unique) and that it is the **first** element when $x \leq y$, $\forall y \in X$. The last (resp. first) element of a subset $Y \subseteq X$ is the **maximum** (resp. **minimum**) of Y.

We say that $x \in X$ is **maximal** when $x \leq y \Rightarrow x = y$, and **minimal** when $y \leq x \Rightarrow y = x$.

We say that $x \in X$ is an **upper** (resp. **lower**) bound of a subset $Y \subseteq X$ when $y \leq x$ (resp. $x \leq y$) for all $y \in Y$. The **supremum** of Y is the first element of the subset of all upper bounds

[1] We assume that any set X admits a unique map $\emptyset \to X$ (in particular there is a unique map $\emptyset \to \emptyset$, the identity), while there is no map $X \to \emptyset$ when $X \neq \emptyset$.

of Y (if it exists, it is unique) and the **infimum** of Y is the last element of the subset of all lower bounds of Y (if it exists, it is unique). An order is a **lattice** when it has first and last element, and any pair $x_1, x_2 \in X$ has supremum and infimum.

An order X is **total** if any pair is comparable ($x \leq y$ or $y \leq x$ for any $x, y \in X$) and it is a **well-order** when every non empty subset has a first element (hence it is a total order).

The **chains** in an order X are the totally ordered subsets of X.

In the founding principles of Mathematics we admit the so named Zorn's lemma:

Zorn's Lemma: *Let X be a non empty order. If any chain of X has an upper bound in X, then X has some maximal element.*

Axiom of Choice: *Any surjective map $p\colon X \to Y$ admits a* ***section*** $s\colon Y \to X$ (a map such that $p \circ s = \mathrm{Id}_Y$).

Proof: Let S be the set of pairs (A, s) where $A \subseteq Y$ and $s\colon A \to X$ is a map such that $p \circ s = \mathrm{Id}_A$.

The set S is not empty (just take a point $y \in Y$, a point $x \in p^{-1}(y)$, and put $s(y) = x$) and we order S as follows:

$$(A, s) \leq (B, t) \text{ when } A \subseteq B \text{ and } s = t|_A.$$

Any chain $\{(A_i, s_i)\}_{i \in I}$ admits the upper bound $(\bigcup_i A_i, s)$, where $s(a) = s_i(a)$ when $a \in A_i$.

Hence, by Zorn's lemma, S has a maximal element (Z, s).

If $Z \neq Y$, and $y \in Y - Z$, we may extend s to $Z \cup \{y\}$ (just put $s(y) = x$, where $x \in p^{-1}(y)$) against the maximal character of (Z, s). Hence $Z = Y$ and s is a section of p.

Corollary: *If $p\colon X \to Y$ is a surjective map, then $|Y| \leq |X|$.*

Proof: Let s be a section of p. Since s is injective, we have $|Y| = |s(Y)| \leq |X|$. q.e.d.

1. $\mathbb{N} \times \mathbb{N}$ is a well-order set with the lexicographical order $00 < 01 < \ldots < 10 < 11 < \ldots$

2. Any set of cardinals is an ordered set.

3. The subsets of a set, subgroups of a group, ideals of a ring, vector subspaces of a vector space, etc., ordered by inclusion, form a lattice.

4. Any ordered set $(X, \leq)$ defines a **dual** order $X^* = (X, \leq^*)$, where we put $x \leq^* y \Leftrightarrow y \leq x$. It is clear that $X^{**} = X$, and that a map $f\colon X \to Y$ is an anti-isomorphism if and only if $f\colon X^* \to Y$ is an isomorphism. Moreover, X is a lattice if and only if so is X^*.

5. Since $f\colon \mathbb{Z} \to \mathbb{N}$, $f(n) = 2n$ when $n \geq 0$ and $f(-n) = 2n - 1$ when $n \geq 1$, is bijective, we see that $|\mathbb{Z}| = \aleph_0$.

6. Since $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(n, m) = 2^n 3^m$, is injective, we see that $\aleph_0^2 = \aleph_0$; hence $\aleph_0^n = \aleph_0$, $n \geq 1$.

7. Since $\mathbb{Q}$ is a quotient of $\mathbb{Z} \times (\mathbb{N} - 0)$, we see that $|\mathbb{Q}| = \aleph_0$; hence $|\mathbb{Q}^n| = \aleph_0$, $n \geq 1$.

8. The set of infinite decimal expressions $0.c_1c_2c_3\ldots$ has cardinal $10^{\aleph_0}$: it is uncountable.

9. Cantor's theorem shows that "the set X of all sets" is non sense: since $\mathcal{P}(X) \subseteq X$, we would have $2^{|X|} = |\mathcal{P}(X)| \leq |X|$. Analogously there is no "set of all groups", etc.

Ordinal Numbers

Definition: Let X be a well-order. A set $Y \subseteq X$ is an **initial ray** of X when $y < x$ for all $y \in Y$, $x \in X - Y$. When Y is incomplete, $Y \neq X$, then $Y = (\leftarrow x) := \{y \in X : y < x\}$, where x is the supremum of the ray, so that X is isomorphic to the set of incomplete initial rays with the inclusion order.

Theorem: *Any set X admits a well-order.*

Proof: Consider the set W of all pairs $(Y, \leq)$ where $Y \subseteq X$ and $\leq$ is a well-order on Y.

The set W is not empty (finite subsets admit a well-order) and we order W as follows:

$$(Y', \leq') \leq (Y, \leq) \text{ when } (Y', \leq') \text{ is an initial ray of } (Y, \leq).$$

In W any chain $\{(Y_i, \leq_i)\}_{i \in I}$ admits an upper bound:

In fact $Y = \bigcup_i Y_i$ has an obvious order $\leq$ inducing $\leq_i$ on Y_i, and Y_i is an initial ray of Y: if $y \in Y, y \notin Y_i$, then y is in some Y_j containing Y_i so that y bounds above Y_i. Finally, Y is a well-ordered set: if $\emptyset \neq Z \subseteq Y$, we have $\emptyset \neq Y_i \cap Z$ for some index i, and the first element of $Y_i \cap Z$ is the first element of Z, because Y_i is an initial ray of Y.

According to Zorn's lemma, W has a maximal element $(Y, \leq)$.

If $Y \neq X$, and $x \in X - Y$, we may extend the well-order of Y to $Y \cup \{x\}$ so that $y \leq x$, $\forall y \in Y$, against the maximal character of $(Y, \leq)$. Hence $Y = X$ and $\leq$ is a well-order on X.

Zermelo's Theorem: *Let X be a well ordered set. If $f: X \to X$ is a strictly increasing map ($x < y \Rightarrow f(x) < f(y)$) then $x \leq f(x)$, $\forall x \in X$.*

Proof: Otherwise, let x be the first element such that $f(x) < x$, so that $f(x) \leq f(f(x))$.

Since f is increasing, we have $f(f(x)) < f(x)$. Absurd. q.e.d.

1. *Any well-order X has a unique automorphism: the identity.*

   If $\tau: X \to X$ is an automorphism and $x < \tau(x)$, then $\tau^{-1}(x) < x$. Impossible.

2. *If two well-orders are isomorphic, then the isomorphism is unique.* (So, we may identify isomorphic well-orders and name them **ordinal** numbers. The ordinal of $\mathbb{N}$ is ω.)

   If $f, g: X \to X'$ are isomorphisms, then $g^{-1}f$ is an automorphism; hence $g^{-1}f = \mathrm{Id}_X$.

3. *Any two initial rays $R_1 \subset R_2$ of a well-order never are isomorphic.*

   If $f: R_2 \to R_1 \subset R_2$ is an isomorphism and $x \in R_2 - R_1$, then $f(x) < x$. Impossible.

Definition: Let α, β be two ordinal numbers. We put $\beta < \alpha$ when β is isomorphic to an incomplete initial ray of α. By the above results, $\leq$ defines an order on any set of ordinal numbers, and the set $(\leftarrow \alpha)$ of ordinal numbers $\beta < \alpha$ is a well-order, since it is just α.

Lemma: *Let X, Y be two well-orders. If any incomplete initial ray of X is isomorphic to an incomplete initial ray of Y, then $X \leq Y$.*

Proof: Any incomplete initial ray $(\leftarrow x)$ of X is isomorphic to a unique incomplete initial ray $\big(\leftarrow f(x)\big)$ of Y. If $x < x'$, by the above results it is clear that $f(x) < f(x')$, and that f is an isomorphism of X onto an initial ray of Y (eventually complete).

Lemma: *If α and β are two ordinal numbers, then $\alpha \leq \beta$ or $\beta \leq \alpha$.*

Proof: Otherwise, by the lemma we may consider the first incomplete initial ray $(\leftarrow x)$ of α (resp. $(\leftarrow, y)$ of β) not isomorphic to an incomplete initial ray of β (resp. of α).

Now $(\leftarrow x)$ and $(\leftarrow, y)$ satisfy the hypothesis of the above lemma, so that $(\leftarrow x) \leq (\leftarrow y)$ and $(\leftarrow y) \leq (\leftarrow x)$. Hence $(\leftarrow x)$ and $(\leftarrow y)$ are isomorphic. Absurd.

Theorem: *If S is a set of ordinal numbers, then $(S, \leq)$ is a well-order.*

Proof: If $R \subseteq S$ is not empty, we consider an ordinal $\alpha \in R$.

The initial rays of α give all the ordinals $\beta < \alpha$, so that $\{\beta \in R \colon \beta \leq \alpha\}$ has a first element β_0 and, by the lemma, $\beta_0 \leq \alpha < \gamma$ for any other $\gamma \in R$.

Cardinal Numbers: Let ω be the ordered set of all finite ordinal numbers, so that $|\omega| = \aleph_0$.

Let ω_1 be the ordered set of all ordinal numbers of cardinal $\leq \aleph_0$, so that ω_1 is the first ordinal number of cardinal $> \aleph_0$. We put $\aleph_1 := |\omega_1|$, so that $\aleph_0 < \aleph_1$ and there is no intermediate cardinal: if $|\sigma| < \aleph_1$, then $\sigma < \omega_1$, so that $|\sigma| \leq \aleph_0$.

Let ω_2 be the ordered set of all ordinal numbers of cardinal $\leq \aleph_1$, so that ω_2 is the first ordinal number of cardinal $> \aleph_1$, and we put $\aleph_2 := |\omega_2|$. So we obtain an increasing sequence $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$ of cardinals with no intermediate cardinal. Now we put $\aleph_\omega := |\bigcup_n \omega_n|$, and so on:

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_\omega < \aleph_{\omega+1} < \ldots$$

In general, if $\alpha = \beta + 1$, we consider the ordered set ω_α of all ordinal numbers of cardinal $\leq \aleph_\beta$, and we put $\aleph_\alpha := |\omega_\alpha|$. Otherwise, we consider $\omega_\alpha = \bigcup_{\beta<\alpha} \omega_\beta$, and we put $\aleph_\alpha := |\omega_\alpha|$.

Now, given an infinite cardinal $|\sigma|$, if we consider the first ordinal α such that $|\sigma| \leq \aleph_\alpha$, then $|\sigma| = \aleph_\alpha$. Otherwise $\sigma < \omega_\alpha$, so that $|\sigma| \leq \aleph_\beta$ for some $\beta < \alpha$. Absurd.

So we see how the ordinal numbers correspond to the infinite cardinal numbers, so that any set of infinite cardinal numbers is a well-order.

1.3 Real and Complex Numbers

Now we consider the ring (commutative with unity) of all sequences $(q_n) = (q_0, q_1, q_2, \ldots)$ of rational numbers, with the termwise addition and product.

Definition: A sequence of rational numbers (q_n) is a **Cauchy** sequence when for every positive rational $\varepsilon \in \mathbb{Q}_+$ there is an index n_ε such that $|q_n - q_m| < \varepsilon$, $\forall m, n \geq n_\varepsilon$; and it is a **null** sequence when for every $\varepsilon \in \mathbb{Q}_+$ there is an index n_ε such that $|q_n| < \varepsilon$, $\forall n \geq n_\varepsilon$.

Constant sequences $(q) = (q, q, q, \ldots)$ are Cauchy sequences.

Any null sequence is a Cauchy sequence, and any Cauchy sequence is bounded: there exists $c \in \mathbb{N}$ such that $|q_n| \leq c$ for any index n (just take $c \geq \varepsilon + \max\{|q_0|, |q_1|, \ldots, |q_{n_\varepsilon}|\}$.).

Any **subsequence** (q_{i_n}), $i_0 < i_1 < i_2 < \ldots$, of a Cauchy (resp. null) sequence (q_n) also is a Cauchy (resp. null) sequence, and the difference $(q_n - q_{i_n})$ is a null sequence.

Theorem: *Cauchy sequences of rational numbers form a subring, and the null sequences define an ideal of this subring.*

Proof: First we prove that the Cauchy sequences form a subring and, since constant sequences are Cauchy sequences, we only have to prove that the sum $(q_n) + (q'_n) = (q_n + q'_n)$ and product $(q_n) \cdot (q'_n) = (q_n q'_n)$ of two Cauchy sequences (q_n), (q'_n) also are Cauchy sequences.

The sum is clear, since $|(q_n + q'_n) - (q_m + q'_m)| \leq |q_n - q_m| + |q'_n - q'_m|$.

For the product, consider a constant $c \in \mathbb{N}$ bounding (q_n) and (q'_n). Given $\varepsilon \in \mathbb{Q}_+$, there is an index n_ε such that $|q_n - q_m|, |q'_n - q'_m| < \varepsilon$, $\forall m, n \geq n_\varepsilon$, so that

$$|q_n q'_n - q_m q'_m| = |q_n(q'_n - q'_m) + (q_n - q_m)q'_m| \leq |q_n| \cdot |q'_n - q'_m| + |q_n - q_m| \cdot |q'_m| < 2c\varepsilon.$$

Finally the null sequences clearly form an additive subgroup and, when (q'_n) is a null sequence, we consider a constant c bounding (q_n) and an index n_ε such that $|q'_n| < \frac{\varepsilon}{c}$, $\forall n \geq n_\varepsilon$, so that $|q_n q'_n| \leq c|q'_n| < \varepsilon$, and we see that $(q_n q'_n)$ also is a null sequence.

Definition: The ring of **real** numbers $\mathbb{R}$ is the quotient of the ring of Cauchy sequences of rational numbers by the ideal of null sequences.

A real number is ≥ 0 when it may be represented by some Cauchy sequence (q_n) with all the terms $q_n \geq 0$. If $a, b \in \mathbb{R}$, we put $a \leq b$ when $b - a \geq 0$.

So we reduce statements on $\mathbb{R}$ to statements on $\mathbb{Q}$. *If the natural numbers are free from contradiction, so are the real numbers.*

We have a canonical injective ring morphism $\mathbb{Q} \to \mathbb{R}$, $q \mapsto [(q)]$, (preserving inequalities).

A Cauchy sequence (q_n) and any subsequence (q_{i_n}) represent the same real number.

Lemma: *Given a real number $x \neq 0$, there exists $k \in \mathbb{N}$ such that x may be represented by a Cauchy sequence (q_n) with all the terms $q_n \geq \frac{1}{k}$, or with all the terms $q_n \leq -\frac{1}{k}$.*

Proof: By definition, any non-null Cauchy sequence admits a subsequence (q_n) with all the terms $|q_n| \geq \frac{1}{k}$ for some constant $k \in \mathbb{N}$.

Now (q_n) admits a subsequence of positive terms or a subsequence of negative terms.

Theorem: *The ring $\mathbb{R}$ is a field and $\leq$ is total order compatible with the additive and multiplicative structure:*

1. *If $a \leq b$, then $a + c \leq b + c$.*
2. *If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.*

Proof: Clearly $\leq$ is reflexive and transitive. It is antisymmetric: If $a \leq b$ and $b \leq a$, then $b - a$ and $a - b$ are represented by Cauchy sequences (q_n) and (q'_n) with all the terms $q_n, q'_n \geq 0$. Now $(q_n + q'_n)$ is a null sequence and so is (q_n), since $|q_n| \leq |q_n + q'_n|$, so that $b - a = 0$.

It is a total order: If $a \neq b$, by the above lemma $b - a > 0$ or $a - b > 0$; hence $a < b$ or $b < a$.

Finally we prove that $\mathbb{R}$ is a field: By the lemma, any non-zero real number is represented by a Cauchy sequence (q_n) with all the terms $|q_n| \geq \frac{1}{k}$ for some constant $k \in \mathbb{N}$. If (q_n^{-1}) is a Cauchy sequence, then $1 = [(q_n)] \cdot [(q_n^{-1})]$, and we conclude. Now,

$$\left| \frac{1}{q_n} - \frac{1}{q_m} \right| = \left| \frac{q_m - q_n}{q_n q_m} \right| \leq \frac{|q_m - q_n|}{k^2}.$$

Definition: The **absolute value** of $x \in \mathbb{R}$ is defined to be $|x| = \max\{x, -x\}$.

Corollary: $|a + b| \leq |a| + |b|$.

Proof: We have $a \leq |a|$ and $b \leq |b|$; hence $a + b \leq |a| + b \leq |a| + |b|$. Moreover $-a \leq |a|$ and $-b \leq |b|$; hence $-a - b \leq |a| - b \leq |a| + |b|$.

Corollary: $a \leq b \Rightarrow -b \leq -a$; $a \leq b,\ 0 \leq c \Rightarrow ac \leq bc$; $0 < a \leq b \Rightarrow 0 < b^{-1} \leq a^{-1}$.

Proof: If $a \leq b$, then $-b = a - (a+b) \leq b - (a+b) = -a$.

If $0 \leq c$ and $a \leq b$, then $0 \leq b - a$ and $0 \leq (b-a)c = bc - ac$, so that $ac \leq bc$.

If $0 < a$, the above proof shows that $0 < a^{-1}$; hence $b^{-1} = a(ab)^{-1} \leq b(ab)^{-1} = a^{-1}$.

Definitions: A sequence of real numbers (x_n) is a **Cauchy** sequence when for every positive real number $\varepsilon \in \mathbb{R}_+$ there is an index n_ε such that $|x_n - x_m| < \varepsilon$, $\forall m, n \geq n_\varepsilon$; and it **converges** to a **limit point** $x \in \mathbb{R}$ (and we put $x = \lim x_n$) when there is an index n_ε such that $|x_n - x| < \varepsilon$, $\forall n \geq n_\varepsilon$.

Any convergent sequence is a Cauchy sequence.

Theorem: *For all $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{Q}_+$ there exists $q \in \mathbb{Q}$ with $|q - x| < \varepsilon$.*

Proof: If (q_n) represents x, then there exists $n_0 \in \mathbb{N}$ such that $|q_n - q_{n_0}| < \frac{\varepsilon}{2}$, $\forall n \geq n_0$.

Hence $|x - q_{n_0}| \leq \frac{\varepsilon}{2} < \varepsilon$.

Theorem: *Any Cauchy sequence (x_n) of real numbers converges to a unique real limit.*

Proof: Let us see the *existence* of the limit:

Pick $q_n \in \mathbb{Q}$ such that $|q_n - x_n| < \frac{1}{n}$. Then (q_n) is a Cauchy sequence of rational numbers:

Given $\varepsilon \in \mathbb{Q}_+$, there exists an index n_0 such that $n_0 > \frac{3}{\varepsilon}$ and $|x_n - x_m| < \frac{\varepsilon}{3}$, $\forall n \geq n_0$, so that we have $|q_n - q_m| \leq |q_n - x_n| + |x_n - x_m| + |x_m - q_m| < \varepsilon$.

In particular, if we put $x = [(q_n)]$, we have $|q_n - x| \leq \varepsilon$, $\forall n \geq n_0$.

Now, given $\varepsilon > 0$, there exists an index n_ε such that $n_\varepsilon > \frac{2}{\varepsilon}$ and $|q_n - x| < \frac{\varepsilon}{2}$, $\forall n \geq n_\varepsilon$, so that we have $|x_n - x| \leq |x_n - q_n| + |q_n - x| < \frac{1}{n} + \frac{\varepsilon}{2} \leq \varepsilon$, $\forall n \geq n_\varepsilon$. Hence $x = \lim x_n$.

If $y \in \mathbb{R}$ is a different limit, there is an index n_0 such that $|x_n - x| < \frac{|y-x|}{2}$ and $|x_n - y| < \frac{|y-x|}{2}$, $\forall n \geq n_0$, and we get a contradiction: $|y - x| \leq |y - x_n| + |x_n - x| < \frac{|y-x|}{2} + \frac{|y-x|}{2} = |y - x|$.

Corollary: *Any non empty bounded above set of real numbers has a supremum, and any non empty bounded below set of real numbers has an infimum.*

Proof: If $X \subset \mathbb{R}$ is bounded above, for each natural number n we consider the first fraction $q_n = \frac{a}{2^n}$, $a \in \mathbb{Z}$, which is an upper bound of X. Now (q_n) is a Cauchy sequence, because $|q_n - q_m| < 2^{-n}$ when $m \geq n$, and $x := \lim q_n$ is the supremum of X.

If $X \subset \mathbb{R}$ is bounded below, then $-X$ is bounded above.

If x is the supremum of $-X$, then $-x$ is the infimum of X.

Corollary: *Any bounded above non-decreasing sequence (x_n) converges, hence so does any bounded below non-increasing sequence.*

Proof: Put $x = \sup\{x_n\}$. Given $\varepsilon > 0$, there exists and index n_ε such that $x - \varepsilon < x_{n_\varepsilon} \leq x$; hence $x - \varepsilon < x_n \leq x$ for all $n \geq n_\varepsilon$, since the sequence is non-decreasing.

Corollary: *Any positive real number b has a unique positive square root.*

Proof: If $a > 0$ and $a^2 > b$, then $a > \frac{1}{2}\big(a + \frac{b}{a}\big) > 0$ and $\big(\frac{1}{2}\big(a + \frac{b}{a}\big)\big)^2 = \frac{1}{4}\big(\big(a - \frac{b}{a}\big)^2 + 4b\big) > b$.

If we put $a_0 = a$ and $a_{n+1} = \frac{1}{2}\big(a_n + \frac{b}{a_n}\big)$, we obtain a decreasing sequence of positive real numbers with limit $l = \frac{1}{2}\big(l + \frac{b}{l}\big)$. Hence $l^2 = b$.

The positive square root is unique because $0 < x < y \Rightarrow x^2 < y^2$.

Complex Numbers: The set $\mathbb{C}$ of **complex** numbers is the set of pairs $x+yi$ of real numbers $x,y\in\mathbb{R}$, endowed with the operations ($i^2=-1$)

$$(x_1+y_1i)+(x_2+y_2i) := (x_1+x_2)+(y_1+y_2)i,$$
$$(x_1+y_1i)\cdot(x_2+y_2i) := (x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)i.$$

The **conjugate** of $z=x+yi$ is the complex number $\bar z = x-yi$, and the **modulus** of z is the real number $|z| = +\sqrt{z\cdot\bar z} = +\sqrt{x^2+y^2}\geq 0$.

Clearly $|z|>0$ when $z\neq 0$; hence $\mathbb{C}$ is a field, the inverse of z being $\bar z/|z|$. Moreover

$$\overline{z+u}=\bar z+\bar u \quad , \quad \overline{zu}=\bar z\bar u \quad , \quad \bar{\bar z}=z \quad , \quad |z|=|\bar z| \quad ,$$
$$|zu|=|z|\cdot|u| \quad , \quad z+\bar z\leq 2|z| \quad , \quad |z+u|\leq |z|+|u| \quad .$$

These properties are easy to check, except the **triangle inequality** $|z+u|\leq|z|+|u|$. Now,

$$\begin{aligned}|z+u|^2 &= (z+u)(\overline{z+u}) = (z+u)(\bar z+\bar u) = |z|^2+|u|^2+z\bar u+\bar z u\\ &= |z|^2+|u|^2+z\bar u+\overline{z\bar u} \leq |z|^2+|u|^2+2|z\bar u|\\ &= |z|^2+|u|^2+2|z|\cdot|\bar u| = |z|^2+|u|^2+2|z|\cdot|u| = (|z|+|u|)^2.\end{aligned}$$

Theorem: $|\mathbb{R}|=|\mathbb{C}|=2^{\aleph_0}$.

Proof: The set S of all sequences of rational numbers has cardinal $\aleph_0^{\aleph_0}=2^{\aleph_0}$.

Since $\mathbb{R}$ is a quotient of a subset of S, we see that $|\mathbb{R}|\leq 2^{\aleph_0}$.

On the other hand, any infinite decimal $0.c_0c_1c_2\ldots$, where $0\leq c_i\leq 9$, defines a Cauchy sequence of rational numbers $(0.c_0, 0.c_0c_1, 0.c_0c_1c_2,\ldots)$. When we only consider digits $c_i\in\{0,1\}$, we obtain an injection $2^{\mathbb{N}}\hookrightarrow\mathbb{R}$; hence $2^{\aleph_0}\leq|\mathbb{R}|$ and we conclude that $|\mathbb{R}|=2^{\aleph_0}$.

Finally, $|\mathbb{C}|=|\mathbb{R}\times\mathbb{R}|=|\mathbb{R}|^2=2^{2\aleph_0}=2^{\aleph_0}$.

1.4 Metric Spaces and Topological Spaces

Definition: A **metric** d on a set X is a map $\mathrm{d}\colon X\times X\to\mathbb{R}_{\geq 0}$ such that

1. $\mathrm{d}(x,y)=\mathrm{d}(y,x)$.
2. $\mathrm{d}(x,z)\leq\mathrm{d}(x,y)+\mathrm{d}(x,z)$. (*Triangle inequality*).
3. $\mathrm{d}(x,y)=0$ if and only if $x=y$.

and a **metric space** is a set X endowed with a metric d.

Given two metric spaces (X,d) and (X',d'), a map $f\colon X\to X'$ is a **metric morphism** when $\mathrm{d}(x,y)=\mathrm{d}\big(f(x),f(y)\big)$, $\forall x,y\in X$; and it is an **isometry**, or **metric isomorphism**, if moreover it is bijective (so that $f^{-1}\colon X'\to X$ also is an isometry).

Compositions of metric morphisms also are metric morphisms.

Examples: The usual metric on $\mathbb{R}$ is $\mathrm{d}(x,y)=|y-x|$.

The usual metric on $\mathbb{C}$ is $\mathrm{d}(x,y)=|y-x|$.

Any scalar product defines a metric $\mathrm{d}(p,q):=\|q-p\|=$ (see p. 57). The usual scalar product in $\mathbb{R}^n$ induces the metric

$$\mathrm{d}\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big)=\sqrt{(y_1-x_1)^2+\ldots+(y_n-x_n)^2}$$

and the existence of orthonormal bases shows that any Euclidean vector space of dimension n is metrically isomorphic to $\mathbb{R}^n$ in the real case, and to $\mathbb{R}^{2n}$ in the complex case.

A metric on the set of all bounded real-valued functions on a set X is

$$\mathrm{d}(f,h) = \sup_{x\in X}\{|h(x)-f(x)|\}.$$

A metric d on a set X obviously induces a metric $\mathrm{d}_Y(y_1,y_2) := \mathrm{d}(y_1,y_2)$ on any subset $Y \subseteq X$, and we say that (Y, d_Y) is a **subspace** of the metric space (X, d).

Definition: In a metric space X, the **ball** with center at a point $x \in X$ and radius $r \in \mathbb{R}_+$ is

$$B_r(x) = B(x,r) = B_{\mathrm{d}}(x,r) := \{y \in X \colon \mathrm{d}(x,y) < r\}$$

and $U \subseteq X$ is said to be an **open** set in X when for any point $x \in U$ there exists a ball $B_\varepsilon(x) \subseteq U$ (so that the empty set $\emptyset$ and X are open sets).

A subset $Y \subseteq X$ is a **closed** set in X when the complement $X - Y$ is an open set.

Theorem: *Arbitrary unions and finite intersections of open sets are open sets.*

Proof: Arbitrary unions of open sets are obviously open, and so are finite intersections since

$$B(x,\varepsilon_1) \cap \ldots \cap B(x,\varepsilon_n) = B(x,\varepsilon)\ , \quad \varepsilon = \min\{\varepsilon_1,\ldots,\varepsilon_n\}.$$

Definition: A **topology** on a set X is a family of subsets, named **open** sets, such that

1. The subsets $\emptyset$ and X are open sets.
2. Arbitrary unions of open sets are open sets.
3. Finite intersections of open sets are open sets.

and we say that $Y \subseteq X$ is a **closed** set when $X - Y$ is open, so that arbitrary intersections and finite unions of closed sets are closed sets, and X and $\emptyset$ are closed sets.

A **topological space** is a set X endowed with a topology, and it is **metrizable** if the topology is defined by a (non necessarily unique) metric on X.

Definitions: Given a topology on X, the **interior** of $Y \subseteq X$ is the biggest open set $\overset{\circ}{Y} \subseteq Y$ (the union of all the open sets contained in Y), and Y is a **neighborhood** of a subset Z, when Z is contained in the interior of Y.

The **closure** of $Y \subseteq X$ is the smallest closed set $\bar{Y}$ containing Y (the intersection of all the closed sets containing Y), so that $x \in \bar{Y}$ just when any neighborhood of x intersects Y.

The **boundary** of Y is the set ∂Y of points in the closure $\bar{Y}$ not in the interior of Y.

When $\bar{Y} = X$, we say that Y is **dense** in X.

A sequence of points (x_n) in X **converges** to a **limit point** $x \in X$ (and we put $x = \lim x_n$ or $x_n \to x$) when for any neighborhood U of x there is an index n_U such that $x_n \in U$, $\forall n \geq n_U$ (in a metric space, this condition means that $\lim \mathrm{d}(x, x_n) = 0$). We say that a sequence (x_n) in X has an **adherent** point $x \in X$ when for any neighborhood U of x and any index n we have $x_m \in U$ for some index $m \geq n$. When all the points x_n are in a subset Y, then clearly $x \in \bar{Y}$.

1. In a metric space X, any ball $B_r(x)$ is open and any open set is a (may be infinite) union of balls, while $\{y \in X \colon \mathrm{d}(x,y) = r\}$ and $\{y \in X \colon \mathrm{d}(x,y) \leq r\}$ are closed subspaces. Moreover, a point $x \in X$ is in the closure of a subspace Y if and only if $x = \lim y_n$ for some sequence $y_n \in Y$. In fact, if $x \in \bar{Y}$, there are points $y_n \in B\big(x, \frac{1}{n}\big) \cap Y$, so that $y_n \to x$.

2. Let X be a topological space and $Y \subseteq X$. The subsets $U \cap Y$, where U is an open set in X, clearly define a topology on Y and, when endowed with this topology, we say that Y is a **subspace** of X. When X is a metric space, this topology is defined by the induced metric d_Y, because $B_{\mathrm{d}_Y}(y,r) = B_{\mathrm{d}_X}(y,r) \cap Y$.

3. In $\mathbb{R} = (-\infty, \infty)$, the intervals $(a,b) := \{x \in \mathbb{R} \colon a < x < b\}$, $(-\infty, b)$ and (a, ∞) are open, while the intervals $[a,b] := \{x \in \mathbb{R} \colon a \leq x \leq b\}$, $(-\infty, b]$ and $[a, \infty)$ are closed. When $a < b$, in fact (a,b) is the interior of $[a,b]$ and $[a,b]$ is the closure of (a,b).

4. In $\mathbb{R}^d$, any product $U_1 \times \ldots \times U_d$ of open sets $U_i \subseteq \mathbb{R}$ is an open set, and any product $Y_1 \times \ldots \times Y_d$ of closed sets $Y_i \subseteq \mathbb{R}$ is a closed set.

5. The rational numbers are dense in $\mathbb{R}$ (see page 22).

6. If all the subsets of a set X are defined to be open sets, we obtain the **discrete** topology on X. If only $\emptyset$ and X are defined to be open sets, we obtain the **trivial** topology on X.

Definition: If a topological space X has a unique open closed set, then $X = \emptyset$. If X only has two open closed sets (namely $\emptyset$ and $X \neq \emptyset$) we say that X is **connected**. If X admits an open closed set U different from $\emptyset$ and X (so that $X = U \cup U^c$ is a disjoint union of two non-empty open sets) we say that X is **disconnected**. A subset Y of X is connected (or disconnected) when so it is with the induced topology.

A subset $I \subseteq \mathbb{R}$ is an **interval** (eventually with infinite ends) if $[a,b] \subseteq I$ whenever $a, b \in I$.

Theorem: *A subset $X \subseteq \mathbb{R}$ is connected if and only if it is a non-empty interval.*

Proof: Assume that X is connected and $a, b \in X$. If $x \in [a,b]$ is not in X, then $X \cap [x, \infty) = X \cap (x, \infty)$ is an open closed set in X. Hence it is empty (and $b < x$), or it is X (and $x < a$). Absurd, and we conclude that X is an interval.

Now let us see that any bounded closed interval $[a,b]$ is connected. If $[a,b] = U \cup U^c$ is a disjoint union of two non-empty open closed sets, we may assume that $a \in U$. Let $c = \inf U^c$, so that $[a,c) \subseteq U$.Then $c \in U^c$ and $c \in U$, because U^c and U are closed. Absurd.

Finally, any non-empty interval is an increasing union of bounded closed intervals, so that it is easy to check that it also is connected.

Definition: An open **cover** of X is a family of open sets $\{U_i\}_{i \in I}$ such that $X = \bigcup_i U_i$, and X is **compact** if any open cover $X = \bigcup_i U_i$ has a finite subcover: $X = U_{i_1} \cup \ldots \cup U_{i_n}$. Equivalently, if a family of closed sets $\{Y_i\}_{i \in I}$ has empty intersection, $\emptyset = \bigcap_i Y_i$, then so does some finite family $\emptyset = Y_{i_1} \cap \ldots \cap Y_{i_n}$. A subset Y of a topological space space X is said to be compact when so it is with the induced topology: If $\{U_i\}_{i \in I}$ is a family of open sets in X and $Y \subseteq \bigcup_i U_i$, then $Y \subseteq U_{i_1} \cup \ldots \cup U_{i_n}$ for some finite set of indices.

Proposition: *Any sequence (x_n) in a compact space K has an adherent point $x \in K$.*

Proof: Let Y_n be the closure of $\{x_n, x_{n+1}, \ldots\}$, so that $Y_{n+1} \subseteq Y_n$. If $\bigcap_n Y_n = \emptyset$, then so does a finite family $\emptyset = Y_{n_1} \cap \ldots \cap Y_{n_r} = Y_{\max n_1, \ldots, n_r}$. Absurd.

Hence $\bigcap_n Y_n \neq \emptyset$, and any point $x \in \bigcap_n Y_n$ is adherent to (x_n).

Lemma: *Any closed subspace Y of a compact space K also is compact.*

Proof: If a family of closed subsets of Y has empty intersection, $\emptyset = \bigcap_i Y_i$, then so does some finite family because the sets Y_i are closed in K and K is compact.

Lemma: *Any compact subspace K of a metric space X is closed and* ***bounded*** (i.e. contained in a ball).

Proof: If $x \in X - K$, for each point $y \in K$ we may find disjoint balls $x \in B_y$ and $y \in B'_y$.
Then $K \subseteq \bigcup_y B'_y$, so that K admits a finite subcover, $K \subseteq B'_{y_1} \cup \ldots \cup B'_{y_n}$.
Hence $x \in B_{y_1} \cap \ldots \cap B_{y_n}$ and $(B_{y_1} \cap \ldots \cap B_{y_n}) \cap K = \emptyset$, so that $X - K$ is open.

Finally, fix a point $x \in X$. The open cover $K \subseteq \bigcup_{r \in \mathbb{N}} B_r(x)$ admits a finite subcover $K \subseteq B_{r_1}(x) \cup \ldots \cup B_{r_n}(x) = B_r(x)$, so that K is bounded.

Heine-Borel Theorem: *Compact subsets of $\mathbb{R}^d$ are just bounded closed subsets.*

Proof: If $K \subset \mathbb{R}^d$ is compact, it is a closed bounded set by the lemma.
Conversely, any closed bounded set $K \subset \mathbb{R}^d$ is contained in a cube

$$I_0 = [a_1, a_1 + l] \times \ldots \times [a_d, a_d + l]$$

and, by the above lemma, we only have to show that I_0 is compact.

Assume that I_0 is not compact, and let $I_0 \subseteq \bigcup_i U_i$ be an open cover with no finite subcover. Bisecting the edges of I_0, we obtain 2^d cubes, and at least one of such cubes, say I_1, does not admit a finite subcover. So we obtain a decreasing sequence of cubes

$$I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_n \supset \ldots ,$$

the size of I_n being $l/2^n$. Let us consider a sequence (y_n) such that $y_n \in I_n$.

The coordinates $x_i(y_n)$ of these points define d Cauchy sequences of real numbers, since $|x_i(y_n) - x_i(y_m)| \leq 2^{-n_0} l$ when $m, n \geq n_0$. Hence (y_n) converges to a point $y \in \mathbb{R}^d$, and $y \in I_n$ since all the terms of the sequence, up to a finite number, are in I_n.
Since $y \in I_0$, we have $y \in U_i$ for some index i, and there is a ball $B_r(y) \subseteq U_i$.
Since $y \in I_n$, when n is big enough (namely $\sqrt{d} l/2^n < r$) we have $I_n \subseteq B_r(y) \subseteq U_i$.
Absurd, I_n does not admit any finite subcover.

Alternative Proof in the crucial case $I = [a, b]$: Let $I \subseteq \bigcup_i U_i$ be an open cover. Since I is connected, it is enough to show that the set C of all points $x \in I$ such that $[a, x]$ admits a finite subcover is a closed open set in I.

Now, given a point $x \in I$, we have $x \in B_r(x) \subseteq U_i$ for some index i and some $r > 0$. It is clear that $B_r(x) \subseteq C$ whenever $x \in C$ (hence C is open) and that $B_r(x) \subseteq I - C$ whenever $x \in I - C$ (hence C is closed).

1.4.1 Continuous Functions

Definition: A map $f \colon X \to Y$ between topological spaces is **continuous at a point** $x \in X$ when $f^{-1}(V)$ is a neighborhood of x for any neighborhood V of $f(x)$. We say that f is **continuous** when it is continuous at any point of X: when $f^{-1}(V)$ is an open set in X for any open set V in Y; i.e. $f^{-1}(Z)$ is a closed set in X for any closed set Z in Y. We say that a continuous bijective map f is an **homeomorphism** if $f^{-1} \colon Y \to X$ also is continuous.

1. Compositions of continuous maps also are continuous.

2. Any continuous map f preserves limits: if $x = \lim x_n$, then $f(x) = \lim f(x_n)$.

   Conversely, when X and Y are metric spaces, any map $f \colon X \to Y$ preserving limits is continuous: if $Z \subseteq Y$ is closed and a sequence $x_n \in f^{-1}(Z)$ converges, then $\lim x_n \in f^{-1}(Z)$ because $f(\lim x_n) = \lim f(x_n) \in Z$.

3. The functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x + y$, and $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto xy$, are continuous; hence, sums and products of real-valued continuous functions are continuous.

   The function $\mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$, $x \mapsto 1/x$, is continuous; hence the inverse of a continuous function (without zeroes) also is continuous.

Theorem: *Let $f: X \to Y$ be a continuous map. If X is compact, then $f(X)$ is compact. If X is connected, then $f(X)$ is connected.*

Proof: Let us consider an open cover $f(X) \subseteq \bigcup_i V_i$, so that $X = \bigcup_i f^{-1}(V_i)$.
Now, if X is compact, it admits a finite subcover, $X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$.
Hence $f(X) \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$, and $f(X)$ is compact.

Let V be an open closed set in $f(X)$, so that $f^{-1}(V)$ is an open closed set in X.
If X is connected, then $f^{-1}(V) = \emptyset$ or $f^{-1}(V) = X$. Hence $V = \emptyset$ or $V = f(X)$.

Corollary: *Any continuous function $f: K \to \mathbb{R}$ on a compact topological space $K \neq \emptyset$ attains a maximum and a minimum* (there exist $p, q \in K$ such that $f(p) \leq f(x) \leq f(q)$ for all $x \in K$).

Proof: Since $f(K) \subset \mathbb{R}$ is bounded, it admits an infimum a and a supremum b.
Since $f(K)$ is closed, we have that $a, b \in f(K)$, and we conclude.

Corollary: *If a continuous function $f: X \to \mathbb{R}$ on a connected topological space X attains two values $a < b$, then it also attains any value $a < c < b$* (i.e. $c = f(x)$ for some $x \in X$.)

Intermediate Value Theorem: *Any continuous function $f: [a, b] \to \mathbb{R}$ attains any value between $f(a)$ and $f(b)$.*

Corollary: *If ρ is a positive real number and $n \geq 1$ is a natural number, there exists a unique positive real number $\sqrt[n]{\rho}$ such that $(\sqrt[n]{\rho})^n = \rho$.*

Proof: Consider the continuous function $f: [0, b] \to \mathbb{R}$, $f(x) = x^n$, where $b^n > \rho$.
The existence of a positive n-th root follows from the above corollary, since $f(0) < \rho < f(b)$, and it is unique since $x < y \Rightarrow x^n < y^n$.

Bolzano's Theorem: *Let $f: [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)$ and $f(b)$ have opposite sign, then we have $f(x) = 0$ at some point $a < x < b$.*

1.4.2 Completion of a Metric Space

Definition: A map $f: X \to Y$ between metric spaces is **uniformly continuous** when for every $\varepsilon \in \mathbb{R}_+$ there exists $\delta \in \mathbb{R}_+$ such that $f\big(B_\delta(x)\big) \subseteq B_\varepsilon\big(f(x)\big)$, $\forall x \in X$; i.e. such that $\mathrm{d}(x, y) < \delta \Rightarrow \mathrm{d}(f(x), f(y)) < \varepsilon$.
Compositions of uniformly continuous maps also are uniformly continuous.

Proposition: *If Y is a subset of a metric space X, then the function*

$$f: X \to \mathbb{R},\ f(x) = \mathrm{d}(x, Y) = \inf\{\mathrm{d}(x, y);\ y \in Y\}$$

is uniformly continuous. Moreover, $f(\bar{Y}) = 0$ and $f(x) > 0$ whenever $x \notin \bar{Y}$.

Proof: Since $\mathrm{d}(x', y) \leq \mathrm{d}(x', x) + \mathrm{d}(x, y)$, $\forall y \in Y$, we have $|f(x') - f(x)| \leq \mathrm{d}(x', x)$.
Moreover, $f(x) = 0$ if and only if any ball $B_\varepsilon(x)$ intersects Y, i.e. $x \in \bar{Y}$.

Theorem: *Any continuous map f from a compact metric space K to a metric space Y is uniformly continuous.*

Proof: Since f is continuous, given $\varepsilon > 0$, at any point $x \in K$ we have a ball $B(x, \delta_x)$ such that $f\big(B(x,\delta_x)\big) \subseteq B\big(f(x), \varepsilon\big)$, and the balls $B(x, \frac{1}{2}\delta_x)$ define an open cover of K.

Since K is compact, it admits a finite subcover, $K = B(x_1, \frac{1}{2}\delta_{x_1}) \cup \ldots \cup B(x_n, \frac{1}{2}\delta_{x_n})$.

Put $\delta := \min\{\frac{1}{2}\delta_{x_1}, \ldots, \frac{1}{2}\delta_{x_n}\} > 0$.

Now, if $\mathrm{d}(x,y) < \delta$, there is an index i such that $\mathrm{d}(x_i, x) < \frac{1}{2}\delta_{x_i}$, so that

$$\mathrm{d}(x_i, y) \le \mathrm{d}(x_i, x) + \mathrm{d}(x, y) < \tfrac{1}{2}\delta_{x_i} + \delta \le \delta_{x_i}.$$

Hence $\mathrm{d}\big(f(x_i), f(y)\big) < \varepsilon$, and we conclude:

$$\mathrm{d}\big(f(x), f(y)\big) \le \mathrm{d}\big(f(x), f(x_i)\big) + \mathrm{d}\big(f(x_i), f(y)\big) < \varepsilon + \varepsilon = 2\varepsilon.$$

Definitions: A sequence of points (x_n) of a metric space X is a **Cauchy** sequence when for every $\varepsilon \in \mathbb{R}_+$ there is an index n_ε such that $\mathrm{d}(x_n, x_m) < \varepsilon$, $\forall m, n \ge n_\varepsilon$.

Any convergent sequence is a Cauchy sequence, and a metric space X is **complete** when every Cauchy sequence in X converges to a limit $x \in X$ (clearly unique).

Uniformly continuous maps preserve Cauchy sequences.

1. *Any Euclidean vector space is complete, hence $\mathbb{C}$ is complete.* In fact, a sequence of points converges (resp. is a Cauchy sequence) if and only if the coordinates converge (resp. define Cauchy sequences) in $\mathbb{R}$.

2. An infinite **series** of complex numbers $\sum_n a_n$ is always understood as the sequence of partial sums $s_n = a_0 + \ldots + a_n$. Hence, it converges if for any $\varepsilon > 0$ there exists and index n_ε such that $|a_n + \ldots + a_m| < \varepsilon$ whenever $m \ge n \ge n_\varepsilon$.

3. *If a series $\sum_n a_n$ converges, then $\lim a_n = 0$.*

   We have $\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = 0$.

4. *If $|a_n| \le b_n$ when $n \gg 0$, and $\sum_n b_n$ converges, then so does $\sum_n a_n$.*

   When $n \gg 0$, we have $|a_n + \ldots + a_m| \le b_n + \ldots + b_m = |b_n + \ldots + b_m|$.

5. *If $\lim \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_n |a_n|$ (hence $\sum_n a_n$) converges.*

   We may assume that $\frac{|a_{n+1}|}{|a_n|} \le r < 1$. Hence $|a_n| \le |a_0| r^n$, and $\sum_n |a_0| r^n = |a_0| \frac{1}{1-r}$.

Proposition: *Any closed subspace Y of a complete metric space X is complete.*

Proof: Any Cauchy sequence (y_n) in Y converges to a point $x \in X$, because X is complete, and $x \in Y$ when Y is closed.

Proposition: *Any complete subspace Y of a metric space X is a closed subspace.*

Proof: If $x \in \bar{Y}$, then we have points $y_n \in B\big(x, \frac{1}{n}\big) \cap Y$, so that $y_n \to x$.

Now (y_n) is a Cauchy sequence because $\mathrm{d}(y_n, y_m) \le \mathrm{d}(y_n, x) + \mathrm{d}(x, y_m) < \frac{1}{n} + \frac{1}{m}$.

Hence (y_n) converges to a point $y \in Y$, and $y = x$ because the limit point is unique.

Proposition: *Any compact metric space K is complete.*

Proof: Let (x_n) be a Cauchy sequence in K.

Given $\varepsilon > 0$, there is an index n_ε such that $\mathrm{d}(x_n, x_m) < \varepsilon$, $\forall n, m \geq n_\varepsilon$.

Now, let $x \in K$ be an adherent point to (x_n). We have $\mathrm{d}(x, x_n) < \varepsilon$ for some $n \geq n_\varepsilon$, so that $\mathrm{d}(x, x_m) \leq \mathrm{d}(x, x_n) + \mathrm{d}(x_n, x_m) < 2\varepsilon$, $\forall m \geq n$. We conclude that $x_n \to x$.

Definition: A topological space is **Baire** when the intersection of any countable family of dense open sets is dense (the union of any countable family of closed sets with empty interior has dense complement). In particular, if a non-empty Baire space is a countable union $X = \bigcup_n Y_n$ of closed sets, then some Y_n has non-empty interior.

Baire Theorem: *Any complete metric space X is Baire.*

Proof: Let $\{U_n\}$ be a countable family of dense open sets in X, and let us see that any non-empty open set $V \subseteq X$ intersects $\bigcap_n U_n$.

Since U_1 is dense, there is a closed ball $\bar{B}(x_1, r_1) \subseteq U_1 \cap V$.

Recursively, there are closed balls $\bar{B}(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \cap U_n$, $0 < r_n < \frac{1}{n}$.

Since $x_m \in B(x_n, r_n)$ when $m \geq n$, we have that (x_n) is a Cauchy sequence.

Hence $x_n \to x \in X$. Since $\bar{B}(x_n, r_n)$ is closed, we conclude that $x \in \bar{B}(x_n, r_n) \subseteq U_n \cap V$.

Definition: In a metric space X, two Cauchy sequences $(x_n), (x'_n)$ are said to be **equivalent** if $\lim \mathrm{d}(x_n, x'_n) = 0$, and it is an equivalence relation on the set of all Cauchy sequences.

The quotient set will be denoted $\widehat{X}$ and it is the **completion** of the metric space X.

Theorem: *The completion $\widehat{X}$ is a complete metric space with the metric*

$$\mathrm{d}(\hat{x}, \hat{y}) := \lim \mathrm{d}(x_n, y_n),$$

where (x_n) and (y_n) are Cauchy sequences representing $\hat{x}, \hat{y} \in \widehat{X}$ respectively.

We have a canonical metric morphism $i\colon X \to \widehat{X}$, $i(x) = [(x, x, \ldots)]$, with dense image, and it is an isometry when X is complete.

Proof: First we see that $\mathrm{d}(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ (hence it converges):

$$\mathrm{d}(x_n, y_n) - \mathrm{d}(x_m, y_m) \leq \mathrm{d}(x_n, x_m) + \mathrm{d}(x_m, y_m) + \mathrm{d}(y_m, y_n) - \mathrm{d}(x_m, y_m) = \mathrm{d}(x_n, x_m) + \mathrm{d}(y_m, y_n)$$

hence $|\mathrm{d}(x_n, y_n) - \mathrm{d}(x_m, y_m)| \leq \mathrm{d}(x_n, x_m) + \mathrm{d}(y_m, y_n)$, and we conclude.

1. d *is well-defined*:

   If (x'_n) also represents $\hat{x}$, then $\mathrm{d}(x'_n, y_n) \leq \mathrm{d}(x'_n, x_n) + \mathrm{d}(x_n, y_n)$; hence $\lim \mathrm{d}(x'_n, y_n) \leq \lim \mathrm{d}(x_n, y_n)$, since $\lim \mathrm{d}(x'_n, x_n) = 0$. By symmetry $\lim \mathrm{d}(x_n, y_n) \leq \lim \mathrm{d}(x'_n, y_n)$ and we conclude.

2. d *is a metric on* $\widehat{X}$:

   Clearly $\mathrm{d}(\hat{x}, \hat{y}) = \mathrm{d}(\hat{y}, \hat{x})$ and, if $\mathrm{d}(\hat{x}, \hat{y}) = 0$, then (x_n) and (y_n) are equivalent, and $\hat{x} = \hat{y}$. If a Cauchy sequence (z_n) represents a point $\hat{z} \in \widehat{X}$, then $\mathrm{d}(x_n, z_n) \leq \mathrm{d}(x_n, y_n) + \mathrm{d}(y_n, z_n)$, and taking limits we see that $\mathrm{d}(\hat{x}, \hat{z}) \leq \mathrm{d}(\hat{x}, \hat{y}) + \mathrm{d}(\hat{y}, \hat{z})$.

3. *The metric morphism i has dense image*:

   Given $\hat{x} \in \widehat{X}$ and $\varepsilon > 0$, let (x_n) be a Cauchy sequence representing $\hat{x}$. There is an index n_0 such that $\mathrm{d}(x_n, x_m) < \frac{1}{2}\varepsilon$, $\forall n, m \geq n_\varepsilon$. Hence $\mathrm{d}\big(\hat{x}, i(x_{n_0})\big) \leq \frac{1}{2}\varepsilon < \varepsilon$, and $i(x_{n_0}) \in B_\varepsilon(\hat{x})$.

   If moreover X is complete, then $\hat{x} = \lim x_n \in X$, so that i is bijective.

Finally, $\widehat{X}$ *is complete* (compare with the proof in p. 22, where $X = \mathbb{Q}$ and $\widehat{X} = \mathbb{R}$):

Given a Cauchy sequence (x_n) in $\widehat{X}$, pick $q_n \in X$ such that $\mathrm{d}(q_n, x_n) < \frac{1}{n}$.

Then (q_n) is a Cauchy sequence in X: Given $\varepsilon > 0$, there is an index n_0 such that $n_0 > \frac{3}{\varepsilon}$ and $\mathrm{d}(x_n, x_m) < \frac{\varepsilon}{3}$, $\forall n, m \geq n_0$, so that $\mathrm{d}(q_n, q_m) \leq \mathrm{d}(q_n, x_n) + \mathrm{d}(x_n, x_m) + \mathrm{d}(x_m, q_m) < \varepsilon$.

In particular, if we put $x = [(q_n)] \in \widehat{X}$, we have $\mathrm{d}(q_n, x) \leq \varepsilon$, $\forall n \geq n_0$.

Now, given $\varepsilon > 0$, there exists an index n_ε such that $n_\varepsilon > \frac{2}{\varepsilon}$ and $\mathrm{d}(q_n, x) < \frac{\varepsilon}{2}$, $\forall n \geq n_\varepsilon$, so that $\mathrm{d}(x_n, x) \leq \mathrm{d}(x_n, q_n) + \mathrm{d}(q_n, x) < \frac{1}{n} + \frac{\varepsilon}{2} \leq \varepsilon$, $\forall n \geq n_\varepsilon$. Hence $x = \lim x_n$.

Theorem: *Let X, Y be two metric spaces. Any uniformly continuous map $f\colon X \to Y$ induces a uniformly continuous map $\hat{f}\colon \widehat{X} \to \widehat{Y}$ such that the following square is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \widehat{X} & \xrightarrow{\hat{f}} & \widehat{Y} \end{array}$$

Proof: If $s = (x_n)$ is a Cauchy sequence in X, then $f(s) = (f(x_n))$ is a Cauchy sequence in Y, because f is uniformly continuous.

If (x'_n) is equivalent to (x_n), for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathrm{d}(f(x_n), f(x'_n)) < \varepsilon$ whenever $\mathrm{d}(x_n, x'_n) < \delta$. Hence $(f(x_n))$ is equivalent to $(f(x'_n))$, and f induces a natural map $\hat{f}\colon \widehat{X} \to \widehat{Y}$, $\hat{f}\big([s]\big) = \big[f(s)\big]$, such that the square is commutative.

Let us see that $\hat{f}$ is uniformly continuous:

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathrm{d}(x, y) < \delta \Rightarrow \mathrm{d}\big(f(x), f(y)\big) < \varepsilon$.

Take $\hat{x} = [(x_n)], \hat{y} = [(y_n)] \in \widehat{X}$ such that $\mathrm{d}(\hat{x}, \hat{y}) = \lim \mathrm{d}(x_n, y_n) < \delta$. When $n \gg 0$, we have $\mathrm{d}\big(f(x_n), f(y_n)\big) < \varepsilon$, so that $\mathrm{d}\big(\hat{f}(\hat{x}), \hat{f}(\hat{y})\big) = \lim \mathrm{d}\big(f(x_n), f(y_n)\big) \leq \varepsilon < 2\varepsilon$.

Universal Property: *If $f\colon X \to Y$ is a uniformly continuous map and Y is complete, there exists a unique uniformly continuous extension $\phi\colon \widehat{X} \to Y$* (a map such that $f = \phi \circ i$.)

Proof: The existence follows from the above theorem, since $i\colon Y \to \widehat{Y}$ is an isometry.

The uniformly continuous extension is unique since it is determined on the dense set $i(X)$.

Corollary: *If X is complete, the completion of any subspace $Y \subseteq X$ is just the closure $\bar{Y}$.*

Proof: The natural map $\phi\colon \widehat{Y} \to \bar{Y}$ preserves distances, $\mathrm{d}(\hat{x}, \hat{y}) = \mathrm{d}(\phi(\hat{x}), \phi(\hat{y}))$, so that in particular it is injective, and it is surjective because any point $\bar{y} \in \bar{Y}$ is the limit of some Cauchy sequence (y_n) in Y: just take points $y_n \in Y \cap B(\bar{y}, 1/n)$.

1.5 Differential Calculus

In this section U will be a non empty open set in $\mathbb{R}$, and $a \in U$.

Definition: A function $f\colon U \to \mathbb{R}$ is **differentiable** at a when the quotients $\frac{f(x)-f(a)}{x-a}$ have a finite limit as $x \to a$, so that we have

$$f(x) = f(a) + (x - a)u(x)$$

for some function u continuous at $x = a$ (in particular so is f, and such function u is clearly unique), and the **derivative** of f at $x = a$ is

$$f'(a) := u(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\,,$$

(for any $\varepsilon \in \mathbb{R}_+$ there is $\delta \in \mathbb{R}_+$ such that $\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \varepsilon$ when $0 < |x-a| < \delta$).

We say that f is differentiable when so it is at any point of U (in particular f is continuous), and the function $f' \colon U \to \mathbb{R}$ is the **derivative** of f.

Theorem: *Constant functions have null derivative, and the derivative of x is 1. Sums and products of differentiable functions also are differentiable and, if a differentiable function f does not vanish at any point of U, then $\frac{1}{f}$ also is differentiable. Moreover*

$$(f+g)' = f' + g' \quad , \quad (fg)' = f'g + fg' \quad , \quad (1/f)' = -f'/f^2 \quad , \quad (g/f)' = (g'f - f'g)/f^2.$$

Proof: We have $f = f(a) + (x-a)u$, $g = g(a) + (x-a)v$ where u, v are continuous at a and $u(a) = f'(a)$, $v(a) = g'(a)$. Now,

$$f + g - f(a) - g(a) = (x-a)(u+v),$$
$$fg - f(a)g(a) = (x-a)\big(f(a)v + g(a)u + (x-a)uv\big),$$
$$\frac{1}{f} - \frac{1}{f(a)} = \frac{f(a)-f}{f(a)f} = \frac{-u}{f(a)f}(x-a).$$

Chain Rule: *Let U, V be non-empty open sets in $\mathbb{R}$. If $y \colon U \to V$ is differentiable at a point $a \in U$ and $f \colon V \to \mathbb{R}$ is differentiable at $b = y(a) \in V$, then the composition $z = f(y)$ is differentiable at a and $z'(a) = f'(b)y'(a)$.*

Hence, if y and f are differentiable, so is $z = f(y)$ and $z' = f'(y)y'$.

Proof: We have $y(x) = b + (x-a)u(x)$, for all $x \in U$, and $f(y) = f(b) + (y-b)v(y)$, for all $y \in V$. Hence,

$$z(x) = f\big(y(x)\big) = f(b) + \big(y(x) - b\big)v\big(y(x)\big) = z(a) + (x-a)u(x)v\big(y(x)\big),$$

where the function $uv(y)$ is continuous at a, since so are u and y, and v is continuous at $y(a)$.

Hence z is differentiable at a and $z'(a) = u(a)v\big(y(a)\big) = u(a)v(b) = y'(a)f'(b)$.

Definition: A function $f \colon U \to \mathbb{R}$ has a **local maximum** (resp. **local minimum**) at $a \in U$ if there is a neighborhood V of a where $f(x) \leq f(a)$, $\forall x \in V$ (resp. $f(x) \geq f(a)$, $\forall x \in V$), and we say that it is **strict** if moreover $f(x) \neq f(a)$, $\forall a \neq x \in V$.

Theorem: *If a function $f \colon U \to \mathbb{R}$ has a local maximum or minimum at a point $a \in U$ and it is differentiable at a, then $f'(a) = 0$.*

Proof: We have $f(x) - f(a) = (x-a)u(x)$, where u is continuous at a, and $f'(a) = u(a)$.

If f has a local minimum, in a neighborhood of a we have $(x-a)u(x) \geq 0$. Hence $u(x) \geq 0$ when $x > a$ (so that and $u(a) \leq 0$) and $u(x) \leq 0$ when $x < a$ (so that $u(a) \leq 0$). We conclude that $u(a) = 0$.

If f has a local maximum, then $-f$ has a local minimum, so that $-f'(a) = 0$.

Theorem: *If f is differentiable at a and $f'(a) > 0$ (resp. $f'(a) < 0$), then there exists $\varepsilon > 0$ such that $f(a-t) < f(a) < f(a+t)$ (resp. $f(a-t) > f(a) > f(a+t)$) for all $0 < t < \varepsilon$.*

Proof: We have $f = f(a) + (x-a)u$ where u is continuous at a and $u(a) > 0$; hence u is positive on a certain interval $(a-\varepsilon, a+\varepsilon)$ and we conclude.

Theorem: *If f is differentiable on an open interval U and $f' > 0$ (resp. $f' < 0$), then f is a **strictly increasing** function: $x < y \Rightarrow f(x) < f(y)$ (resp. a **strictly decreasing** function: $x < y \Rightarrow f(x) > f(y)$).*

Proof: Now, if $f(x) \geq f(y)$ for some $x < y \in U$, since f is continuous, it attains a maximum at an interior point ξ of $[x, y]$ and $f'(\xi) = 0$, so contradicting that f' is positive.

Rolle's Theorem: *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, differentiable in (a,b). If $f(a) = f(b)$, then $f'(\xi) = 0$ at some point $a < \xi < b$.*

Proof: Since f attains a maximum and a minimum in $[a, b]$ and $f(a) = f(b)$, it attains a maximum or a minimum at an interior point $a < \xi < b$, so that $f'(\xi) = 0$.

Cauchy's Mean Value Theorem: *Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions, differentiable in (a,b). There exists $a < \xi < b$ such that $\big[f(b) - f(a)\big]g'(\xi) = \big[g(b) - g(a)\big]f'(\xi)$.*

Proof: Apply Rolle's theorem to $h(x) = \begin{vmatrix} 1 & f(x) & g(x) \\ 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \end{vmatrix}$, and use $h'(x) = \begin{vmatrix} 0 & f'(x) & g'(x) \\ 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \end{vmatrix}$.

When $g(x) = x$, we obtain:

Lagrange's Mean Value Theorem: *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, differentiable in (a,b). There exists $a < \xi < b$ such that $f(b) - f(a) = f'(\xi)(b - a)$.*

Theorem: *If $f\colon (a,b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant.*

Proof: Apply Lagrange's theorem to every closed interval $[c, d] \subset (a, b)$.

Definition: A function f is of **class** $\mathcal{C}^0$ if it is continuous, of class $\mathcal{C}^m$ if it is differentiable and f' is of class $\mathcal{C}^{m-1}$, and of class $\mathcal{C}^\infty$ when it is of class $\mathcal{C}^m$, $\forall m \in \mathbb{N}$. We denote by $\mathcal{C}^m(U)$ the ring of all functions of class $\mathcal{C}^m$ on U.

The n-th iterated derivative of a function f will be denoted $f^{(n}$.

Theorem: *Let $f\colon U \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. If $f'(a) > 0$ at a point $a \in U$, then f is a strictly increasing function on a neighborhood V of a.*

If $f'(a) < 0$, then f is a strictly decreasing function on a neighborhood V of a.

Proof: Since f' is continuous, it is positive on a neighborhood of a.

Lemma: *Let U be an interval and f a function of class $\mathcal{C}^n$ on U, $n \geq 1$. If $f'(a) = \ldots = f^{(n-1}(a) = 0$, then for any point $x \in U$ there is an intermediate point ξ between a and x such that*

$$f(x) - f(a) = \frac{f^{(n}(\xi)}{n!}(x - a)^n \cdot$$

Proof: By induction on n, and it is Lagrange's theorem when $n = 1$.

When $n > 1$, applying Cauchy's mean value theorem to $f(x)$ and $g(x) = (x - a)^n$, we obtain

$$\frac{f(x) - f(a)}{(x - a)^n} = \frac{f'(\xi)}{n(\xi - a)^{n-1}} = \frac{1}{n}\frac{f'(\xi) - f'(a)}{(\xi - a)^{n-1}}$$

where $a < \xi < x$ (or $x < \xi < a$). Now f' is a function of class $\mathcal{C}^{n-1}$ and the first $n-2$ derivatives at the point $x = a$ vanish. By induction, there is a point $a < \xi' < \xi < x$ (or...) such that

$$\frac{1}{n}\frac{f'(\xi) - f'(a)}{(\xi - a)^{n-1}} = \frac{1}{n}\frac{f^{(n}(\xi')}{(n-1)!} = \frac{f^{(n}(\xi')}{n!}.$$

Corollary: *Let f be a function of class $\mathcal{C}^n$ such that $f'(a) = \ldots = f^{(n-1}(a) = 0$, $n \geq 2$.*
If n is even and $f^{(n}(a) > 0$, then f has a strict local minimum at $x = a$.
If n is even and $f^{(n}(a) < 0$, then f has a strict local maximum at $x = a$.
If n is odd and $f^{(n}(a) \neq 0$, then f has not an extremum at $x = a$.

Proof: Since $f^{(n}$ is continuous, in a neighborhood of a we have $f(x) - f(a) = \frac{1}{n}f^{(n}(\xi)(x-a)^n$, where the sign of $f^{(n}(\xi)$ coincides with the sign of $f^{(n}(a)$. Now the result is clear.

Definition: The n-th **Taylor polynomial** at a point $x = a$ of a function f of class $\mathcal{C}^n$ is

$$(T_a^n f)(x) := f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n}(a)}{n!}(x-a)^n.$$

Remark that $P(a) = f(a)$, $P'(a) = f'(a)$, $P''(a) = f''(a), \ldots, P^{(n}(a) = f^{(n}(a)$.

Taylor's Formula: *Let f be a function of class $\mathcal{C}^{n+1}$. For any point $x \in U$ there is an intermediate point ξ between a and x such that*

$$f(x) = (T_a^n f)(x) + \frac{f^{(n+1}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Proof[2]: The function $h(x) = f(x) - (T_a^n f)(x)$ is of class $\mathcal{C}^{n+1}$ and $h'(a) = \ldots = h^{(n}(a) = 0$. Moreover $h(a) = 0$ and $h^{(n+1} = f^{(n+1}$. We conclude by the lemma.

Primitives: Let U be an interval, and $f\colon U \to \mathbb{R}$ be an arbitrary function. If $F\colon U \to \mathbb{R}$ is differentiable and $F' = f$, we say that F is a **primitive** of f, and we put $\mathrm{d}F = f\mathrm{d}x$ (because infinitesimally the increments of F at $x = a$ are the product of $f(a)$ by the increments of x). If it exists, the primitive is unique up to the addition of a constant c and (in view of the forthcoming Barrow's rule) we put $\int f(x)\mathrm{d}x = F(x) + c$, so that $\int \mathrm{d}F = F + c$.

If $u, v\colon U \to \mathbb{R}$ are differentiable, we have $(uv)' = u'v + uv'$, so that $\mathrm{d}(uv) = v\mathrm{d}u + u\mathrm{d}v$, and we obtain the **integration by parts** formula:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$$

Moreover, the chain's rule gives the **integration by substitution** formula:

$$\int f(y)\mathrm{d}y = \int f\big(y(x)\big)y'(x)\mathrm{d}x,$$

(if F is a primitive of $f\colon V \to \mathbb{R}$, and $y\colon U \to V$ is differentiable, then a primitive of $f(y)y'$ is $F(y)\colon U \to \mathbb{R}$). Therefore, when $y = y(x)$ is bijective and the inverse $x(y)$ also is differentiable, if $G(x)$ is a primitive of $f\big(y(x)\big)y'(x)$, then $G\big(x(y)\big)\colon V \to \mathbb{R}$ is a primitive of f.

[2]The proofs of the lemma and this theorem only use the existence of the last derivative, not its continuity.

1.6 Integral Calculus

Let $f\colon [a,b] \to \mathbb{R}$ be a **bounded** function (there is $c \in \mathbb{R}_+$ such that $|f(x)| \le c$, $\forall x \in [a,b]$).

Definitions: A **partition** of the interval $[a,b]$ is any finite subset P containing both ends. We usually consider it as an increasing sequence $a = x_0 < x_1 < \ldots < x_n = b$, and we put

$$I_i = [x_{i-1}, x_i],\ \mu(I_i) = x_i - x_{i-1},\ m_i = \inf\{f(x);\ x \in I_i\},\ M_i = \sup\{f(x);\ x \in I_i\},$$

and the lower and upper **Riemann sums** of f over $[a,b]$ with partition P are

$$s_f(P) = \sum_{i=1}^{n} m_i \mu(I_i) \quad , \quad S_f(P) = \sum_{i=1}^{n} M_i \mu(I_i).$$

Lemma: *Given partitions P, P' of $[a,b]$, if we put $P'' = P \cup P'$, then*

$$s_f(P) \le s_f(P'') \le S_f(P'') \le S_f(P').$$

Definition: By the lemma, the supremum of the lower sums is bounded above by the infimum of the upper sums. We say that a bounded function f is **Riemann integrable** when both coincide, and we put

$$\int_a^b f \mathrm{d}x = \sup_P \{s_f(P)\} = \inf_P \{S_f(P)\}$$

because it is "the sum of the product of f by the infinitesimal differences of x". Hence, f is Riemann integrable when for any $\varepsilon > 0$ there are partitions P, P' such that $S_f(P') - s_f(P) < \varepsilon$ (or, by the lemma, when there is a partition P such that $S_f(P) - s_f(P) < \varepsilon$).

1. *If $f, h\colon [a,b] \to \mathbb{R}$ are Riemann integrable, then so is any $\mathbb{R}$-linear combination $\lambda f + \mu h$ and*

$$\int_a^b (\lambda f + \mu h)\mathrm{d}x = \lambda \int_a^b f \mathrm{d}x + \mu \int_a^b h \mathrm{d}x\ .$$

   In fact we have $s_f(P) + s_h(P) \le s_{f+h}(P) \le S_{f+h}(P) \le S_f(P) + S_h(P)$.

   Moreover $s_{\lambda f}(P) = \lambda s_f(P)$ and $S_{\lambda f}(P) = \lambda S_f(P)$ when $\lambda \ge 0$. And finally, we also have $s_{-f}(P) = -S_f(P)$ and $S_{-f}(P) = -s_f(P)$.

2. *If $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable and $m \le f \le M$, then $m(b-a) \le \displaystyle\int_a^b f \mathrm{d}x \le M(b-a)$.*

3. *If $f, h\colon [a,b] \to \mathbb{R}$ are Riemann integrable and $f \le h$, then $\displaystyle\int_a^b f \mathrm{d}x \le \int_a^b h \mathrm{d}x$.*

   If $f \le h$, then $0 \le h - f$, so that $0 \le \displaystyle\int_a^b (h-f)\mathrm{d}x = \int_a^b h \mathrm{d}x - \int_a^b f \mathrm{d}x$.

4. *If $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then so is $|f|$ and $\left|\displaystyle\int_a^b f \mathrm{d}x\right| \le \int_a^b |f| \mathrm{d}x$.*

   If $\bar{m}_i$ and $\bar{M}_i$ are the infimum and supremum of $|f|$ on an interval $[x_{i-1}, x_i]$ of a partition P of $[a,b]$, then $\bar{M}_i - \bar{m}_i \le M_i - m_i$ because $|y| - |x| \le |y - x|$.

   Hence $S_{|f|}(P) - s_{|f|}(P) \le S_f(P) - s_f(P)$ and $|f|$ is integrable. Moreover $f \le |f|$ and $-f \le |f|$; hence $\int_a^b f \mathrm{d}x \le \int_a^b |f| \mathrm{d}x$ and $-\int_a^b f \mathrm{d}x \le \int_a^b |f| \mathrm{d}x$, so that $\left|\int_a^b f \mathrm{d}x\right| \le \int_a^b |f| \mathrm{d}x$.

5. *Any continuous function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable.*

   Since f is uniformly continuous, given $\varepsilon > 0$, there is $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ when $|y - x| < \delta$, so that we have $M_i - m_i < \varepsilon$ when P is a partition with intervals of length $< \delta$. Hence $S_f(P) - s_f(P) < \sum_i \varepsilon\mu(I_i) = \varepsilon(b-a)$, and f is integrable.

6. *Any **increasing** function ($x \le y \Rightarrow f(x) \le f(y)$) is Riemann integrable; hence so is any **decreasing** function ($x \le y \Rightarrow f(x) \ge f(y)$).*

   Since $m_i = f(x_{i-1})$ and $M_i = f(x_i)$, when we consider the partition P_n of $[a,b]$ in n intervals of equal length, we have

$$S_f(P_n) - s_f(P_n) \le \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\frac{b-a}{n} = \big(f(b) - f(a)\big)\frac{b-a}{n}\cdot$$

7. *If $a < c < b$, then f is Riemann integrable if and only if so are the restrictions of f to $[a,c]$ and $[c,b]$, and we have*

$$\int_a^b f\mathrm{d}x = \int_a^c f\mathrm{d}x + \int_c^b f\mathrm{d}x.$$

   Any partition of $[a,b]$ is contained in a partition $P = P' \cup P''$, where P' is a partition of $[a,c]$ and P'' of $[c,b]$, and we have $s_f(P) = s_f(P') + s_f(P'')$, $S_f(P) = S_f(P') + S_f(P'')$.

Theorem: *If a bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable, then $F(x) = \int_a^x f\mathrm{d}t$ is continuous. If moreover, f is continuous at an interior point $a < c < b$, then $F(x)$ is differentiable at the point $x = c$ and $F'(c) = f(c)$.*

Proof: If $|f| \le c$, then $|F(y) - F(x)| = |\int_x^y f\mathrm{d}t| \le \int_x^y |f|\mathrm{d}t \le c|y-x|$.

Hence F is continuous and $F(c+t) - F(c) = \int_c^{c+t} f\mathrm{d}x = u(t)t$, where

$$\inf\{f(x);\ |x-c| \le t\} \le u(t) \le \sup\{f(x);\ |x-c| \le t\}.$$

If f is continuous at $x = c$, then $u(t)$ is continuous at $t = 0$, and $u(0) = f(c)$.

Hence $F(x)$ is differentiable at $t = c$ and $F'(c) = u(0) = f(c)$.

Barrow's Rule: *Let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function. If $F\colon [a,b] \to \mathbb{R}$ is a continuous function, differentiable in (a,b), and $F' = f$, then*

$$\int_a^b f\mathrm{d}x = F(b) - F(a).$$

Proof: Given a partition P, by the mean value theorem there exist $\xi_i \in (x_{i-1}, x_i)$ such that

$$F(b) - F(a) = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

so that $s_f(P) \le F(b) - F(a) \le S_f(P)$. Since f is integrable, $F(b) - F(a) = \int_a^b f\mathrm{d}x$.

Corollary: *If $F\colon U \to \mathbb{R}$ is of class C^1 and $[a,b] \subset U$, then* $\displaystyle\int_a^b \mathrm{d}F = F(b) - F(a)$.

Change of Variable Formula: *Let $y(x)$ be a function of class $\mathcal{C}^1$ on a neighborhood of $[a,b]$. If f is a continuous function on $y\big([a,b]\big)$, then*

$$\int_{y(a)}^{y(b)} f(y)\mathrm{d}y = \int_a^b f\big(y(x)\big)y'(x)\mathrm{d}x.$$

Proof: If $F' = f$, by the chain's rule $F\big(y(x)\big)$ is a primitive of $f\big(y(x)\big)y'(x)$, so that

$$\int_a^b f\big(y(x)\big)y'(x)\mathrm{d}x = F\big(y(b)\big) - F\big(y(a)\big) = \int_{y(a)}^{y(b)} f(y)\mathrm{d}y.$$

Definition: When $f(x_1, \dots, x_n)$ is a function defined on a subset of $\mathbb{R}^n$, the **partial derivative** $\frac{\partial f}{\partial x_i}(a_1, \dots, a_n)$ denotes the derivative, if it exists, of $f(a_1, \dots, x_i, \dots, a_n)$ at $x_i = a_i$ (in particular f must be defined at $(a_1, \dots, x_i, \dots, a_n)$ when x_i is in a small interval around a_i):

$$\frac{\partial f}{\partial x_i}(a_1, \dots, a_n) = \lim_{t \to 0} \frac{f(a_1, \dots, a_i + t, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{t}.$$

Differentiation under the Integral Sign: *If $f(x,t)$ is a continuous function on a rectangle $[a,b] \times (c,d)$, then $F(t) = \int_a^b f(x,t)\mathrm{d}x$ is continuous. If moreover $\frac{\partial f}{\partial t}(x,t)$ exists and defines a continuous function on the rectangle, then $F(t)$ is differentiable and*

$$\frac{\partial}{\partial t}\int_a^b f(x,t)\mathrm{d}x = \int_a^b \frac{\partial f}{\partial t}(x,t)\mathrm{d}x.$$

Proof: Since $f(x,t)$ is continuous, it is uniformly continuous on a compact set $[a,b] \times [t-c, t+c]$ and, given $\varepsilon > 0$, there exists $\delta > 0$ such that, when $|h| < \delta$, we have $|f(x,t+h) - f(x,t)| < \varepsilon$ for any $x \in [a,b]$. Integrating we conclude:

$$|F(t+h) - F(t)| = \left| \int_a^b f(x,t+h)\mathrm{d}x - \int_a^b f(x,t)\mathrm{d}x \right| < \varepsilon(b-a).$$

Now assume that $\frac{\partial f}{\partial t}(x,t)$ exists and it is continuous.

Fix a point $(x,t) \in [a,b] \times (c,d)$. By the mean value theorem, when h is a small real number we have $f(x,t+h) - f(x,t) = h\frac{\partial f}{\partial t}(x, t+\xi)$, where $|\xi| \le |h|$. Hence,

$$\frac{f(x,t+h) - f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t) = \frac{\partial f}{\partial t}(x,t+\xi) - \frac{\partial f}{\partial t}(x,t),$$

and, since $\frac{\partial f}{\partial t}$ is uniformly continuous on a compact set $[a,b] \times [t-c, t+c]$, given $\varepsilon > 0$, there exists $\delta > 0$ such that, when $|h| < \delta$, we have $|\frac{\partial f}{\partial t}(x,t+\xi) - \frac{\partial f}{\partial t}(x,t)| < \varepsilon$ for any $x \in [a,b]$. Integrating we conclude:

$$\left| \frac{\int_a^b f(x,t+h)\mathrm{d}x - \int_a^b f(x,t)\mathrm{d}x}{h} - \int_a^b \frac{\partial f}{\partial t}(x,t)\mathrm{d}x \right| < \varepsilon(b-a).$$

Definition: A set $A \subset \mathbb{R}$ has **null measure** when for any $\varepsilon \in \mathbb{R}_+$ there is a countable cover $A \subseteq \bigcup_n I_n$ by open intervals with total measure $\sum_n \mu(I_n) \le \varepsilon$ (in the sense that all the finite sums are bounded above by ε). In particular, any subset $B \subseteq A$ also has null measure.

Lemma: *Any countable union $A = \bigcup_n A_n$ of sets of null measure also has null measure.*

Proof: Given $\varepsilon > 0$, each set A_n admits a countable open cover by open intervals with total measure $\le 2^{-n}\varepsilon$. Now, the family of all these intervals define a countable cover of A by open intervals with total measure $\le \sum_n 2^{-n}\varepsilon = \varepsilon$.

Example: Let us divide $[0,1]$ in three parts, and pick both extremes, $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now we divide both in three parts, and pick the extremes, $A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Repeating the process we obtain a decreasing sequence $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ of closed sets, and $A = \bigcap_n A_n$ is the **Cantor set**. It is a closed set (of cardinal $2^{\aleph_0}$) with null measure.

In fact $A \subset A_n$ and A_n is a disjoint union of 2^n closed intervals with total measure $2^n/3^n$, so thatit may be covered with open intervals of total measure $2^{n+1}/3^n$.

Definition: Let $f\colon X \to \mathbb{R}$ be a bounded function on a topological space X. The **oscillation** of f on a subset $Z \subseteq X$ is $o(f, Z) = \sup_{x\in Z} f(x) - \inf_{x\in Z} f(x)$, and the oscillation of f at a point $x \in X$ is the infimum $o(f, x)$ of the oscillations of f on the neighborhoods of x.

By definition, f is continuous at the point x if and only if $o(f, x) = 0$.

The set $\{x \in X : o(f, x) < \frac{1}{n}\}$ is open; hence $A_n = \{x \in X : o(f, x) \geq \frac{1}{n}\}$ is closed.

Lebesgue's Theorem: *A bounded function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set $A = \{x \in [a, b] : o(f, x) \neq 0\}$ of points of discontinuity has null measure.*

Proof: If f is integrable, since $A = \bigcup_n A_n$, we only have to show that A_n has null measure.

Given $\varepsilon > 0$, take a partition P such that $S_f(P) - s_f(P) < \frac{\varepsilon}{n}$, and consider the intervals $U_j = (x_{j-1}, x_j)$ of the partition intersecting A_n, so that $o(f, U_j) \geq \frac{1}{n}$. We have

$$\frac{1}{n}\sum_j \mu(U_j) \leq \sum_j o(f, U_j)\mu(I_j) \leq \sum_{I_i\in P} o(f, I_i)\mu(I_i) = S_f(P) - s_f(P) < \frac{1}{n}\varepsilon$$

so that $\sum_j \mu(U_j) < \varepsilon$. Now, these intervals U_j cover A_n up to a finite number of points, and we conclude that A_n admits a finite cover by open intervals with total measure $< 2\varepsilon$.

Conversely, if A has null measure, then so does $A_\varepsilon = \{x \in [a, b] : o(f, x) \geq \varepsilon\} \subseteq A$ and, since A_ε is compact, it admits a finite open cover $A_\varepsilon \subset I_1 \cup \ldots \cup I_n$ by disjoint intervals with total measure $\mu(I_1) + \ldots + \mu(I_n) < \varepsilon$.

The end points of these intervals define a partition P', and any point of the compact set $K = (I_1 \cup \ldots \cup I_n)^c$ has a neighborhood where the oscillation of f is $< \varepsilon$. Hence K admits a finite cover by open intervals where the oscillation of f is $< \varepsilon$, and we may add points of K so as to obtain a partition P such that the oscillation of f on the remaining intervals I_i is $< \varepsilon$.

We conclude that f is integrable because, if $|f| \leq c$, then we have

$$S_f(P) - s_f(P) = \sum_{j=1}^{n} o(f, I_j)\mu(I_j) + \sum_i o(f, I_i)\mu(I_i) \leq \sum_{j=1}^{n} 2c\mu(I_j) + \sum_i \varepsilon\mu(I_i) < 2c\varepsilon + (b - a)\varepsilon.$$

Corollary: *If $f, h\colon [a, b] \to \mathbb{R}$ are Riemann integrable functions, then so are fh, $\max(f, h)$ and $\min(f, h)$. Hence so are $|f|$ and $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$.*

Proof: The points of discontinuity are contained in $\{\text{disc. of } f\} \cup \{\text{disc. of } h\}$.

Corollary: *If $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable and $g\colon f([a, b]) \to \mathbb{R}$ is a bounded continuous function, then the composition $g \circ f$ is Riemann integrable.*

1.7 Power Series

Definition: Let Y be a metric space. A sequence of maps $f_n\colon X \to Y$ **converges uniformly** to a limit map $f\colon X \to Y$ if for any $\varepsilon \in \mathbb{R}_+$ there is an index n_ε such that $\mathrm{d}\big(f(x), f_n(x)\big) < \varepsilon$, $\forall n \geq n_\varepsilon$, $x \in X$. When Y is complete, we say that (f_n) converges uniformly if for any $\varepsilon > 0$ there is an index n_ε such that $\mathrm{d}\big(f_m(x), f_n(x)\big) < \varepsilon$, $\forall n, m \geq n_\varepsilon$, $x \in X$, so that at any point $x \in X$ we have a Cauchy sequence $\big(f_n(x)\big)$ in Y, with a limit $f(x)$, and (f_n) converges uniformly to $f\colon X \to Y$ because $\mathrm{d}\big(f(x), f_n(x)\big) = \lim \mathrm{d}\big(f_m(x), f_n(x)\big) \leq \varepsilon$, $\forall n \geq n_\varepsilon$.

Example: If $f_n\colon X \to \mathbb{C}$ are complex-valued functions, the infinite series $\sum_n f_n$ converges uniformly when $|f_n(x) + \ldots + f_m(x)| < \varepsilon$, for all $x \in X$, $0 \ll n \leq m$. In particular, since $|f_n + \ldots + f_m| \leq |f_n| + \ldots + |f_m|$, if $\sum_n |f_n|$ converges uniformly, so does $\sum_n f_n$.

Theorem: *Let X be a topological space. If a sequence of maps $f_n\colon X \to Y$ converges uniformly on X to a limit map $f\colon X \to Y$ and all the maps f_n are continuous at a point $p \in X$, then f also is continuous at p.*

Proof: Given $\varepsilon > 0$ there is an index n such that $\mathrm{d}\big(f(x), f_n(x)\big) < \varepsilon$, $\forall x \in X$. Since f_n is continuous at p, in a neighborhood U of p we have $\mathrm{d}\big(f_n(x), f_n(p)\big) < \varepsilon$, $\forall x \in U$; hence

$$\mathrm{d}\big(f(x), f(p)\big) \leq \mathrm{d}\big(f(x), f_n(x)\big) + \mathrm{d}\big(f_n(x), f_n(p)\big) + \mathrm{d}\big(f_n(p), f(p)\big) < \varepsilon + \varepsilon + \varepsilon.$$

Theorem: *Let $f_n\colon [a,b] \to \mathbb{R}$ be Riemann integrable functions. If (f_n) converges uniformly to a function f, then f is Riemann integrable, and $F_n(x) = \int_a^x f_n \mathrm{d}t$ converge uniformly to $F(x) = \int_a^x f \mathrm{d}t$. In particular,*

$$\lim \int_a^b f_n \mathrm{d}x = \int_a^b (\lim f_n) \mathrm{d}x.$$

Proof: Given $\varepsilon > 0$, there is an index n such that $|f(x) - f_n(x)| < \varepsilon$, $\forall x \in [a,b]$, and there is a partition P such that $S_{f_n}(P) - s_{f_n}(P) < \varepsilon$. Hence,

$$S_f(P) - s_f(P) \leq |S_f(P) - S_{f_n}(P)| + |S_{f_n}(P) - s_{f_n}(P)| + |s_{f_n}(P) - s_f(P)| < (b-a)\varepsilon + \varepsilon + (b-a)\varepsilon$$

and we see that f is integrable. Moreover,

$$|F(x) - F_n(x)| \leq \int_a^x |f(t) - f_n(t)| \mathrm{d}t \leq (b-a)\varepsilon.$$

Theorem: *If some differentiable functions $f_n\colon (a,b) \to \mathbb{R}$ uniformly converge to a function f, and the derivatives f'_n uniformly converge to a function h, then f is differentiable and $f' = h$.*

Proof: Fix a point $c \in (a,b)$, and we may assume $c = 0$.

Since f_n is differentiable, we have $f_n(t) = f_n(0) + u_n(t)t$, where $u_n(t)$ is continuous and $u_n(0) = f'_n(0)$. By the mean value theorem, we have

$$u_n(t) - u_m(t) = \frac{f_n(t) - f_m(t) - (f_n(0) - f_m(0))}{t - 0} = f'_n(\xi) - f'_m(\xi) \quad , \quad a < \xi < b.$$

Since (f'_n) converges uniformly, (u_n) converges uniformly to a continuous function u.

Since $(f_n(0))$ converges, $f_n = f_n(0) + u_n t$ converge uniformly to $f(t) = f(0) + u(t)t$.

Now, $f(t)$ is differentiable at the point $t = 0$, because $u(t)$ is continuous, and we have $f'(0) = u(0) = \lim u_n(0) = \lim f'_n(0) = h(0)$; i.e. $f'(c) = h(c)$. q.e.d.

In fact, only the convergence of $\big(f_n(c)\big)$ at a point $c \in (a,b)$ is required: in such case this argument shows that (f_n) uniformly converges to a function f.

Abel's Theorem: *Let $\sum_n a_n x^n$ be a power series with complex coefficients. If $\sum_n a_n b^n$ converges for some $b \in \mathbb{C}$, then $\sum_n |a_n z^n|$ (hence $\sum_n a_n z^n$) converges when $|z| < |b|$.*

Proof: If $\sum_n a_n b^n$ converges, then $\lim |a_n b^n| = 0$. Let c be an upper bound of $\{|a_n b^n|\}$.

Now $\sum_n |a_n z^n| \leq c \sum_n \left|\frac{z}{b}\right|^n$, and the geometric series converges when $\left|\frac{z}{b}\right| < 1$.

Definition: The **radius of convergence** of a power series $\sum_n a_n(x-a)^n$ with complex coefficients is the supremum r of all non-negative real numbers c such that $\sum_n |a_n|c^n$ converges (eventually $r=0$ or $r=\infty$), so that $\sum_n |a_n(z-a)^n|$ converges when $|z-a|<r$ and, by Abel's theorem, $\sum_n a_n(z-a)^n$ does not converge when $|z-a|>r$.

When $0<r\leq\infty$, we have a well-defined complex function $s(z)=\sum_n a_n(z-a)^n$ on the open ball $B_r(a)$, and the series $\sum_n a_n(x-a)^n$ converges uniformly to the function s on any compact set $K\subset B_r(a)$. In fact, if M is the maximum of $|z-a|$ on K, then we have $|a_n(z-a)^n|\leq|a_n|M^n$ on K, and $\sum_n |a_n|M^n$ converges because $M<r$.

Proposition: *The radius of convergence r of a power series $\sum_n a_n(x-a)^n$ is*

$$r^{-1}=\overline{\lim}\sqrt[n]{|a_n|}:=\inf\{k\in\mathbb{R}_+:\ \sqrt[n]{|a_n|}\leq k \text{ for any index } n, \text{ up to a finite number}\}.$$

Proof: If $c\big(\overline{\lim}\sqrt[n]{|a_n|}\big)<\theta<1$, then $c\sqrt[n]{|a_n|}<\theta$ for any index n, up to a finite number and, since the geometric series $\sum_n\theta^n$ converges, so does $\sum_n |a_n|c^n$.

If $c\big(\overline{\lim}\sqrt[n]{|a_n|}\big)>1$, then $\sum_n |a_n|c^n$ does not converge, since it has infinite terms >1.

Corollary: *$\sum_n a_n(x-a)^n$ and $\sum_n a_n n(x-a)^{n-1}$ have the same radius of convergence.*

Proof: Let us see that $\lim\sqrt[n]{n}=1$. If $1<\theta$, when $n\gg 0$ we have

$$1>\frac{1+\frac{1}{n}}{\theta}=\frac{(n+1)\theta^{-(n+1)}}{n\theta^{-n}}.$$

Hence $\sum_n n\theta^{-n}$ converges and $\lim n\theta^{-n}=0$.

When $n\gg 0$, we have $n\theta^{-n}<1$, hence $n<\theta^n$, hence $1<\sqrt[n]{n}<\theta$.

Now, the series $\sum_n a_n n(x-a)^{n-1}$ and $\sum_n a_n n(x-a)^n=(x-a)\sum_n a_n n(x-a)^{n-1}$ have the same radius of convergence, and

$$\overline{\lim}\sqrt[n]{n|a_n|}=\big(\lim\sqrt[n]{n}\big)\big(\overline{\lim}\sqrt[n]{|a_n|}\big)=\overline{\lim}\sqrt[n]{|a_n|}.$$

Theorem: *Let $\sum_n a_n(x-a)^n$ be a power series with real coefficients and radius of convergence $0<r\leq\infty$. Then $s(x)=\sum_n a_n(x-a)^n$ is a function of class $\mathcal{C}^\infty$ on $(a-r,a+r)$, and $s'(x)=\sum_n na_n(x-a)^{n-1}$.*

Corollary: *Let $\sum_n a_n(x-a)^n$ and $\sum_n b_n(x-a)^n$ be power series with real coefficients and positive radius of convergence. If both define the same function $f(x)$ on some neighborhood of the point $x=a$, then $a_n=b_n$ for any index n.*

Proof: We have $n!a_n=f^{(n}(a)=n!b_n$.

Definition: Let $U\subseteq\mathbb{R}$ be an open set. A function $f\colon U\to\mathbb{R}$ is **analytic** when each point $a\in U$ has an open neighborhood where $f(x)$ is defined by some power series $\sum_n a_n(x-a)^n$ (unique by the above corollary) with positive radius of convergence.

Mertens' Theorem: *Let $\sum_n a_n$, $\sum_n b_n$ be two series of complex numbers and put $c_n=\sum_{i=0}^n a_ib_{n-i}$. If $\sum_n |a_n|$ and $\sum_n b_n$ converge, then so does $\sum_n c_n$ and*

$$\sum_{i=0}^{\infty}c_n=\left(\sum_{i=0}^{\infty}a_n\right)\left(\sum_{i=0}^{\infty}b_n\right).$$

Proof: Put $A_n=\sum_{i=0}^n a_i$, $B_n=\sum_{i=0}^n b_i$, $C_n=\sum_{i=0}^n c_i=\sum_{i=0}^n a_iB_{n-i}$.

If $A = \sum_n a_n = \lim A_n$, and $B = \sum_n b_n = \lim B_n$, then we have

$$|C_n - AB| \leq \left|\sum_{i=0}^{n} a_i B_{n-i} - B \sum_{i=0}^{n} a_i\right| + \left|B \sum_{i=0}^{n} a_i - AB\right| \leq \sum_{i=0}^{n} |a_i| \cdot |B - B_{n-i}| + |B| \cdot \left|A - \sum_{i=0}^{n} a_i\right|.$$

Since the second term converges to 0, we only have to show that so does the first term.
Put $c = \sup\{|B - B_n|\}$, $k = \sum_n |a_n|$.
Given $\varepsilon > 0$, there is an index m such that $\sum_{i=m}^{\infty} |a_i| < \varepsilon$ and $|B - B_n| < \varepsilon$, $\forall n \geq m$.
Hence, when $n \geq 2m$, we have

$$\sum_{i=0}^{n} |a_i||B - B_{n-i}| = \sum_{i=0}^{m-1} |a_i||B - B_{n-i}| + \sum_{i=m}^{n} |a_i||B - B_{n-i}| < k\varepsilon + \varepsilon c.$$

Corollary: *Any product of real analytic functions also is analytic.*

1.7.1 Elementary Functions

Let $U \subseteq \mathbb{R}$ be an open set. A function $f\colon U \to \mathbb{C}$, $f(t) = x(t) + iy(t)$ is continuous if and only if so are $x(t)$ and $y(t)$, and we say that $f(t)$ is differentiable (integrable, analytic etc.) when so are $x(t)$ and $y(t)$, and we put $f'(t) := x'(t) + iy'(t)$, $\int_a^b f(t)\mathrm{d}t := \int_a^b x(t)\mathrm{d}t + i\int_a^b y(t)\mathrm{d}t$.

Definition: The series $\sum_n \frac{1}{n!}c^n$ (where $0! := 1$) converges for any positive real number c, since $\lim \frac{n!c^{n+1}}{(n+1)!c^n} = \lim \frac{c}{n+1} = 0$. Hence, the power series $\sum_n \frac{1}{n!}x^n$ has infinite radius of convergence and it defines a continuous function

$$\exp\colon \mathbb{C} \longrightarrow \mathbb{C}\ , \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

Now, by Mertens' theorem, for any $a, b \in \mathbb{C}$ we have

$$\exp(a) \cdot \exp(b) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{a^m}{m!} \frac{b^{n-m}}{(n-m)!} \right) = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \exp(a+b).$$

Since $\exp(z)\exp(-z) = \exp(0) = 1$, we have $\exp(-z) = \frac{1}{\exp(z)}$ (in particular $\exp(z) \neq 0$).
Now we put $e := \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$.
It is an irrational number, since $n!e = n! + \ldots + \frac{n!}{n!} + \frac{n!}{(n+1)!} + \ldots$ never is an integer, because $n! + \ldots + \frac{n!}{n!}$ is a natural number and

$$0 < \frac{1}{n+1} + \ldots + \frac{1}{(n+1)\ldots(n+r)} + \ldots < \sum_{r=1}^{\infty} \frac{1}{2^r} = 1.$$

For any rational number $\frac{n}{m}$ we have $\exp\left(\frac{n}{m}\right)^m = \exp(n) = e^n$.
Hence $\exp\left(\frac{n}{m}\right) = e^{\frac{n}{m}}$, and for any complex number we put $e^z := \exp(z)$.
The analytic function $e^t\colon \mathbb{R} \to \mathbb{R}_+$ is an increasing function, since so are the functions $\frac{t^n}{n!}$, and the derivative is $(e^t)' = e^t$. Since $e > 2$, we have that e^t is arbitrarily large when $t \to \infty$ (hence e^t is arbitrarily small when $t \to -\infty$) and we see that $e^t\colon \mathbb{R} \to \mathbb{R}_+$ is a group isomorphism.
Since $\overline{e^z} = e^{\bar{z}}$, we have $|e^{it}|^2 = e^{it}e^{-it} = 1$ when $t \in \mathbb{R}$. Hence $|e^{it}| = 1$, and we put

$$e^{it} = \cos t + i \sin t,$$
$$\cos t = \frac{e^{it} + e^{-it}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n},$$
$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1},$$

so that $\cos^2 t + \sin^2 t = |e^{it}|^2 = 1$.

Moreover, $\cos(-t) = \cos t$, $\sin(-t) = -\sin t$ because $e^{-it} = \overline{e^{it}}$, and $e^{it}e^{is} = e^{i(t+s)}$ gives

$$\cos(t+s) = (\cos t)(\cos s) - (\sin t)(\sin s),$$
$$\sin(t+s) = (\cos t)(\sin s) + (\sin t)(\cos s).$$

Since $(e^{it})' = ie^{it}$, we have $(\cos t)' = -\sin t$, $(\sin t)' = \cos t$.

Now we have $\cos 0 = 1$, $\sin 0 = 0$ and

$$\cos 2 = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \ldots < 1 - \frac{2^2}{2!} + \frac{2^4}{4!} = -\frac{1}{3}$$

because the terms are decreasing with alternate signs.

Let t_0 be the minimum positive real number where $\cos t_0 = 0$. We put $\pi := 2t_0$.

Since $\sin'(t) = \cos t$ is positive in $[0, t_0)$, we have $\sin t_0 > 0$. Since $\sin^2 t_0 = 1$, we have

$$e^{\frac{\pi}{2}i} = i \quad , \quad \cos(t + \tfrac{\pi}{2}) = -\sin t \ , \ \sin(t + \tfrac{\pi}{2}) = \cos t,$$
$$e^{\pi i} = -1 \quad , \quad \cos(t + \pi) = -\cos t \ , \ \sin(t + \pi) = -\sin t,$$
$$e^{2\pi i} = 1 \quad , \quad \cos(t + 2\pi) = \cos t \ , \ \sin(t + 2\pi) = \sin t.$$

Now, $\cos t$ is a decreasing function on $[0, \frac{\pi}{2}]$, while $\sin t$ is increasing, and $\cos 0 = \sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = \sin 0 = 0$. Hence, *any real number* $-1 \leq x \leq 1$ *is* $x = \cos\alpha$ *for a unique* $0 \leq \alpha \leq \pi$.

Moreover, *if* $a^2 + b^2 = 1$, *then we have* $a + bi = e^{i\theta}$ *for some real number* θ (well defined up to the addition of an integer multiple of 2π). Summarizing:

Theorem: *The exponential* $e^t\colon \mathbb{R} \to \mathbb{R}_+$ *is a group isomorphism.*

The exponential $e^{it}\colon \mathbb{R} \to \{z \in \mathbb{C}\colon |z| = 1\}$ *is a group epimorphism with kernel* $2\pi i\mathbb{Z}$.

The exponential $e^z\colon \mathbb{C} \to \mathbb{C} - \{0\}$ *is a group epimorphism with kernel* $2\pi i\mathbb{Z}$.

Definition: If z is a complex number of modulus $\rho \neq 0$, then $z = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$ for some real number θ (well defined up to the addition of an integer multiple of 2π) named **argument** of z. We have $\arg(zz') = \arg z + \arg z'$ because $e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}$.

Hence, *any non null complex number* $z = \rho e^{i\theta}$ *has* n *complex* n*-th roots, namely*

$$\sqrt[n]{\rho}\, e^{i\frac{\theta+2k\pi}{n}} \quad , \quad k = 0, \ldots, n-1.$$

Lemma: *Let* $P(x) = c_0x^n + c_1x^{n-1} + \ldots + c_n \in \mathbb{C}[x]$. *If we put* $c = \max\{|c_1|, \ldots, |c_n|\}$, *then for any complex number* z *of modulus* ρ *we have*

$$\rho^n \left(|c_0| - \tfrac{c}{\rho-1}\right) \leq |P(z)|.$$

Proof: $|P(z)| \geq |c_0z^n| - |c_1z^{n-1} + \ldots + c_n| \geq |c_0|\rho^n - c(\rho^{n-1} + \ldots + 1) =$
$= |c_0|\rho^n - c(\rho^n - 1)/(\rho - 1) \geq |c_0|\rho^n - c\rho^n/(\rho - 1) = \rho^n(|c_0| - c/(\rho - 1))$.

D'Alembert's Theorem: *Any non constant polynomial* $P(x) \in \mathbb{C}[x]$ *has a complex root.*

Proof: The disk $D_\rho := \{z \in \mathbb{C}\colon |z| \leq \rho\}$ is a closed bounded set in $\mathbb{C}$; hence it is compact and the continuous function $f\colon D_\rho \to \mathbb{R}$, $f(z) = |P(z)|$, attains a minimum at some point $a \in D_\rho$.

By the above lemma, we may fix ρ so that $|P(z)| > |P(0)|$ whenever $|z| = \rho$, so that a must be an interior point of D_ρ.

If $|P(a)| > 0$, then we consider the first monomial with non null coefficient in $P(a+z)$:

$$P(a+z) = P(a) + c_d z^d + c_{d+1} z^{d+1} + \ldots + c_n z^n \quad , \quad c_d \neq 0.$$

Now we may choose z so that $|z| < 1$ and $|P(a) + c_d z^d| = |P(a)| - |c_d z^d|$ (just take z of argument $\frac{1}{d}\big(\pi + \arg P(a) - \arg c_d\big)$ and small modulus). Hence

$$|P(a+z)| \leq |P(a)| - |c_d z^d| + |c_{d+1} z^{d+1} + \ldots + c_n z^n| \leq |P(a)| - |c_d| \cdot |z^d| + c|z|^{d+1},$$

where $c := (n-d) \max\{|c_{d+1}|, \ldots, |c_n|\}$. It is clear that we may choose z of so small modulus that $|P(a+z)| < |P(a)|$ and $a + z \in D_\rho$. Absurd.

Hence $|P(a)| = 0$, and $z = a$ is a complex root of P.

Chapter 2

Linear Algebra

2.1 Groups

Definitions: A **permutation** of n elements is a bijection $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$. The product of permutations is the composition and S_n is the set of permutations of n elements.

Given different numbers $1 \leq a_1,\dots,a_r \leq n$, the **cycle** $(a_1 \dots a_r) \in S_n$ is the permutation transforming a_i into a_{i+1} (and a_r into a_1) and fixing any other number.

The cycles $(a_1 a_2)$ of length 2 are named **transpositions**.

Two cycles $\sigma = (a_1 \dots a_r)$, $\tau = (b_1 \dots b_s)$ are **disjoint** when $a_i \neq b_j$ for any pair of indices i, j; so that $\sigma\tau = \tau\sigma$. Any permutation is a product $\sigma = \alpha_1 \dots \alpha_r$ of disjoint cycles, uniquely up to the order of the factors. If d_i is the length of α_i, the **form** of σ is $d_1, \dots, d_r$.

Lemma: *Any permutation is a product of transpositions.*

Proof: $(a_1 \dots a_r) = (a_1 a_2)(a_2 a_3) \dots (a_{r-1} a_r)$.

Proposition: *Two permutations $\sigma, \tau \in S_n$ are **conjugate** ($\tau = \gamma\sigma\gamma^{-1}$ for some $\gamma \in S_n$) if and only if both have equal form.*

Proof: Let $\sigma = (a_1 \dots a_{d_1})(b_1 \dots b_{d_2}) \dots (c_1 \dots c_{d_r})$ be a product of disjoint cycles.

If $\gamma \in S_n$, and we put $i' = \gamma(i)$, then $\gamma\sigma\gamma^{-1} = (a'_1 \dots a'_{d_1})(b'_1 \dots b'_{d_2}) \dots (c'_1 \dots c'_{d_r})$.

Definition: Let $\Delta = \prod_{i<j}(x_j - x_i)$. If $\sigma \in S_n$, then the factors of $\sigma\Delta = \prod_{i<j}(x_{\sigma(j)} - x_{\sigma(i)})$ coincide, up to a sign, with the factors of Δ, and the **sign** of σ is defined to be

$$(*) \qquad \sigma\Delta = (\operatorname{sgn}\sigma)\Delta \ , \quad \operatorname{sgn}\sigma = \pm 1.$$

Proposition: $\operatorname{sgn}\tau\sigma = (\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)$, *and* $\operatorname{sgn}(a_1 \dots a_r) = (-1)^{r-1}$.

Proof: To show that $\operatorname{sgn}\tau\sigma = (\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)$, just apply τ to the variables in $(*)$,

$$\tau(\sigma\Delta) = (\operatorname{sgn}\sigma)(\tau\Delta) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)\Delta.$$

A direct calculation shows that $\operatorname{sgn}(12) = -1$, and for any other transposition τ we have $\tau = \gamma(12)\gamma^{-1}$, so that $\operatorname{sgn}\tau = (\operatorname{sgn}\gamma)(\operatorname{sgn}(12))(\operatorname{sgn}\gamma^{-1}) = \operatorname{sgn}(12) = -1$.

Now, $(a_1 \dots a_r)$ is a product of $r-1$ transpositions, hence the sign is $(-1)^{r-1}$.

Definition: A map $G \times G \xrightarrow{\cdot} G$ defines a **group** structure on the set G if

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$; $\forall a, b, c \in G$.

2. There exists[1] $1 \in G$ such that $1 \cdot a = a \cdot 1 = a,\ \forall a \in G$.

3. For any $a \in G$ there exists[2] $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

If $a \cdot b = b \cdot a$, $\forall a, b \in G$, the group is said to be **abelian**, the operation is denoted by $+$, the neutral element 0, and the inverse $-a$ (and it is named **opposite**).

Examples: $\mathbb{Z}$ is an abelian group with the sum.
S_n is a group (non abelian when $n \geq 3$) with the product of permutations.
Groups allow simplification: if $ab = ac$, then $a^{-1}ab = a^{-1}ac$, and $b = c$.

Definitions: Let G, G' be two groups. A map $f \colon G \to G'$ is a **group morphism** if

$$f(a \cdot b) = f(a) \cdot f(b),\ \forall a, b \in G.$$

If moreover f is bijective, it is a group **isomorphism**, (**automorphism** when $G' = G$) and in such a case so is f^{-1}, and we put $G \simeq G'$.
Isomorphic groups have the same properties in Group Theory.

1. *If f is a group morphism, then $f(1) = 1$ and $f(a^{-1}) = f(a)^{-1}$.*
   $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$, and $f(a) \cdot f(a^{-1}) = f(aa^{-1}) = f(1) = 1$.

2. *Compositions of group morphisms are group morphisms.*
   $(gf)(ab) = g(f(ab)) = g(f(a) \cdot f(b)) = g(f(a)) \cdot g(f(b)) = (gf)(a) \cdot (gf)(b)$.

3. *If $f \colon G \to G'$ is a group isomorphism, so is the bijection $f^{-1} \colon G' \to G$.*
   $f\left(f^{-1}(a') \cdot f^{-1}(b')\right) = f\left(f^{-1}(a')\right) \cdot f\left(f^{-1}(b')\right) = a' \cdot b' = f\left(f^{-1}(a'b')\right)$, and f is injective.

Definition: A subset H of a group G is a **subgroup** if it is a group with the product of G,

1. $a, b \in H \Rightarrow a \cdot b \in H$.

2. $1 \in H$.

3. $a \in H \Rightarrow a^{-1} \in H$.

Examples: The subgroups 1 and G are the **trivial** subgroups of G.
The intersection of any family of subgroups is also a subgroup.
The subgroup $(a_1, \ldots, a_n)$ **generated** by $a_1, \ldots, a_n \in G$ is the minimal subgroup containing them (the intersection of all subgroups containing them).
The subgroup generated by an element $a \in G$ is $(a) = \{\ldots, a^{-2}, a^{-1}, 1, a, a^2, \ldots\}$.
The subgroup of $\mathbb{Z}$ generated by $a, b \in \mathbb{Z}$ is $a\mathbb{Z} + b\mathbb{Z} = \{ax + by \colon x, y \in \mathbb{Z}\}$.

Theorem: *Any subgroup H of $\mathbb{Z}$ is generated by a unique natural number, $H = n\mathbb{Z}$.*

Proof: If $H = 0$, take $n = 0$.
If $H \neq 0$, let n be the first positive number in H (it exists since $-H = H$).
Now $n\mathbb{Z} \subseteq H$, because H is a subgroup, and, if $m \in H$, we divide m by n,

$$m = cn + r; \quad 0 \leq r \leq n - 1,$$
$$r = m - cn \in H;$$

[1]This **neutral** element is unique: if e is also neutral, then $e = e \cdot 1 = 1$.
[2]This **inverse** element is unique: if $a \cdot b = 1$, then $a^{-1} = a^{-1} \cdot 1 = a^{-1} \cdot (a \cdot b) = (a^{-1} \cdot a) \cdot b = 1 \cdot b = b$.

hence $r = 0$, by the choice of n, and $m \in n\mathbb{Z}$. Therefore $H = n\mathbb{Z}$.
Uniqueness is evident.

Examples: The inclusion $n\mathbb{Z} \subseteq m\mathbb{Z}$ means that m divides n. Since $p \in \mathbb{N}$ is **prime** when it has just two divisors (namely 1 and $p \neq 1$), this condition means that there are just two subgroups of $\mathbb{Z}$ containing $p\mathbb{Z}$, namely $\mathbb{Z}$ and $p\mathbb{Z}$.

Given $n_1, n_2 \in \mathbb{N}$, we have $n_1\mathbb{Z}+n_2\mathbb{Z} = d\mathbb{Z}$, $n_1\mathbb{Z}\cap n_2\mathbb{Z} = m\mathbb{Z}$ for some $d, m \in \mathbb{N}$, the greatest common divisor and the least common multiple of n_1 and n_2.

Hence n_1 and n_2 are **coprime** (the unique common divisor is 1) if and only if $n_1\mathbb{Z}+n_2\mathbb{Z} = \mathbb{Z}$.

Proposition: *Let $f\colon G \to \bar{G}$ be a group morphism. If H is a subgroup of G, then $f(H)$ is a subgroup of $\bar{G}$. If $\bar{H}$ is a subgroup of $\bar{G}$, then $f^{-1}(\bar{H})$ is a subgroup of G. Hence the* ***image*** $\operatorname{Im} f = f(G)$ *is a subgroup of $\bar{G}$, and the* ***kernel*** $\operatorname{Ker} f = f^{-1}(1)$ *is a subgroup of G.*

Proof: $1 \in f(H)$ because $f(1) = 1$ and $1 \in H$.
If $f(h'), f(h) \in f(H)$, then $f(h')f(h) = f(h'h) \in f(H)$, and $f(h)^{-1} = f(h^{-1}) \in f(H)$.
$1 \in f^{-1}(\bar{H})$, because $f(1) = 1 \in \bar{H}$.
If $g', g \in f^{-1}(\bar{H})$, then $f(g'g) = f(g')f(g) \in \bar{H}$, and $f(g^{-1}) = f(g)^{-1} \in \bar{H}$.

Proposition: *A group morphism f is injective if and only if Ker$(f) = 1$.*

Proof: If f is injective, and $f(g) = 1 = f(1)$, then $g = 1$.
If $\operatorname{Ker} f = 1$, and $f(a) = f(b)$, then $f(a^{-1}b) = 1$; hence $a^{-1}b = 1$, and $a = b$.

2.1.1 The Quotient Group

Definitions: Two elements $a, b \in G$ are **congruent** modulo a subgroup H when $a^{-1}b \in H$; i.e., $b \in aH$, and we put $a \equiv b$ (mod. H). This is an equivalence relation,

1. $a \equiv a$, because $a^{-1}a = 1 \in H$ for all $a \in G$.
2. If $a \equiv b$, then $a^{-1}b \in H$; hence $b^{-1}a = \left(a^{-1}b\right)^{-1} \in H$, and $b \equiv a$.
3. If $a \equiv b$ and $b \equiv c$, then $a^{-1}b, b^{-1}c \in H$; hence $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$, and $a \equiv c$.

G/H denotes the quotient set, and the class of $a \in G$ is just $aH = \{ah\colon\ h \in H\}$.
The **order** of G is the cardinal $|G|$, and the **index** of H is the cardinal of G/H.

Lagrange's Theorem: *Let H be a subgroup of a finite group G. The order of H divides the order of G, and the quotient is the index of H,*

$$|G/H| = |G| \,/\, |H|\,.$$

Proof: All the equivalence classes aH have equal cardinal, because the obviously surjective maps $H \to aH$, $h \mapsto ah$, are injective: If $ax = ay$, then $x = a^{-1}(ax) = a^{-1}(ay) = y$.
Hence $|G|$ is the product of $|H|$ by the number $|G/H|$ of equivalence classes.

Definition: A subgroup H of G is **normal**, and we put $H \lhd G$, when $gHg^{-1} \subseteq H,\ \forall g \in G$.

The kernel of any group morphism $f\colon G \to G'$ is a normal subgroup:
If $h \in \operatorname{Ker} f$, then $f(ghg^{-1}) = f(g)f(h)f(g)^{-1} = f(g)f(g)^{-1} = 1$, hence $ghg^{-1} \in \operatorname{Ker} f$.

Theorem: *If H is a normal subgroup of G, there exists a unique group structure on the quotient set G/H such that $\pi\colon G \to G/H$ is a group morphism. Moreover,* $\operatorname{Ker} \pi = H$.

Proof: The only possible product, $[a] \cdot [b] = [ab]$ is well-defined, and it defines a group structure,

1. If $[a'] = [a]$, then $a' = ah$, $a'b = ahb = ab(b^{-1}hb) \in abH$; hence $[a'b] = [ab]$.

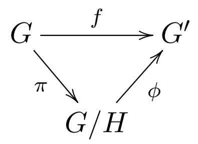
2. $([a] \cdot [b]) \cdot [c] = [ab] \cdot [c] = [(ab)c] = [a(bc)] = [a] \cdot [bc] = [a] \cdot ([b] \cdot [c])$.

3. $[a] \cdot [1] = [a \cdot 1] = [a]$, $[1] \cdot [a] = [1 \cdot a] = [a]$.

4. $[a] \cdot [a^{-1}] = [a \cdot a^{-1}] = [1]$, $[a^{-1}] \cdot [a] = [a^{-1} \cdot a] = [1]$.

5. $\mathrm{Ker}\pi = \{a \in G \colon [a] = [1]\} = [1] = H$.

Universal Property: *Let H be a normal subgroup of G. If $f \colon G \to G'$ is a group morphism and $f(H) = 1$, then there is a unique group morphism $\phi \colon G/H \to G'$ such that $\phi([a]) = f(a)$,*



$$f = \phi\pi$$

Proof: The map $\phi \colon G/H \to G'$, $\phi([g]) = f(g)$, is well-defined,

$$[g'] = [g], \ g' = gh \in gH \subseteq g(\mathrm{Ker}\, f), \ f(g') = f(g)f(h) = f(g) \cdot 1 = f(g),$$

and it is a group morphism: $\phi([a] \cdot [b]) = \phi([ab]) = f(ab) = f(a) \cdot f(b) = \phi([a]) \cdot \phi([b])$.

Isomorphism Theorem: *If $f \colon G \to G'$ is a group morphism, then we have a group isomorphism $\phi \colon G/\mathrm{Ker}\, f \xrightarrow{\sim} \mathrm{Im}\, f$, $\phi([g]) = f(g)$.*

Proof: By the universal property we have an epimorphism $\phi \colon G/\mathrm{Ker}\, f \to \mathrm{Im}\, f$, $\phi([g]) = f(g)$, and it is injective: If $1 = \phi([g]) = f(g)$, then $g \in \mathrm{Ker}\, f$ and $[g] = 1$.

Definition: A group G is **cyclic** if it is generated by some element,

$$G = (g) = \{\dots, g^{-n}, \dots, g^{-1}, g^0 = 1, g, g^2, \dots, g^n, \dots\}.$$

For example, $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group, generated by the class $[1]$.

Classification of Cyclic groups: *Any cyclic group G is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$.*

Proof: If $G = (g)$, the morphism $f \colon \mathbb{Z} \to G$, $f(m) = g^m$, is surjective.
Since $\mathrm{Ker}\, f = n\mathbb{Z}$ for some $n \in \mathbb{N}$, we have an isomorphism $\phi \colon \mathbb{Z}/n\mathbb{Z} \simeq G$, $\phi([m]) = g^m$.

Definition: The **order** of an element $g \in G$ is the order of the subgroup (g).

1. *The order of g is the first natural number $r \neq 0$ such that $g^r = 1$ (if it exists), and in such a case $g^m = 1$ if and only if m is a multiple of r.*

   In fact, $\phi \colon \mathbb{Z}/r\mathbb{Z} \to (g)$, $\phi([m]) = g^m$, is an isomorphism.

2. *If G is a finite group of order n, then $g^n = 1$ for all $g \in G$.*

3. *The order of a permutation of form $d_1, \dots, d_r$ is the* l.c.m.$(d_1, \dots, d_r)$.

4. *The generators of the group $\mathbb{Z}/n\mathbb{Z}$ are just the classes $[m]$ of numbers coprime to n.*

   In fact $[m]$ generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $\pi^{-1}([m\mathbb{Z}]) = m\mathbb{Z} + n\mathbb{Z}$ coincides with $\mathbb{Z}$.

5. The **alternate subgroup** A_n is the kernel of the group morphism $\mathrm{sgn} \colon S_n \to \{\pm 1\}$. It is a normal subgroup of index 2, because $S_n/A_n \simeq \{\pm 1\}$; hence $|A_n| = n!/2$.

2.2 Rings

Definition: Two maps $A \times A \xrightarrow{+} A$, $A \times A \xrightarrow{\cdot} A$ define a structure of **ring** (commutative with unit) on the set A if

1. $(A, +)$ is a commutative group.
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c,\ \forall a, b, c \in A$.
3. $a \cdot (b + c) = a \cdot b + a \cdot c,\ \forall a, b, c \in A$.
4. $a \cdot b = b \cdot a,\ \forall a, b \in A$.
5. There exists[3] $1 \in A$ such that $a \cdot 1 = a$, $\forall a \in A$.

By (3), the map $A \xrightarrow{a\cdot} A$ is a group morphism; hence $a \cdot 0 = 0$, $a \cdot (-b) = -(a \cdot b)$.

Let A, B be two rings. A group morphism $f \colon A \to B$ is a **ring morphism** if

$$f(a \cdot b) = f(a) \cdot f(b),$$
$$f(1) = 1.$$

If moreover f is bijective, it is a ring **isomorphism**, (**automorphism** when $B = A$) and in such a case so is f^{-1}, and we put $A \simeq B$.

Isomorphic rings have the same properties in Ring Theory.

Compositions of ring morphisms are also ring morphisms.

Definition: An element $a \in A$ is a **unit** or **invertible** if $ab = 1$ for some $b \in A$.

A ring $k \neq 0$ is a **field** if any non null element is invertible. In this notes, unless otherwise stated, k always denotes a field, and the elements of k are named **scalars**.

Definition: An additive subgroup B of a ring A is a **subring** if it is stable under products and $1 \in B$, so that B also is a ring with the operations of A.

An additive subgroup $\mathfrak{a} \subseteq A$ is an **ideal** if it is closed under products by arbitrary elements of A ($a \in A$, $b \in \mathfrak{a} \Rightarrow ab \in \mathfrak{a}$).

Examples: $\mathbb{Z}$ is a ring, while $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are fields.

Polynomials in a variable x with coefficients in a ring A form a ring $A[x]$.

The invertible elements of a ring A form a multiplicative abelian group A^*.

If $f \colon A \to B$ is a ring morphism and a is invertible in A, then $f(a^{-1}) \cdot f(a) = f(a^{-1}a) = f(1) = 1$; hence $f(a)$ is invertible in B. Moreover, $f \colon A^* \to B^*$ is a group morphism.

If an ideal $\mathfrak{a}$ contains an invertible element, then $1 \in \mathfrak{a}$ and $\mathfrak{a} = A$. In particular, the unique ideals of a field k are 0 and k.

Intersections of ideals of A are also ideals of A.

Any ideal of $\mathbb{Z}$ is $n\mathbb{Z}$, where n is a natural number, because so is any subgroup.

The kernel of a ring morphism $A \to B$ is an ideal of A, and the image is a subring of B.

Theorem: *Let $\mathfrak{a}$ be an ideal of a ring A. The quotient group $A/\mathfrak{a}$ has a unique ring structure such that the canonical projection $\pi \colon A \to A/\mathfrak{a}$ is a ring morphism. Moreover,* $\operatorname{Ker} \pi = \mathfrak{a}$.

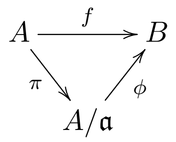
Proof: The unique possible product, $[a] \cdot [b] = [ab]$, is well-defined (and it is easy to check that it defines a ring structure in $A/\mathfrak{a}$, the unity being $[1]$):

$$[a] = [a'],\ a' = a + c \in a + \mathfrak{a},\ a'b = ab + cb \in ab + \mathfrak{a},\ [a'b] = [ab].$$

[3]This **identity** element is unique: if e is another identity, then $e = e \cdot 1 = 1$.

Example: If p is a prime number, then $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is a field. In fact, if $0 < n < p$, then $p\mathbb{Z} \subset n\mathbb{Z} + p\mathbb{Z}$; hence $n\mathbb{Z} + p\mathbb{Z} = \mathbb{Z}$, and we have $nm + pa = 1$, so that $[n] \cdot [m] = 1$.

Universal Property: *If a ring morphism $f\colon A \to B$ vanishes on an ideal $\mathfrak{a}$ of A, then there is a unique ring morphism $\phi\colon A/\mathfrak{a} \to B$ such that $\phi([a]) = f(a)$,*



$$f = \phi\,\pi$$

Proof: By the universal property of the quotient group, there is a unique group morphism $\phi\colon A/\mathfrak{a} \to B$ such that $f = \phi\pi$, and ϕ is a ring morphism,

$$\phi([a] \cdot [b]) = \phi([ab]) = f(ab) = f(a)f(b) = \phi([a])\phi([b]),$$
$$\phi(1) = \phi(\pi(1)) = f(1) = 1.$$

Isomorphism Theorem: *If $f\colon A \to B$ is a ring morphism, then we have a ring isomorphism $\phi\colon A/\mathrm{Ker}\, f \xrightarrow{\sim} \mathrm{Im}\, f$, $\phi([a]) = f(a)$.*

Proof: It is a group isomorphism, and a ring morphism by the universal property.

Definitions: An element $a \in A$ is a **zero divisor** if $ab = 0$ for some $b \neq 0$; i.e., the group morphism $A \xrightarrow{a\cdot} A$ is not injective. A ring A is **integral** (or a **domain**) when for any null finite product $\prod_{i\in I} a_i = 0$ we have $a_i = 0$ for some index $i \in I$ (when $I = \emptyset$, we see that $1 \neq 0$ in A).

A **proper** element (non null nor invertible) of a domain is **irreducible** if it is not a product of two proper elements.

Any field is a domain: If $ab = 0$ and $b \neq 0$, then $0 = abb^{-1} = a$.

If A is an integral ring, and $P, Q \in A[x]$ are non null, then

$$PQ = (a_n x^n + \ldots + a_0)(b_m x^m + \ldots + b_0) = a_n b_m x^{n+m} + \ldots + \Big(\sum_{i+j=d} a_i b_j\Big) x^d + \ldots + a_0 b_0 \neq 0$$

because $a_n b_m \neq 0$ when $a_n, b_m \neq 0$. Moreover $\deg(PQ) = \deg P + \deg Q$. Hence $A[x]$ *also is a domain and any invertible polynomial is constant*: $A[x]^* = A^*$.

Division Theorem: *Let $Q \neq 0$ be a polynomial with coefficients in a field k. For any $P \in k[x]$ there is a unique pair $C, R \in k[x]$ (named* ***quotient*** *and* ***remainder****) such that*

$$P = CQ + R, \quad \deg R < \deg Q \quad \text{or} \quad R = 0.$$

Proof: The existence is given by the usual division algorithm.

Now, if $P = C_1 Q + R_1$, where $\deg R_1 < \deg Q$ or $R_1 = 0$, then $Q(C_1 - C) = R - R_1$.

Since $\deg(R - R_1) < \deg Q$, then $C_1 - C = 0$, and $R - R_1 = 0$.

Ruffini's Rule: *A polynomial $P(x) \in k[x]$ is a multiple of $x - a$ if and only if $P(a) = 0$.*

Proof: If $P(x) = C(x)(x - a) + r$, then $P(a) = C(a) \cdot 0 + r = r$. q.e.d.

1. *$a \in k$ is a root of $P \in k[x]$ if and only if P is a multiple of $x - a$ in the ring $k[x]$.*

2. *Any irreducible polynomial in $k[x]$ of degree > 1 has no root in k.*

3. *A polynomial of degree 2 or 3 is irreducible in $k[x]$ if and only if it has no root in k.*

   If $P = Q_1Q_2$, then some factor has degree 1, and it has some root in k.

4. *If $a_1, \dots, a_r \in k$ are different roots of $P \in k[x]$, then P is a multiple of $(x-a_1)\dots(x-a_r)$.*

   By Ruffini's rule, $P = (x-a_1)Q$, and $a_2, \dots, a_r$ are roots of Q. Hence $(x-a_2)\dots(x-a_r)$ divides Q by induction on r, and $(x-a_1)\dots(x-a_r)$ divides P.

5. *Any polynomial of degree 2 with complex coefficients has a complex root.*

   It is enough to show that any complex number z has a complex square root, and we may assume that $|z| = 1$. Now, if $z \neq -1$, we have $\left(\frac{1+z}{|1+z|}\right)^2 = \frac{(1+z)^2}{(1+z)(1+\bar{z})} = \frac{1+z}{1+\bar{z}} = \frac{z\bar{z}+z}{1+\bar{z}} = z$.

6. *The number of roots of a polynomial $0 \neq P \in k[x]$ in k is bounded by the degree of P.*

7. *Let $a_1, \dots, a_r \in k$ be different elements of a field. Given $b_1, \dots, b_r \in k$, there is a unique polynomial $P(x) \in k[x]$, $\deg P < r$, such that $P(a_i) = b_i$.*

   In fact we have the **Lagrange's interpolation formula**:

$$P(x) = \sum_{j=1}^{r} b_j \frac{Q_j(x)}{Q_j(a_j)}\ , \quad Q_j(x) = \frac{(x-a_1)\dots(x-a_r)}{(x-a_j)}.$$

2.3 Vector Spaces

Definition: Let k be a field. A structure of **k-vector space** on an abelian group E (the elements of E are named **vectors**) is defined by a map $k \times E \xrightarrow{\cdot} E$ such that

1. $\lambda(e_1 + e_2) = \lambda e_1 + \lambda e_2$, $\forall \lambda \in k$, $e_1, e_2 \in E$.
2. $(\lambda_1 + \lambda_2)e = \lambda_1 e + \lambda_2 e$, $\forall \lambda_1, \lambda_2 \in k$, $e \in E$.
3. $(\lambda\mu)e = \lambda(\mu e)$, $\forall \lambda, \mu \in k$, $e \in E$.
4. $1 \cdot e = e$ for all $e \in E$.

and a subgroup $V \subseteq E$ is a **vector subspace** if $\lambda V \subseteq V$, $\forall \lambda \in k$, so that V also is a k-vector space.

By (1) and (2), the maps $E \xrightarrow{\lambda\cdot} E$ and $k \xrightarrow{\cdot e} E$ are group morphisms; hence

$$\lambda \cdot 0 = 0,\ \lambda \cdot (-e) = -(\lambda e),\ 0 \cdot e = 0,\ (-\lambda)e = -(\lambda e).$$

Moreover, if $\lambda e = 0$, and $\lambda \neq 0$, then $0 = \lambda^{-1}(\lambda e) = 1 \cdot e = e$.

Definition: A group morphism $f\colon E \to F$ between k-vector spaces is **k-linear** if

$$f(\lambda \cdot e) = \lambda \cdot f(e);\ \forall \lambda \in k,\ e \in E.$$

If moreover it is bijective, it is a linear **isomorphism** (**automorphism** when $F = E$) and in such a case so is f^{-1}, and we put $E \simeq F$.

Isomorphic vector spaces have the same properties in Linear Algebra.

Compositions of linear maps are also linear maps.

Proposition: *Let $f\colon E \to F$ be a linear map. If $V \subseteq$ is a vector subspace, then $f(V)$ is a vector subspace of F. If $W \subseteq F$ is a vector subspace, then $f^{-1}(W)$ is a vector subspace of E. Hence* $\operatorname{Im} f = f(E)$ *is a vector subspace of F and* $\operatorname{Ker} f = f^{-1}(0)$ *is a vector subspace of E.*

Proof: $f(V)$ and $f^{-1}(W)$ are subgroups (p. 45) and $f(v) \in f(V) \Rightarrow \lambda f(v) = f(\lambda v) \in f(V)$.

Moreover, if $e \in f^{-1}(W)$, then $f(\lambda e) = \lambda f(e) \in W$; hence $\lambda e \in f^{-1}(W)$.

Examples: k^n is a k-vector space with the product $\lambda(\mu_1, \ldots, \mu_n) = (\lambda\mu_1, \ldots, \lambda\mu_n)$.

E and 0 are the trivial vector subspaces of E.

If V and W are vector subspaces of E, so are $V \cap W$ and the **sum**

$$V + W = \{v + w\colon v \in V \text{ and } w \in W\}.$$

Let A be a matrix $m \times n$ with coefficients in k. The map $f\colon k^n \to k^m$, $f(X) = AX$, is linear, the kernel being the space of solutions of the homogeneous system $AX = 0$, and $B \in \operatorname{Im} f$ just when the system of linear equations $AX = B$ is consistent.

If $e_1, \ldots, e_n \in E$, the map $f\colon k^n \to E$, $f(\lambda_1, \ldots, \lambda_n,) = \lambda_1 e_1 + \ldots + \lambda_n e_n$, is linear, the image being the minimal vector subspace containing $e_1, \ldots, e_n$,

$$\langle e_1, \ldots, e_n \rangle = ke_1 + \ldots + ke_n = \{\lambda_1 e_1 + \ldots + \lambda_n e_n\colon\ \lambda_1, \ldots, \lambda_n \in k\}.$$

Theorem: *If $f, g\colon E \to F$ are linear maps, then so are the maps*

$$\lambda f\colon E \longrightarrow F,\ (\lambda f)(e) = \lambda \cdot f(e),$$
$$f + g\colon E \longrightarrow F,\ (f+g)(e) = f(e) + g(e),$$

where $\lambda \in k$; and the set $\operatorname{Hom}_k(E, F)$ of all k-linear maps $E \to F$ is a k-vector space.

Proof: It follows directly from the definitions.

Theorem: *If V is a vector subspace of E, there exists a unique vector space structure on the quotient group E/V such that $\pi\colon E \to E/V$ is a linear map. Moreover,* $\operatorname{Ker} \pi = V$.

Proof: The unique possible product, $\lambda \cdot [e] = [\lambda e]$, is well-defined,

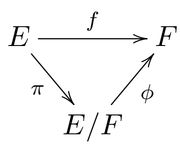
$$[e] = [e'],\ e' - e \in V,\ \lambda e' - \lambda e = \lambda(e' - e) \in V,\ [\lambda e] = [\lambda e'],$$

and it is easy to check that it defines a vector space structure on E/V.

Definitions: $X = p + V$ is the **linear subvariety** of E with **direction** V passing through the point p, and linear subvarieties of direction V form the vector space E/V.

Two linear subvarieties $p + V$, $q + W$ are **parallel** when they have equal direction, $V = W$.

Universal Property: *Let V be a vector subspace of E. Any linear map $f\colon E \to F$ vanishing on V uniquely factors through a linear map $\phi\colon E/V \to F$ such that $\phi([e]) = f(e)$,*



$$f = \phi\,\pi$$

Proof: By the universal property of the quotient group, there is a unique group morphism $\phi\colon E/V \to F$ such that $f = \phi \circ \pi$, and ϕ is linear,

$$\phi(\lambda[e]) = \phi([\lambda e]) = f(\lambda e) = \lambda f(e) = \lambda \phi([e]).$$

Isomorphism Theorem: *If $f\colon E \to F$ is a k-linear map, then we have a k-linear isomorphism $\phi\colon E/\operatorname{Ker} f \xrightarrow{\sim} \operatorname{Im} f$, $\phi([e]) = f(e)$.*

Proof: It is a group isomorphism by the isomorphism theorem for group morphisms, and it is linear by the universal property of $E/\operatorname{Ker} f$.

Theorem: *If $p\colon E \to \bar{E}$ is a linear surjective map, we have a lattice isomorphism*

$$\begin{bmatrix}\text{Vector sub-}\\ \text{spaces of } \bar{E}\end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix}\text{Vector subspaces of}\\ E \text{ containing } \operatorname{Ker} p\end{bmatrix}, \ \bar{V} \mapsto p^{-1}(\bar{V}),$$

the inverse isomorphism being the map $V \mapsto p(V)$. If $V = p^{-1}(\bar{V})$, then we have a natural isomorphism $E/V \simeq \bar{E}/\bar{V}$.

Proof: We have $p(p^{-1}\bar{V}) = \bar{V}$ because p is surjective and, when $\operatorname{Ker} p \subseteq V$, we also have

$$p^{-1}\big(p(V)\big) = V + \operatorname{Ker} p = V,$$

so that the kernel of the epimorphism $E \xrightarrow{p} \bar{E} \xrightarrow{\pi} \bar{E}/\bar{V}$ is just V; hence $E/V \simeq \bar{E}/\bar{V}$.

Definition: A sequence $\ldots \to E_{n-1} \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} E_{n+1} \to \ldots$ of linear maps is **exact** when $\operatorname{Im} f_{n-1} = \operatorname{Ker} f_n$ for any index n.

A sequence $0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0$ is exact when i is injective and p is a surjective map with kernel $\operatorname{Im} i$, so that $E'' \simeq E/\operatorname{Im} i$. These are named **short** exact sequences.

Definitions: A vector space $E \neq 0$ is **simple** if 0 and E are the only vector subspaces of E.

A **flag** in E of **length** n is a maximal increasing sequence $E_0 \subset E_1 \subset \ldots \subset E_n$ of vector subspaces ($E_0 = 0$, $E_n = E$ and the quotients E_i/E_{i-1} are simple). E is **finite-dimensional** if it admits some flag, and the **dimension** of E is the minimal length of all flags.

The unique vector space of dimension 0 is $E = 0$.

Vector spaces of dimension 1 are just simple spaces, and k is simple.

Theorem: *Let E be a vector space of finite dimension.*

1. $\dim V \leq \dim E$ *for any subspace $V \subseteq E$; and $V = E$ when the equality holds.*
2. *All flags in E have equal length, namely the dimension of E.*
3. *If $0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0$ is exact, then* $\dim E = \dim E' + \dim E''$.

Proof: (1) Let $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ be a flag, $n = \dim E$, and put $V_i = E_i \cap V$.

The kernel of $V_i \to E_i/E_{i-1}$ is $E_{i-1} \cap V_i = V_{i-1}$, so that we have an injective linear map $V_i/V_{i-1} \hookrightarrow E_i/E_{i-1}$ and we see that V_i/V_{i-1} is simple or null (i.e., $V_{i-1} = V_i$).

Eliminating repetitions in $0 \subseteq V_1 \ldots \subseteq V_n = V$ we obtain a flag; hence $\dim V \leq \dim E$.

If $\dim V = n$, then all the inclusions $V_{i-1} \subset V_i$ are strict.

Now, if $E_{i-1} = V_{i-1} \subset V_i \subseteq E_i$, then $V_i = E_i$ because E_i/E_{i-1} is simple.

Since $V_0 = E_0$, we see that $V = V_n = E_n = E$.

(2) If $0 \subset \bar{E}_1 \subset \ldots \subset \bar{E}_d = E$ is a flag, then $d \leq \dim E$ because $\dim \bar{E}_{i-1} < \dim \bar{E}_i$.

By definition $\dim E \leq d$; hence $d = \dim E$.

(3) Since E' is isomorphic to $\operatorname{Im} i$, we may assume that E' is a vector subspace of E and that i is the inclusion map. Given flags $0 \subset E'_1 \subset \ldots \subset E'_m = E'$ and $0E''_1 \subset\subset \ldots \subset E''_d = E''$, then

$$0 \subset \ldots \subset E'_m = p^{-1}(0) \subset \ldots \subset p^{-1}(E''_d) = E$$

is a flag in E by the former theorem; hence $\dim E = m + d = \dim E' + \dim E''$.

Corollary: $\dim E/V = \dim E - \dim V$.

$\dim E = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f)$, *for any linear map* $f \colon E \to F$.

$\dim(E \times F) = \dim E + \dim F$.

$\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$.

Proof: $0 \to V \xrightarrow{i} E \xrightarrow{\pi} E/V \to 0$ is exact, because $V = \operatorname{Ker} \pi$.

$0 \to \operatorname{Ker} f \xrightarrow{i} E \xrightarrow{f} \operatorname{Im} f \to 0$ is exact.

$0 \to E \xrightarrow{i_1} E \times F \xrightarrow{\pi_2} F \to 0$ is exact; $i_1(e) = (e, 0)$, $\pi_2(e, v) = v$.

$0 \to V \cap W \xrightarrow{i} V \times W \xrightarrow{s} V + W \to 0$ is exact; $i(e) = (e, -e)$, $s(v, w) = v + w$.

Corollary: *Any injective or surjective k-linear map* $f \colon E \to F$ *between k-vector spaces of equal dimension (finite, of course) is an isomorphism.*

Proof: $\dim(\operatorname{Ker} f) = \dim F - \dim(\operatorname{Im} f)$; hence, $\operatorname{Ker} f = 0$ if and only if $\operatorname{Im} f = F$.

2.3.1 Bases

Any family of vectors $e_1, \ldots, e_n \in E$ defines a linear map

$$f \colon k^n \longrightarrow E, \; f(\lambda_1, \ldots, \lambda_n) = \lambda_1 e_1 + \ldots + \lambda_n e_n.$$

Definitions: The vectors $e_1, \ldots, e_n$ form a **base** of E if f is an isomorphism (any vector $e \in E$ may be uniquely written as a linear combination $e = x_1 e_1 + \ldots + x_n e_n$).

In such a case $n = \dim E$, and $(x_1, \ldots, x_n) \in k^n$ are the **coordinates** of e in such base.

The **usual base** of k^n is $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots,$ $e_n = (0, \ldots, 0, 1)$.

The vectors $e_1, \ldots, e_n$ **generate** or **span** E if f is surjective, $ke_1 + \ldots + ke_n = E$.

In such a case $n \geq \dim E$, and they form a base of E when the equality holds.

The vectors $e_1, \ldots, e_n$ are **linearly independent** if f is injective (the unique null linear combination, $\lambda_1 e_1 + \ldots + \lambda_n e_n = 0$, is the trivial one, $\lambda_1 = \ldots = \lambda_n = 0$).

In such a case $n \leq \dim E$, and they form a base of E when the equality holds.

Proposition: *Any generating system* $e_1, \ldots, e_n$ *of* E *contains a base of* E.

Proof: Put $E_i = ke_1 + \ldots + ke_i$. Eliminating e_i when $E_{i-1} = E_i$, we may assume that we have $0 \subset E_1 \subset \ldots \subset E_n = E$, so that $n \leq \dim E$. Hence $n = \dim E$, and $e_1, \ldots, e_n$ is a base of E.

Proposition: *In a finite dimensional vector space* E*, any linearly independent family* $e_1, \ldots, e_n$ *may be extended to a base of* E.

Proof: Put $E_i = ke_1 + \ldots + ke_i$. The inclusions $E_{i-1} \subset E_i$ are strict because $\dim E_i = i$, and we may fix $e_{n+1}, \ldots, e_{n+r} \in E$ so that $0 \subset E_1 \subset \ldots \subset E_{n+r} = E$, with $E_{n+j} = ke_1 + \ldots + ke_{n+j}$.

Hence $n + r \leq \dim E$, and the generating system $e_1, \ldots, e_n, \ldots, e_{n+r}$ is a base of E.

Theorem: *Any vector space* $E \neq 0$ *of finite dimension* n *has a base,* $E \simeq k^n$*. Hence, two finite dimensional k-vector spaces are isomorphic if and only if both have equal dimension.*

Proof: Any non null vector of E may be extended to a base of E.

Rouché-Frobënius Theorem: *A system of linear equations $AX = B$ is consistent if and only if* $\operatorname{rk} A = \operatorname{rk}(A|B)$.

Proof: Let $A_1, \ldots, A_n \in k^m$ be the columns of A. The system $x_1A_1+\ldots+x_nA_n = B$ is consistent if and only if $B \in \langle A_1, \ldots, A_n\rangle$; i.e., $\langle A_1, \ldots, A_n\rangle = \langle A_1, \ldots, A_n, B\rangle$.

This is equivalent to the coincidence of the dimensions (p. 51).

Now, the **rank** of a matrix (the maximum number of linearly independent columns) is just the dimension of the vector subspace of k^m spanned by the columns.

Proposition: $\dim(\operatorname{Hom}_k(E, F)) = (\dim E)(\dim F)$.

In fact, if $e_1, \ldots, e_n$ is a base of E, then we have an isomorphism

$$\operatorname{Hom}_k(E, F) \xrightarrow{\sim} F\times \overset{n}{\cdots} \times F,\ f \mapsto (f(e_1), \ldots, f(e_n)),$$

Proof: If $f(e_1) = \ldots = f(e_n) = 0$, then $f = 0$ because

$$f(E) = f(ke_1 + \ldots + ke_n) = k \cdot f(e_1) + \ldots + k \cdot f(e_n) = 0.$$

Moreover, the map $f(x_1e_1 + \ldots + x_ne_n) = x_1v_1 + \ldots + x_nv_n$ is linear, and $f(e_i) = v_i$.

Now $\dim(\operatorname{Hom}_k(E, F)) = \dim F^n = n(\dim F)$.

Definition: Let $f\colon E \to F$ be a linear map. If $(e_1, \ldots, e_n)$ is a base of E, and $(v_1, \ldots, v_m)$ is a base of F, then we have

$$f(e_j) = a_{1j}v_1 + \ldots + a_{mj}v_m.$$

and $A = (a_{ij})$ is the **matrix** of f in such bases. Since the j-th column of A are the coordinates of $f(e_j)$; the rank of A is just the dimension of $\langle f(e_1), \ldots, f(e_n)\rangle = \operatorname{Im} f$,

$$\dim(\operatorname{Im} f) = \operatorname{rk} A,$$
$$\dim(\operatorname{Ker} f) = n - \operatorname{rk} A.$$

If we write the coordinates of a vector $e = \sum_j x_je_j \in E$ as a column $X = (x_j)$, then

$$f(e) = \textstyle\sum_j x_jf(e_j) = \sum_j \sum_i x_ja_{ij}e'_i = \sum_i \big(\sum_j a_{ij}x_j\big)e'_i,$$

so that the coordinates Y of $f(e)$ in the base of F are just

$$Y = AX. \tag{2.1}$$

If $h\colon F \to W$ is linear, $(w_1, \ldots, w_r)$ is a base of W, and $B = (b_{ki})$ is the matrix of h, then

$$h(v_i) = \textstyle\sum_k b_{ki}w_k$$
$$(hf)(e_j) = h\big(\textstyle\sum_i a_{ij}v_i\big) = \sum_i a_{ij}h(v_i) = \sum_{i,j} a_{ij}b_{ki}w_k = \sum_k \big(\sum_i b_{ki}a_{ij}\big)w_k.$$

The matrix of $h \circ f$ in the bases $e_1, \ldots, e_n$ and $w_1, \ldots, w_r$ is $BA := \big(\sum_i b_{ki}a_{ij}\big)$.

The matrix product is associative (so is the composition of maps) and *the matrix of any isomorphism f is invertible, the inverse being the matrix of f^{-1}.*

Definition: Let $(e'_1, \ldots, e'_n)$ be a new base of E. The **base change** matrix is the matrix $B = (b_{ij}) \in M_{n\times n}(k)$ of the identity map $E \to E$ in the bases $(e'_1, \ldots, e'_n)$ and $(e_1, \ldots, e_n)$,

$$e'_j = b_{1j}e_1 + \ldots + b_{nj}e_n.$$

By 2.1, if X and X' are the coordinates of $e \in E$ in these bases,

$$X = BX' \quad , \quad X' = B^{-1}X.$$

If we also fix a new base $(v'_1, \ldots, v'_m)$ in F, we have a base change matrix $C \in M_{m\times m}(k)$, and the commutative square

$$\begin{array}{ccccc} e'_1,\ldots,e'_n & F & \xrightarrow{f} & F & v'_1,\ldots,v'_m \\ & \mathrm{Id}\downarrow & & \uparrow \mathrm{Id} & \\ e_1,\ldots,e_n & E & \xrightarrow{f} & E & v_1,\ldots,v_m \end{array}$$

shows that the matrix $A' \in M_{m\times n}(k)$ of f in the new bases is just

$$A' = C^{-1}AB. \tag{2.2}$$

Definition: The sum of some vector subspaces $V_1, \ldots, V_r$ of a vector space is **direct**, and we put $V_1 \oplus \ldots \oplus V_r$, if the always surjective linear map

$$s\colon V_1 \times \ldots \times V_r \longrightarrow V_1 + \ldots + V_r,\ \ s(v_1, \ldots, v_r) = v_1 + \ldots + v_r,$$

is an isomorphism, so that $\dim(V_1 \oplus \ldots \oplus V_r) = \dim V_1 + \ldots + \dim V_r$.

Examples: If $e_1, \ldots, e_n$ is a base of E, then $E = ke_1 \oplus \ldots \oplus ke_n$.

If $r = 2$, then $\operatorname{Ker} s = \{(v, -v)\colon v \in V_1 \cap V_2\}$, and the sum is direct $\Leftrightarrow$ $V_1 \cap V_2 = 0$.

If $E = V \oplus W$, $(V + W = E,\ V \cap W = 0)$ we say that W is a **supplement** of V in E.

Theorem: *Any vector subspace $V \subseteq E$ admits a supplement in E.*

Proof: Let X be the set of all vector subspaces $W \subseteq E$ such that $V \cap W = 0$, ordered by inclusion. It is not empty because $V \cap 0 = 0$.

Any chain $\{W_i\}$ has an upper bound: the crux is to show that $W = \bigcup_i W_i$ is a subgroup, since $\lambda W = \bigcup_i \lambda W_i \subseteq \bigcup_i W_i = W$, $V \cap W = \bigcup_i (V \cap W_i) = 0$ and $W_i \subseteq W$.

Now, if $u, w \in W$, then $u, w \in W_i$ by the chain condition; hence $u + w, -u \in W_i \subseteq W$.

By Zorn's lemma, X has a maximal element W, and $V \cap W = 0$ since $W \in X$.

Let us show that $E = V + W$. If $e \in E$, and $e \notin W$, then $W \subset W + ke$, so that $V \cap (W + ke)$ is not zero. There is a vector $0 \neq v = w + \lambda e$, where $v \in V$, $w \in W$.

Now $\lambda \neq 0$, because $V \cap W = 0$, and we see that $e = \lambda^{-1}(v - w) \in V + W$.

Theorem: *Any exact sequence $0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0$ of linear maps admits a linear **section*** (a map $s\colon E'' \to E$ such that $ps = \mathrm{Id}_{E''}$). *Hence $i + s\colon E' \oplus E'' \to E$ is an isomorphism and the following diagram commutes, where $i_1(e') = (e', 0)$, $\pi_2(e', e'') = e''$,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \xrightarrow{i_1} & E' \oplus E'' & \xrightarrow{\pi_2} & E'' & \longrightarrow & 0 \\ & & \| & & \wr|\, i+s & & \| & & \\ 0 & \longrightarrow & E' & \xrightarrow{i} & E & \xrightarrow{p} & E'' & \longrightarrow & 0 \end{array}$$

Proof: Let W be a supplement of $\operatorname{Im} i = \operatorname{Ker} p$ in E, so that $p\colon W \xrightarrow{\sim} E''$.

The required map is $s = p^{-1}\colon E'' \to W \subseteq E$, and the diagram clearly commutes.

Let us see that $i + s\colon E' \oplus E'' \to E$ is an isomorphism.

If $0 = i(e') + s(e'')$, then $0 = pi(e') + ps(e'') = e''$, $i(e') = 0$; hence $e' = 0$ since i is injective.

If $e \in E$, we put $e'' = p(e)$, so that $e - s(e'') \in \operatorname{Ker} p = \operatorname{Im} i$; hence $e - s(e'') = i(e')$, and we conclude that $e = i(e') + s(e'')$.

2.4 The Dual Space

Definitions: Linear forms on a k-vector space E are k-linear maps $\omega\colon E \to k$.

The **dual space** of E is $E^* = \mathrm{Hom}_k(E,k)$.

Theorem: *If $e_1,\dots,e_n$ is a base of E, there is a unique base $\omega_1,\dots,\omega_n$ of E^*, named **dual base**, such that $\omega_i(e_j)=\delta_{ij}$,* (where $\delta_{ij}=1$ when $i=j$, and 0 otherwise).

Proof: The map $\omega_i(x_1e_1+\ldots+x_ne_n)=x_i$ is a linear form and $\omega_i(e_j)=\delta_{ij}$.

The linear forms $\omega_1,\dots,\omega_n$ span E^* because, for any linear form ω, we have

$$\omega = \omega(e_1)\omega_1 + \ldots + \omega(e_n)\omega_n \tag{2.3}$$

since both coincide on the base $e_1,\dots,e_n$. Hence $\omega_1,\dots,\omega_n$ form a base of E^*.

Note: 2.3 states that the coordinates of ω in the dual base are $Z=(\omega(e_1),\dots,\omega(e_n))$, which is the matrix of $\omega\colon E\to k$ when we consider in k the base defined by the unity.

If $(e'_1,\dots,e'_n)$ is another base of E, and $(\omega'_1,\dots,\omega'_n)$ is the dual base, the coordinates Z' of ω in the new base are $Z'=ZB$ by 2.2, where B is the base change matrix, and Z, Z' are rows; hence $Z^t=(B^{-1})^t(Z')^t$ and the base change matrix in E^* is just $(B^{-1})^t$.

Lemma: *If $e\in E$ is non null, there exists $\omega\in E^*$ such that $\omega(e)=1$.*

Proof: Let V be a supplement of ke. The linear form $\omega\colon E=ke\oplus V\to k$, $\omega(\lambda e+v)=\lambda$, is well defined (since $e\neq 0$) and $\omega(e)=1$. q.e.d.

Any vector $e\in E$ defines a linear form $\psi(e):E^*\longrightarrow k$, $\psi(e)(\omega)=\omega(e)$; so that we have a natural linear map $\psi\colon E\to E^{**}$.

Reflexivity Theorem: *$E=E^{**}$, for any finite dimensional vector space E.*

Proof: Since $\dim E^{**}=\dim E^*=\dim E$, we only have to show that ψ is injective.

If $\psi(e)=0$, then $\omega(e)=\psi(e)(\omega)=0$, $\forall\omega\in E^*$, and $e=0$ by the above lemma.

Definition: The **incident** of a vector subspace $V\subseteq E$ is the vector subspace of E^*

$$V^{\circ}=\{\omega\in E^*\colon \omega(v)=0,\ \text{for all } v\in V\}.$$

Lemma: *If E is finite dimensional, then $V=(V^{\circ})^{\circ}$ via the identification $E=E^{**}$.*

Proof: Clearly $V\subseteq (V^{\circ})^{\circ}=\{e\in E\colon \omega(e)=0,\ \text{for all } \omega\in V^{\circ}\}$.

Conversely, if $e\in(V^{\circ})^{\circ}$, and $\pi\colon E\to E/V$ is the canonical map, we have $0=\bar\omega(\pi e)$ for all $\bar\omega\in(E/V)^*$, since $\bar\omega\circ\pi\in V^{\circ}$. Hence $\pi(e)=0$, and $e\in V$.

Theorem: *If E is finite dimensional, we have a natural lattice anti-isomorphism*

$$\begin{bmatrix}\text{Vector sub-}\\ \text{spaces of } E\end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix}\text{Vector sub-}\\ \text{spaces of } E^*\end{bmatrix},\ V\mapsto V^{\circ}.$$

Proof: It is a bijective map according to the lemma, the inverse being the incidence in E^*.

Moreover, if $V_1^{\circ}\subseteq V_2^{\circ}$, then $(V_1^{\circ})^{\circ}\supseteq(V_2^{\circ})^{\circ}$, and $V_1\supseteq V_2$ by the lemma.

Corollary: $(V+W)^{\circ}=V^{\circ}\cap W^{\circ}$.

$$(V \cap W)^{\rm o} = V^{\rm o} + W^{\rm o}.$$
$$\dim V^{\rm o} = \dim E - \dim V.$$

Proof: These properties are lattice properties.

Example: The equations $a_1x_1 + \ldots + a_nx_n = 0$ of the hyperplanes passing through V form the incident subspace $V^{\rm o}$. If $L = k\omega_1 + \ldots + k\omega_m$ is a vector subspace of E^*, then $L^{\rm o}$ is the space of solutions of the homogeneous system

$$\left.\begin{array}{c} \omega_1 = a_{11}x_1 + \ldots . + a_{1n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_m = a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{array}\right\}$$

Definition: The **transpose** of a linear map $f\colon F \to E$ is the linear map

$$f^*\colon E^* \longrightarrow F^*, \;\; f^*(\omega) = \omega \circ f.$$

It is clear that $(g \circ f)^* = f^* \circ g^*$, and $(\lambda f_1 + \mu f_2)^* = \lambda f_1^* + \mu f_2^*$.
If $i\colon V \to E$ is the inclusion, $i(v) = v$, then $i^*(\omega)$ is the restriction of ω to V.

If (a_{ij}) is the matrix of f in some bases $v_1, \ldots, v_m$ of F and $e_1, \ldots, e_n$ of E, then the matrix of f^* in the dual bases $\theta_1, \ldots, \theta_m$ of F^* and $\omega_1, \ldots, \omega_n$ of E^*, is the transpose matrix

$$(f^*\omega_j)(v_i) = \omega_j(f(v_i)) = a_{ji}.$$

Frobënius Theorem: *If* $0 \longrightarrow E_1 \xrightarrow{\;j\;} E \xrightarrow{\;p\;} E_2 \longrightarrow 0$ *is an exact sequence of linear maps, so is the sequence of transpose maps:*

$$0 \longrightarrow E_2^* \xrightarrow{\;p^*\;} E^* \xrightarrow{\;j^*\;} E_1^* \longrightarrow 0.$$

Proof: It is enough (p. 54) to consider the exact sequence

$$0 \longrightarrow E_1 \xrightarrow{\;i_1\;} E_1 \oplus E_2 \xrightarrow{\;\pi_2\;} E_2 \longrightarrow 0$$

We have that i_1^* is surjective, π_2^* is injective, and $\operatorname{Im}\pi_2^* \subseteq \operatorname{Ker} i_1^*$ because

$$i_1^*\pi_1^* = (\pi_1 i_1)^* = \mathrm{Id},$$
$$i_2^*\pi_2^* = (\pi_2 i_2)^* = \mathrm{Id},$$
$$i_1^*\pi_2^* = (\pi_2 i_1)^* = 0.$$

Finally, if $\omega \in \operatorname{Ker} i_1^*$, then $\omega = (i_1\pi_1 + i_2\pi_2)^*\omega = (\pi_1^* i_1^* + \pi_2^* i_2^*)\omega = \pi_2^*(i_2^*\omega) \in \operatorname{Im}\pi_2^*$.

Corollary: *If* $f\colon E \to F$ *is a linear map, then* $\operatorname{Ker} f^* = (\operatorname{Im} f)^{\rm o}$, $\operatorname{Im} f^* = (\operatorname{Ker} f)^{\rm o}$.

Proof: The exact sequence $0 \to \operatorname{Ker} f \xrightarrow{i} E \xrightarrow{f} F \xrightarrow{\pi} F/\operatorname{Im} f \to 0$ induces an exact sequence

$$0 \longrightarrow (F/\operatorname{Im} f)^* \xrightarrow{\;\pi^*\;} F^* \xrightarrow{\;f^*\;} E^* \xrightarrow{\;i^*\;} (\operatorname{Ker} f)^* \longrightarrow 0.$$

Hence $\operatorname{Im} f^* = \operatorname{Ker} i^* = (\operatorname{Ker} f)^{\rm o}$ and $\operatorname{Ker} f^* = \operatorname{Im}\pi^*$.
Finally, $\operatorname{Im}\pi^* = (\operatorname{Im} f)^{\rm o}$ by the universal property of the quotient space $F/\operatorname{Im} f$.

Corollary: $\operatorname{Im} f^* = (\operatorname{Im} f)^*$, *(column and row ranks of any matrix coincide).*

Proof: By the universal property of the quotient space, $(E/\operatorname{Ker} f)^* \simeq (\operatorname{Ker} f)^{\rm o} = \operatorname{Im}(f^*)$, and by the Isomorphism theorem, $(E/\operatorname{Ker} f)^* \simeq (\operatorname{Im} f)^*$.

Note: The exact sequence $0 \to (E/V)^* \xrightarrow{\pi^*} E^* \xrightarrow{i^*} V^* \to 0$, where $V^{\rm o} = \operatorname{Ker} i^*$, proves again that $\dim V^{\rm o} = \dim E - \dim V$; hence that $V = (V^{\rm o})^{\rm o}$.

2.5 Euclidean Vector Spaces

In this section we put $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and a group morphisms $f\colon E \to E'$ between $\mathbb{K}$-vector spaces is said to be **semilinear** when $f(\lambda e) = \bar{\lambda} f(e)$, $\lambda \in \mathbb{K}$.

When $\mathbb{K} = \mathbb{R}$ semilinear maps are just linear maps and, when $\mathbb{K} = \mathbb{C}$, semilinear maps are just linear maps if E' is endowed with the conjugate complex structure $\lambda \cdot e' := \bar{\lambda} e'$.

Definition: A **scalar product** on a $\mathbb{K}$-vector space E is a map $E \times E \dot{\longrightarrow} \mathbb{K}$ which is

1. *Left-semilinear and right-linear*:
$$(\lambda e + \mu e') \cdot v = \bar{\lambda}(e \cdot v) + \bar{\mu}(e' \cdot v)\ ,\ e \cdot (\lambda v + \mu v') = \lambda(e \cdot v) + \mu(e \cdot v').$$
2. $v \cdot e = \overline{e \cdot v}$.
3. *Positive definite*: $e \cdot e \geq 0$, and $e = 0$ if the equality holds.

The **modulus** of a vector is the non-negative real number $\|e\| = +\sqrt{e \cdot e}$, and we have $\|\lambda e\| = \sqrt{\lambda^2 e \cdot e} = |\lambda| \cdot \|e\|$, so that $\|\frac{e}{\|e\|}\| = 1$.

Two vectors e, v are **orthogonal** when $e \cdot v = 0$, and the **Pythagorean theorem** holds: *If* $e \cdot v = 0$, *then* $\|e + v\|^2 = \|e\|^2 + e \cdot v + v \cdot e + \|v\|^2 = \|e\|^2 + \|v\|^2$.

Lemma: $|e \cdot v| \leq \|e\| \cdot \|v\|$, (**Cauchy-Schwarz inequality**).
$\|e + v\| \leq \|e\| + \|v\|$, (**triangle inequality**).

Proof: Put $v = \alpha e + w$ with $e \cdot w = 0$ (just take $\alpha = \frac{e \cdot v}{e \cdot e}$), so that $\|\alpha e\| \leq \|v\|$ by the Pythagorean theorem, and
$$|e \cdot v| = |\alpha(e \cdot e)| = |\alpha| \cdot \|e\|^2 \leq \|e\| \cdot \|v\|.$$

Alternative proof: Since $|(\lambda e) \cdot (\mu v)| = |\bar{\lambda}\mu| \cdot |e \cdot v| \leq |\lambda| \cdot |\mu| \cdot \|e\| \cdot \|v\| = \|\lambda e\| \cdot \|\mu v\|$, if the inequality holds for two vectors e, v, so it does for λe and μv. Hence, we may assume that $\|e\| = \|v\| = 1$, and multiplying e by a factor $e^{i\theta}$ we may also assume that $e \cdot v$ is a non-negative real number.

In such case we conclude: $0 \leq (v - e) \cdot (v - e) = v^2 + e^2 - 2e \cdot v = 2 - 2e \cdot v$, $0 \leq e \cdot v \leq 1 = \|e\| \cdot \|v\|$.

Finally, the triangle inequality directly follows from the Cauchy-Schwarz inequality:
$$\|e + v\|^2 = e^2 + v^2 + e \cdot v + \overline{e \cdot v} \leq e^2 + v^2 + 2|e \cdot v| \leq \|e\|^2 + \|v\|^2 + 2\|e\| \cdot \|v\| = (\|e\| + \|v\|)^2.$$

Definition: In the real case, the (measure in radians of the) **angle** formed by two non-null vectors e, v is defined to be (p. 41) the unique real number $\alpha \in [0, \pi]$ such that $\cos \alpha = \frac{e \cdot v}{\|e\| \cdot \|v\|}$, so that $e \cdot v = 0$ means that $\alpha = \pi/2$.

Definition: A **Euclidean** vector space is a finite dimensional real or complex vector space E endowed with a scalar product[4], so that we have a semilinear map (named **polarity**)
$$\phi\colon E \longrightarrow E^*,\ (\phi e)(v) = e \cdot v.$$

Theorem: *The polarity* $\phi\colon E \to E^*$ *is a semilinear isomorphism.*

Proof: If $\phi(e) = 0$, then $e \cdot e = (\phi e)(e) = 0$; hence $e = 0$, and ϕ is injective.

Since $\dim E = \dim E^*$, we conclude that ϕ is an isomorphism.

[4]In fact, in the classical Euclidean Geometry the scalar product is well-defined up to a positive factor, and to fix it is just to fix the length unit.

Definitions: The **orthogonal** of a vector subspace $V \subseteq E$ is the vector subspace

$$V^\perp = \{e \in E \colon e \cdot v = 0 \text{ for all } v \in V\} = \{e \in E \colon \phi(e) \in V^{\mathrm{o}}\}.$$

Corollary: *We have a lattice anti-automorphism*

$$\begin{bmatrix}\text{Vector sub-}\\ \text{spaces of } E\end{bmatrix} \xrightarrow{\sim} \begin{bmatrix}\text{Vector sub-}\\ \text{spaces of } E\end{bmatrix}, \ V \mapsto V^\perp.$$

so that $(V+W)^\perp = V^\perp \cap W^\perp$, $(V \cap W)^\perp = V^\perp + W^\perp$ *and* $\dim V^\perp = \dim E - \dim V$.

Proof: It follows from the properties of the incidence, since $\phi(V^\perp) = V^{\mathrm{o}}$.

Corollary: $(V^\perp)^\perp = V$.

$E = V^\perp \oplus V$.

Proof: We have $V = (V^\perp)^\perp$ because $V \subseteq (V^\perp)^\perp$, and

$$\dim (V^\perp)^\perp = \dim E - \dim V^\perp = \dim E - (\dim E - \dim V) = \dim V.$$

Finally, $V \cap V^\perp = 0$ since $v \cdot v = 0 \Rightarrow v = 0$, and $V + V^\perp = E$ because

$$\dim (V + V^\perp) = \dim V + \dim V^\perp = \dim E.$$

Examples: In the 3 following examples, we consider a real Euclidean space:

1. In any triangle, the three medians intersect at the barycenter $g = \frac{a+b+c}{3}$, mutually dividing at the ratio 2:1, as the following identities show:

$$\frac{a+b+c}{3} = \frac{a}{3} + \frac{2}{3}\frac{b+c}{2} = \frac{b}{3} + \frac{2}{3}\frac{a+c}{2} = \frac{c}{3} + \frac{2}{3}\frac{a+b}{2}.$$

2. Moreover, if h is the intersection point of the heights trough the vertices a and c, then

$$(b-a) \cdot (h-c) = 0 \tag{2.4}$$
$$(c-b) \cdot (h-a) = 0 \tag{2.5}$$

and adding them we obtain $(c-a) \cdot (h-b) = 0$, so that the third height also passes through the orthocenter h. On the other hand, if f is the intersection point of the bisectors of the sides ab and bc, then

$$0 = 2(b-a) \cdot \left(f - \tfrac{a+b}{2}\right) = (b-a) \cdot 2f + a \cdot a - b \cdot b \tag{2.6}$$
$$0 = 2(c-b) \cdot \left(f - \tfrac{b+c}{2}\right) = (c-b) \cdot 2f + b \cdot b - c \cdot c \tag{2.7}$$

and adding them we obtain $0 = (c-a) \cdot 2f + a \cdot a - c \cdot c = 2(c-a) \cdot \left(f - \frac{a+c}{2}\right)$, so that the bisector of the side ab also passes through the circumcenter f.

3. Now, adding 2.4 with 2.6, and 2.5 with 2.7, we obtain (recall that $g = \frac{a+b+c}{3}$)

$$(b-a) \cdot (h + 2f - 3g) = 0,$$
$$(c-b) \cdot (h + 2f - 3g) = 0.$$

Since the vectors $b-a$, $c-b$ are linearly independent (because a, b, c are not collinear) and the polarity is an isomorphism, we see that $h + 2f - 3g = 0$. The barycenter divides the segment determined by the orthocenter and the circumcenter in the ratio 2:1,

$$g = \frac{h}{3} + \frac{2f}{3} = h + \frac{2}{3}(f - h).$$

Definition: A base $e_1, \ldots, e_n$ of E is **orthonormal** when $e_i \cdot e_j = \delta_{ij}$, so that

$$(x_1 e_1 + \ldots + x_n e_n) \cdot (y_1 e_1 + \ldots + y_n e_n) = \bar{x}_1 y_1 + \ldots + \bar{x}_n y_n.$$

Theorem: *Any Euclidean vector space $E \neq 0$ admits orthonormal bases.*

Proof: If $E = \mathbb{K}v$, an orthonormal base is $e = \frac{v}{\|v\|}$, since $e \cdot e = 1$.

If $n = \dim E > 1$, we take $e_n \in E$ of module 1. Now $\dim (\mathbb{K}e_n)^\perp = n-1$, and

$$E = (\mathbb{K}e_n)^\perp \oplus \mathbb{K}e_n.$$

By induction $(\mathbb{K}e_n)^\perp$ admits an orthonormal base $e_1, \ldots, e_{n-1}$.

Now $e_1, \ldots, e_n$ span E and $e_i \cdot e_j = \delta_{ij}$, hence they form an orthonormal base of E.

2.6 Diagonalization of Endomorphisms

Definition: Let E be a k-vector space of dimension n. The **endomorphisms** of E are the linear maps $T\colon E \to E$, and $\mathrm{End}_k(E) = \mathrm{Hom}_k(E, E)$ is a vector space of dimension n^2.

The product of endomorphisms is the composition, $ST = S \circ T$.

In general, given a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_d x^d \in k[x]$, we put

$$p(T) = a_0 \mathrm{Id}_E + a_1 T + a_2 T^2 + \ldots + a_d T^d.$$

Definitions: A scalar $\alpha \in k$ is an **eigenvalue** of T if there exists a non null vector $e \in E$ such that $T(e) = \alpha e$. We say that e is an **eigenvector** of T of eigenvalue α, and that $V_\alpha := \mathrm{Ker}\,(\alpha \mathrm{Id} - T) = \{e \in E \colon T(e) = \alpha e\}$ is an **eigenspace** of T.

The linear map $k[x] \to \mathrm{End}_k(E)$, $p(x) \mapsto p(T)$, is not injective, since $\dim k[x] = \infty$, and the **annihilator** polynomial of T is the unitary generator $\phi_T(x) = x^d + \ldots$ of the ideal $\{p(x) \in k[x] \colon p(T) = 0\}$. In $k[x]$, it is the polynomial of least degree annihilating T, and it divides any other polynomial in $k[x]$ annihilating T.

Theorem: *The roots in k of the annihilator polynomial $\phi_T(x)$ are just the eigenvalues of T.*

Proof: If $e \neq 0$ and $T(e) = \alpha e$, then $0 = \phi(T)e = \phi(\alpha)e$; hence $\phi(\alpha) = 0$.

Conversely, if $\phi(x) = (x - \alpha)p(x)$, then $p(T) \neq 0$, and $p(T)v \neq 0$ for some vector v.

Hence $(T - \alpha)\big[p(T)v\big] = \phi(T)v = 0$, and α is an eigenvalue.

Corollary: *When $k = \mathbb{C}$, any endomorphism admits an eigenvector.*

Corollary: *When $k = \mathbb{C}$, there is a basis of E such that the matrix of T is upper-triangular, with the entries in the main diagonal being eigenvalues of T.*

Proof: Fix an eigenvector e of T, and consider the quotient vector space $\bar{E} = E/\langle e \rangle$. The endomorphism $\bar{T}\colon \bar{E} \to \bar{E}$, $\bar{T}(\bar{e}) = [T(e)]$, is well-defined and, since $\phi_T(x)$ annihilates $\bar{T}$, any eigenvalue of $\bar{T}$ also is an eigenvalue of T.

By induction on the dimension of E, we may assume the existence of a basis $\bar{e}_2, \ldots, \bar{e}_n$ of $\bar{E}$ such that the matrix of $\bar{T}$ is upper-triangular, with the entries of the main diagonal being eigenvalues of $\bar{T}$, hence of T. It is easy to check that $e, e_2, \ldots, e_n$ is the required basis.

Examples: The symmetry S of $\mathbb{R}^3$ with respect to a plane has the eigenvalues $x = 1, -1$. Since $S^2 = \mathrm{Id}$, the annihilator divides $x^2 - 1$; hence $\phi_S(x) = x^2 - 1$.

A rotation T of right angle in $\mathbb{R}^3$ only has the eigenvalue $x = 1$. Since $T^4 = \mathrm{Id}$, the annihilator divides $x^4 - 1 = (x-1)(x+1)(x^2+1)$; hence $\phi_T(x) = (x-1)(x^2+1)$.

Definition: An endomorphism T of an Euclidean vector space E is **selfadjoint** or **hermitian** (or **symmetric** in the real case) when $T(e) \cdot v = e \cdot T(V)$, $\forall e, v \in E$.

If A is the matrix of T in an orthonormal base of E, this condition states that A is an hermitian matrix, $\bar{A}^t = A$. Hence, in the real case, that A is symmetric, $A^t = A$.

Spectral Theorem: *Any eigenvalue of a selfadjoint endomorphism $T\colon E \to E$ is real, and eigenvectors corresponding to different eigenvalues are orthogonal. Moreover, any complex root of the annihilator polynomial $\phi(x)$ is real, and E admits an orthonormal base of eigenvectors.*

Proof: If $0 \neq e \in E$ is an eigenvector, $T(e) = \alpha e$, then $e \cdot e \neq 0$ and α is real:

$$\bar{\alpha}(e \cdot e) = (\alpha e) \cdot e = T(e) \cdot e = e \cdot T(e) = e \cdot (\alpha e) = \alpha(e \cdot e).$$

Now, if e_α, e_β are eigenvector with eigenvalues $\alpha \neq \beta$, then $\bar{\alpha} = \alpha$, and $e_\alpha \cdot e_\beta = 0$ because

$$\alpha(e_\alpha \cdot e_\beta) = \bar{\alpha}(e_\alpha \cdot e_\beta) = (\alpha e_\alpha) \cdot e_\beta = T(e_\alpha) \cdot e_\beta = e_\alpha \cdot T(e_\beta) = e_\alpha \cdot (\beta e_\beta) = \beta(e_\alpha \cdot e_\beta).$$

Moreover, in the complex case, any complex root of $\phi(x)$, being an eigenvalue, is real; while in the real case, it is enough to prove that the annihilator $\phi(x)$ has no quadratic factor

$$\big(x - (a+bi)\big)\big(x - (a-bi)\big) = (x-a)^2 + b^2 \ , \quad (\text{with } a, b \in \mathbb{R} \text{ and } b \neq 0).$$

If $\phi(x) = \big((x-a)^2 + b^2\big)q(x)$, then $q(T) \neq 0$, so that $\big((T-a)^2 + b^2\big)e = 0$ for some non null vector $e = q(T)v$, and we obtain a contradiction[5]:

$$0 = \big((T-a)^2 e + b^2 e\big) \cdot e = \big((T-a)e\big) \cdot \big((T-a)e\big) + b^2(e \cdot e) \geq b^2(e \cdot e) > 0.$$

Finally, fix an eigenvector e and, dividing e by $\|e\|$, we may assume that $\|e\| = 1$.

The orthogonal subspace $V := \langle e \rangle^\perp$ has dimension $n-1$ and $T(V) \subseteq V$: If $v \in V$, then $T(v) \cdot e = v \cdot T(e) = v \cdot (\alpha e) = \alpha(v \cdot e) = 0$, so that $T(v) \in V$.

By induction on the dimension of E, we may assume that V admits an orthonormal base $e_2, \ldots, e_n$ of eigenvectors of the selfadjoint endomorphism $T|_V\colon V \to V$, hence of T.

Now $e, e_2, \ldots, e_n$ is an orthonormal base of E, of eigenvectors of T.

Corollary: *Let T be an endomorphism of an euclidean vector space E. There exists an orthonormal base $u_1, \ldots, u_n$ of E such that $T(u_i) \cdot T(u_j) = 0$ when $i \neq j$.*

Proof: Since the scalar product induces an isomorphism $\phi\colon E \to E^*$, we have that $T(e) \cdot T(v) = S(e) \cdot v$ for some selfadjoint endomorphism S:

$$e \cdot S(v) = \overline{S(v) \cdot e} = \overline{T(v) \cdot T(e)} = T(e) \cdot T(v) = S(e) \cdot v.$$

By the spectral theorem, there exists an orthonormal base $u_1, \ldots, u_n$ of E such that $S(u_i) = \alpha_i u_i$; hence $T(u_i) \cdot T(u_j) = S(u_i) \cdot u_j = (\alpha_i u_i) \cdot u_j = 0$ when $i \neq j$.

[5] An alternative more natural proof, reducing the real case to the obvious complex case, is to show that the annihilator polynomial $\phi(x)$ of a matrix with real entries A also is the annihilator polynomial $\phi_{\mathbb{C}}(x)$ of A as a complex matrix. In fact, since $\phi(A) = 0$, it is clear that $\phi_{\mathbb{C}}$ divides ϕ. Moreover, if $A^m = (a_1 + b_1 i)A^{m-1} + \ldots + (a_m + b_m i)I$, where $a_1, b_1, \ldots, a_m, b_m \in \mathbb{R}$, considering the real parts, we see that $A^m = a_1 A^{m-1} + \ldots + a_m I$. Hence, $\deg \phi_{\mathbb{C}} \geq \deg \phi$ and, both polynomials being monic, we see that $\phi_{\mathbb{C}}(x) = \phi(x)$.

Example: In Quantum Mechanics, the state space of a system is a complex vector space E (tipically of infinite dimension) with a scalar product $\langle\ |\ \rangle$, the states being the unidimensional vector subspaces (normally represented by a vector of norm 1), and any observable magnitude is described by an hermitian endomorphism $T\colon E \to E$. The possible values of the magnitude are just the eigenvalues of T (hence real numbers). If the state of the system is represented by an eigenvector ψ of eigenvalue α, then the measurement of such magnitude is certainly α; but if the state is represented by a sum $\psi = \psi_1 + \ldots + \psi_r$ of eigenvectors with different eigenvalues $\alpha_1, \ldots, \alpha_r$ (and such decomposition is unique because the sum $V_{\alpha_1} + \ldots + V_{\alpha_r}$ always is a direct sum, see p. 82, and it exists at least when E is finite dimensional) then the possible values of the measurement are $\alpha_1, \ldots, \alpha_r$, with probability

$$\mathcal{P}(\alpha_i) = \frac{\|\psi_i\|^2}{\|\psi\|^2} = \frac{\langle\psi_i|\psi_i\rangle}{\langle\psi|\psi\rangle}$$

and, the states $\psi_1, \ldots, \psi_r$ being mutually orthogonal, by the Pythagorean theorem we have that $\|\psi\|^2 = \|\psi_1\|^2 + \ldots + \|\psi_r\|^2$, $\mathcal{P}(\alpha_1) + \ldots + \mathcal{P}(\alpha_r) = 1$.

In particular, the mean value of the measurements in the state $\mathbb{C}\psi$ is just

$$\textstyle\sum_i \alpha_i \mathcal{P}(\alpha_i) = \dfrac{\langle \sum_i \psi_i | \sum_i \alpha_i \psi_i\rangle}{\langle\psi|\psi\rangle} = \dfrac{\langle\psi|T(\psi)\rangle}{\langle\psi|\psi\rangle}.$$

Definition: An endomorphism $T\colon E \to E$ is **diagonalizable** if E admits a base $e_1, \ldots, e_n$ of eigenvectors, $T(e_j) = \alpha_j e_j$, so that the matrix of T is diagonal,

$$\begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix}$$

Lemma: *An endomorphism of eigenvalues $\alpha_1, \ldots, \alpha_r$ is diagonalizable if and only if $V_{\alpha_1} + \ldots + V_{\alpha_r} = E$.*

Proof: If T is diagonalizable, then $V_{\alpha_1} + \ldots + V_{\alpha_r}$ is E, since it contains a base of E.

If $V_{\alpha_1} + \ldots + V_{\alpha_r} = E$, considering bases in V_{α_i} we see that E admits a generating system of eigenvectors, hence a base.

Lemma: $\dim(V_{\alpha_1} + \ldots + V_{\alpha_m}) = \dim V_{\alpha_1} + \ldots + \dim V_{\alpha_m}$, $(\ \alpha_i \neq \alpha_j)$.

Proof: $\mathrm{Ker}\,(T - \alpha_1) \ldots (T - \alpha_m) = \mathrm{Ker}\,(T - \alpha_1) \oplus \ldots \oplus \mathrm{Ker}\,(T - \alpha_m)$, by p. 82.

Corollary: *The number r of different eigenvalues of an endomorphism T of a vector space of dimension n always is $r \leq n$; and if $r = n$, then T is diagonalizable.*

Proof: $r \leq \dim V_{\alpha_1} + \ldots + \dim V_{\alpha_r} = \dim(V_{\alpha_1} + \ldots + V_{\alpha_r}) \leq \dim E = n$.

Diagonalization Criterion: *An endomorphism T is diagonalizable if and only if all the roots of the annihilator polynomial $\phi_T(x)$ are in k and are simple roots.*

Proof: If all the roots of $\phi_T(x)$ are in k and are simple, $\phi_T(x) = (x - \alpha_1) \ldots (x - \alpha_r)$, then we have $0 = (T - \alpha_1) \ldots (T - \alpha_r)$ and T is diagonalizable (p. 82),

$$E = \mathrm{Ker}\,(T - \alpha_1) \oplus \ldots \oplus \mathrm{Ker}\,(T - \alpha_r) = V_{\alpha_1} \oplus \ldots \oplus V_{\alpha_r}.$$

Conversely, if $\alpha_1, \dots, \alpha_r$ are the eigenvalues of T, then $(T-\alpha_1)\dots(T-\alpha_r)$ vanishes on a base when T is diagonalizable; hence it is null and $\phi_T(x)$ divides $(x-\alpha_1)\dots(x-\alpha_r)$, so that any root of $\phi_T(x)$ is simple and it is in k.

Examples: By 2.15, an endomorphism T of matrix A is diagonalizable if and only if there is an invertible matrix B such that $D = B^{-1}AB$ is diagonal. In such a case

$$A^m = (BDB^{-1})(BDB^{-1})\dots = BD^mB^{-1} = B \begin{pmatrix} \alpha_1^m & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n^m \end{pmatrix} B^{-1}$$

and so we obtain the general solution $X_m = A^m X_0$ of the system of difference equations

$$X_{m+1} = AX_m \quad , \quad \begin{cases} x_{m+1} = a_{11}x_m + \dots + a_{1n}z_m \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots \\ z_{m+1} = a_{n1}x_m + \dots + a_{nn}z_m \end{cases}$$

When $k = \mathbb{R}$ or $\mathbb{C}$, the general solution of the system of differential equations

$$X' = AX \quad , \quad \begin{cases} x_1' = a_{11}x_1 + \dots + a_{1n}x_n \\ \quad \dots\dots\dots\dots\dots\dots\dots \\ x_n' = a_{n1}x_1 + \dots + a_{nn}x_n \end{cases}$$

is $X = B\bar{X}$, where $\bar{X}$ is the general solution of the system $\bar{X}' = D\bar{X}$. In fact,

$$X' = B\bar{X}' = BD\bar{X} = BDB^{-1}X = AX.$$

Now, the differential equations $\bar{x}_i' = \alpha_i \bar{x}_i$ of the system $\bar{X}' = D\bar{X}$ are solved in p. 84, so that $\bar{x}_i(t) = c_i\, e^{\alpha_i t}$, $c_i \in k$ $(= \mathbb{R}$ or $\mathbb{C})$, and the solutions of the initial system are

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B \begin{pmatrix} c_1\, e^{\alpha_1 t} \\ \vdots \\ c_n\, e^{\alpha_n t} \end{pmatrix}.$$

Theorem: *Let $\alpha_1, \dots, \alpha_r$ be the eigenvalues of T. When $k = \mathbb{C}$, the matrix A of T in a base of E has the form*

$$A = \begin{pmatrix} \alpha_1 & * & * & & & & \\ 0 & \ddots & * & & & 0 & \\ 0 & 0 & \alpha_1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & \alpha_r & * & * \\ & 0 & & & 0 & \ddots & * \\ & & & & 0 & 0 & \alpha_r \end{pmatrix}.$$

Proof: The annihilator polynomial is $\phi(x) = (x-\alpha_1)^{n_1}\dots(x-\alpha_r)^{n_r}$; hence

$$E = \operatorname{Ker}\phi = \big(\operatorname{Ker}(x-\alpha_1)^{n_1}\big) \oplus \dots \oplus \big(\operatorname{Ker}(x-\alpha_r)^{n_r}\big),$$

where $E_i := \operatorname{Ker}(x-\alpha_i)^{n_i}$ is a T-invariant vector subspace. The restriction $T|_{E_i}$, being annihilated by $(x-\alpha_i)^{n_i}$, has a unique eigenvalue α_i; hence there is a base of V_i such that the matrix of $T|_{E_i}$ is upper triangular with all the entries in the main diagonal equal to α_i.

2.7 Tensors

Definition: Let $E_1, \ldots, E_r, F$ be k-vector spaces. A map $T\colon E_1 \times \ldots \times E_r \to F$ is k-**multilinear** if it is k-linear in any variable,

$$T(\ldots, e_i + v_i, \ldots) = T(\ldots, e_i, \ldots) + T(\ldots, v_i, \ldots),$$
$$T(\ldots, \lambda e_i, \ldots) = \lambda T(\ldots, e_i, \ldots).$$

Multilinear maps $E_1 \times \ldots \times E_r \to F$ form a k-vector space:

$$(T + \bar{T})(e_1, \ldots, e_r) = T(e_1, \ldots, e_r) + \bar{T}(e_1, \ldots, e_r),$$
$$(\lambda T)(e_1, \ldots, e_r) = \lambda \cdot T(e_1, \ldots, e_r).$$

Definitions: Tensors of type (p, q) on a k-vector space E of finite dimension are multilinear maps $T\colon E \times \overset{p}{\ldots} \times E \times E^* \times \overset{q}{\ldots} \times E^* \longrightarrow k$, and they form a vector space $T_p^q E$. Tensors of type $(p, 0)$ are **covariant**[6] of order p, and $(0, q)$-tensors are **contravariant** of order q.

In particular, $T_1^0 E = E^*$, and $T_0^1 E = E^{**} = E$, and we agree that $T_0^0 E = k$.

Any linear map $f\colon F \to E$ naturally induces linear maps

$$f^*\colon T_p^0 E \longrightarrow T_p^0 F \ , \ \ (f^* T)(v_1, \ldots, v_p) = T\big(f(v_1), \ldots, f(v_p)\big),$$
$$f_*\colon T_0^q F \longrightarrow T_0^q E \ , \ \ (f_* T)(\omega_1, \ldots, \omega_q) = T\big(f^*(\omega_1), \ldots, f^*(\omega_q)\big).$$

If $g\colon E \to V$ is also linear, then $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$.

Definition: The **tensor product** of a (p, q)-tensor T with a (p', q')-tensor T' is the following $(p + p', q + q')$-tensor

$$(T \otimes T')(e_1, \ldots, e_{p+p'}, \omega_1, \ldots, \omega_{q+q'}) = T(e_1, \ldots, e_p, \omega_1, \ldots, \omega_q) \cdot T'(e_{p+1}, \ldots e_{p+p'}, \omega_{q+1}, \ldots \omega_{q+q'}).$$

1. $(\lambda T + \mu \bar{T}) \otimes T' = \lambda (T \otimes T') + \mu (\bar{T} \otimes T')$.
   $T \otimes (\lambda T' + \mu \bar{T}') = \lambda (T \otimes T') + \mu (T \otimes \bar{T}')$.
2. $(T \otimes T') \otimes T'' = T \otimes (T' \otimes T'')$.
3. $f^*(T \otimes T') = f^*(T) \otimes f^*(T')$.

Proof: Put $\underline{e} = (e_1, \ldots, e_p)$, $\underline{e}' = (e_{p+1}, \ldots, e_{p+p'})$, $\underline{\omega} = (\omega_1, \ldots, \omega_q)$, $\underline{\omega}' = (\omega_{q+1}, \ldots, \omega_{q+q'})$.

$$\big((\lambda T + \mu \bar{T}) \otimes T'\big)(\underline{e}, \underline{e}', \underline{\omega}, \underline{\omega}') = (\lambda T + \mu \bar{T})(\underline{e}, \underline{\omega}) T'(\underline{e}', \underline{\omega}')$$
$$= \lambda T(\underline{e}, \underline{\omega}) T'(\underline{e}', \underline{\omega}') + \mu \bar{T}(\underline{e}, \underline{\omega}) T'(\underline{e}', \underline{\omega}') = \big(\lambda (T \otimes T') + \mu (\bar{T} \otimes T')\big)(\underline{e}, \underline{e}', \underline{\omega}, \underline{\omega}')$$

and analogously $T \otimes (\lambda T' + \mu \bar{T}') = \lambda (T \otimes T') + \mu (T \otimes \bar{T}')$.

The second property is obvious, and finally we prove the third:

$$(f^*(T \otimes T'))(\underline{e}, \underline{e}') = (T \otimes T')(f(\underline{e}), f(\underline{e}')) = T(f(\underline{e})) \cdot T'(f(\underline{e}'))$$
$$= (f^* T)(\underline{e}) \cdot (f^* T')(\underline{e}') = (f^*(T) \otimes f^*(T'))(\underline{e}, \underline{e}').$$

Note: We also put $T^q E = T_0^q E$ and $T^p E^* = T_p^0 E$. The tensor product defines a structure of (non commutative) k-algebra on the vector space $T_\bullet^\bullet E = \bigoplus_{p,q} T_p^q E$ of all finite formal sums of tensors $\sum_{p,q} T_p^q$, and f^* is a morphism of k-algebras, only defined on the subalgebra $T^\bullet E^* = \bigoplus_p T^p E^*$.

[6] They have p covariant indices or variables.

Theorem: $\dim T_p^q E = (\dim E)^{p+q}$. *In fact, if* $e_1, \dots, e_n$ *is a base of* E, *and* $\omega_1, \dots, \omega_n$ *is the dual base, then a base of* $T_p^q E$ *is defined by the tensors*

$$\omega_{i_1} \otimes \ldots \otimes \omega_{i_p} \otimes e_{j_1} \otimes \ldots \otimes e_{j_q}.$$

Proof: Let us fix a sequence $i_1, \dots, i_p, j_1, \dots, j_q$. Since $\omega_i(e_j) = \delta_{ij}$,

$$(\omega_{i_1} \otimes \ldots \otimes \omega_{i_p} \otimes e_{j_1} \otimes \ldots \otimes e_{j_q})(e_{i_1}, \dots, e_{i_p}, \omega_{j_1}, \dots, \omega_{j_q}) = 1,$$

while the remaining tensors in the considered family vanish on $(e_{i_1}, \dots, e_{i_p}, \omega_{j_1}, \dots, \omega_{j_q})$.

Hence $\omega_{i_1} \otimes \ldots \otimes \omega_{i_p} \otimes e_{j_1} \otimes \ldots \otimes e_{j_q}$ is not a linear combination of the remaining tensors: it is a family of linearly independent tensors.

To show that this family spans $T_p^q E$, we prove that any (p,q)-tensor T is

$$T = \sum_{\substack{1 \leq i_1, \dots, i_p \leq n \\ 1 \leq j_1, \dots, j_q \leq n}} T(e_{i_1}, \dots, e_{i_p}, \omega_{j_1}, \dots, \omega_{j_q})\, \omega_{i_1} \otimes \ldots \otimes \omega_{i_p} \otimes e_{j_1} \otimes \ldots \otimes e_{j_q}.$$

In fact, the difference vanishes on any sequence $(e_{i_1}, \dots, e_{i_p}, \omega_{j_1}, \dots, \omega_{j_q})$; hence it is null.

The coordinates of a (p,q)-tensor T are just $\lambda_{i_1 \dots i_p}^{j_1 \dots j_q} = T(e_{i_1}, \dots, e_{i_p}, \omega_{j_1}, \dots, \omega_{j_q})$.

Example: Any endomorphism T defines a $(1,1)$-tensor $T(e,\omega) = \omega(Te)$, and this linear map $\mathrm{End}_k(E) \to T_1^1 E$ is injective: if $\omega(Te) = 0$, $\forall \omega \in E^*$, then $T(e) = 0$ (p. 55).

Since both vector spaces have dimension $(\dim E)^2$, it is an isomorphism, $T_1^1 E = \mathrm{End}_k(E)$.

If (a_{ij}) is the matrix of T, i.e., $T(e_j) = \sum_i a_{ij} e_i$, then the coordinates of the corresponding tensor $T = \sum_{ij} \lambda_i^j \omega_i \otimes e_j$ are $\lambda_i^j = T(e_i, \omega_j) = \omega_j(Te_i) = a_{ji}$.

Universal Property: *For any multilinear map* $T\colon E^* \times \overset{p}{\dots} \times E^* \times E \times \overset{q}{\dots} \times E \to F$ *there exists a unique linear map* $f\colon T_p^q E \to F$ *such that*

$$f(\omega_1 \otimes \ldots \otimes \omega_p \otimes e_1 \otimes \ldots \otimes e_q) = T(\omega_1, \dots, \omega_p, e_1, \dots, e_q). \tag{2.8}$$

Proof: We fix a base of E, and we define $f\colon T_p^q E \to F$ in the corresponding base of $T_p^q E$ by 2.8. Now both terms in 2.8 are multilinear maps coinciding on the sequences of vectors of a base; hence they coincide. Uniqueness is obvious.

Corollary: *We have canonical isomorphisms* $(T_p^q E)^* = T_q^p E$.

Proof: Put $F = k$ in the universal property.

Index Contraction: *There exists a unique linear map* $C_1^1\colon T_p^q E \to T_{p-1}^{q-1} E$ *such that*

$$C_1^1(\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_p \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_q) = \omega_1(e_1)\, \omega_2 \otimes \ldots \otimes \omega_p \otimes e_2 \otimes \ldots \otimes e_q.$$

Proof: Just apply the universal property to the multilinear map

$$T\colon E^* \times \overset{p}{\dots} \times E^* \times E \times \overset{q}{\dots} \times E \longrightarrow T_{p-1}^{q-1} E$$
$$T(\omega_1, \dots, \omega_p, e_1, \dots, e_q) = \omega_1(e_1)\, \omega_2 \otimes \ldots \otimes \omega_p \otimes e_2 \otimes \ldots \otimes e_q.$$

Note: We have a contraction $C_i^j\colon T_p^q E \to T_{p-1}^{q-1} E$ for any other pair of indices i, j.

The **trace** of an endomorphism $T = \sum_{ij} a_{ji}\, \omega_i \otimes e_j$ is $\mathrm{tr} T = C_1^1 T = \sum_i a_{ii}$.

2.7.1 Alternate Tensors

If $T \in T^pE^*$ is a covariant p-tensor and $\sigma \in S_p$, we put

$$(\sigma T)(e_1, \dots, e_p) = T(e_{\sigma(1)}, \dots, e_{\sigma(p)}). \tag{2.9}$$

(the permutation σ is applied to the places, not to the indices).

1. *The map* $\sigma \colon T^pE^* \longrightarrow T^pE^*$ *is linear.*
2. *For any linear map* $f \colon F \to E$*, we have* $f^*(\sigma T) = \sigma(f^*T)$.
3. $\tau(\sigma T) = (\tau\sigma)T;\ \ \sigma, \tau \in S_p$.
4. $\sigma(\omega_1 \otimes \dots \otimes \omega_p) = \omega_{\sigma^{-1}(1)} \otimes \dots \otimes \omega_{\sigma^{-1}(p)};\ \ \omega_1, \dots, \omega_p \in E^*$.

Proof: The two first properties are obvious. Let us see the third and fourth,

$$\begin{aligned}\big(\tau(\sigma T)\big)(e_1, \dots, e_p) &= (\sigma T)(e_{\tau(1)}, \dots, e_{\tau(p)}) = T(e_{\tau(\sigma 1)}, \dots, e_{\tau(\sigma p)}) \\ &= T(e_{(\tau\sigma)1}, \dots, e_{(\tau\sigma)p}) = \big((\tau\sigma)T\big)(e_1, \dots, e_p), \\ (\sigma(\omega_1 \otimes \dots \otimes \omega_p))(e_1, \dots, e_p) &= \omega_1(e_{\sigma(1)}) \cdot \ldots \cdot \omega_p(e_{\sigma(p)}) \\ &= \omega_{\sigma^{-1}(1)}(e_1) \cdot \ldots \cdot \omega_{\sigma^{-1}(p)}(e_p) = (\omega_{\sigma^{-1}(1)} \otimes \dots \otimes \omega_{\sigma^{-1}(p)})(e_1, \dots, e_p).\end{aligned}$$

Definition: A covariant tensor $\Omega_p \in T^pE^*$ is **alternate**, or a p**-form**, when

$$\Omega_p(\dots, e, \dots, e, \dots) = 0\ , \quad \forall e \in E.$$

The alternate tensors of order p form a vector subspace $\Lambda^pE^* \subseteq T^pE^*$, and we agree that $\Lambda^0E^* = k$ and $\Lambda^1E^* = E^*$. Analogously we have a subspace $\Lambda^qE \subseteq T^qE$ of alternate contravariant q-tensors.

If $f \colon F \to E$ is linear, then $f^*\Omega_p$ is alternate, and $f^* \colon \Lambda^pE^* \to \Lambda^pF^*$ is linear.

Lemma: $\sigma\Omega_p = (\operatorname{sgn}\sigma)\Omega_p$, *for any* p*-form* Ω_p.

Proof: We may assume that $\sigma = (ij)$ is a transposition,

$$\begin{aligned}0 &= \Omega_p(\dots, e_i + e_j, \dots, e_i + e_j, \dots) \\ &= \Omega(\dots e_i \dots e_i \dots) + \Omega(\dots e_i \dots e_j \dots) + \Omega(\dots e_j \dots e_i \dots) + \Omega(\dots e_j \dots e_j \dots) \\ &= \Omega_p(\dots, e_i, \dots, e_j, \dots) + \Omega_p(\dots, e_j, \dots, e_i, \dots).\end{aligned}$$

Definition: The **anti-symmetrization** of a covariant p-tensor $T \in T^pE^*$ is the tensor

$$h(T) = \sum_{\sigma \in S_p} (\operatorname{sgn}\sigma)(\sigma T)$$

and we agree that $h(T) = T$ when $p = 0$ or 1. By property 4,

$$h(\omega_1 \otimes \dots \otimes \omega_p) = \sum_{\sigma \in S_p} (\operatorname{sgn}\sigma)(\omega_{\sigma(1)} \otimes \dots \otimes \omega_{\sigma(p)}). \tag{2.10}$$

Lemma: $h \colon T^\bullet E^* \longrightarrow \Lambda^\bullet E^*$ *is a surjective linear map, and the kernel is a two-sided ideal,*

$$h(T) = 0 \ \Rightarrow\ h(T \otimes T') = h(T' \otimes T) = 0. \tag{2.11}$$

Proof: If τ transposes two repeated terms in a sequence $(\dots, e, \dots, e, \dots)$, then

$$\sum_{\sigma\in A_p}(\operatorname{sgn}\sigma)(\sigma T) \quad\text{and}\quad \tau\Big(\sum_{\sigma\in A_p}(\operatorname{sgn}\sigma)(\sigma T)\Big) = -\sum_{\sigma\in A_p}\operatorname{sgn}(\tau\sigma)(\tau\sigma T),$$

coincide on $(\ldots,e,\ldots,e,\ldots)$, and the difference $h(T)$ vanishes; i.e., $h(T)\in\Lambda^pE^*$.

Let us see that h is surjective. Let $e_1,\ldots,e_n$ be a base of E, and $\omega_1,\ldots,\omega_n$ the dual base. Put $\lambda_{i_1\ldots i_p}=\Omega_p(e_{i_1},\ldots,e_{i_p})$, so that $\lambda_{\sigma(i_1)\ldots\sigma(i_p)}=(\operatorname{sgn}\sigma)\lambda_{i_1\ldots i_p}$, and

$$\begin{aligned}\Omega_p &= \sum_{1\le i_1,\ldots,i_p\le n}\lambda_{i_1\ldots i_p}\omega_{i_1}\otimes\ldots\otimes\omega_{i_p}\\ &= \sum_{i_1\le\ldots\le i_p}\Big(\sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)\lambda_{i_1\ldots i_p}\omega_{\sigma(i_1)}\otimes\ldots\otimes\omega_{\sigma(i_p)}\Big)\\ &= \sum_{i_1\le\ldots\le i_p}\lambda_{i_1\ldots i_p}h(\omega_{i_1}\otimes\ldots\otimes\omega_{i_p}) = h\Big(\sum_{i_1\le\ldots\le i_p}\lambda_{i_1\ldots i_p}\omega_{i_1}\otimes\ldots\otimes\omega_{i_p}\Big).\end{aligned}\tag{2.12}$$

Finally, we identify any permutation $\sigma\in S_p$ with a permutation $\sigma\in S_{p+q}$ fixing the numbers $p+1,\ldots,p+q$, so that for any covariant q-tensor T' we have

$$\sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)\sigma(T\otimes T') = h(T)\otimes T' = 0,$$
$$\sum_{\sigma\in S_p}(\operatorname{sgn}\tau\sigma)\big(\tau\sigma(T\otimes T')\big) = 0,\ \text{ for all } \tau\in S_{p+q},$$

and the sum in $h(T\otimes T')$ corresponding to any equivalence class τS_p is null.

Hence the total sum vanishes. Analogously $h(T'\otimes T)=0$.

Definition: Since $\Lambda^\bullet E^*$ is the quotient of $T^\bullet E^*$ by an ideal, we have a product on $\Lambda^\bullet E^*$, induced by the tensor product. The **exterior product** of two alternate tensors Ω_p, Ω_q is

$$\Omega_p\wedge\Omega_q = h(T_p\otimes T_q);\quad \text{where } \Omega_p=h(T_p),\ \Omega_q=h(T_q).$$

Since $h(\Omega_p)=(p!)\Omega_p$, then $\Omega_p=h\big(\frac{1}{p!}\Omega_p\big)$ when $\operatorname{char}k=0$, and

$$(\operatorname{char}k=0)\qquad\qquad \Omega_p\wedge\Omega_q = \frac{1}{p!q!}\,h(\Omega_p\otimes\Omega_q).$$

1. $(\lambda\Omega_p+\mu\bar\Omega_p)\wedge\Omega_q = \lambda(\Omega_p\wedge\Omega_q)+\mu(\bar\Omega_p\wedge\Omega_q)$.
   $\Omega_p\wedge(\lambda\Omega_q+\mu\bar\Omega_q) = \lambda(\Omega_p\wedge\Omega_q)+\mu(\Omega_p\wedge\bar\Omega_q)$.
2. $(\Omega_p\wedge\Omega_q)\wedge\Omega_r = \Omega_p\wedge(\Omega_q\wedge\Omega_r)$.
3. $\Omega_p\wedge\Omega_q = (-1)^{pq}\Omega_q\wedge\Omega_p$.
4. $\omega\wedge\omega=0$, $\omega\in E^*$.
5. $f^*(\Omega_p\wedge\Omega_q) = f^*(\Omega_p)\wedge f^*(\Omega_q)$.
6. $\omega_1\wedge\ldots\wedge\omega_p = h(\omega_1\otimes\ldots\otimes\omega_p) = \sum_{\sigma\in S_p}(\operatorname{sgn}\sigma)(\omega_{\sigma(1)}\otimes\ldots\otimes\omega_{\sigma(p)})$.
7. $\dim\Lambda^pE^*=\binom{n}{p}$, *so that* $\Lambda^pE^*=0$ *when* $p>n$. *In fact, if* $\omega_1,\ldots,\omega_n$ *is a base of* E^*, *then the* p*-forms* $\omega_{i_1}\wedge\ldots\wedge\omega_{i_p}$, $i_1<\ldots<i_p$, *form a base of* Λ^pE^*, *and*
   $$\Omega_p = \sum_{i_1\le\ldots\le i_p}\Omega_p(e_{i_1},\ldots,e_{i_p})\,\omega_{i_1}\wedge\ldots\wedge\omega_{i_p}.$$
8. $\omega_1,\ldots,\omega_p\in E^*$ *are linearly dependent if and only if* $\omega_1\wedge\ldots\wedge\omega_p=0$.

9. $e_1, \ldots, e_p \in E$ *are linearly dependent if and only if* $\Omega(e_1, \ldots, e_p) = 0$ *for any* p*-form* Ω.

Proof: Properties 1, 2 and 5 follow from the corresponding properties of the tensor product.

To prove 4, when $\tau = (12)$, we have,

$$\omega \wedge \omega = h(\omega \otimes \omega) = \omega \otimes \omega - \tau(\omega \otimes \omega) = \omega \otimes \omega - \omega \otimes \omega = 0.$$

(3) First we prove the case $p = q = 1$. Given two 1-forms ω, ω',

$$0 = (\omega + \omega') \wedge (\omega + \omega') = \omega \wedge \omega + \omega \wedge \omega' + \omega' \wedge \omega + \omega' \wedge \omega' = \omega \wedge \omega' + \omega' \wedge \omega\,;$$

hence $\omega \wedge \omega' = -\omega' \wedge \omega$. Now the case $\Omega_p = \omega_1 \wedge \ldots \wedge \omega_p$, $\Omega_q = \theta_1 \wedge \ldots \wedge \theta_q$ follows directly, and we may easily conclude the general case.

(6) It follows from 2.10.

(7) The p-forms $\omega_{i_1} \wedge \ldots \wedge \omega_{i_p}$ span $\Lambda^p E^*$ since, by 2.12,

$$\Omega_p = \sum_{i_1 \leq \ldots \leq i_p} \Omega_p(e_{i_1}, \ldots, e_{i_p})\, \omega_{i_1} \wedge \ldots \wedge \omega_{i_p},$$

and they are linearly independent since $(\omega_{i_1} \wedge \ldots \wedge \omega_{i_p})(e_{j_1}, \ldots, e_{j_p}) = 0$, except if $j_1, \ldots, j_p$ is a reordering of $i_1 < \ldots < i_p$.

(8) If $\omega_1, \ldots, \omega_p$ are linearly dependent, someone is a linear combination of the remaining, and property 4 shows that $\omega_1 \wedge \ldots \wedge \omega_p = 0$.

Any linearly independent family $\omega_1, \ldots, \omega_p$ may be extended to a base of E^*. Hence $\omega_1 \wedge \ldots \wedge \omega_p$ is in a base of $\Lambda^p E^*$, and it is non null.

(9) If $e_1, \ldots, e_p$ are linearly dependent, someone is a linear combination of the remaining, and $\Omega(e_1, \ldots, e_p) = 0$ for any alternate p-tensor by definition.

Any linearly independent family $e_1, \ldots, e_p$ may be extended to a base of E. If we consider the dual basis $\omega_1, \ldots, \omega_p, \ldots$, we have that $(\omega_1 \wedge \ldots \wedge \omega_p)(e_1, \ldots, e_p) = 1$.

Definition: The **determinant** of a matrix $A = (a_{ij})$ with n rows and columns is

$$|A| = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\, a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

According to property (6), for any linear forms $\omega_1, \ldots, \omega_p \in E^*$ and vectors $e_1, \ldots, e_p \in E$, we have

$$(\omega_1 \wedge \ldots \wedge \omega_p)(e_1, \ldots, e_p) = |\omega_i(e_j)| = |\omega_j(e_i)|.$$

Moreover if we consider the dual base $\omega_1, \ldots, \omega_n$ of a base of E, then $(\omega_{i_1} \wedge \ldots \wedge \omega_{i_p})(e_1, \ldots, e_p)$ is just the minor of order p formed with the rows $i_1, \ldots, i_p$ of the matrix whose columns are the coordinates of $e_1, \ldots, e_p$ in the fixed base.

Hence, property (9) is just the **Rank Theorem:** *Some columns of a matrix are linearly dependent if and only if all the minors formed with these columns vanish.*

Note: The kernel of $h\colon T^\bullet E \to \Lambda^\bullet E$ is the ideal I generated by the tensors $e \otimes e$.

In fact we have a surjective morphism $T^\bullet E/I \to \Lambda^\bullet E$, since $h(e \otimes e) = 0$. Moreover, in $T^\bullet E/I$ we have $[e] \cdot [e'] = -[e'] \cdot [e]$, because $[e] \cdot [e] = 0$. If we fix a base, the products $[e_{i_1}] \ldots [e_{i_p}]$, $i_1 < \ldots < i_p$, span $T^\bullet E/I$. Hence $T^\bullet E/I \to \Lambda^\bullet E$ is an isomorphism: $I = \operatorname{Ker} h$.

Definition: If $\Omega_p \in \Lambda^p E^*$, $e \in E$, the **interior contraction** is the $(p-1)$-form

$$(i_e \Omega_p)(e_2, \ldots, e_p) = \Omega_p(e, e_2, \ldots, e_p).$$

Theorem: $i_e(\Omega_p \wedge \Omega_q) = (i_e\Omega_p) \wedge \Omega_q + (-1)^p \Omega_p \wedge (i_e\Omega_q)$.

Proof: Let us fix a base $e = e_1, \ldots, e_n$ and the dual base $\omega_1, \ldots, \omega_n$.

We may assume that $\Omega_p = \omega_{i_1} \wedge \ldots \wedge \omega_{i_p} = \omega_{i_1} \wedge \omega_I$, and $\Omega_q = \omega_{j_1} \wedge \ldots \wedge \omega_{j_q} = \omega_{j_1} \wedge \omega_J$.

When $i_1 > 1$ and $j_1 > 1$, both terms are 0. When $i_1 = 1$ and $j_1 > 1$,

$$i_e(\Omega_p \wedge \Omega_q) = i_e(\omega_1 \wedge \omega_I \wedge \Omega_q) = \omega_I \wedge \Omega_q,$$
$$(i_e\Omega_p) \wedge \Omega_q + (-1)^p \Omega_p \wedge (i_e\Omega_q) = \omega_I \wedge \Omega_q + (-1)^p \Omega_p \wedge 0 = \omega_I \wedge \Omega_q,$$

and analogously when $i_1 > 1$ and $j_1 = 1$. When $i_1 = 1$ and $j_1 = 1$,

$$i_e(\Omega_p \wedge \Omega_q) = i_e(\omega_1 \wedge \omega_I \wedge \omega_1 \wedge \omega_J) = i_e(0) = 0,$$
$$(i_e\Omega_p) \wedge \Omega_q + (-1)^p \Omega_p \wedge (i_e\Omega_q) = \omega_I \wedge \Omega_q + (-1)^p \omega_1 \wedge \omega_I \wedge \omega_J = \omega_I \wedge \Omega_q - \omega_I \wedge \Omega_q = 0.$$

Universal Property: *For any alternate multilinear map $H\colon E \times \overset{p}{\ldots} \times E \to F$ there exists a unique linear map $f\colon \Lambda^p E \to F$ such that*

$$f(e_1 \wedge \ldots \wedge e_p) = H(e_1, \ldots, e_p). \tag{2.13}$$

Proof: We fix a base $v_1, \ldots, v_n$ of E, and we define a linear map $f\colon \Lambda^p E \to F$ imposing 2.13 on the corresponding base $\{v_{i_1} \wedge \ldots \wedge v_{i_p}\}$ of $\Lambda^p E$. Now both terms in 2.13 are multilinear maps $E \times \overset{p}{\ldots} \times E \to F$ coinciding on the sequences of vectors of a base; hence both coincide.

Uniqueness is clear, since the products $e_1 \wedge \ldots \wedge e_p$ span $\Lambda^p E$.

Corollary: *We have canonical isomorphisms* $(\Lambda^p E)^* = \Lambda^p E^*$,

$$(\omega_1 \wedge \ldots \wedge \omega_p)(e_1 \wedge \ldots \wedge e_p) = (\omega_1 \wedge \ldots \wedge \omega_p)(e_1, \ldots, e_p).$$

Proof: Just put $F = k$ in the above universal property.

2.7.2 Volume Forms and Characteristic Polynomial

Definition: Since $\dim \Lambda^n E^* = 1$, $n = \dim E$, any endomorphism $\Lambda^n E^* \to \Lambda^n E^*$ is a homothety.

Hence any endomorphism $T\colon E \to E$ induces an endomorphism $T^*\colon \Lambda^n E^* \to \Lambda^n E^*$, the product by some scalar $\det(T)$, named **determinant** of T.

Proposition: *The determinant of an endomorphism T coincides with the determinant of the matrix $A = (a_{ij})$ in any base,* $\det(T) = |A|$.

Proof: $T^*(\omega_1 \wedge \ldots \wedge \omega_n) = (T^*\omega_1) \wedge \ldots \wedge (T^*\omega_n)$, and $T^*(\omega_i) = \sum_j a_{ij}\omega_j$.

Theorem: $\det(TS) = (\det T)(\det S)$.

Proof: $(TS)^* = S^*T^*$.

Lemma: *Let $n = \dim E$ and $0 \neq \Omega_E \in \Lambda^n E^*$. Some vectors $e_1, \ldots, e_n$ form a base of E if and only if* $\Omega_E(e_1, \ldots, e_n) \neq 0$.

Proof: If $e_1, \ldots, e_n$ form a base of E, the coordinate of Ω_E in the corresponding base of $\Lambda^n E^*$ is $\Omega_E(e_1, \ldots, e_n)$, and it is not 0 since $\Omega_E \neq 0$.

If $e_1, \ldots, e_n$ are linearly dependent, someone is a linear combination of the remaining; hence $\Omega_E(e_1, \ldots, e_n) = 0$, because Ω_E is alternate.

Theorem: *An endomorphism T is invertible if and only if* $\det(T) \neq 0$.

Proof: Fix a base $e_1, \dots, e_n$ of E, and a non-zero n-form Ω_E.

$$\Omega_E\big(T(e_1), \dots, T(e_n)\big) = (T^*\Omega_E)(e_1, \dots, e_n) = (\det T)\,\Omega_E(e_1, \dots, e_n). \tag{2.14}$$

Hence $T(e_1), \dots, T(e_n)$ is a base if and only if $\det(T) \neq 0$.

Definitions: Let E be a real vector space of dimension n. A non null n-form Ω_E is a **volume form**, and $\Omega_E(e_1, \dots, e_n)$ is the **volume** (with sign) of the parallelepiped determined by $e_1, \dots, e_n$. Two volume forms Ω_E, Ω'_E define the same **orientation** of E when $\Omega'_E = \lambda\Omega_E$ for some $\lambda > 0$. Equivalence classes are orientations of E. Once we fix an orientation $[\Omega_E]$, a base $e_1, \dots, e_n$ of E is **direct** if $\Omega_E(e_1, \dots, e_n) > 0$. By 2.14,

$$\begin{bmatrix} \text{Volume of} \\ Te_1, \dots, Te_n \end{bmatrix} = (\det T) \begin{bmatrix} \text{Volume of} \\ e_1, \dots, e_n \end{bmatrix}$$

If we fix a scalar product in E, the polarity $\phi\colon E \to E^*$ induces isomorphisms

$$\phi\colon \Lambda^p E^* \longrightarrow \Lambda^p E = (\Lambda^p E^*)^*,$$

hence a scalar product in $\Lambda^p E^*$: $\Omega_p \cdot \Omega'_p = (\phi\Omega_p)(\Omega'_p)$.

If $\omega_1, \dots, \omega_n$ is the dual base of an orthonormal base $e_1, \dots, e_n$, the p-forms $\omega_{i_1} \wedge \dots \wedge \omega_{i_p}$ define an orthonormal base of $\Lambda^p E^*$:

$$\phi(\omega_{i_1} \wedge \dots \wedge \omega_{i_p})(\omega_{j_1} \wedge \dots \wedge \omega_{j_p}) = (e_{i_1} \wedge \dots \wedge e_{i_p})(\omega_{j_1} \wedge \dots \wedge \omega_{j_p}) = \delta_{i_1 j_1} \dots \delta_{i_p j_p}.$$

Theorem: *Let E be an Euclidean oriented vector space E. There is a unique volume form Ω_E such that the volume of any direct orthonormal base is 1.*

Proof: Since $\dim \Lambda^n E^* = 1$, the orientation has a unique volume form Ω_E of module 1 and, if $e_1, \dots, e_n$ is a direct orthonormal base, we have $\Omega_E = \omega_1 \wedge \dots \wedge \omega_n$; hence $\Omega_E(e_1, \dots, e_n) = 1$.

Definition: Once we fix a base $e_1, \dots, e_n$ of E, any endomorphism T has a $n \times n$ matrix $A = (a_{ij})$,

$$T(e_j) = \sum_{i=1}^{n} a_{ij} e_i \;;\; j = 1, \dots, n,$$

and the matrix A' of T in another base of E is (p. 54)

$$A' = B^{-1}AB. \tag{2.15}$$

where B is the base change matrix. Hence, if I is the unit $n \times n$ matrix, using properties of the determinants (that we shall prove in the next section) we may see that the polynomial $c_T(x) = |xI - A|$ is intrinsic: it does not depend on the base

$$\begin{aligned} |xI - A'| &= |xB^{-1}IB - B^{-1}AB| = |B^{-1}(xI - A)B| \\ &= |B^{-1}| \cdot |xI - A| \cdot |B| = |B|^{-1}|B| \cdot |xI - A| = |xI - A| \end{aligned}$$

The **characteristic** polynomial $c_T(x)$ of the endomorphism T is defined to be

$$c_T(x) = \begin{vmatrix} x - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & x - a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & x - a_{nn} \end{vmatrix} = x^n - \Big(\sum_{i=1}^{n} a_{ii}\Big)x^{n-1} + \dots + (-1)^n |A|.$$

Theorem: *The roots in k of the characteristic polynomial are just the eigenvalues of T.*

Proof: A scalar α is an eigenvalue of T if and only if

$$0 \neq \dim(\operatorname{Ker}(\alpha \mathrm{Id} - T)) = n - \operatorname{rk}(\alpha I - A);$$

or equivalently $\operatorname{rk}(\alpha I - A) < n$; that is to say, $c_T(\alpha) = |\alpha I - A| = 0$.

Corollary: *All the roots of the characteristic polynomial of a selfadjoint endomorphism are real.*

Proof: In the complex case, any root of $|xI - A|$, being an eigenvalue, is real by the Spectral theorem; hence also in the real case, since any real symmetric matrix A is a complex hermitian matrix.

Diagonalization Criterion: *An endomorphism T is diagonalizable if and only if all the roots of the characteristic polynomial $c_T(x)$ are in k and the multiplicity m_i of any root α_i is*

$$m_i = \dim\big(\operatorname{Ker}(T - \alpha_i \mathrm{Id})\big).$$

Proof: If T is diagonalizable, in some base the matrix is diagonal

$$D = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix}$$

and any root of $c_T(x) = |xI - D| = (x - \alpha_1)\ldots(x - \alpha_n)$ is in k.

The multiplicity m_i of any root α_i is the number of repetitions in $\alpha_1, \ldots, \alpha_n$; hence

$$m_i = n - \operatorname{rk}(D - \alpha_i I) = \dim\big(\operatorname{Ker}(T - \alpha_i \mathrm{Id})\big) = \dim V_{\alpha_i}.$$

Conversely, if all roots of $c_T(x)$ are in k, then $\sum_i m_i = \deg c_T(x) = \dim E$.

Now, if $m_i = \dim V_{\alpha_i}$, then $\dim\big(\sum_i V_{\alpha_i}\big) = \sum_i \dim V_{\alpha_i} = \dim E$, and $\sum_i V_{\alpha_i} = E$.

Hamilton-Cayley Theorem: *The characteristic polinomial $c(x) = x^n + \ldots + c_1 x + c_0$ annihilates the endomorphism, $c(T) = T^n + \ldots + c_1 T + c_0 \mathrm{Id} = 0$.*

Proof: It is obvious when $n = 1$, since $T = a\mathrm{Id}$, and $c(x) = x - a$.

If $n > 1$, then the matrix of $c(T)$ is $c(A) = A^n + \ldots + c_0 I$ and, to prove that $c(A) = 0$, we may consider an extension of k.

By Kronecker's theorem, we may assume that $c(x)$ has some root $\alpha \in k$.

Let $e_1, \ldots, e_n$ be a base of E, where $T(e_1) = \alpha e_1$,

$$A = \begin{pmatrix} \alpha & \ldots \\ 0 & \bar{A} \end{pmatrix} \quad , \quad A^r = \begin{pmatrix} \alpha^r & \ldots \\ 0 & \bar{A}^r \end{pmatrix}.$$

Now $c(x) = (x - \alpha)\bar{c}(x)$, where $\bar{c}(x) = |xI - \bar{A}|$ and, by induction, $\bar{c}(\bar{A}) = 0$.

$$c(A) = (A - \alpha I)\bar{c}(A) = \begin{pmatrix} 0 & \ldots \\ 0 & B \end{pmatrix} \begin{pmatrix} \bar{c}(\alpha) & \ldots \\ 0 & \bar{c}(\bar{A}) \end{pmatrix} = \begin{pmatrix} 0 & \ldots \\ 0 & B \end{pmatrix} \begin{pmatrix} \bar{c}(\alpha) & \ldots \\ 0 & 0 \end{pmatrix} = 0.$$

Chapter 3

Algebra I

3.1 The Quotient Ring

Definitions: An ideal $\mathfrak{m}$ of a ring A is **maximal** when there are just two ideals containing it, namely $\mathfrak{m}$ and $A \neq \mathfrak{m}$.

An ideal $\mathfrak{p}$ is **prime** when for any finite product $\prod_{i \in I} a_i \in \mathfrak{p}$ we have $a_i \in \mathfrak{p}$ for some index $i \in I$ (when $I = \emptyset$, we see that $1 \notin \mathfrak{p}$).

The **sum** of ideals $\mathfrak{a} + \mathfrak{b} = \{a + b\colon\ a \in \mathfrak{a},\ b \in \mathfrak{b}\}$ is the minimal ideal containing them.

The ideal **generated** by $a_1, \dots, a_n \in A$ is the ideal $(a_1, \dots, a_n) = a_1A + \ldots + a_nA$.

An ideal $\mathfrak{a}$ is **principal** if it is generated by some element, $\mathfrak{a} = aA$.

A **principal ideal domain** is a domain where any ideal is principal.

The **product** of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is the ideal

$$\mathfrak{ab} = \{a_1b_1 + \ldots + a_nb_n;\ a_1, \dots, a_n \in \mathfrak{a},\ b_1, \dots, b_n \in \mathfrak{b}\}$$

Theorem: *An ideal $\mathfrak{a}$ of a ring A is prime $\Leftrightarrow$ $A/\mathfrak{a}$ is an integral ring.*
An ideal $\mathfrak{a}$ of a ring A is maximal $\Leftrightarrow$ $A/\mathfrak{a}$ is a field.

Proof: If $\mathfrak{a}$ is prime and $[a] \cdot [b] = 0$, then $ab \in \mathfrak{a}$, and $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$; hence, $[a] = 0$ or $[b] = 0$.

Conversely, if $A/\mathfrak{a}$ is integral and $ab \in \mathfrak{a}$, then $[a] \cdot [b] = [ab] = 0$, and $[a] = 0$ or $[b] = 0$; hence, $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$.

If $\mathfrak{a}$ is maximal and $[a] \neq 0$, then the inclusion $\mathfrak{a} \subset \mathfrak{a} + aA$ is strict; hence $A = \mathfrak{a} + aA$, and $1 = x + ab$, where $x \in \mathfrak{a}$, $b \in A$. Therefore $[a] \cdot [b] = 1$ and $[a]$ is invertible in $A/\mathfrak{a}$.

Conversely, if $A/\mathfrak{a}$ is a field and $\mathfrak{a} \subset \mathfrak{b}$, we take $b \in \mathfrak{b}$ not in $\mathfrak{a}$. Since $[b] \neq 0$ in $A/\mathfrak{a}$, there exists $[a] \in A/\mathfrak{a}$ such that $[a][b] = 1$. Hence $1 \in ab + \mathfrak{a} \subseteq \mathfrak{b}$, and $\mathfrak{b} = A$.

Corollary: *Any maximal ideal is a prime ideal.*

Proof: Any field is a domain.

Euclid's Lemma: *If a prime number p divides a product, it divides some factor.*

Proof: Since $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, it is a prime ideal.

Theorem: *If $p\colon A \to \bar{A}$ is a surjective ring morphism, we have a lattice isomorphism*

$$(I := \operatorname{Ker} p) \qquad \begin{bmatrix}\text{Ideals} \\ \text{of } \bar{A}\end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix}\text{Ideals of } A \\ \text{containing } I\end{bmatrix},\ \bar{\mathfrak{a}} \mapsto p^{-1}(\bar{\mathfrak{a}}),$$

the inverse map being $\mathfrak{a} \mapsto p(\mathfrak{a})$*. If* $\bar{\mathfrak{a}} = p(\mathfrak{a})$*, we have a natural ring isomorphism* $A/\mathfrak{a} \simeq \bar{A}/\bar{\mathfrak{a}}$*; hence* $\mathfrak{a}$ *is a maximal (resp. prime) ideal if and only if so is* $\bar{\mathfrak{a}}$*.*

Proof: We have $p(p^{-1}\bar{\mathfrak{a}}) = \bar{\mathfrak{a}}$ because p is surjective and, when $I \subseteq \mathfrak{a}$, we also have

$$p^{-1}\big(p(\mathfrak{a})\big) = \mathfrak{a} + \operatorname{Ker} p = \mathfrak{a},$$

so that the kernel of the epimorphism $A \xrightarrow{p} \bar{A} \xrightarrow{\pi} \bar{A}/\bar{\mathfrak{a}}$ is just $\mathfrak{a}$; hence $A/\mathfrak{a} \simeq \bar{A}/\bar{\mathfrak{a}}$.

Chinese Remainder Theorem: *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of a ring* A*. If* $\mathfrak{a} + \mathfrak{b} = A$*, then* $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$*, and we have a natural ring isomorphism*

$$\phi\colon A/\mathfrak{a} \cap \mathfrak{b} \xrightarrow{\sim} (A/\mathfrak{a}) \times (A/\mathfrak{b}),\ \ \phi([x]_{\mathfrak{a}\mathfrak{b}}) = ([x]_{\mathfrak{a}}, [x]_{\mathfrak{b}}).$$

Proof: Let $1 = a + b \in \mathfrak{a} + \mathfrak{b}$. If $c \in \mathfrak{a} \cap \mathfrak{b}$, then $c = c(a+b) = ca + cb \in \mathfrak{a}\mathfrak{b}$.

Hence $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, since the inclusion $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ always holds.

Moreover, the kernel of the morphism $f\colon A \to (A/\mathfrak{a}) \times (A/\mathfrak{b})$, $f(x) = ([x]_{\mathfrak{a}}, [x]_{\mathfrak{b}})$, is $\mathfrak{a} \cap \mathfrak{b}$, and f is surjective because $f(bx + ay) = ([x]_{\mathfrak{a}}, [y]_{\mathfrak{b}})$,

$$\begin{aligned} x = (a+b)x \equiv bx \equiv bx + ay \quad (\text{mod. } \mathfrak{a}) \\ y = (a+b)y \equiv ay \equiv bx + ay \quad (\text{mod. } \mathfrak{b}) \end{aligned}$$

Corollary: $\mathbb{Z}/mn\mathbb{Z} = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$*, when* m *and* n *are coprime.*

Definition: The **Euler indicator** $\phi(n)$ is the number of integers between 1 and n which are prime to n; i.e. the number of generators of any cyclic group of order n, and the order of the group of units $(\mathbb{Z}/n\mathbb{Z})^*$ of the ring $\mathbb{Z}/n\mathbb{Z}$. In fact, a class $[m]$ is invertible, $[1] = [a] \cdot [m] = a[m]$, if and only if it generates the group $\mathbb{Z}/n\mathbb{Z}$.

Proposition: $\phi(p^r) = (p-1)p^{r-1}$, when p is prime.
$\phi(n \cdot m) = \phi(n) \cdot \phi(m)$, when n and m are coprime.

Proof: The integers between 1 and p^r which are not prime to p^r are just $p, 2p, \ldots, p^{r-1}p$.

With respect to the second formula, by the chinese remainder theorem

$$(\mathbb{Z}/nm\mathbb{Z})^* = (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z})^* = (\mathbb{Z}/n\mathbb{Z})^* \times (\mathbb{Z}/m\mathbb{Z})^*.$$

Euler's Congruence: $a^{\phi(n)} \equiv 1$ (mod. n)*, when* a, n *are coprime.*

Proof: If $[a] \in (\mathbb{Z}/n\mathbb{Z})^*$, then $[1] = [a]^{\phi(n)} = [a^{\phi(n)}]$, (p. 46).

Fermat's Congruence: $n^{p-1} \equiv 1$ (mod. p)*, when a prime* p *does not divide* n*.*

Corollary: $n^p \equiv n$ (mod. p)*, when* p *is prime.*

Wilson's congruence: $(p-1)! \equiv -1$ (mod. p) *when* p *is a prime number.*

Proof: $x^{p-1} - 1 = (x-1)(x-2)\ldots(x-(p-1))$ since $1, \ldots, p-1$ are roots of $x^{p-1} - 1$ in $\mathbb{F}_p$.

Definition: If $a \in \mathbb{Z}$ is not a multiple of an odd prime p, the **Legendre's symbol** $\left(\frac{a}{p}\right)$ is 1 when $\bar{a} \in \mathbb{F}_p^*$ is a square, and -1 otherwise.

Corollary: *If p is odd, then $\mathbb{F}_p^{*2}$ is a subgroup of $\mathbb{F}_p^*$ of order $\frac{p-1}{2}$, and* $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

Proof: The kernel of the group morphism $f\colon \mathbb{F}_p^* \to \mathbb{F}_p^*$, $f(x) = x^2$, is $\{\pm 1\}$ because $x^2 - 1 = (x+1)(x-1)$; and $-1 \neq 1$ since $p \neq 2$.

By the isomorphism theorem, the image $\mathbb{F}_p^{*2}$ of f has order $\frac{p-1}{2}$.

Now, $\mathbb{F}_p^*/\mathbb{F}_p^{*2} \simeq \{\pm 1\}$ since it is a group of order 2, and the Legendre's symbol is just the canonical projection $\pi\colon \mathbb{F}_p^* \to \mathbb{F}_p^*/\mathbb{F}_p^{*2} \simeq \{\pm 1\}$; hence it is a group morphism.

Corollary: $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}}$ (mod. p).

Proof: If $a \in \mathbb{F}_p^*$, then $a^{\frac{p-1}{2}} = \pm 1$ because $(a^{\frac{p-1}{2}})^2 = a^{p-1} = 1$.

If a is a square, $a = b^2$, then $a^{\frac{p-1}{2}} = b^{p-1} = 1$. Since the number of roots of $x^{\frac{p-1}{2}} - 1$ is bounded by the degree, we see that $a^{\frac{p-1}{2}} = 1$ if and only if a is a square in $\mathbb{F}_p$.

Corollary: -1 *is a quadratic residue modulo $p \neq 2$ if and only if $p \equiv 1$ (mod. 4).*

Gauss Lemma: *If $P \in \mathbb{Z}[x]$ is a product of polynomials with rational coefficients, $P = Q_1 Q_2$, multiplying both factors by some constants we have a decomposition $P = Q_1' Q_2'$ in $\mathbb{Z}[x]$.*

Proof: We may assume that the coefficients of Q_1 and Q_2 have a common denominator

$$P = \frac{1}{a}(a_0 x^n + \ldots + a_n) \cdot \frac{1}{b}(b_0 x^m + \ldots + b_m)$$
$$abP = (a_0 x^n + \ldots + a_n)(b_0 x^m + \ldots + b_m)$$

where $a, a_0, \ldots, a_n, b, b_0, \ldots, b_m \in \mathbb{Z}$.

If a prime p divides ab, by the following lemma p divides some factor of the second term.

So eliminating all the prime factors of ab, we get a decomposition in $\mathbb{Z}[x]$,

$$P = (a_0' x^n + \ldots + a_n')(b_0' x^m + \ldots + b_m').$$

Lemma: *If $\mathfrak{p}$ is a prime ideal of a ring A, then $\mathfrak{p}[x]$ is a prime ideal of $A[x]$*

Proof: The ideal $\mathfrak{p}[x]$ is the kernel of the surjective ring morphism

$$\phi\colon A[x] \longrightarrow (A/\mathfrak{p})[x],\ \phi\big(\textstyle\sum_i a_i x^i\big) = \sum_i \bar{a}_i x^i.$$

Hence $A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})[x]$ is a domain, and $\mathfrak{p}[x]$ is a prime ideal.

Corollary: *If $P \in \mathbb{Z}[x]$ is irreducible of degree ≥ 1, then $P(x)$ is irreducible in $\mathbb{Q}[x]$.*

Note: Given $P \in \mathbb{Z}[x]$, let us fix different integers $a_0, a_1, \ldots, a_d$. If $Q \in \mathbb{Z}[x]$ divides P, then $Q(a_i)$ divides $P(a_i)$. Since any integer has a finite number of divisors, interpolating (p. 49) we obtain all the possible divisors of $P(x)$ in $\mathbb{Z}[x]$ of a given degree d. So we may obtain the irreducible factor decomposition of P in $\mathbb{Z}[x]$; hence in $\mathbb{Q}[x]$ by Gauss lemma.

Reduction Criterion: *Let $Q = c_0 x^n + \ldots + c_n \in \mathbb{Z}[x]$, and let p be a prime not dividing c_0. If Q has a factor of degree d in $\mathbb{Z}[x]$, then the reduction $\bar{Q} = \bar{c}_0 x^n + \ldots + \bar{c}_n \in \mathbb{F}_p[x]$ has a factor of degree d. Hence, if $\bar{Q}$ is irreducible in $\mathbb{F}_p[x]$, then so is Q in $\mathbb{Q}[x]$.*

Proof: We have $\deg \bar{Q} = \deg Q$, since $\bar{c}_0 \neq 0$.

If we have a decomposition $Q = AB$ in $\mathbb{Z}[x]$, then $\bar{Q} = \bar{A}\bar{B}$ in $\mathbb{F}_p[x]$ and

$$\deg \bar{A} + \deg \bar{B} = \deg \bar{Q} = \deg Q = \deg A + \deg B.$$

Since $\deg \bar{A} \leq \deg A$, $\deg \bar{B} \leq \deg B$, then $\deg \bar{A} = \deg A$, and $\deg \bar{B} = \deg B$.
Finally, if $\bar{Q}$ is irreducible, then Q has no factor of degree $1, \ldots, n-1$ in $\mathbb{Z}[x]$.
Hence Q has no factor of degree $1, \ldots, n-1$ in $\mathbb{Q}[x]$ by Gauss lemma.

Eisenstein's Criterion: *A polynomial $Q(x) = c_0 x^n + \ldots + c_n \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$ when there is a prime p not dividing c_0 such that p divides $c_1, \ldots, c_n$ and p^2 does not divide c_n.*

Proof: By Gauss lemma, if Q is not irreducible, it is a product of non constant polynomials with integer coefficients, and reducing modulo p we have

$$\begin{aligned} Q &= (a_0 + a_1 x + \ldots + a_r x^r)(b_0 + b_1 x + \ldots + b_{n-r} x^{n-r}), \\ \bar{c}_0 x^n &= (\bar{a}_0 + \bar{a}_1 x + \ldots + \bar{a}_r x^r)(\bar{b}_0 + \bar{b}_1 x + \ldots + \bar{b}_{n-r} x^{n-r}), \\ \bar{c}_0 x^n &= (\bar{a}_r x^r)(\bar{b}_{n-r} x^{n-r}). \end{aligned}$$

Hence $\bar{a}_0 = \bar{b}_0 = 0$, and $c_n = a_0 b_0$ is a multiple of p^2, a contradiction.

Corollary: *The polynomial $x^{p-1} + \ldots + x + 1$ is irreducible in $\mathbb{Q}[x]$ when p is prime.*

Proof: $\frac{x^p - 1}{x-1} = x^{p-1} + \ldots + x + 1$ is irreducible if and only if so is

$$\frac{(x+1)^p - 1}{x} = x^{p-1} + \binom{p}{1} x^{p-2} + \ldots + \binom{p}{i} x^{p-i-1} + \ldots + \binom{p}{p-1}.$$

The number $\binom{p}{i} = \frac{p(p-1)\ldots(p-i+1)}{i!}$ is a multiple of p when $1 \leq i \leq p-1$, and $\binom{p}{p-1} = p$ is not a multiple of p^2. Hence it is irreducible by the Eisenstein's criterion.

3.2 Principal Ideal Domains

Definitions: Let A be a principal ideal domain. If $a, b \in A$, then we have $aA + bA = dA$ and $aA \cap bA = mA$ for some $d, m \in A$, well-defined up to invertible factors. Then d is the **greatest common divisor** of a and b (a common divisor which is a multiple of any other common divisor), and m is the **least common multiple**, (a common multiple dividing any other common multiple). The equality $dA = aA + bA$ proves

Bézout's Identity: *$d = \alpha a + \beta b$, for some $\alpha, \beta \in A$.*

Euclid's Lemma: *Let $0 \neq p \in A$. The following conditions are equivalent:*

1. *p is irreducible.*
2. *pA is a maximal ideal.*
3. *pA is a prime ideal.*

Proof: ($1 \Rightarrow 2$) If p is irreducible and $pA \subseteq bA$, then $p = bc$; hence b or c is invertible in A. If so is b, then $bA = A$; and if so is c, then $bA = bcA = pA$.

($2 \Rightarrow 3$) Any maximal ideal is prime (p. 71).

($3 \Rightarrow 1$) If pA is prime, and $ab = p \in pA$, then $a \in pA$ or $b \in pA$. If $a \in pA$, then

$$a = pc, \ p = ab = pcb, \ bc = 1,$$

and b is invertible. Analogously, a is invertible if $b \in pA$. Hence p is irreducible.

Decomposition Theorem: *Any proper element of a principal ideal domain A decomposes, uniquely up to the order and units, as a product of irreducible elements, $a = p_1 \dots p_r$, $r \geq 1$.*

Proof: First we prove the uniqueness. If $a = p_1 \dots p_r$, then the number of factors coinciding, up to units, with a given factor p is the greatest exponent n such that p^n divides a.

In fact, if $p^n|a$, by Euclid's lemma p divides some factor p_i; hence p coincides with p_i up to a unit and p^{n-1} divides $p_1 \dots \widehat{p}_i \dots p_r$. Iterating we see that n factors coincide with p.

Now we prove the existence. If a proper element $a \in A$ does not decompose as a product of irreducible elements, then we have $a = bc$, where $aA \subset bA$, $aA \subset cA$, and some factor is proper and does not decompose as a product of irreducible elements.

So we obtain an infinite increasing sequence of ideals of A, against the following

Lemma: *Any chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ of A stabilizes, $I_n = I_{n+1} = \dots$.*

Proof: If $a, b \in \cup_i I_i$, then $a, b \in I_i$ for some index i, so that $a + b, xa \in I_i \subseteq \cup_i I_i$, $\forall x \in A$, and we see that $\cup_i I_i$ is an ideal of A. Let c be a generator of $\cup_i I_i$.

Now $c \in I_n$ for some index n, and $I_n \subseteq I_{n+j} \subseteq \cup_i I_i = cA \subseteq I_n$; hence $I_n = I_{n+j}$, $\forall j > 0$.

Definition: A domain A is **Euclidean** if there is a map $\delta : A - \{0\} \to \mathbb{N}$ such that

1. If $0 \neq a, b \in A$, then $b = ac + r$ for some $c, r \in A$, where $\delta(r) < \delta(a)$ or $r = 0$.
2. $\delta(a) \leq \delta(ab)$ for any non null $a, b \in A$.

Theorem: *Any ideal $\mathfrak{a}$ of an Euclidean ring A is principal, $\mathfrak{a} = aA$.*

Proof: If $\mathfrak{a} = 0$, we take $a = 0$. If $\mathfrak{a} \neq 0$, we take $a \in \mathfrak{a}$ with $\delta(a)$ minimal, and we have $aA \subseteq \mathfrak{a}$ because $a \in \mathfrak{a}$. Now, if $b \in \mathfrak{a}$, dividing b by a

$$b = ac + r, \quad \text{where } \delta(r) < \delta(a) \text{ or } r = 0,$$
$$r = b - ac \in \mathfrak{a};$$

we see that $r = 0$, by the choice of a. Hence $b = ac \in aA$, and $\mathfrak{a} = aA$.

Euclid's Algorithm: *If $a = cb + r$, then* g.c.d.$(a, b) =$ g.c.d.(b, r).

Proof: $(a, b) \subseteq (b, r)$ since $a = cb + r$, and $(b, r) \subseteq (a, b)$ since $r = a - cb$. q.e.d.

1. The ring $\mathbb{Z}$ is Euclidean, with $\delta(n) = |n|$.
2. If k is a field, the ring $k[x]$ is Euclidean, with $\delta(P) = \deg P$. Hence, any non null ideal $\mathfrak{a}$ is generated by a **unitary** polynomial $x^n + a_1 x^{n-1} + \dots + a_n$, clearly unique.

3. The ring of **gaussian integers** $A = \mathbb{Z}[\,i\,] = \mathbb{Z} + \mathbb{Z}i$ is Euclidean, with $\delta(z) = z \cdot \bar{z} = |z|^2$.

   In fact, for any complex number $u + vi$ there is $x + yi \in A$ such that $|(u + vi) - (x + yi)| < 1$ (just take $|u - x|$, $|v - y| \leq 1/2$). Hence, if $a, b \in A$, $a \neq 0$, there is $c \in A$ such that $|\frac{b}{a} - c| < 1$; so that $r = b - ac \in A$, and $|r| = |a\big(\frac{b}{a} - c\big)| < |a|$.

4. In a Euclidean ring, iterated divisions let us calculate d =g.c.d.(a, b) as the last non null remainder, and the coefficients of Bézout's Identity. In fact, if $d = \alpha r + \beta b$, then

$$d = \alpha r + \beta b = \alpha(a - cb) + \beta b = \alpha a + (\beta - c\alpha)b.$$

5. The Diophantine equation $ax + by = c$ has integer solution if and only if $c \in a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}$, and in such a case, if $d = \alpha a + \beta b$, a solution is $x_o = \alpha c/d$, $y_o = \beta c/d$.

3.2.1 Simple Fractions

Lemma: $\dfrac{P}{Q_1 \dots Q_r} = \dfrac{B_1}{Q_1} + \cdots + \dfrac{B_r}{Q_r}$ *when $Q_1, \dots, Q_r \in k[x]$ are pairwise coprime.*

Proof: Put $Q = Q_2 \dots Q_r$. By Bézout's identity $1 = AQ_1 + BQ$, where $A, B \in k[x]$.
Hence $P = PAQ_1 + PBQ$, and we conclude by induction on r,

$$\frac{P}{Q_1 Q} = \frac{PBQ}{Q_1 Q} + \frac{PAQ_1}{Q_1 Q} = \frac{B_1}{Q_1} + \frac{PA}{Q_2 \dots Q_r} = \frac{B_1}{Q_1} + \frac{B_2}{Q_2} + \cdots + \frac{B_r}{Q_r}.$$

Lemma: *Let $Q \in k[x]$ be of degree $d \geq 1$. If $B \in k[x]$, then there exist $A_0, \dots, A_n \in k[x]$, of degree $< d$, such that $B = A_0 + A_1 Q + A_2 Q^2 + \ldots + A_n Q^n$.*

Proof: Let $C, A_0 \in k[x]$ such that $B = A_0 + QC$, $\deg A_0 < d$. By induction on the degree, there are $A_1, A_2, \ldots \in k[x]$ of degree $< d$ such that $B = A_0 + Q(A_1 + A_2 Q + A_3 Q^2 + \ldots)$.

Definition: A rational fraction is **simple** if it is a monomial ax^n or $\frac{P}{Q^n}$, where Q is irreducible and $\deg P < \deg Q$.

Theorem: *Any rational fraction $\frac{P(x)}{Q(x)}$ with coefficients in a field k decomposes, uniquely up to the order, as a sum of simple fractions.*

Proof: If $Q = Q_1^{n_1} \dots Q_r^{n_r}$ is the irreducible factor decomposition, then

$$\frac{P}{Q} = \frac{B_1}{Q_1^{n_1}} + \ldots + \frac{B_r}{Q_r^{n_r}}$$

and there are polynomials A_{i0}, A_{i1}, $\ldots \in k[x]$, $\deg A_{ij} < \deg Q_i$, such that

$$\frac{B_i}{Q_i^{n_i}} = \frac{A_{i0} + A_{i1} Q_i + A_{i2} Q_i^2 + \ldots}{Q_i^{n_i}} = \sum_{j=0}^{n_i - 1} \frac{A_{ij}}{Q_i^{n_i - j}} + \text{Polynomial.}$$

Now we prove the uniqueness. Given two decompositions

$$\frac{A}{Q_1^{n_1}} + \ldots = \frac{B}{Q_1^{n_1}} + \ldots, \; A \neq 0,$$

where $Q_1^{n_1}$ is the greatest power of Q_1 in both decomposotions (eventually $B = 0$), multiplying by the greatest common multiple of all denominators we obtain an equality

$$(A - B) Q_2^{n_2} \dots Q_r^{n_r} = Q_1 C.$$

Hence Q_1 divides $A - B$ and, since $\deg(A - B) < \deg Q_1$, we have $A = B$.
We conclude by induction on the number of terms of a decomposition.

3.3 Extensions and Roots

Definitions: Let k be a field. A k-**algebra** is a ring A endowed with a ring morphism $j\colon k \to A$, so that A is a k-vector space, $\lambda a := j(\lambda)a$, and we identify λ with $j(\lambda)$.

A **morphism** of k-algebras $f\colon A \to B$ is a ring morphisms such that $f(\lambda) = \lambda$, $\lambda \in k$, and it is an **isomorphism** if it is bijective.

The dimension of A as a k-vector space is the **degree** $[A : k]$ of A over k, and A is a **finite** k-algebra if so is the degree.

Definitions: A k-algebra $k \to L$ is an **extension** of k if L is a field, and it is a **finite** extension if so is the degree $[L : k]$. It is a **trivial** extension if $k \xrightarrow{\sim} L$; i.e. $[L : k] = 1$.

The extension **generated** by $\alpha_1, \ldots, \alpha_n \in L$ is

$$k(\alpha_1, \ldots, \alpha_n) = \{a/b\colon\ a, b \in k[\alpha_1, \ldots, \alpha_n],\ b \neq 0\}\,.$$

A **root** of $P = c_0x^n + \ldots + c_n \in k[x]$ is an element α of some extension L of k such that $P(\alpha) = c_0\alpha^n + \ldots + c_n = 0$, and the **multiplicity** is the greatest exponent m such that $(x-\alpha)^m$ divides P in $L[x]$. Let us consider the irreducible factor decomposition of P in $L[x]$:

$$P = c_0(x-\alpha_1)^{m_1} \ldots (x-\alpha_r)^{m_r} Q_1^{n_1} \ldots Q_s^{n_s}$$

where $\deg Q_i > 1$. The roots of P in L are $\alpha_1, \ldots, \alpha_r$, with multiplicities $m_1, \ldots m_r$.

The number of roots in an extension, counted with multiplicities, is bounded by the degree,

$$m_1 + \ldots + m_r \ \leq \deg P(x),$$

and we say that P has all the roots in L if the equality holds. In such a case

$$c_0x^n + c_1x^{n-1} + \ldots + c_{n-1}x + c_n = c_0(x-\alpha_1)\ldots(x-\alpha_n),$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of P in L, repeated with their multiplicities, and identifying coefficients we obtain **Cardano Formulae**:

$$(-1)^r \tfrac{c_r}{c_0} = \sum_{i_1<\ldots<i_r} \alpha_{i_1} \ldots \alpha_{i_r},\ 1 \leq r \leq n.$$

Definition: A polynomial $P(x_1, \ldots, x_n)$ with coefficients in a ring A is said to be **symmetric** when $P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = P(x_1, \ldots, x_n)$, for any permutation $\sigma \in S_n$.

The **elementary symmetric functions** are the polynomials

$$s_r(x_1, \ldots, x_n) = \sum_{i_1<\ldots<i_r} x_{i_1} \ldots x_{i_r}\ , \quad 1 \leq r \leq n.$$

For example, $s_1(x_1, \ldots, x_n) = x_1 + \ldots + x_n$, and $s_n(x_1, \ldots, x_n) = x_1 \ldots x_n$.

Symmetric Functions Theorem: *If $P(x_1, \ldots, x_n)$ is a symmetric polynomial, then there exists a unique polynomial $Q(y_1, \ldots, y_n)$ with coefficients in A such that*

$$P(x_1, \ldots, x_n) = Q(s_1, \ldots, s_n), \quad s_r = s_r(x_1, \ldots, x_n).$$

Proof: We may assume that P is homogeneous of degree d, and we shall prove that $P(x_1, \ldots, x_n) = Q(s_1, \ldots, s_n)$ for some polynomial $Q(y_1, \ldots, y_n)$ homogeneous of degree d when we consider y_i of degree i. We prove the existence by induction on n and d, and it is obvious when $n = 1$ or $d = 0$.

Since $P(x_1,\dots,x_{n-1},0)$ is symmetric, there is a polynomial $\bar{Q}(y_1,\dots,y_{n-1})$, homogeneous of degree d when $\deg y_i = i$, such that

$$P(x_1,\dots,x_{n-1},0)=\bar{Q}(\bar{s}_1,\dots,\bar{s}_{n-1}),\ \ \bar{s}_r = s_r(x_1,\dots,x_{n-1}).$$

Now $P(x_1,\dots,x_n)-\bar{Q}(s_1,\dots,s_{n-1})$ vanishes when we put $x_n=0$, so that it is a multiple of x_n and, being symmetric, a multiple of $s_n=x_1\dots x_n$,

$$P(x_1,\dots,x_n)-\bar{Q}(s_1,\dots,s_{n-1})=s_nP'(x_1,\dots,x_n).$$

Since $P'(x_1,\dots,x_n)$ is symmetric and homogeneous of degree $d-n$, by induction

$$P'(x_1,\dots,x_n)=Q'(s_1,\dots,s_n)$$

for some polynomial $Q'(y_1,\dots,y_n)$, so that $P(x_1,\dots,x_n)=\bar{Q}(s_1,\dots,s_{n-1})+s_nQ'(s_1,\dots,s_n)$.
Uniqueness directly follows from the following theorem:

Theorem: *Let $Q\in A[y_1,\dots,y_n]$. If $Q(s_1,\dots,s_n)=0$, then $Q(y_1,\dots,y_n)=0$.*

Proof: It is clear when $n=1$. If $n\geq 2$ and the theorem is false, we take $Q(y_1,\dots,y_n)\neq 0$ of minimum degree in y_n such that $Q(s_1,\dots,s_n)=0$,

$$Q=Q_0(y_1,\dots,y_{n-1})+Q_1(y_1,\dots,y_{n-1})y_n+\dots+Q_d(y_1,\dots,y_{n-1})y_n^d\,.$$

If we put $y_n=0$ in the identity $Q(s_1,\dots,s_n)=0$, we obtain $Q_0(\bar{s}_1,\dots,\bar{s}_{n-1})=0$.
By induction $Q_0=0$, and

$$Q(y_1,\dots,y_n)=y_n\cdot(Q_1+\dots+Q_dy_n^{d-1})=y_n\cdot R(y_1,\dots,y_n)\,.$$

Hence $R(y_1,\dots,y_n)\neq 0$ and $R(s_1,\dots,s_n)=0$, against the choice of Q.

Definition: The **derivative** of a polynomial $P(x)=a_0+a_1x+\dots+a_nx^n\in k[x]$ is

$$P'(x)=a_1+2a_2x+\dots+na_nx^{n-1},$$

where we put $ia_i=a_i+\overset{i}{\dots}+a_i\in k$. It may be that $\deg P'<n-1$, since $1+\overset{i}{\dots}+1$ may be zero in k. For example, when $k=\mathbb{F}_p$, then the derivative of $P=x^p+1$ is $P'=0$.

The following equalities hold, since they hold when $P=x^i$, $Q=x^j$,

$$(aP+bQ)'=aP'+bQ';\quad a,b\in k$$
$$(P\cdot Q)'=P'\cdot Q+P\cdot Q'.$$

Now, if $P(x)=c_0x^n+\dots+c_n=c_0(x-\alpha_1)\dots(x-\alpha_n)$, we shall determine the sums $\sigma_r=\alpha_1^r+\dots+\alpha_n^r$, and we agree that $\sigma_0=n$.

$$\frac{P'}{P}=\sum_{i=1}^n\frac{1}{x-\alpha_i}=\sum_{i=1}^n\left(\frac{1}{x}+\frac{\alpha_i}{x^2}+\frac{\alpha_i^2}{x^3}+\dots\right)$$

and we obtain **Girard Formula**: $\dfrac{P'}{P}=\dfrac{\sigma_0}{x}+\dfrac{\sigma_1}{x^2}+\dfrac{\sigma_2}{x^3}+\dots$

$$nc_0x^{n-1}+\dots+c_{n-1}=(c_0x^n+\dots+c_n)(\sigma_0x^{-1}+\sigma_1x^{-2}+\sigma_2x^{-3}+\dots)$$

Equating the coefficients of x^{n-r-1}, we obtain

$$(n-r)c_r = \sum_{i+j=r} c_i\sigma_j = \sum_{i=0}^{r} c_i\sigma_{r-i}\ , \quad 1 \le r \le n-1,$$

$$0 = \sum_{i+j=r} c_i\sigma_j = \sum_{i=0}^{n} c_i\sigma_{r-i}\ , \quad r \ge n,$$

Newton Formulae: $$\begin{cases} 0 = c_0\sigma_r + c_1\sigma_{r-1} + \ldots + c_{r-1}\sigma_1 + rc_r, & r \le n \\ 0 = c_0\sigma_r + c_1\sigma_{r-1} + c_2\sigma_{r-2} + \ldots + c_n\sigma_{r-n}, & r \ge n \end{cases}$$

3.3.1 Kronecker's Theorem

Lemma: *$k[x]/(P)$ is a k-algebra of degree $d = \deg P$, and $1, \bar{x}, \ldots, \bar{x}^{d-1}$ form a base.*

Proof: If $V = k \oplus \ldots \oplus kx^{d-1}$, we have to show that the linear map

$$\pi\colon V \to k[x]/(P),\ \pi(Q) = [Q],$$

is an isomorphism. It is injective because $V \cap (P) = 0$, and it is surjective because any polynomial coincides in $k[x]/(P)$ with the remainder of the division by P.

Kronecker's Theorem: *If $P \in k[x]$ is irreducible, then a root of P is $\bar{x} \in k[x]/(P)$; and if $\alpha \in L$ is another root, there is an isomorphism of k-algebras $k[x]/(P) \simeq k(\alpha)$, $\bar{x} \mapsto \alpha$.*

Proof: The ideal (P) is maximal by Euclid's lemma; hence $k[x]/(P)$ is an extension of k.

Let us see that $\bar{x}$ is a root of $P = \sum_i a_i x^i$,

$$P(\bar{x}) = \textstyle\sum_i a_i[\,x\,]^i = \big[\sum_i a_i x^i\,\big] = [P(x)] = 0.$$

If $\alpha \in L$ is another root, the image of the morphism of k-algebras $k[x] \to L$, $x \mapsto \alpha$, is $k[\alpha]$, and the kernel contains the maximal ideal (P); hence it is (P).

By the isomorphism theorem, we have an isomorphism

$$k[x]/(P) \xrightarrow{\ \sim\ } k[\alpha],\ [Q(x)] \mapsto Q(\alpha).$$

Therefore $k[\alpha]$ is a field, because so is $k[x]/(P)$, and $k[\alpha] = k(\alpha)$.

Corollary: *Let $P \in k[x]$ be an irreducible polynomial of degree d. If α is a root of P, then P divides any other polynomial with coefficients in k admitting the root α, and*

$$k(\alpha) = k \oplus k\alpha \oplus k\alpha^2 \oplus \ldots \oplus k\alpha^{d-1}.$$

Examples: The polynomial $x^n - 2$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein criterion; hence $\mathbb{Q}(\sqrt[n]{2}\,)$ is a finite extension of $\mathbb{Q}$ of degree n.

The equality $k(\alpha) = k[\alpha]$ is just the rationalization of algebraic expressions: if $Q \in k[x]$ and $Q(\alpha) \neq 0$, we put $x = \alpha$ in Bézout's identity, $1 = AP + BQ$, and we obtain $\frac{1}{Q(\alpha)} = B(\alpha)$.

Theorem: *Any non constant polynomial $P \in k[x]$ has all the roots in a finite extension of k.*

Proof: $P(x)$ has a root α in a finite extension K; hence $P = (x-\alpha)Q$, where $Q \in K[x]$.

By induction on the degree, Q has all the roots in finite extension L of K.

Hence $P = (x-\alpha)Q$ has all the roots in L, a finite extension of k by the following result:

Degree Theorem: *If $k \to K \to L$ are finite extensions, then*

$$[L : k] = [L : K] \cdot [K : k].$$

Proof: If $K = ku_1 \oplus \ldots \oplus ku_n$, and $L = Kv_1 \oplus \ldots \oplus Kv_m$, then

$$L = (ku_1 \oplus \ldots \oplus ku_n)v_1 \oplus \ldots \oplus (ku_1 \oplus \ldots \oplus ku_n)v_m = \bigoplus_{i,j} ku_i v_j .$$

Theorem: *Let $P \in k[x]$ and let K, L be two extensions of k where P has all the roots. If $\alpha_1, \ldots, \alpha_n \in K$ and $\beta_1, \ldots, \beta_n \in L$ are the roots of P, there is an isomorphism of k-algebras $\tau\colon k(\alpha_1, \ldots, \alpha_n) \to k(\beta_1, \ldots, \beta_n)$ such that (reordering the roots) $\tau(\alpha_i) = \beta_i$; $i = 1, \ldots, n$.*

Proof: By induction on $[k(\alpha_1, \ldots, \alpha_n) : k]$, and it is obvious when $k(\alpha_1, \ldots, \alpha_n) = k$.

If some root $\alpha_i \notin k$, then we have an isomorphism $k(\alpha_i) \to k(\beta_i)$ by Kronecker's theorem, so that both K and L are extensions of $k(\alpha_i)$.

We conclude because $[k(\alpha_1, \ldots, \alpha_n) : k(\alpha_i)] < [k(\alpha_1, \ldots, \alpha_n) : k]$.

D'Alembert's Theorem: *Any non constant polynomial $P \in \mathbb{C}[x]$ has a complex root.*

Proof: If $P \in \mathbb{R}[x]$ has degree $n = 2^d m$, with m odd, we proceed by induction on d.

It holds when $d = 0$ by Bolzano's theorem.

When $d > 0$, by Kronecker's theorem, $P = (x - \alpha_1) \ldots (x - \alpha_n)$ in some extension L of $\mathbb{C}$.

Given $a \in \mathbb{R}$, the polynomial with roots $\alpha_i + \alpha_j + a\alpha_i\alpha_j$ has degree $\binom{n}{2} = 2^{d-1}m(n-1)$ and real coefficients (they are symmetric functions of the roots α_i). By induction this polynomial has a complex root. Hence there are indices i, j and real numbers $a \neq b$ such that

$$\alpha_i + \alpha_j + a\alpha_i\alpha_j,\ \ \alpha_i + \alpha_j + b\alpha_i\alpha_j \in \mathbb{C}.$$

Therefore $\alpha_i + \alpha_j, \alpha_i\alpha_j \in \mathbb{C}$, so that α_i, α_j are roots of a polynomial of degree 2 with complex coefficients; hence $\alpha_i, \alpha_j \in \mathbb{C}$.

If $P \in \mathbb{C}[x]$, and $\bar{P}$ denotes the conjugate polynomial, then $P\bar{P} \in \mathbb{R}[x]$, and $P\bar{P}$ has a complex root α. It is a root of P or $\bar{P}$, in which case $\bar{\alpha}$ is a root of P.

3.3.2 Quadratic Irrationals

Definitions: An element α of an extension L of k is **algebraic** over k if it is a root of some non null polynomial with coefficients in k, hence of an irreducible factor P_α (the **minimal polynomial** of α). An extension $k \to L$ is **algebraic** if any element of L is algebraic over k.

P_α divides any other polynomial with coefficients in k admitting the root α (p. 79) and

$$[k(\alpha) : k] = \deg P_\alpha.$$

Lemma: *$\alpha \in L$ is algebraic over k if and only if $k(\alpha)$ is a finite extension of k.*

Proof: If $k(\alpha)$ is a finite extension of degree d, then $1, \alpha, \alpha^2, \ldots, \alpha^d$ are linearly dependent, $a_0 + a_1\alpha + \ldots + a_d\alpha^d = 0$, where $a_i \in k$, and α is algebraic over k.

Corollary: *If $\alpha_1, \ldots, \alpha_n \in L$ are algebraic over k, then $k \to k(\alpha_1, \ldots, \alpha_n)$ is a finite extension.*

Proof: By induction on n, the following extensions are finite

$$k \longrightarrow k(\alpha_1, \ldots, \alpha_{n-1}) \longrightarrow k(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n).$$

Theorem: *The elements of L algebraic over k form an extension of k.*

Proof: If α, β are algebraic over k, then $k(\alpha, \beta)$ is a finite extension of k.
Any element of $k(\alpha, \beta)$ is algebraic; hence so are $\alpha + \beta$, $\alpha\beta$, and α/β when $\beta \neq 0$.

Theorem: *Let $k \to K$ be an algebraic extension, and let L be an extension of K. If $\alpha \in L$ is algebraic over K, then α is algebraic over k.*

Proof: If α is a root of $c_0 x^n + \ldots + c_n \in K[x]$, then $k(c_0, \ldots, c_n, \alpha)$ is a finite extension of $k(c_0, \ldots, c_n)$, which is a finite extension of k, and α is algebraic over k.

Definition: A field $K \subset \mathbb{C}$ is an extension of $\mathbb{Q}$ by **quadratic radicals** if $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i^2 \in \mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})$, $1 \leq i \leq n$. A complex number is a **quadratic irrational** if it is in some extension of $\mathbb{Q}$ by quadratic radicals.

Lemma: *If $a \in k$, then the degree of $k(\sqrt{a}\,)$ over k is 1 or 2.*

Proof: Put $\alpha = \sqrt{a}$. Then $[k(\alpha) : k] = \deg P_\alpha = 1$ or 2, since P_α divides $x^2 - a$.

Theorem: *The degree of any extension $\mathbb{Q} \to K$ by quadratic radicals is a power of 2.*

Proof: Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})$, $1 \leq i \leq n$.
Now $[K : \mathbb{Q}]$ is a power of 2 because we have extensions of degree 1 or 2

$$\mathbb{Q} \longrightarrow \mathbb{Q}(\alpha_1) \longrightarrow \mathbb{Q}(\alpha_1, \alpha_2) \longrightarrow \ldots \longrightarrow \mathbb{Q}(\alpha_1 \ldots, \alpha_{n-1}) \longrightarrow \mathbb{Q}(\alpha_1 \ldots, \alpha_n) = K.$$

Corollary: *If α is a quadratic irrational, then the degree of P_α is a power of 2.*

Proof: Let K be an extension of $\mathbb{Q}$ by quadratic radicals.
If $\alpha \in K$, then $\mathbb{Q}(\alpha) \subseteq K$, so that $\deg P_\alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ divides $[K : \mathbb{Q}] = 2^n$.

Definitions: The complex roots of $x^n - 1$ are the complex n-th **roots of unity**.
Since $\left(e^{\frac{2\pi k}{n} i}\right)^n = e^{2\pi k i} = 1$ when $k \in \mathbb{Z}$, and the degree of a polynomial bounds the number of roots, the complex n-th roots of unity form a cyclic group μ_n of order n,

$$\mu_n = \{\varepsilon_n, \varepsilon_n^2, \ldots, \varepsilon_n^n = 1\}, \quad \varepsilon_n = e^{\frac{2\pi}{n} i} = \cos\tfrac{2\pi}{n} + i \sin\tfrac{2\pi}{n}.$$

The **primitive** n-th roots of unity are the generators of the cyclic group μ_n. They are ε_n^m where m, n are coprime (p. 46), and they are the roots of the n-th **cyclotomic polynomial**

$$\Phi_n(x) = \prod_{\substack{1 \leq m \leq n \\ \text{g.c.d.}(m,n)=1}} \left(x - e^{\frac{2\pi i}{n} m}\right).$$

Since the order d of any element of μ_n divides n, and the elements of μ_n of order d are just the primitive d-th roots of unity, we have

$$x^n - 1 = \prod_{\alpha \in \mu_n} (x - \alpha) = \prod_{d|n} \Phi_d(x) = \Phi_1(x) \ldots \Phi_n(x).$$

So we may calculate $\Phi_n(x)$ inductively and, being unitary, it has integer coefficients[1].

[1]and we shall see (p. 154) that $\Phi_n(x)$ is irreducible in $\mathbb{Q}[x]$.

For example, $\Phi_1 = x - 1$, $\Phi_2 = x + 1$, $\Phi_3 = x^2 + x + 1$, $\Phi_4 = x^2 + 1$, $\Phi_6 = x^2 - x + 1$, $\Phi_8 = x^4 + 1$, $\Phi_9 = x^6 + x^3 + 1$ and $\Phi_p = x^{p-1} + x^{p-2} + \ldots + x + 1$ when p is prime.

Ruler-and-Compass Constructions: Fix a segment in a plane, identify the end points with 0 and 1, and the points of the plane with complex numbers. Since quadratic irrationals are obtained from 0 and 1 by means of sums, subtractions, products, quotients and quadratic radicals, they are constructible using straightedge and compass. Conversely constructible points are quadratic irrationals, because intersections of straight lines and circles are determined in terms of sums, subtractions, products, quotients and quadratic radicals[2], as well as the line passing through two given points, and the circle with given center and radius.

1. The roots $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ of a polynomial $ax^2 + bx + c$ with rational coefficients are quadratic irrationals, as well as the roots of $ax^4 + bx^2 + c$ and $ax^4 + bx^3 + cx^2 + bx + a$, since the substitutions $y = x^2$ and $y = x^{-1} + x$ transform them into quadratic equations.

2. If $e^{\frac{2\pi i}{n}}$ is a quadratic irrational, then so is $e^{\frac{2\pi i}{2n}} = \sqrt{e^{\frac{2\pi i}{n}}}$. Moreover, $e^{\frac{2\pi i}{3}}$, $e^{\frac{2\pi i}{5}}$ are quadratic irrationals, since $\Phi_3 = x^2 + x + 1$ and $\Phi_5 = x^4 + x^3 + x^2 + x + 1$. Hence so is
$$e^{\frac{2\pi i}{15}} = (e^{\frac{2\pi i}{15}})^6 (e^{\frac{2\pi i}{15}})^{-5} = (e^{\frac{2\pi i}{5}})^2 (e^{\frac{2\pi i}{3}})^{-1}.$$
The regular polygons of 2^n, $2^n 3$, $2^n 5$ and $2^n 15$ sides are ruler-and-compass constructible.

3. $\sqrt[3]{2}$ is not a quadratic irrational, since it is a root of the irreducible polynomial $x^3 - 2$.
Doubling the cube is impossible using straight-edge and compass.

4. If p is prime, the irreducible polynomial of $e^{\frac{2\pi i}{p}}$ is $\Phi_p = x^{p-1} + \ldots + 1$ (p. 74); hence $e^{\frac{2\pi i}{p}}$ is not a quadratic irrational when $p - 1$ has an odd factor.
The regular polygons of 7, 11, 13, 19,... sides are not ruler-and-compass constructible.

5. The reduction of $\Phi_9 = x^6 + x^3 + 1$ modulo 2 is irreducible: it has no root in $\mathbb{F}_2$ and no irreducible polynomial of degree 2 or 3, $x^2 + x + 1$, $x^3 + x + 1$, $x^3 + x^2 + 1$, divides $\bar{\Phi}_9$. Hence Φ_9 is irreducible in $\mathbb{Q}[x]$, and $e^{\frac{2\pi i}{9}}$ is not a quadratic irrational.
Trisection of the angle of $\frac{2\pi}{3}$ radians is impossible using straight-edge and compass.

3.3.3 Operator Theory

Theorem: *Let T be an endomorphism of a k-vector space E. If $P, Q \in k[x]$ are coprime,*
$$\mathrm{Ker}\,(PQ)(T) = \mathrm{Ker}\,P(T) \oplus \mathrm{Ker}\,Q(T).$$

Proof: Bézout's identity $1 = AP + BQ$ shows that any vector $e \in E$ decomposes
$$e = AP(T)e + BQ(T)e.$$

If $PQ(T)e = 0$, then $AP(T)e \in \mathrm{Ker}\,Q(T)$, and $BQ(T)e \in \mathrm{Ker}\,P(T)$.
Hence $\mathrm{Ker}\,PQ(T) = \mathrm{Ker}\,P(T) + \mathrm{Ker}\,Q(T)$.
If $e \in \mathrm{Ker}\,P(T) \cap \mathrm{Ker}\,Q(T)$, then $e = AP(T)e + BQ(T)e = 0 + 0$.

Corollary: *If T is an endomorphism of a complex vector space, and $P \in \mathbb{C}[x]$ is a polynomial with complex roots $\alpha_1, \ldots, \alpha_r$ of multiplicities $m_1, \ldots, m_r$, then*
$$\mathrm{Ker}\,P(T) = \mathrm{Ker}\,(T - \alpha_1)^{m_1} \oplus \ldots \oplus \mathrm{Ker}\,(T - \alpha_r)^{m_r}.$$

[2] The intersection points of two circles $P = x^2 + y^2 + ax + by + c = 0$, $P' = x^2 + y^2 + a'x + b'y + c' = 0$ are just the intersection points of one circle with the straight line $P' - P = 0$.

Difference Equations

Let E be the vector space of all complex sequences $(s_0, s_1, \ldots)$. Let $\nabla\colon E \to E$ be the operator $\nabla s_n = s_{n+1}$, and let $\Delta = \nabla - 1$ be the difference operator, $\Delta s_n = s_{n+1} - s_n$.

Commutation Formula: $P(\nabla)(\alpha^n s_n) = \alpha^n P(\alpha\nabla)s_n;\ P \in \mathbb{C}[x],\ \alpha \in \mathbb{C}.$

Proof: We may assume that $P(x) = x^k$, and we have

$$\nabla^k(\alpha^n s_n) = \alpha^{n+k} s_{n+k} = \alpha^n(\alpha\nabla)^k s_n.$$

Corollary: *The sequences* α^n, $n\alpha^n, \ldots, n^{k-1}\alpha^n$ *form a base of* $\operatorname{Ker}(\nabla - \alpha)^k$.

Proof: Since $(\nabla - \alpha)^k \alpha^n s_n = \alpha^n(\alpha\nabla - \alpha)^k s_n = \alpha^{n+k}\Delta^k s_n$, we have to show that $\operatorname{Ker}\Delta^k$ is defined by all polynomial sequences $a_0 + a_1 n + \ldots + a_{k-1}n^{k-1}$.

We proceed by induction on k, and we use that any sequence $s = (s_n)$ is

$$s_n = (\nabla^n s)_0 = \big((1+\Delta)^n s\big)_0 = \sum_{i=0}^{n} \binom{n}{i} (\Delta^i s)_0.$$

If $\Delta^k s = 0$, then s is a polynomial sequence of degree $< k$, because $\binom{n}{i} = \frac{n(n-1)\ldots(n-i+1)}{i!}$ is a polynomial function of n of degree i,

$$s_n = s_0 + \tbinom{n}{1}(\Delta s)_0 + \ldots + \tbinom{n}{k-1}(\Delta^{k-1}s)_0.$$

Finally, $\Delta^{d+1}n^d = 0$ because we have $\Delta^{d+1}n^d = \Delta^d(\Delta n^d) = \Delta^d((n+1)^d - n^d) = 0$, and $(n+1)^d - n^d$ is a polynomial function of degree $d-1$.

Particular Solution: The complex roots of P solve the equation $P(\nabla)x_n = 0$, and solutions of $P(\nabla)x_n = y_n$ are obtained adding a particular solution. Iterating, the search of a solution is reduced to the equation $(\nabla - \alpha)x_n = y_n$. By the commutation formula,

$$y_n = (\nabla - \alpha)x_n = (\nabla - \alpha)\alpha^n\alpha^{-n}x_n = \alpha^{n+1}\Delta(\alpha^{-n}x_n),$$

and $\Delta(v_0 + \ldots + v_{n-1}) = v_n$. Hence a solution of $(\nabla - \alpha)x_n = y_n$ is $x_n = \alpha^{n-1}\sum\limits_{i=0}^{n-1}\alpha^{-i}y_i$.

In case that $Q(\nabla)y_n = 0$ for some polynomial $Q(x)$ coprime to $P(x)$, Bézout's identity $\mathrm{Id} = P(\nabla)A(\nabla) + B(\nabla)Q(\nabla)$ shows that $x_n = A(\nabla)y_n$ is a solution of $P(\nabla)x_n = y_n$.

Example: Fibonacci's sequence 0, 1, 1, 2, 3, 5, 8, 13,... satisfies the equation $x_{n+2} = x_{n+1} + x_n$, and the roots of $x^2 - x - 1$ are ϕ, $-\phi^{-1}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Hence

$$x_n = c_1\phi^n + c_2(-\phi)^{-n},$$

and the constants c_1, c_2 may be determined with the initial terms

$$\left.\begin{aligned} c_1 + c_2 = x_0 = 0 \\ c_1\phi - c_2\phi^{-1} = x_1 = 1 \end{aligned}\right\} \quad c_1 = \frac{1}{\phi + \phi^{-1}} = \frac{1}{\sqrt{5}},\ c_2 = -c_1.$$

Differential Equations

Let E be the complex vector space of all infinitely derivable functions $f\colon \mathbb{R} \to \mathbb{C}$ and let D be the derivative operator, $Df = D\big(x(t) + y(t)i\big) = x'(t) + y'(t)i$.

Commutation Formula: $P(D)(e^{\alpha t}f) = e^{\alpha t}P(D+\alpha)f;\ P \in \mathbb{C}[x],\ \alpha \in \mathbb{C}$.

Proof: We may assume that $P(x) = x^k$, and we proceed by induction on k.
If $k = 1$, then $D(e^{\alpha t}f) = \alpha e^{\alpha t}f + e^{\alpha t}Df = e^{\alpha t}(D+\alpha)f$. In general,

$$\begin{aligned} D^k(e^{\alpha t}f) &= D(D^{k-1}e^{\alpha t}f) = D(e^{\alpha t}(D+\alpha)^{k-1}f) \\ &= \alpha e^{\alpha t}(D+\alpha)^{k-1}f + e^{\alpha t}D(D+\alpha)^{k-1}f = e^{\alpha t}(D+\alpha)^k f. \end{aligned}$$

Corollary: *The functions* $e^{\alpha t}, te^{\alpha t}, \dots, t^{k-1}e^{\alpha t}$ *form a base of* $\operatorname{Ker}(D-\alpha)^k$.

Proof: $(D-\alpha)^k(e^{\alpha t}f) = e^{\alpha t}D^k f$, and $\operatorname{Ker} D^k = \langle 1, t, \dots, t^{k-1}\rangle$.

Particular Solution: The complex roots of P solve the equation $P(D)f = 0$, and the solutions of $P(D)f = g$ are obtained adding a particular solution. Iterating, the search of a solution is reduced to the equation $(D-\alpha)f = g$. By the commutation formula,

$$(D-\alpha)f = (D-\alpha)e^{\alpha t}e^{-\alpha t}f = e^{\alpha t}D(e^{-\alpha t}f),$$

and a particular solution of $(D-\alpha)f = g$ is just $f(t) = e^{\alpha t}\int e^{-\alpha t}g(t)\mathrm{d}t$.
We may also use the decomposition as a sum of simple fractions

$$\begin{aligned} f(t) &= \frac{1}{(D-\alpha_1)^{m_1}\dots(D-\alpha_r)^{m_r}}\, g = \sum_{i,j} \frac{a_{ij}}{(D-\alpha_i)^j}\, g \\ &= \sum_{i,j} a_{ij}e^{\alpha_i t}\frac{1}{D^j}\, e^{-\alpha_i t}g = \sum_{i,j} a_{ij}e^{\alpha_i t}\int\!\!\int \overset{j}{\dots} \int e^{-\alpha_i t}g(t)\,\mathrm{d}t. \end{aligned}$$

3.3.4 Root Separation

Definitions: Let $P, Q \in \mathbb{R}[x]$ be coprime. We say that $f(x) = \frac{P(x)}{Q(x)}$ has a **pole** of order m at $a \in \mathbb{R}$ if Q has a root of multiplicity m at this point, and that the **excess** of f at $x = a$ is

$$E_a\frac{P}{Q} = \begin{cases} +1 & \text{if } \frac{P}{Q} \text{ switches from } -\infty \text{ to } +\infty \text{ at } x = a. \\ -1 & \text{si } \frac{P}{Q} \text{ switches from } +\infty \text{ to } -\infty \text{ at } x = a. \\ \ \ 0 & \text{otherwise.} \end{cases}$$

The excess $E_a^b f$ between a and b is defined to be the sum of excesses of f at all points of the interval $[a,b]$, assuming that the end points a, b are not poles of f.

The sign variation $V(c_1, \dots, c_n)$ of a sequence is the number of sign changes, after eliminating null terms, and the **variation** between a and b of some polynomials $P_1, \dots, P_n$ is

$$V_a^b(P_1, \dots, P_n) = V(P_1(a), \dots, P_n(a)) - V(P_1(b), \dots, P_n(b)).$$

1. $E_a^b(\text{Polynomial}) = 0$.
2. $E_a^b(f_1 + f_2) = E_a^b f_1 + E_a^b f_2$, *when* f_1 *and* f_2 *have no common pole.*
3. $E_a^b(\lambda f) = (\operatorname{sgn}\lambda)E_a^b f$, *for all non null* $\lambda \in \mathbb{R}$.

4. $E_a^b \frac{1}{Q} = \frac{1}{2}(\operatorname{sgn} Q(b) - \operatorname{sgn} Q(a))$.

5. $E_a^b \frac{P}{Q} + E_a^b \frac{Q}{P} = V_a^b(P, Q)$.

Proof: Only property 5 is not obvious. Since $\frac{P}{Q}$ and $\frac{Q}{P}$ have no common pole,

$$E_a^b \frac{P}{Q} + E_a^b \frac{Q}{P} = E_a^b \frac{P^2+Q^2}{PQ} = E_a^b \frac{1}{PQ} = \frac{1}{2}(\operatorname{sgn} P(b)Q(b) - \operatorname{sgn} P(a)Q(a)) = V_a^b(P, Q).$$

Excess Calculation: Applying Euclid's algorithm (changing the remainders sign)

$$\begin{array}{lll}
P = Q_1 Q - R_1 & \frac{P}{Q} = Q_1 - \frac{R_1}{Q} & E_a^b \frac{P}{Q} = -E_a^b \frac{R_1}{Q} \\
Q = Q_2 R_1 - R_2 & \frac{Q}{R_1} = Q_2 - \frac{R_2}{R_1} & E_a^b \frac{Q}{R_1} = -E_a^b \frac{R_2}{R_1} \\
\dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\
R_{n-1} = Q_{n+1} R_n & \frac{R_{n-1}}{R_n} = Q_{n+1} & E_a^b \frac{R_{n-1}}{R_n} = 0
\end{array}$$

$$\begin{aligned}
E_a^b \frac{Q}{P} &= E_a^b \frac{Q}{P} + E_a^b \frac{P}{Q} + E_a^b \frac{R_1}{Q} + E_a^b \frac{Q}{R_1} + E_a^b \frac{R_2}{R_1} + \ldots + E_a^b \frac{R_n}{R_{n-1}} + E_a^b \frac{R_{n-1}}{R_n} \\
&= V_a^B(P, Q) + V_a^b(Q, R_1) + \ldots + V_a^b(R_{n-1}, R_n) = V_a^b(P, Q, R_1, \ldots, R_n)
\end{aligned}$$

Sturm's Theorem: $\begin{bmatrix} \text{Number of roots of} \\ P \text{ between } a \text{ and } b \end{bmatrix} = V_a^b(P, P', R_1, \ldots, R_n)\ ; \quad P(a)P(b) \neq 0\,.$

Proof: Let $P = (x-a_1)^{m_1} \ldots (x-a_r)^{m_r} Q$, where Q has no real root. Then

$$\frac{P'}{P} = \frac{m_1}{x-a_1} + \ldots + \frac{m_r}{x-a_r} + \frac{Q'}{Q}\cdot$$

Since $E_a^b \frac{Q'}{Q} = 0$, because $\frac{Q'}{Q}$ has no pole, $E_a^b \frac{P'}{P}$ coincides with the number of roots of P between a and b (counted without multiplicity).

Note: This proof, and the excess calculation, require that $Q, R_1, \ldots, R_n$ do not vanish at a nor at b. If someone vanishes, we move a little a and b. First we remark that no consecutive remainders may vanish at a (or b), since a would be a root of the g.c.d., hence of P. Moreover, if $R_i(a) = 0$, since $R_{i-1} = Q_i R_i - R_{i+1}$, then $R_{i-1}(a)$ and $R_{i+1}(a)$ have opposite sign.

Modifying the end points, $a' = a + \varepsilon$, $b' = b + \varepsilon$, so that no remainder vanishes and excesses do not change, nor signs of non null remainders, we are in the former case,

$$E_a^b \frac{Q}{P} = E_{a'}^{b'} \frac{Q}{P} = V_{a'}^{b'}(P, Q, R_1, \ldots, R_n) = V_a^b(P, Q, R_1, \ldots, R_n).$$

3.3.5 Multiple Roots

Theorem: *A root α of $P \in k[x]$ is multiple if and only if it is a root of $P'(x)$.*

Proof: Let $P = (x-\alpha)^m Q$, where $Q(\alpha) \neq 0$.

If $m = 1$, then $P' = Q + (x-\alpha)Q'$; hence $P'(\alpha) = Q(\alpha) \neq 0$.

If $m \geq 2$, then $P' = m(x-\alpha)^{m-1} + (x-\alpha)^m Q'$; hence $P'(\alpha) = 0$.

Corollary: *The multiple roots of $P(x)$ are just the roots of $D(x) = $ g.c.d.$\big(P, P'\big)$.*

Proof: Roots of D are roots of P and P'; hence multiple roots of P.

Conversely, by Bézout's identity $D = AP + BP'$, if α is a multiple root of P (hence a root of P'), then α is a root of D.

Corollary: *All the roots of an irreducible polynomial $P \in k[x]$ are simple, or $P'(x) = 0$.*

Proof: $D = \text{g.c.d.}(P, P')$ is 1 or P, since P is irreducible.
If $D = 1$, any root of P is simple by the former result.
If $D = P$, since it divides P' and $\deg P' < \deg P$, we have $P' = 0$.

Definition: Let A be a ring. The kernel of the unique ring morphism $\mathbb{Z} \to A$ is an ideal $d\mathbb{Z}$, and d is the **characteristic** of A.
It is 0 when $1+ \overset{n}{\ldots} +1 \neq 0$, $\forall n \geq 1$, and it is positive when $1+ \overset{n}{\ldots} +1 = 0$ for some $n \geq 1$.

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ have null characteristic, and the characteristic of $\mathbb{Z}/n\mathbb{Z}$ is n.
The formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for the roots of a quadratic polynomial $ax^2+bx+c \in k[x]$ only holds when the characteristic of the field k is not 2 (so that $2a \neq 0$).

Null Characteristic

Theorem: *If* $\operatorname{char} k = 0$*, any root of an irreducible polynomial $P \in k[x]$ is simple.*

Proof: $P'(x) \neq 0$, because $\deg P' = \deg P - 1$.

Theorem: *If* $\operatorname{char} k = 0$*, any root of multiplicity m of $P \in k[x]$ has multiplicity $m-1$ in P'.*

Proof: If α is a root of multiplicity m, then $P = (x-\alpha)^m Q$, where $Q(\alpha) \neq 0$. Now

$$P' = (x-\alpha)^{m-1}(mQ + (x-\alpha)Q'),$$

where $mQ(\alpha) + (\alpha-\alpha)Q'(\alpha) = mQ(\alpha) \neq 0$, since $m \neq 0$ when $\operatorname{char} k = 0$.

Descartes Rule: *The number of positive roots $r_+(P)$ of a polynomial with real coefficients $P(x) = a_0 + a_1x + \ldots + a_nx^n$, counted with multiplicity, is bounded by the number of sign changes $V(P)$ in the sequence of coefficients,*

$$r_+(P) \leq V(a_0, a_1, \ldots, a_n),$$

and both coincide if all the roots of $P(x)$ are real.

Proof: Eliminating the null roots, we may assume that $a_0 > 0$, and we may also assume that the rule holds for polynomials of degree $< n$, so that $r_+(P') \leq V(P')$.

$$V(P') = \begin{cases} V(P) \text{ if } a_0 \text{ has the sign of the consecutive non-zero coefficient } a_j & (I) \\ V(P)-1 \text{ if } a_0 \text{ and } a_j \text{ has opposite sign} & (II) \end{cases}$$

Let us compare $r_+(P')$ and $r_+(P)$. If the positive roots of P are $\alpha_1 < \ldots < \alpha_r$, with multiplicities $m_1, \ldots, m_r$, then α_i is a root of P' of multiplicity $m_i - 1$ and, by Rolle's theorem, P' vanishes between any two consecutive roots, so that $r_+(P') \geq r_+(P) - 1$.
In case (II) the rule follows.
In case (I), the derivative P' has one more root between 0 and α_1, because $P(0) = a_0 > 0$ and the first non null derivative of P at $x = 0$ is positive. Hence $r_+(P') \geq r_+(P)$ and we conclude.

The number of negative roots of P is $r_-(P) = r_+(\bar{P})$, where $\bar{P}(x) = P(-x)$.
We have $r_+(P) \leq V(P)$, $r_-(P) \leq V(\bar{P})$, and $V(P) + V(\bar{P}) \leq n$.
If all the roots of P are real, then $r_+(P) + r_-(P) = n$, since $a_0 \neq 0$.
Hence $r_+(P) = V(P)$, and $r_-(P) = V(\bar{P})$.

Positive Characteristic

Theorem: *The characteristic of any domain is 0 or a prime number.*

Proof: Let A be a domain of positive characteristic $d = nm$.

In A we have $nm = 0$; hence $n = 0$ or $m = 0$, and we conclude that $n = d$ or $m = d$ in $\mathbb{Z}$.

Lemma[3]**:** *$(a+b)^p = a^p + b^p$, when the characteristic is a prime number p.*

Proof: The number $\binom{p}{i} = \frac{p(p-1)\dots(p-i+1)}{i!}$ is a multiple of p when $0 < i < p$, since p does not divide the denominator. Hence

$$(a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} = a^p + b^p.$$

Theorem: *If $Q \in \mathbb{F}_p[x]$ is irreducible, then any root of $Q(x)$ is simple.*

Proof: If $Q = \sum_i a_i x^i$ has a multiple root, then $0 = Q' = \sum_i i a_i x^{i-1}$. Hence $a_i = 0$ when i is not a multiple of p, and Q is not irreducible by Fermat's congruence,

$$Q(x) = a_0 + a_p x^p + a_{2p} x^{2p} + \dots = (a_0 + a_p x + a_{2p} x^2 + \dots)^p.$$

Example: Let $k = \mathbb{F}_2(t)$. The polynomial $x^2 - t$ is irreducible in $k[x]$, since it has no root in k; but any root α is multiple, because $x^2 - t = x^2 - \alpha^2 = (x-\alpha)^2$.

3.4 Rings of Fractions

Definition: Let A be a ring. $S \subseteq A$ is a **multiplicative system** if $1 \in S$, and $s, t \in S \Rightarrow st \in S$.

We consider in $A \times S$ the equivalence relation, clearly reflexive and symmetric,

$$(a,s) \equiv (b,t) \;\Leftrightarrow\; \text{there are } u, v \in S \text{ such that } au = bv,\; su = tv.$$

It is transitive: if $(a,s) \equiv (b,t)$, $(b,t) \equiv (c,r)$, there are $u, v, u', v' \in S$ such that $au = bv$, $su = tv$, $bu' = cv'$, $tu' = rv'$. Hence $auu' = bvu' = cvv'$, $suu' = tvu' = rvv'$, and $(a,s) \equiv (c,r)$.

Definition: The **localization** A_S of A at S is the quotient set with the ring structure

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st} \quad , \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

where $\frac{a}{s}$ is the class of (a,s). These operations are well-defined: if we replace $\frac{a}{s}$ by $\frac{au}{su}$,

$$\frac{au}{su} + \frac{b}{t} = \frac{(at+bs)u}{stu} = \frac{at+bs}{st} \quad , \quad \frac{au}{su} \cdot \frac{b}{t} = \frac{abu}{stu} = \frac{ab}{st}$$

In A_S; we have $0 = \frac{0}{1}$, $1 = \frac{1}{1}$, and $-\frac{a}{s} = \frac{-a}{s}$. Moreover, $\frac{a}{s} = 0$ if and only if $ua = 0$ for some $u \in S$; hence $\frac{a}{s} = \frac{b}{t}$ if and only if $u(at - bs) = 0$ for some $u \in S$.

The canonical ring morphism $\gamma\colon A \to A_S$, $\gamma(a) = \frac{a}{1}$, is the **localization morphism**, and $\gamma(s) = \frac{s}{1}$ is invertible in A_S for all $s \in S$, the inverse being $\frac{1}{s}$.

Theorem: *If A is a domain, then $S = A - \{0\}$ is a multiplicative system, A_S is a field (the **field of fractions** of A) and $\gamma\colon A \to A_S$ is injective.*

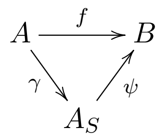
[3] This lemma proves again **Fermat's congruence**: $n^p = (1 + \dots + 1)^p = 1^p + \dots + 1^p = n$ in $\mathbb{F}_p$.

Proof: S is a multiplicative system because $1 \neq 0$ and $a, b \neq 0 \Rightarrow ab \neq 0$.

If $\frac{a}{1} = 0$, then $sa = 0$, where $s \neq 0$; hence $a = 0$, so that γ is injective and $A_S \neq 0$.

Now, if $\frac{a}{s} \neq 0$, then $a \neq 0$, and $\frac{s}{a} \in A_S$ is the inverse of $\frac{a}{s}$.

Universal Property: *If $f\colon A \to B$ is a ring morphism and $f(s)$ is invertible in B for all $s \in S$, then there is a unique ring morphism $\psi\colon A_S \to B$ such that $\psi(\frac{a}{1}) = f(a)$,*



$f = \psi\,\gamma$

Proof: The unique possible morphism, $\psi\colon A_S \to B$, $\psi(\frac{a}{s}) = f(a)f(s)^{-1}$, is well-defined,

$$\psi\big(\tfrac{au}{su}\big) = f(au)f(su)^{-1} = f(a)f(u)f(s)^{-1}f(u)^{-1} = f(a)f(s)^{-1}.$$

Theorem: *If J is an ideal of A_S, then $I = A \cap J = \{a \in A\colon \frac{a}{1} \in J\}$ is an ideal of A, and $J = IA_S = \{\frac{a}{s};\ a \in I\}$.*

Proof: It is clear that $A \cap J$ is an ideal of A and that $IA_S \subseteq J$.

Now, if $\frac{b}{t} \in J$, then $\frac{b}{1} = \frac{t}{1}\frac{b}{t} \in J$; hence $b \in I$ and $\frac{b}{t} \in IA_S$.

Definition: A **unique factorization domain** is an integral ring A where any proper element can be written as a product of irreducible elements, uniquely up to the order and units.

Euclid's lemma holds in such rings: *If an irreducible element p divides a product, then it divides some factor (pA is a prime ideal).* In fact, if $bc = pa$, writing b and c as a product of irreducibles, some factor coincides, up to a unit, with p; hence b or c is a multiple of p.

In general, if $a \in A$ divides a product bc and it has no common irreducible factor with b, then it divides c. Moreover, Gauss lemma and its proof hold in $A[x]$.

Proposition: *Let A be a unique factorization domain. Any root of $c_0x^n + \ldots + c_n \in A[x]$, $c_0c_n \neq 0$, in the field of fractions of A is $x = \frac{a}{b}$, where $a \in A$ divides c_n and $b \in A$ divides c_0.*

Proof: If $x = \frac{a}{b}$ is a root, then $0 = c_0a^n + c_1a^{n-1}b + \ldots + c_{n-1}ab^{n-1} + c_nb^n$, and

$$c_nb^n = -a(c_0a^{n-1} + \ldots + c_{n-1}b^{n-1}).$$

If a and b have no common irreducible factor, then a divides c_n.

Analogously b divides c_0a^n, and we conclude that b divides c_0.

Lemma: *If A is a domain, then so is $A[x]$, and $A[x]^* = A^*$.*

Proof: $(a_nx^n + \ldots)(b_mx^m + \ldots) = a_nb_mx^{n+m} + \ldots$, so that $\deg(PQ) = \deg P + \deg Q$.

Now it is clear that $A[x]$ is integral and that invertible polynomials have degree 0.

Theorem: *If A is a unique factorization domain, then so is $A[x]$.*

Proof: We prove the existence of irreducible factor decompositions by induction on the degree, and we put $P = dQ$, where the coefficients of Q have no common irreducible factor.

If Q is irreducible, $P = dQ$ is a product of irreducibles. Otherwise $Q = Q_1Q_2$, where Q_1 and Q_2 are products of irreducibles by induction, hence so is $P = dQ_1Q_2$.

To prove the uniqueness, we consider two irreducible factor decompositions,

$$p_1 \dots p_r P_1(x) \dots P_s(x) = q_1 \dots q_m Q_1(x) \dots Q_n(x),$$

where $p_i, q_j \in A$; $\deg P_i$, $\deg Q_j \geq 1$. Let Σ be the field of fractions of A.

The ring $\Sigma[x]$ is Euclidean, and all factors P_i, Q_j are irreducible in $\Sigma[x]$ by Gauss lemma. Hence $s = n$, and reordering $Q_i = \frac{a_i}{b_i} P_i$ (where a_i, b_i have no common irreducible factor); $b_i Q_i = a_i P_i$, and any irreducible factor of b_i (resp. a_i) divides P_i (resp. Q_i), which is irreducible: a_i and b_i are invertible, and $p_1 \dots p_r = u q_1 \dots q_m$, where $u \in A$ is invertible.

Hence $r = m$, and reordering, $p_i = q_i$ up to units.

Corollary: *$\mathbb{Z}[x_1, \dots, x_n]$ and $k[x_1, \dots, x_n]$ are unique factorization domains.*

Example: Let us see the **Vandermonde determinant**,

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

This determinant is a polynomial $V(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$, and it vanishes in the quotient by the ideal $(x_j - x_i)$; hence it is a multiple of $\prod_{i<j}(x_j - x_i)$, this ring being a unique factorization domain, and both coincide up to a constant factor c because both polynomials have degree $0 + 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$.

Let us see that $c = 1$. The diagonal monomial of $V(x_1, \dots, x_n)$ is $x_n^{n-1} \dots x_3^2 x_2$ while, by induction on n, the other term is

$$(x_n - x_1) \dots (x_n - x_{n-1}) \prod_{i<j<n} (x_j - x_i) = (x_n^{n-1} + \dots)(x_{n-1}^{n-2} \dots x_2 + \dots) = x_n^{n-1} \dots x_2 + \dots$$

3.5 The Resultant

Definition: Let us express two polynomials $P, Q \in k[x]$ in terms of their roots:

$$P = a_0 x^n + a_1 x^{n-1} + \dots + a_n = a_0 (x - \alpha_1) \dots (x - \alpha_n)$$
$$Q = b_0 x^m + b_1 x^{n-1} + \dots + b_m = b_0 (x - \beta_1) \dots (x - \beta_m)$$

P and Q have a common root if and only if the **resultant** $R(P, Q)$ vanishes,

$$R(P, Q) = a_0^m b_0^n \textstyle\prod_{i,j} (\alpha_i - \beta_j) \in k.$$

1. *If $m = 0$, then $R(P, Q) = b_0^n$.*

2. $R(Q, P) = (-1)^{nm} R(P, Q)$.

   $R(Q, P) = a_0^m b_0^n \prod_{i,j} (\beta_j - \alpha_i) = (-1)^{nm} a_0^m b_0^n \prod_{i,j} (\alpha_i - \beta_j) = (-1)^{nm} R(P, Q)$.

3. $R(P, Q) = a_0^m \prod_i Q(\alpha_i)$.

   $R(P, Q) = a_0^m b_0^n \prod_i \prod_j (\alpha_i - \beta_j) = a_0^m \prod_i \big(b_0 \prod_j (\alpha_i - \beta_j)\big) = a_0^m \prod_i Q(\alpha_i)$.

Now we consider the polynomial ring $\mathbb{Z}[a_0, \alpha_1 \dots, \alpha_n, b_0, \beta_1, \dots, \beta_m]$.

4. $R(P,Q) \in \mathbb{Z}[a_0,\dots,a_n,b_0,\dots,b_m]$.

   The resultant is a symmetric polynomial in α_i, with coefficients in $\mathbb{Z}[a_0,b_0,\dots,b_m]$; hence it is a polynomial in the elementary symmetric functions of the roots α_i, which are $\pm\frac{a_i}{a_0}$ by Cardano's formulae:

   (∗) $\qquad R(P,Q) = F\Big(a_0,b_0,\dots,b_m,\frac{a_1}{a_0},\dots,\frac{a_n}{a_0}\Big) = \frac{T(a_0,\dots,a_n,b_0,\dots,b_m)}{a_0^r}$

   (∗∗) $\qquad R(P,Q) = \pm R(Q,P) = \frac{\bar{T}(a_0,\dots,a_n,b_0,\dots,b_m)}{b_0^s}$

   Now, $\mathbb{Z}[a_0,\dots,a_n,b_0,\dots,b_m]$ is a polynomial ring (p. 78); hence a unique factorization domain, and comparing (∗) and (∗∗) we see that $r = s = 0$.

5. *$R(P,Q)$ is a homogeneous polynomial of degree m in the variables $a_0,\dots,a_n$, and homogeneous of degree n in the variables $b_0,\dots,b_m$.*

   If we multiply the variables $a_0,\dots,a_n$ with an indeterminate t, then we replace P by $tP = ta_0x^n + ta_1x^{n-1} + \dots + ta_n = ta_0(x-\alpha_1)\dots(x-\alpha_n)$, and

   $$R(tP,Q) = (ta_0)^m \textstyle\prod_i Q(\alpha_i) = t^m R(P,Q),$$

   so that $R(P,Q)$ is homogeneous of degree m in $a_0,\dots,a_n$. Since $R(P,Q) = \pm R(Q,P)$, it is homogeneous of degree n in $b_0,\dots,b_m$.

Theorem: *If $P = CQ + R$, then $R(P,Q) = (-1)^{nm} b_0^{n-r} R(Q,R)$, where $r = \deg R$.*

Proof: Since $P(\beta_j) = Q(\beta_j)C(\beta_j) + R(\beta_j) = R(\beta_j)$, we have

$$R(Q,P) = b_0^n \textstyle\prod_j P(\beta_j) = b_0^n \prod_j R(\beta_j) = b_0^{n-r} R(Q,R)\,.$$

Example: The **discriminant** Δ of a polynomial $P = x^n + a_1x^{n-1} + \dots + a_n$ is

$$\Delta = \prod_{i<j}(\alpha_j-\alpha_i)^2 = (-1)^{\binom{n}{2}}\prod_{i,j}(\alpha_j-\alpha_i) = (-1)^{\binom{n}{2}}\prod_i P'(\alpha_i) = (-1)^{\binom{n}{2}} R(P,P').$$

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & \alpha_1 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \dots & \alpha_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \alpha_n & \dots & \alpha_n^{n-1} \end{vmatrix} = \begin{vmatrix} n & \sigma_1 & \dots & \sigma_{n-1} \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \\ \dots & \dots & \dots & \dots \\ \sigma_{n-1} & \sigma_n & \dots & \sigma_{2n-2} \end{vmatrix}$$

Proposition: *P and Q have no common root if and only if* g.c.d.$(P,Q) = 1$.

Proof: If they are coprime, then $1 = AP + BQ$, so that they have no common root.
Otherwise, any root of g.c.d.(P,Q) is a common root.

Definition: The **Bézout resultant** is $R^b(P,Q) = a_0^m(\det h_Q)$, where

$$h_Q\colon k[x]/(P) \longrightarrow k[x]/(P),\ h_Q(B) = BQ.$$

Theorem: $R^b(P,Q) = R(P,Q)$.

Proof: If we consider $L[x]/(P)$ instead of $k[x]/(P)$, the determinant of h_Q is the same, because so is the matrix of h_Q in the base $1,x,\dots,x^{n-1}$.

$$L[x]/(P) \simeq L[x]/(x-\alpha_1) \oplus \dots \oplus L[x]/(x-\alpha_n) \simeq L \oplus \dots \oplus L$$

where the isomorphism $L[x]/(x-\alpha_i) \simeq L$ replaces x by α_i. Hence the class $[Q]$ corresponds to $(Q(\alpha_1),\dots,Q(\alpha_n))$ and, in the usual base of L^n, the matrix of h_Q is diagonal,

$$R^b(P,Q) = a_0^m(\det h_Q) = a_0^m Q(\alpha_1)\dots Q(\alpha_n) = R(P,Q).$$

Definition: The condition $(P,Q)=k[x]$ means that the linear map

$$f\colon k[x]/(Q)\oplus k[x]/(P) \longrightarrow k[x]/(PQ),\ \ f(A,B) = AP+BQ,$$

is surjective. Both vector spaces have dimension $m+n$, and the **Euler resultant** is the determinant of the matrix of f when we consider the base $(x^{m-1},0),\dots,(1,0),(0,x^{n-1}),\dots,(0,1)$ in $k[x]/(Q)\oplus k[x]/(P)$, and the base $x^{n+m-1},\dots,1$ in $k[x]/(PQ)$,

$$R^e(P,Q) = \begin{vmatrix} a_0 & a_1 & \dots & \dots & a_n & 0 & \dots & \dots & 0 \\ 0 & a_0 & a_1 & \dots & \dots & a_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & a_{n-1} & a_n \\ b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & \dots & 0 \\ 0 & b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & b_{m-1} & b_m \end{vmatrix}$$

where m rows have the coefficients of P and n rows those of Q.

Clearly $R^e(P,Q)=0$ if and only if P and Q have a common root.

When we consider $a_0,a_1\dots,a_n,b_0,b_1,\dots,b_m$ as indeterminates, $R^e(P,Q)$ is a homogeneous polynomial of degree m in $a_0,\dots,a_n$, and homogeneous of degree n in $b_0,\dots,b_m$.

Lemma: *$R(P,Q)$ is not a multiple of b_0 in $\mathbb{Z}[a_0,\dots,a_n,b_0,\dots,b_m]$ when $m\neq 0$.*

Proof: Let $\bar{Q}(x) = b_1x^{m-1}+\ldots+b_m$, so that $Q = b_0x^m+\bar{Q}$.

If $R(P,Q)$ is a multiple of b_0, then we have

$$0 = R(P,Q) = a_0^m\textstyle\prod_i Q(\alpha_i) = a_0^m\prod_i\bar{Q}(\alpha_i) = a_0R(P,\bar{Q}),$$

in the polynomial ring $\mathbb{Z}[a_0,\alpha_1,\dots,\alpha_n,b_1,\dots,b_m] = \mathbb{Z}[a_0,\alpha_1,\dots,\alpha_n,b_0,\dots,b_m]/(b_0)$.

Hence $R(P,\bar{Q})=0$, and any two polynomials of degrees n, $m-1$ would have null resultant.

Theorem: $R^e(P,Q) = R(P,Q)$.

Proof: Let us see $R^e(P,Q)$ in the polynomial ring $\mathbb{Z}[a_0,\alpha_1,\dots,\alpha_n,b_0,\beta_1,\dots,\beta_m]$.

It vanishes in the quotient by $(\alpha_i-\beta_j)$; hence R^e is a multiple of $\prod_{i,j}(\alpha_i-\beta_j)$, since the ring is a unique factorization domain, so that $a_0^mb_0^nR^e(P,Q)$ is a multiple of the resultant,

$$a_0^mb_0^nR^e(P,Q) = F\cdot R(P,Q).$$

F is a symmetric function of the roots α_i and the roots β_j, since so are R and R^e.

Hence F is a polynomial in the elementary symmetric functions which are $\pm\frac{a_i}{a_0}$ and $\pm\frac{b_j}{b_0}$ by Cardano's formulae. Eliminating denominators, we obtain

$$a_0^rb_0^sR^e(P,Q) = \bar{F}\cdot R(P,Q)$$

and the above lemma shows that R^e is a multiple of R. Since both polynomials have degree m in $a_0,\dots,a_n$ and degree n in $b_0,\dots,b_m$, we have $R^e = cR$ for some constant c.

Let us see that $c=1$. A monomial in R^e is the diagonal $a_0^mb_m^n$, while

$$R(P,Q) = a_0^m\prod_i Q(\alpha_i) = a_0^m\textstyle\prod_i(b_m+b_{m-1}\alpha_i+\ldots) = a_0^mb_m^n+\dots$$

Elimination

Let us consider a system of algebraic equations with complex coefficients

$$\begin{cases} 0 = P(x,y) = a_0(y)x^n + a_1(y)x^{n-1} + \ldots + a_n(y) \\ 0 = Q(x,y) = b_0(y)x^m + b_1(y)x^{m-1} + \ldots + b_m(y) \end{cases}$$

Let $R(y)$ be the resultant of P and Q, considered as polynomials in x with coefficients in $\mathbb{C}(y)$. Since the resultant is a polynomial in the coefficients, $R(y) \in \mathbb{C}[y]$.

Theorem: *The roots of $R(y)$ are just the common roots of $a_0(y)$ and $b_0(y)$, and the ordinates of the solutions of the given system of algebraic equations.*

Proof: If $a_0(\beta) = b_0(\beta) = 0$, clearly β is a root of the Euler resultant.

If $a_0(\beta) \neq 0$ (and analogously if $b_0(\beta) \neq 0$), the Bézout resultant shows that $R(\beta)$ is the determinant of the endomorphism

$$h_{Q(x,\beta)} \colon \mathbb{C}[x]/\big(P(x,\beta)\big) \longrightarrow \mathbb{C}[x]/\big(P(x,\beta)\big).$$

Hence $R(\beta) = 0$ if and only if $Q(x,\beta)$ is not invertible in $\mathbb{C}[x]/\big(P(x,\beta)\big)$, a condition stating that $Q(x,\beta)$ and $P(x,\beta)$ have a common root $\alpha \in \mathbb{C}$: the system has a solution $x = \alpha$, $y = \beta$.

Theorem: *If $P(x,y)$ and $Q(x,y)$ have no common irreducible factor, then the above system of algebraic equations only has a finite number of complex solutions.*

Proof: According to Gauss lemma, both polynomials have no common irreducible factor in $\mathbb{C}(y)[x]$; hence the resultant $R(y)$ is not 0, and it has a finite number of complex roots.

The complex solutions of the system have a finite number of possible ordinates.

Analogously, replacing y by x, they have a finite number of possible abscissas.

The system has a finite number of complex solutions.

Bézout's Theorem: *Let $P, Q \in \mathbb{C}[x,y]$ be polynomials of degrees n and m, with no common irreducible factor. Then nm bounds the number of complex solutions of*

$$\left.\begin{aligned} P(x,y) = 0 \\ Q(x,y) = 0 \end{aligned}\right\}$$

Proof: Fixing suitable axes we may assume that the solutions have different ordinates (so that the degree of the resultant $R(y)$ bounds the number of solutions) and that

$$P(x,y) = a_0x^n + a_1(y)x^{n-1} + \ldots + a_n(y),$$
$$Q(x,y) = b_0x^m + b_1(y)x^{m-1} + \ldots + b_m(y).$$

The coefficients $p_{ij}(y)$ of the Euler resultant are polynomials of degree

$$\deg p_{ij} \leq n + i - j,\ 1 \leq i \leq n,$$
$$\deg p_{m+i,j} \leq m + i - j,\ 1 \leq i \leq n.$$

For the terms $p_{1,\sigma(1)} \cdots p_{m+n,\sigma(m+n)}$ of the Euler resultant we have

$$\begin{aligned} \deg p_{1,\sigma(1)} &\leq n + 1 - \sigma(1) \\ \ldots\ &\ \ldots\ldots \\ \deg p_{m,\sigma(m)} &\leq n + m - \sigma(m) \\ \deg p_{m+1,\sigma(m+1)} &\leq m + 1 - \sigma(m+1) \\ \ldots\ &\ \ldots\ldots \\ \deg p_{m+n,\sigma(m+n)} &\leq m + n - \sigma(m+n) \end{aligned}$$

so that the degree of any term is bounded above by

$$\begin{aligned}
&\big[(n+1)+\ldots+(n+m)\big]+\big[(m+1)+\ldots+(m+n)\big]-\textstyle\sum_i\sigma(i) = \\
&\quad = \big[(n+1)+\ldots+(n+m)\big]+\big[(m+1)+\ldots+(m+n)\big]-\big[1+\ldots+(m+n)\big] \\
&\quad = ((n+1)+(n+m))\tfrac{m}{2}+((m+1)+(m+n))\tfrac{n}{2}-(1+(m+n))\tfrac{m+n}{2} = nm.
\end{aligned}$$

Part II

Second Year

Chapter 4

Analysis II

4.1 Topological Spaces

Definitions: Recall that a **topology** $\mathcal{T}$ on a set X is any family of subsets (named open sets) such that $\emptyset$ and X are open sets, and arbitrary unions and finite intersections of open sets are open sets. Given two topologies $\mathcal{T}, \mathcal{T}'$ on X, if $\mathcal{T} \subseteq \mathcal{T}'$, we say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ or that $\mathcal{T}'$ is **finer** than $\mathcal{T}$.

Let $\{\mathcal{T}_i\}$ be a family of topologies on X. Then $\mathcal{T} = \bigcap_i \mathcal{T}_i$ also is a topology on X.

Hence, any family of subsets $\{U_i\}$ of X **generates** a topology: the coarsest topology (the intersection of the topologies) where all the sets U_i are open. The open sets are the arbitrary unions of finite[1] intersections $U_{i_1} \cap \ldots \cap U_{i_n}$.

A family $\{V_i\}$ of neighborhoods of a point $x \in X$ is a **base of neighborhoods** of x when for any other neighborhood V of x, we have $V_i \subseteq V$ for some index i. A family $\{U_i\}$ of open sets is a **base** of the topology when any other open set is a (possibly infinite) union of open sets in the family ($\{U_i \colon x \in U_i\}$ is a base of neighborhoods of x, for any point $x \in X$).

Let X be a set. Given a family of topological spaces $\{Y_i\}$ and maps $f_i \colon Y_i \to X$, the **final topology** is the finest topology on X such that all the maps f_i are continuous: $U \subseteq X$ is open when $f_i^{-1}(U_i)$ is an open set in Y_i for any index i.

Given maps $f_i \colon X \to Y_i$, the **initial topology** is the coarsest topology on X such that all the maps f_i are continuous: it is the topology generated by the subsets $f_i^{-1}(V)$, where $V \subseteq Y_i$ is open. In particular, when X is a subset of a topological space Y, the induced topology is just the initial topology of the inclusion $j \colon X \hookrightarrow Y$.

Definitions: Given a family of topological spaces $\{X_i\}$, the direct product $\prod_i X_i$ always is endowed with the initial topology of the natural projections $p_i \colon \prod_i X_i \to X_i$, so that a basis of the topology is defined by the products of open sets $\prod_i U_i$, where $U_i = X_i$ up to a finite number of indices, and for any topological space Y we have that

$$\mathrm{Hom}_{\mathrm{Top}}(Y, \textstyle\prod_i X_i) = \prod_i \mathrm{Hom}_{\mathrm{Top}}(Y, X_i) \ , \quad f \mapsto (f \circ p_i).$$

The disjoint union $\amalg_i X_i$ always is endowed with the final topology of the natural inclusions $j_i \colon X_i \to \coprod_i X_i$, so that $U \subseteq \coprod_i X_i$ is open when $j_i^{-1}(U)$ is open for any index i. For any topological space Y we have that

$$\mathrm{Hom}_{\mathrm{Top}}(\amalg_i X_i, Y) = \textstyle\prod_i \mathrm{Hom}_{\mathrm{Top}}(X_i, Y) \ , \quad f \mapsto (j_i \circ f).$$

If $R \subseteq X \times X$ is an equivalence relation on a topological space X, the quotient set X/R always is endowed with the final topology of the canonical projection $\pi \colon X \to X/R$, so that

[1] By definition, $\bigcap_{i \in I} U_i = X$ and $\bigcup_{i \in I} U_i = \emptyset$ when the set of indices I is empty.

$\bar{U} \subseteq X/R$ is open when $\pi^{-1}(\bar{U})$ is open, and for any topological space Y we have an exact sequence of sets (in the sense that the first map is injective and the image is just the subset where both right arrows coincide):

$$\mathrm{Hom}_{\mathrm{Top}}(X/R, Y) \xrightarrow{\pi} \mathrm{Hom}_{\mathrm{Top}}(X, Y) \overset{p_1,p_2}{\rightrightarrows} \mathrm{Hom}_{\mathrm{Top}}(X, R) .$$

Examples: The product topology in $\mathbb{R}^n$ is the topology induced by the usual metric.

Any point x of a metric space X admits a countable base of neighborhoods $B\big(x, \frac{1}{n}\big)$. If moreover X has a countable dense set $\{x_m\}$, then X admits a countable base $B\big(x_m, \frac{1}{n}\big)$ of open sets. In particular, the topology of $\mathbb{R}^n$ (hence of any subspace) admits a countable base.

Proposition: *Let X be a topological space such that any point has a countable base of neighborhoods. A point $x \in X$ is in the closure of $Y \subset X$ if and only if there is a sequence (y_n) in Y which converges to x.*

Proof: If $x \in \bar{Y}$, take a countable base $\{U_n\}$ of neighborhoods of x. Replacing U_n by $U_1 \cap \ldots \cap U_n$, we may assume that $U_{n+1} \subseteq U_n$. Pick $y_n \in U_n \cap Y$. It is clear that (y_n) converges to x.

Proposition: *Let X be a topological space such that any point has a countable base of neighborhoods. A map $f \colon X \to Y$ is continuous if and only if it preserves limits:* $\lim f(x_n) = f(\lim x_n)$ *for any convergent sequence (x_n) in X.*

Proof: Continuous maps always preserve limits. Conversely, if f is not continuous at a point $x \in X$, there is a neighborhood V of $f(x)$ in Y such that $f^{-1}(V)$ is not a neighborhood of x.

Take a countable base $\{U_n\}$, $U_{n+1} \subseteq U_n$, of neighborhoods of x. For any $n \in \mathbb{N}$ we have a point $x_n \in U_n$ such that $f(x_n) \notin V$, so that $x_n \to x$, while $f(x_n)$ does not converge to $f(x)$.

Proposition: *If any point of a compact space K has a countable base of neighborhoods, then any sequence (x_n) in K has a convergent subsequence.*

Proof: Take an adherent point $x \in X$ and consider a countable base $\{U_m\}$ of neighborhoods of x. We have $x_{n_0} \in U_0$ for some index n_1, and $x_{n_1} \in U_1$ for some index $n_1 > n_0$, and so on.

So we obtain a subsequence $(x_{n_0}, x_{n_1}, x_{n_2} \ldots)$ which converges to x.

Proposition: *A topological space X with a countable base of open sets is compact if and only if any sequence has a convergent subsequence.*

Proof: Assume that X admits an open cover $X = \bigcup_\alpha U_\alpha$ with no finite subcover.

Since U_α is a union of open sets in the countable base, we see that we may assume that it is a countable cover. Pick $x_n \in X - (U_1 \cup \ldots \cup U_n)$.

If (x_n) admits a convergent subsequence $x_{n_i} \to x \in X$, we have $x \in U_m$ for some index m. Now, U_m contains all the terms x_{n_i} up to a finite number. Absurd when $n_i \geq m$.

Definitions: A topological space X is T_0 when for any two points $x \neq y$ there is an open set containing one of them but not the other; i.e. $\bar{x} \neq \bar{y}$.

X is T_1 when for any two points $x \neq y$ there are two open sets, one containing x but not y, and the other containing y but not x; i.e. when any point is a closed set.

X is T_2, **separated** or **Hausdorff** when any two points $x \neq y$ have disjoint neighborhoods.

X is T_3 or **regular** when any closed set Y and any point $x \notin Y$ have disjoint neighborhoods: at any point the closed neighborhoods form a basis of neighborhoods.

X is T_4 or **normal** when any two disjoint closed sets have disjoint neighborhoods.

It is clear that $(T_4 + T_1) \Rightarrow (T_3 + T_1) \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

Proposition: *In a separated space X, the limit of a sequence (x_n) is unique, if it exists, and any compact set $K \subseteq X$ is closed.*

Proof: Given two limits $x \neq y$ of (x_n), any neighborhood of x or y contains all the points x_n when $n \gg 0$. Absurd since x and y have disjoint neighborhoods.

Finally, if $K \subseteq X$ is compact, and $x \in X - K$, for each point $y \in K$ we may find disjoint open neighborhoods $x \in U_y$ and $y \in V_y$.

Then $K \subseteq \bigcup_y V_y$, so that K admits a finite subcover, $K \subseteq V_{y_1} \cup \ldots \cup V_{y_n}$.

Hence $x \in U_{y_1} \cap \ldots \cap U_{y_n}$ and $(U_{y_1} \cap \ldots \cap U_{y_n}) \cap K = \emptyset$, so that $X - K$ is open.

Corollary: *Any continuous bijection $f: K \to K'$ between compact separated spaces is a homeomorphism.*

Proof: If $Y \subseteq K$ is closed, then it is compact (p. 25), so that $f(Y)$ is compact.

Hence $f(Y)$ is closed, because K' is separated, and we conclude.

Theorem: *The diagonal $\Delta = \{(x, x)\}$ is closed in $X \times X$ if and only if X is separated.*

Proof: Δ is closed if and only if for any point $(x, y) \notin \Delta$ there are neighborhoods U_x and U_y of x and y respectively, such that $(U_x \times U_y) \cap \Delta = \emptyset$; i.e. $U_x \cap U_y = \emptyset$.

Corollary: *Any direct product $\prod_i X_i$ of separated spaces is separated.*

Proof: If Δ_i is closed in $X_i \times X_i$, the $\prod_i \Delta_i$ is closed in $\prod_i (X_i \times X_i) = (\prod_i X_i) \times (\prod_i X_i)$.

Corollary: *Let X be a separated topological space. If two continuous maps $f, h: T \to X$ coincide on a dense subset of T, then $f = h$.*

Proof: Let us consider the continuous map $(f, h): T \to X \times X$, $(f, h)(t) = \big(f(t), h(t)\big)$.

Then $\{t \in T : f(t) = h(t)\} = (f, h)^{-1}(\Delta)$ is closed in T and contains a dense set: it is T.

Proposition: *Any compact separated space K is normal.*

Proof: Let X, Y be disjoint closed (hence compact) sets and fix a point $x \in X$.

For any point $y \in Y$ we have disjoint neighborhoods $x \in U_y$, $y \in V_y$.

If $Y \subseteq V_{y_1} \cup \ldots \cup V_{y_n}$, then $V_x = V_{y_1} \cup \ldots \cup V_{y_n}$ and $U_x = U_{y_1} \cap \ldots \cap U_{y_n}$ are disjoint neighborhoods of Y and x respectively. If $X \subseteq U_{x_1} \cup \ldots \cup U_{x_m}$, then $U_{x_1} \cup \ldots \cup U_{x_m}$ and $V_{x_1} \cap \ldots \cap V_{x_m}$ are disjoint neighborhoods of X and Y respectively.

Proposition: *Given disjoint closed sets Y, Z in a metric space X, there is a continuous function $f: X \to [0, 1]$ such that $Y = f^{-1}(0)$, $Z = f^{-1}(1)$. In particular, any metric space is a normal T_1 space.*

Proof: The functions $g(x) = \mathrm{d}(x, Y)$ and $h(x) = \mathrm{d}(x, Z)$ are continuous (p. 27). Just put $f = g/(g + h)$.

Definition: Let X be a topological space. A **connected component** of X is a maximal connected subspace, and X is **locally connected** if any point has a base of connected neighborhoods.

Theorem: (1) *If a set $C \subseteq X$ is connected, then the closure $\bar{C}$ also is connected.*

(2) *If some connected sets $C_i \subseteq X$ have non-empty intersection, then $\bigcup_i C_i$ is connected.*

(3) *Any point of X lies in a unique connected component of X.*

(4) *Any connected component of X is closed and, if X is locally connected, it is also open.*

Proof: (1) Let U be a non-empty open closed set in $\bar{C}$. Then $U \cap C$ is non-empty, because C is dense in $\bar{C}$; hence $U \cap C = C$, and we conclude that $U = \bar{C}$ because U is closed.

(2) Now let U be an open closed set in $\bigcup_i C_i$ and pick a point $x \in \bigcap_i C_i$. We may assume that $x \in U$ (otherwise replace U by U^c). Now $C_i \cap U$ is a non-empty open closed set in C_i; hence $C_i \cap U = C_i$ for any index i, and we conclude that $U = \bigcup_i C_i$.

(3) Given $x \in X$, the union of all connected sets $C_i \subseteq X$ containing x is connected by (2), and it is the unique connected component of X containing x.

(4) If C is a connected component of X, then $C \subseteq \bar{C}$ and $\bar{C}$ is connected by (1); hence $C = \bar{C}$.

If moreover X is locally connected and U is a connected neighborhood of $x \in C$, then $U \cup C$ is connected; hence $U \cup C = C$, so that $U \subseteq C$ and we see that C is open.

Definition: A topological space is **locally compact** if any point has a compact neighborhood, and it is **σ-compact** if moreover it is separated and with a countable base of open sets. A subset of a topological space is **relatively compact** if it has compact closure.

Proposition: *In a locally compact separated space, any point has a base of compact neighborhoods.*

Proof: We may assume that X is a compact separated space.

If U is a neighborhood of $x \in X$, then x and U^c have disjoint open neighborhoods, $x \in U_1$, $U^c \subseteq U_2$, so that $U_2^c \subseteq U$ is a closed (hence compact) neighborhood of x.

Proposition: *Any σ-compact space X admits a countable cover by compact sets, $X = \bigcup_n K_n$, such that K_n is contained in the interior of K_{n+1}.*

Proof: Let $X = \bigcup_n U_n$ be a countable cover by relatively compact open sets.

Put $K_1 = \overline{U}_1$. Given K_{n-1}, it admits a finite open cover, $K_{n-1} \subseteq U_{i_1} \cup \ldots \cup U_{i_r}$, and we put $K_n = \overline{U}_{i_1} \cup \ldots \overline{U}_{i_r} \cup \overline{U}_n$.

Proposition: *If X is a connected and locally connected σ-compact space, then $X = \bigcup_n K_n$, where K_n is a compact set contained in the interior of K_{n+1}, and $X - K_n$ has no relatively compact connected component.*

Proof: If $K \subset X$ is compact, let $K^\flat$ be the union of K with the relatively compact connected components of $X - K$. Then $K^\flat$ is closed, because the connected components of $X - K^\flat$ are just the non relatively compact components of $X - K$, and moreover $K^\flat$ is compact:

Let V be a relatively compact open set containing K, and U a connected component of $X - K$. Since X is connected, $U \cap V \neq \emptyset$ (if $U \subseteq X - V$, then $U = \bar{U}$ because $\partial U \subseteq K$). Since U is connected, $U \subseteq V$ or $U \cap (\partial V) \neq \emptyset$ (otherwise $U \subseteq V \amalg (X - \bar{V})$). Only a finite number $U_1, \ldots, U_n$ of relatively compact components intersect the compact set ∂V, so that $K^\flat \subseteq U_1 \cup \ldots \cup U_n \cup V$ is relatively compact; hence compact, since it is closed.

Finally, put $X = \bigcup_n Q_n$, with Q_n compact and $Q_n \subseteq Q_{n+1}$. Take $K_0 = Q_0^\flat$ and, inductively, $K_n = L_n^\flat$, where L_n is a compact set containing $K_{n-1} \cup Q_{n-1}$ in the interior.

Examples: Any open set in $\mathbb{R}^n$ is a locally compact and locally connected space; hence any **topological manifold** X of dimension n (any point of X has an open neighborhood homeomorphic[2] to an open set in $\mathbb{R}^n$) also is locally compact and locally connected.

Hence any open or closed subset of $\mathbb{R}^n$ is a σ-compact space.

Any separated topological manifold with a countable base of open sets is σ-compact.

Any set X, with the **discrete** topology (any subset is an open set), is separated, locally connected and locally compact, but not σ-compact when $|X| > \aleph_0$.

Let X be a locally compact non-compact separated space. If we consider on the disjoint union $X^* := X \cup \{\infty\}$ the topology such that X is an open subspace and the open neighborhoods of ∞ are the complements of the compact sets $K \subset X$, then X^* is a compact separated space, the **one-point compactification** of X.

Theorem: *Any open subset U of a compact separated space K is Baire.*

Proof: Replacing U by $\bar{U}$, we may assume that U is dense; hence, we only have to show that K is Baire. Let $\{U_n\}$ be a countable family of dense open sets in K, and let us see that any non-empty open set $V \subseteq K$ intersects $\bigcap_n U_n$.

Since $U_1 \cap V \neq \emptyset$ and K is regular, there is a non-empty open set V_1 with $\bar{V}_1 \subseteq U_1 \cap V$.

Recursively, we may find non-empty open sets V_n with $\bar{V}_n \subseteq V_{n-1} \cap U_n$. Since $\{\bar{V}_n\}$ is a decreasing sequence of non-empty closed sets, there is $x \in \bigcap_n \bar{V}_n \subseteq V \cap \left(\bigcap_n U_n\right)$.

Corollary: *Any locally compact separated space X is Baire.*

Proof: X is an open subset of the one-point compactification X^*.

4.2 Differential Calculus

In this section $U \subseteq \mathbb{R}^n$ and $U' \subseteq \mathbb{R}^m$ are open subsets, $a = (a_1, \ldots, a_n) \in U$, and $\partial_i := \frac{\partial}{\partial x_i}$.

Definition: A function $f\colon U \to \mathbb{R}$ is **differentiable** at a when there is a polynomial function P of degree ≤ 1 such that the quotients $\frac{f(x)-P(x)}{\|x-a\|}$ have null limit as $x \to a$ (it is a local concept). Since clearly $P(a) = f(a)$, this means the existence of constants $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - \sum_i c_i(x_i - a_i)}{\|x - a\|} = 0.$$

Theorem: *A function $f\colon U \to \mathbb{R}$ is differentiable at $a \in U$ if and only if there are functions $h_i\colon U \to \mathbb{R}$, continuous at a, such that $f = f(a) + \sum_i h_i \cdot (x_i - a_i)$.*

Moreover, in any such decomposition we have $h_i(a) = \frac{\partial f}{\partial x_i}(a)$.

Proof: If f is differentiable at a, we have $f(x) = f(a) + \sum_i c_i(x_i - a_i) + O(x)\|x - a\|$ for some function $O\colon U \to \mathbb{R}$, continuous at a and vanishing at a. Therefore

$$f(x) = f(a) + \textstyle\sum_i c_i(x_i - a_i) + \sum_i \frac{O(x)(x_i - a_i)}{\|x-a\|}(x_i - a_i) = f(a) + \sum_i h_i(x)(x_i - a_i),$$

where $h_i = c_i + \frac{x_i - a_i}{\|x-a\|} O$ is continuous at $x = a$, because $O(a) = 0$ and $\left|\frac{x_i - a_i}{\|x-a\|}\right| \leq 1$.

[2]This definition only is sensible if an open set in $\mathbb{R}^n$ can not be homeomorphic to an open set in $\mathbb{R}^m$ when $n \neq m$, a fact that will be proved in p. 244.

Conversely, if $f = f(a) + \sum_i h_i \cdot (x_i - a_i)$ for some functions h_i continuous at a, then $h_i = c_i + O_i$, where the function O_i is continuous at a and $O_i(a) = 0$. Hence

$$f(x) = f(a) + \textstyle\sum_i c_i(x_i - a_i) + \left(\sum_i O_i(x) \frac{x_i - a_i}{\|x-a\|}\right) \|x - a\|,$$

where $O := \sum_i O_i \cdot \frac{x_i - a_i}{\|x-a\|}$ is continuous at a and $O(a) = 0$. Now, considering the function

$$\sigma_i(x_i) = f(a_1, \ldots, x_i, \ldots, a_n) = \sigma_i(a_i) + h_i(a_1, \ldots, x_i, \ldots, a_n)(x_i - a_i),$$

we see (p. 30) that $h_i(a)$ is just the derivative of $\sigma_i(x_i)$ at $x_i = a_i$; i.e. $h_i(a) = \frac{\partial f}{\partial x_i}(a)$.

Definition: A map $F = (f_1, \ldots, f_m)\colon U \to U'$ is **differentiable** at a when so are $f_1, \ldots, f_m$;

$$F(x) = F(a) + H_x(x - a) \quad , \quad f_i(x) = f_i(a) + \textstyle\sum_j h_{ij}(x) \cdot (x_j - a_j),$$

for some map $U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$, $x \mapsto H_x = \big(h_{ij}(x)\big)$, continuous at a (hence so is F).

We have seen that the linear map $F_{*,a} := H_a$, named **tangent linear map** of F at a, does not depend on H and is given by the **jacobian matrix**:

$$F_{*,a} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_m}{\partial x_1}(a) & \ldots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}.$$

The map F is **differentiable** when so it is at any point of U (in particular F is continuous).

Remarks: (1) Any affine map F is differentiable, and $F_{*,a} = \vec{F}$ at any point a.

(2) In the case of a function $f\colon U \to \mathbb{R}$, the tangent linear map $\mathrm{d}_a f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be the **differential** of f at a:

$$\mathrm{d}_a f = \frac{\partial f}{\partial x_1}(a)\, \mathrm{d}_a x_1 + \ldots + \frac{\partial f}{\partial x_n}(a)\, \mathrm{d}_a x_n.$$

Clearly $\mathrm{d}_a(f + h) = \mathrm{d}_a f + \mathrm{d}_a h$ and $\mathrm{d}_a(fh) = h(a)\mathrm{d}_a f + f(a)\mathrm{d}_a h$.

(3) A way to see that $F_{*,a}$ doesn't depend on H is to remark that for any vector $e \in \mathbb{R}^n$,

$$F_{*,a}(e) = \lim_{t\to 0} \frac{F(a+te) - F(a)}{t}.$$

In the case of a function f, this limit is said to be the **directional derivative** $(\partial_e f)(a)$ of f at the point a in the direction of the vector e:

$$(\partial_e f)(a) := \lim_{t\to 0} \frac{f(a+te) - f(a)}{t} = \frac{\mathrm{d}(f \circ \sigma)}{\mathrm{d}t}\Big|_{t=0} \ , \quad \sigma(t) = a + te \ .$$

Theorem: *If a differentiable function f has a local extremum at a point $a \in U$, then $\mathrm{d}_a f = 0$.*

Proof: We have $(\mathrm{d}_a f)(e) = 0$, $\forall e \in \mathbb{R}^n$, because $f(a + te)$ has a local extremum at $t = 0$.

Chain Rule: *If $F\colon U \to U'$ is differentiable at a and $G\colon U' \to \mathbb{R}^d$ is differentiable at $b = F(a)$, then $G \circ F$ is differentiable at a and $(G \circ F)_{*,a} = G_{*,b} \circ F_{*,a}$.*

Proof: We have maps $H\colon U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ and $H'\colon U' \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^d)$, continuous at a and b respectively, such that $F_{*,a} = H_a$, $G_{*,b} = H'_b$, and for all $x \in U$, $y \in U'$,

$$\begin{aligned} F(x) &= b + H_x(x - a) \\ G(y) &= G(b) + H'_y(y - b) \\ G\big(F(x)\big) &= G(b) + H'_{F(x)}\big(F(x) - b\big) = (G \circ F)(a) + (H'_{F(x)} \circ H_x)(x - a). \end{aligned}$$

Now, the map $U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$, $x \mapsto H'_{F(x)} \circ H_x$, is continuous at a because so are H and F, and H' is continuous at $F(a)$. Hence $G \circ F$ is differentiable at a and

$$(G \circ F)_{*,a} = H'_{F(a)} \circ H_a = H'_b \circ H_a = G_{*,b} \circ F_{*,a}.$$

Corollary: $\mathrm{d}_a(F^* f) = F_a^*(\mathrm{d}_b f)$ *for any function* $f\colon U' \to \mathbb{R}$ *differentiable at* $b = F(a)$.

Definitions: A function is $\mathcal{C}^0$ when it is continuous, of **class** $\mathcal{C}^r$, $r \geq 1$, when the iterated partial derivatives of order $\leq r$ exist and are continuous, and of class $\mathcal{C}^\infty$ when it is of class $\mathcal{C}^r$ for all $r \geq 0$. We denote by $\mathcal{C}^r(U)$, $0 \leq r \leq \infty$, the $\mathbb{R}$-algebra of all functions of class $\mathcal{C}^r$ on U.

A map $F = (f_1, \ldots, f_m)\colon U \to V$ is of class $\mathcal{C}^r$ when so are the functions $f_1, \ldots, f_m$.

Hadamard's Lemma: *If* $f \in \mathcal{C}^r(U)$, $r \geq 1$, *vanishes at any point* $(a_1, \ldots, a_d, x_{d+1}, \ldots, x_n) \in U$ *and* U *is convex, then there are functions* $h_1, \ldots, h_d \in \mathcal{C}^{r-1}(U)$ *such that*

$$f = h_1 \cdot (x_1 - a_1) + \ldots + h_d \cdot (x_d - a_d).$$

Proof: We may assume that $a = 0$. Now fix a point $x = (x_1, \ldots, x_n) \in U$, and consider the function $h(t) := f(tx_1, \ldots, tx_d, x_{d+1}, \ldots, x_n)$, $-\varepsilon < t < 1 + \varepsilon$. Then

$$\begin{aligned} f(x) = h(1) - h(0) &= \int_0^1 h'(t)\,\mathrm{d}t = \sum_{i=1}^d \int_0^1 x_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_d, x_{d+1}, \ldots, x_n)\,\mathrm{d}t \\ &= \sum_{i=1}^d x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_d, x_{d+1}, \ldots, x_n)\,\mathrm{d}t = \sum_{i=1}^d x_i h_i(x)\ , \end{aligned}$$

where h_i is of class $\mathcal{C}^{r-1}$ by the differentiation rule under the integral sign (p. 36).

Corollary: *Any function of class* $\mathcal{C}^1$ *is differentiable. Hence, a function* f *is of class* $\mathcal{C}^1$ *if and only if there is a continuous family of polynomials* $\{P_a\}_{a \in U}$ *of degree* ≤ 1 *such that*

$$\lim_{x \to a} \frac{f(x) - P_a(x)}{\|x - a\|} = 0.$$

Moreover, in such case, these polynomials are unique: $P_a = f(a) + \sum_i \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$.

Proof: If $f \in \mathcal{C}^1(U)$, then $f - f(a) = \sum_i h_i \cdot (x_i - a_i)$ where the functions h_i are continuous.

Corollary: *A map* $F\colon U \to U'$ *is of class* $\mathcal{C}^r$, $r \geq 1$, *if and only if it is differentiable and the map* $U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$, $x \mapsto F_{*,x}$, *is of class* $\mathcal{C}^{r-1}$.

Corollary: *If* $F\colon U \to U'$ *and* $G\colon U' \to U''$ *are of class* $\mathcal{C}^r$, *then so is* $G \circ F$.

Proof: By induction on r, it follows from the chain rule.

Corollary: *A map* $F\colon U \to U'$ *is of class* $\mathcal{C}^r$ *if and only if* $F^* f \in \mathcal{C}^r(U)$ *for all* $f \in \mathcal{C}^r(U')$.

Schwarz's Theorem: *If* f *is a function of class* $\mathcal{C}^2$, *then* $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$.

Proof: We may assume that f is a function on an open ball around the origin and $f(0) = 0$, so that $f = \sum_k f_k x_k$ for some functions f_k of class $\mathcal{C}^1$, and $f_k(0) = (\partial_k f)(0)$.

Applying ∂_i we see that $\partial_i f = f_i + \sum_k (\partial_i f_k) x_k$, where the functions $(\partial_i f_k)$ are continuous, so that $h := \sum_k (\partial_i f_k) x_k = \partial_i f - f_i$ is of class $\mathcal{C}^1$ and $(\partial_j h)(0) = (\partial_i f_j)(0)$.

Since $\partial_j h = \partial_j\partial_i f - \partial_j f_i$, we have $(\partial_j\partial_i f)(0) = (\partial_j f_i)(0) + (\partial_i f_j)(0)$ and we conclude.

Definition: The bilinear map $H_f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, H_f(e,v) = (\partial_e\partial_v f)(p)$, is the **hessian** of the function $f \in \mathcal{C}^2(U)$ at the point $a \in U$.

The matrix of the hessian is just $\big((\partial_i\partial_j f)(p)\big)$; hence the hessian is a symmetric bilinear form and $H(e,e)$ is just the second derivative of $h(t) = f(a+te)$ at $t=0$.

Corollary: *Let a be a* ***critical*** *point (i.e. $\mathrm{d}_a f = 0$) of a function $f \in \mathcal{C}^2(U)$. If the hessian is positive (resp. negative) definite, then the point a is a strict local minimum (resp. maximum) of f. If the hessian has positive index, then the point a is not a local extremum of f.*

Proof: With the notations of the theorem, since $f_i(0) = (\partial_i f)(0) = 0$, we have that

$$f(x) = \sum_{i=1}^n f_i x_i = \sum_{i=1}^n \big(\sum_{j=1}^n f_{ij}x_j\big)x_i = \tfrac{1}{2}\sum_{i,j=1}^n (f_{ij}+f_{ji})x_ix_j = \tfrac{1}{2}\sum_{i,j=1}^n h_{ij}(x)x_ix_j = \tfrac{1}{2}H_x(x,x),$$

where $H_x = \big(h_{ij}(x)\big)$ is symmetric and $H_0 = \big((f_{ij}+f_{ji})(0)\big) = \big((\partial_i\partial_j f)(0)\big)$ is the hessian of f.

If H_0 is positive definite, so is H_x in a neighborhood, because the positive definite metrics define an open set in the space of metrics (p. 167), and $f(x) = \frac{1}{2}H_x(x,x) > 0$ when $x \neq 0$.

A similar argument works when H_0 is negative definite.

If H_0 has positive index, there are vectors e, v such that $H_0(e,e) > 0$ and $H_0(v,v) < 0$. Hence, the function $f(te)$ has a strict local minimum at $t=0$, and $f(tv)$ has a strict local maximum (p. 33). We conclude that f has not a local minimum nor a local maximum.

Definition: A multi-index of length n is a sequence $\alpha = (\alpha_1,\dots,\alpha_n)$ of n natural numbers, and we put $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = (\alpha_1!)\ldots(\alpha_n!)$, $x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$, and

$$\partial_\alpha f = \frac{\partial^{|\alpha|}f}{\partial x^\alpha} = \frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}.$$

The d-th **Taylor polynomial** $T_a^d f$ of a function $f \in \mathcal{C}^d(U)$ at a point $a \in U$ is the unique polynomial of degree $\leq d$ with the same partial derivatives of order $\leq d$ at such point,

$$T_a^d f := \sum_{|\alpha|\leq d} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(a)\cdot(x_1-a_1)^{\alpha_1}\ldots(x_n-a_n)^{\alpha_n}.$$

Lemma: *Let $U \subseteq \mathbb{R}^n$ be a convex open set, $f \in \mathcal{C}^r(U)$ and $0 \in U$. If $(\partial_\alpha f)(0) = 0$, $\forall|\alpha| \leq d \leq r$, then there are functions $O_\alpha \in \mathcal{C}^{r-d}(U)$ vanishing at p such that $f = \sum_{|\alpha|=d} O_\alpha \cdot x^\alpha$.*

Proof: By Hadamard's lemma, $f = \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\dots,tx_n)\,\mathrm{d}t = \sum_{i=1}^n h_ix_i$, $h_i \in \mathcal{C}^{r-1}(U)$.

By the differentiation rule under the integral sign (p. 36), all the partial derivatives of order $\leq d-1$ of the functions h_i vanish at 0.

By induction on d, we have $h_i = \sum_{|\beta|=d-1} O_{i,\beta}\cdot x^\beta$, $O_{i,\beta} \in \mathcal{C}^{r-d}(U)$, and we conclude.

Taylor's Formula: *Let $U \subseteq \mathbb{R}^n$ be a convex open set, $f \in \mathcal{C}^r(U)$ and $p \in U$. If $d \leq r$, then there exist functions $O_\alpha \in \mathcal{C}^{r-d}(U)$ vanishing at a such that*

$$f = T_a^d + \sum_{|\alpha|=d} O_\alpha\cdot(x_1-a_1)^{\alpha_1}\ldots(x_n-a_n)^{\alpha_n}.$$

Proof: Apply the lemma to the function $f - T_p^d f$.

Corollary: *If $f \in \mathcal{C}^r(U)$, there exists a unique continuous family $\{P^r_a\}_{a\in U}$ of polynomials of degree $\leq r$ (namely $P^r_a = T^r_a f$) such that*

$$\lim_{x\to a} \frac{f(x) - P^r_a(x)}{\|x-a\|^r} = 0.$$

Proof: We may assume that $a = 0$. The functions $O_\alpha \cdot x^\alpha/\|x\|^r$ are continuous because $O_\alpha(0) = 0$ and $|x^\alpha/\|x\|^r| \leq 1$; hence the family $P^r_a = T^r_a f$ has the required property.

To show that the polynomials P^r_a are unique, we may also assume that $f = 0$.

Once we fix a point $x \in U$, we have $0 = f(tx) = P^r_a(tx) + O(t)t^r$, where $O(t)$ is a continuous function on a neighborhood of $[0,1]$ and $O(0) = 0$. Therefore,

$$0 = \lim_{t\to 0} \frac{P^r_a(tx)}{t^r}\,.$$

Since $P^r_a(tx)$ is a polynomial of degree $\leq r$, we see that $P^r_a(tx) = 0$. Hence $P^r_a = 0$.

4.2.1 Inverse Mapping Theorem

Definition: A bijective map $F\colon U \to U'$ of class $\mathcal{C}^r$, $r \geq 1$, is a $\mathcal{C}^r$**-diffeomorphim** when the inverse map $F^{-1}\colon U' \to U$ also is of class $\mathcal{C}^r$.

A map $F\colon U \to U'$ of class $\mathcal{C}^r$, $r \geq 1$, is a **local diffeomorphim** at $a \in U$ if it induces a $\mathcal{C}^r$-diffeomorphism of some open neighborhood of a onto an open neighborhood of $F(a)$.

When $m = n$, the determinant $|\frac{\partial f_i}{\partial x_j}(a)|$ of $F_{*,a}$ is the **jacobian** of F at a.

Hadamard's Lemma (along the diagonal): *When U is convex, a function $f\colon U \to \mathbb{R}$ is of class $\mathcal{C}^r$, $r \geq 1$, if and only if we have*

$$f(y) - f(x) = H_1(x,y)(y_1 - x_1) + \ldots + H_n(x,y)(y_n - x_n)\ ; \quad x, y \in U,$$

for some functions $H_i \in \mathcal{C}^{r-1}(U\times U)$. Moreover, for any such decomposition we have that

$$H_i(x,x) = \tfrac{\partial f}{\partial x_i}(x).$$

Proof: If $f \in \mathcal{C}^r(U)$, consider the coordinates

$$u_1 = y_1 - x_1, \ldots, u_n = y_n - x_n, u_{n+1} = y_1 + x_1, \ldots, u_{2n} = y_n + x_n,$$

on $U \times U$ (defining a diffeomorphism onto a convex open set in $\mathbb{R}^{2n}$ because linear isomorphisms preserve convex sets) and apply Hadamard's lemma at any point of the diagonal.

Now, if $f(y) - f(x) = H_1(x,y)(y_1 - x_1) + \ldots + H_n(x,y)(y_n - x_n)$, and $a = (a_1, \ldots, a_n) \in U$, then $f(x) - f(a) = \sum_i H_i(a,x)(x_i - a_i)$, where the functions $F_i(a,x)$ are continuous.

Hence $(\partial_i f)(a) = F_i(a,a)$, so that $\partial_i f \in \mathcal{C}^{r-1}(U)$ and $f \in \mathcal{C}^r(U)$.

Inverse Mapping Theorem: *A map $F\colon U \to U'$ of class $\mathcal{C}^r$, $r \geq 1$, is a local diffeomorphism at a point $a \in U$ if and only if the linear tangent map $F_{*,a}\colon \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism.*

Proof: If F is a local diffeomorphism at a, then $F_{*,a}$ and $F^{-1}_{*,F(a)}$ are mutually inverse by the Chain rule. Hence $F_{*,a}$ is an isomorphism.

Conversely, if $F_{*,a}$ is an isomorphism, then $m = n$. Put $F = (f_1, \ldots, f_n)$.

We may assume that U is convex and, by Hadamard's lemma, that

$$F(y) - F(x) = H(x,y)(y - x),$$

where $H = (H_{ij})$ is a $n \times n$ matrix of functions $H_{ij} \in \mathcal{C}^{r-1}(U \times U)$, and $H(x,x) = F_{*,x}$.

Since $F_{*,a}$ is invertible, we may also assume that $H(x,y)$ is invertible for all $x, y \in U$.

1. *F is injective*:

   If $0 = F(y) - F(x) = H(x,y)(y-x)$, then $y - x = 0$, because $H(x,y)$ is invertible.

2. *F is an open map, so that $F(U)$ is open and $F\colon U \to F(U)$ is a homeomorphism*:

   Given an open set $V \subseteq U$ and a point $p \in V$, consider a closed ball $\bar{B} \subset V$ with center at p, and let S be the sphere. Since F is continuous and S is compact, $F(S)$ is compact and $p' := F(p) \notin F(S)$ because F is injective. Let us see that $F(V)$ contains the open set

   $$W = \{x' \in U' \colon \mathrm{d}(x', p') < \mathrm{d}\big(x', F(S)\big)\}.$$

   Given a point $x' \in W$, we consider the following function $f\colon U \to \mathbb{R}$ of class $\mathcal{C}^r$:

   $$f(x) = \|F(x) - x'\|^2 = \sum_{j=1}^{n} \big(f_j(x) - x'_j\big)^2.$$

   It attains a minimum at a point $x \in \bar{B}$, and x is not in S because $\mathrm{d}(x', p') < \mathrm{d}\big(x', F(S)\big)$. Hence it is a local minimum and

   $$0 = \frac{\partial f}{\partial x_i}(x) = 2\sum_{j=1}^{n}(f_j(x) - x'_j)\frac{\partial f_j}{\partial x_i}(x).$$

   Since $\left|\frac{\partial f_j}{\partial x_i}(x)\right| \neq 0$, we see that $f_j(x) - x'_j = 0$; hence $x' = F(x)$.

3. *F^{-1} is of class $\mathcal{C}^r$*:

   Given $x, y \in U$, put $x' = F(x), y' = F(y)$. Since $y' - x' = H(x,y)(y-x)$, we have

   $$F^{-1}(y') - F^{-1}(x') = H\big(F^{-1}(x'), F^{-1}(y')\big)^{-1}(y' - x').$$

   If F^{-1} is of class $\mathcal{C}^{r-1}$, so is $H\big(F^{-1}(x'), F^{-1}(y')\big)^{-1}$ and, by Hadamard's lemma, F^{-1} is of class $\mathcal{C}^r$. Since F^{-1} is of class $\mathcal{C}^0$, we conclude that it is of class $\mathcal{C}^r$.

Implicit Function Theorem: *Consider functions $f_1, \dots, f_m$ of class $\mathcal{C}^r$, $r \geq 1$, on an open set $V \subseteq \mathbb{R}^n \times \mathbb{R}^m$, and a point $(a,b) \in Y := \{(x,y) \in V \colon f_1(x,y) = \ldots = f_m(x,y) = 0\}$. If*

$$\begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{vmatrix}(a,b) \neq 0,$$

then there is an open neighborhood U of a in $\mathbb{R}^n$, functions $g_1, \dots, g_m \in \mathcal{C}^r(U)$, and an open neighborhood W of (a,b) such that

$$Y \cap W = \{(x,y) \in W \colon y_1 = g_1(x), \dots, y_m = g_m(x)\}.$$

Proof: The map $F\colon V \to \mathbb{R}^n \times \mathbb{R}^m$, $F(x,y) = \big(x, f_1(x,y), \dots, f_m(x,y)\big)$, is a local diffeomorphism at (a,b), by the inverse function theorem; hence it defines a diffeomorphism $F\colon W \simeq F(W)$ in some neighborhood W, and $F(Y \cap W) = \{(x,0) \in F(W)\}$. Put $U = \{x \in \mathbb{R}^n \colon (x,0) \in F(W)\}$.

Then $F^{-1}(x,0) = \big(x, g_1(x), \dots, g_m(x)\big)$, where $g_1, \dots, g_m$ are the required functions.

4.3 The Lebesgue measure

Definition: Let X be a set. A family of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is an **algebra** (resp. **σ-algebra**) if it is closed under finite (resp. countable) unions and intersections, complement, and $\emptyset, X \in \mathcal{A}$.

A **measurable space** $(X, \mathcal{A})$ is a set endowed with a σ-algebra of subsets. If Y is a subset of X, then $Y \cap \mathcal{A} = \{Y \cap A;\ A \in \mathcal{A}\}$ defines a structure of measurable space on Y.

A map $f\colon (Y, \mathcal{B}) \to (X, \mathcal{A})$ is **measurable** when $f^{-1}(A) \in \mathcal{B}$ for all $A \in \mathcal{A}$.

Compositions of measurable maps are also measurable maps.

Any intersection $\bigcap_i \mathcal{A}_i$ of σ-algebras $\mathcal{A}_i \subseteq \mathcal{P}(X)$ is a σ-algebra; hence any family of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ generates a σ-algebra: the intersection of all σ-algebras containing $\mathcal{B}$. The **Borel** σ-algebra $\mathcal{B}(X)$ of a topological space X is the σ-algebra generated by the open subsets and, when Y is a subspace of X, we have $\mathcal{B}(Y) = Y \cap \mathcal{B}(X)$.

Definition: A **measure** μ on a measurable space $(X, \mathcal{A})$ is a map $\mu\colon \mathcal{A} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$.
2. $\mu\big(\coprod_{n\in\mathbb{N}} A_n\big) = \sum_{n=0}^{\infty} \mu(A_n)$.

and a **measure space** is a measurable space endowed with a measure. When $A \in \mathcal{A}$, then μ also defines a measure on $(A, A \cap \mathcal{A})$.

Proposition: *If A_n is an increasing sequence in $\mathcal{A}$, then $\mu\big(\bigcup_n A_n\big) = \sup \mu(A_n)$.*

Proof: Put $B_n = A_{n-1}^c \cap A_n$. Then $\mu(\bigcup A_n) = \sum_n \mu(B_n) = \lim \mu(B_1 \cup \ldots \cup B_n) = \sup \mu(A_n)$.

Definitions: Given a (bounded) interval $I = (a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$, with $a \leq b$, we put $\mu(I) = b - a$. The empty set (a, a) and a point $[a, a]$ are intervals, and $\mu(\emptyset) = \mu(a) = 0$.

In $\mathbb{R}^d$, $d \geq 2$, a (bounded) **rectangle** is direct product $I = I_1 \times \ldots \times I_d$ of (bounded) intervals and we put $\mu(I) = \mu(I_1) \cdot \ldots \cdot \mu(I_d)$. We admit open and closed rectangles, as well as degenerated rectangles I (at least one factor is empty or a point) with $\mu(I) = 0$.

If $I, J \subset \mathbb{R}$ are intervals, so is $I \cap J$, and $I^c \cap J$ is a finite union of disjoint intervals; hence, if $I, J \subset \mathbb{R}^d$ are rectangles, so is $I \cap J$, and $I^c \cap J$ is a finite union of disjoint rectangles.

Lemma: *Let $I_1, \ldots, I_n, I$ be rectangles in $\mathbb{R}^d$. If $I \subseteq I_1 \cup \ldots \cup I_n$, then $\mu(I) \leq \mu(I_1) + \ldots + \mu(I_n)$. If $I_1, \ldots, I_n$ are disjoint and $I_1 \cup \ldots \cup I_n \subseteq I$, then $\mu(I_1) + \ldots + \mu(I_n) \leq \mu(I)$.*

Proof: By induction on d, and it is obvious when $d = 0$ (assuming that $\mu(\mathbb{R}^0) = \mu(0) = 1$).

When $d \geq 1$, any rectangle $I \subset \mathbb{R}^d$ defines a family $I_x = \{y \in \mathbb{R}^{d-1}\colon (x, y) \in I\}$ of rectangles in $\mathbb{R}^{d-1}$, and

$$\mu(I) = \int_{\mathbb{R}} \mu(I_x) \mathrm{d}x.$$

By induction $\mu(I_x) \leq \mu\big((I_1)_x\big) + \ldots + \mu\big((I_n)_x\big)$, and the result follows from the properties of the Riemann integral[3] on $\mathbb{R}$.

Definition: The (d-dimensional) **exterior measure** of $A \subseteq \mathbb{R}^d$ is the infimum of the total volume of all countable covers of A by rectangles:

$$m_e(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(I_n)\,;\ A \subseteq \bigcup_n I_n \right\}.$$

[3]Since a logical dependence of the Lebesgue measure on the Riemann integral is disappointing, let us give an elementary proof of this intuitively obvious lemma: Projecting the vertices of $I_1, \ldots, I_n$ onto the edges of I we obtain partitions of such edges which define a partition of I into smaller rectangles. Replacing each rectangle I_i by the rectangles of such partition contained in it, we may assume that all the rectangles $I_1, \ldots, I_n$ belong to such partition, and in that case the lemma is obvious.

Given $\varepsilon > 0$, it is clear that I_n is contained in an open (resp. closed) rectangle J_n with $\mu(J_n) < \mu(I_n) + \varepsilon/2^n$, so that $\sum_n \mu(J_n) < \varepsilon + \sum_n \mu(I_n)$. Hence, in the definition of $m_e(A)$ we may only consider countable covers by open (resp. closed) rectangles.

When $m_e(A) = 0$, we say that A has **null measure**.

1. $m_e(\emptyset) = 0$.

2. $m_e(A) \leq m_e(B)$ *whenever* $A \subseteq B$.

3. $m_e\big(\bigcup_n A_n\big) \leq \sum_n m_e(A_n)$.

   Given $\varepsilon > 0$, we consider covers $A_n \subseteq \bigcup_m I_{n,m}$ such that $\sum_m \mu(I_{n,m}) < \varepsilon/2^n + m_e(A_n)$. Now $\bigcup_n A_n \subseteq \bigcup_{n,m} I_{n,m}$, and

   $$m_e\big(\bigcup_n A_n\big) \leq \sum_n \big(\sum_m \mu(I_{n,m})\big) < \sum_n \big(\varepsilon/2^n + m_e(A_n)\big) = \varepsilon + \sum_n m_e(A_n).$$

4. $m_e(I) = \mu(I)$ *for any rectangle* I.

   It is clear that $m_e(I) \leq \mu(I)$. Now, given a cover $I \subseteq \bigcup_n I_n$ and $\varepsilon > 0$, pick a closed rectangle $K \subseteq I$ such that $\mu(K) > \mu(I) - \varepsilon$ and open rectangles $U_n \supseteq I_n$ such that $\mu(U_n) < \mu(I_n) + \varepsilon/2^n$. Since K is compact, it admits a finite cover $K \subset U_{n_1} \cup \ldots \cup U_{n_r}$, so that $\mu(K) \leq \sum_n \mu(U_n)$ by the lemma. Hence,

   $$\mu(I) < \varepsilon + \mu(K) \leq \varepsilon + \sum_n \mu(U_n) < 2\varepsilon + \sum_n \mu(I_n).$$

5. *The exterior measure is invariant under translations*: $m_e(a + A) = m_e(A)$.

Examples: The argument of p. 36 shows that *any countable union of sets of null measure also has null measure.* Hence, countable sets have null measure (proving again that the interval $[0, 1]$ is uncountable), as well as hyperplanes $x_i = a_i$, and the set of points with some rational coordinate.

In $\mathbb{R}$, we have $m_e(A) \leq m_e(\mathbb{Q} \cap A) + m_e(\mathbb{Q}^c \cap A) = m_e(\mathbb{Q}^c \cap A) \leq m_e(A)$ for any set $A \subseteq \mathbb{R}$; hence $m_e(\mathbb{Q}^c \cap A) = m_e(A)$. In particular $m_e(\mathbb{Q}^c \cap [a, b]) = b - a$, and $m_e(\mathbb{Q}^c) = \infty$.

The set of algebraic numbers is countable, hence, if T is the set of transcendent real numbers, we have $m_e(T \cap [a, b]) = b - a$, and if we consider in $\mathbb{C}$ the set $\bar{T}$ of transcendent complex numbers, we have $m_e(\bar{T} \cap A) = m_e(A)$ for any subset $A \subseteq \mathbb{C}$.

Caratheodory's Lemma: *If L is a subset of $\mathbb{R}^d$, the following conditions are equivalent*:

1. $\mu(I) = m_e(L \cap I) + m_e(L^c \cap I)$, *for any rectangle* I.

2. $m_e(A) = m_e(L \cap A) + m_e(L^c \cap A)$, *for any set* $A \subseteq \mathbb{R}^d$.

Proof: ($1 \Rightarrow 2$) We always have $m_e(A) = m_e\big((L \cap A) \cup (L^c \cap A)\big) \leq m_e(L \cap A) + m_e(L^c \cap A)$.

Given $\varepsilon > 0$, consider a cover $A \subseteq \bigcup_n I_n$ such that $\sum_n \mu(I_n) < \varepsilon + m_e(A)$.

We have $(L \cap A) \subseteq \bigcup_n (L \cap I_n)$, $(L^c \cap A) \subseteq \bigcup_n (L^c \cap I_n)$; hence

$$m_e(L \cap A) + m_e(L^c \cap A) \leq \sum_n m_e(L \cap I_n) + \sum_n m_e(L^c \cap I_n) = \sum_n m_e(I_n) < \varepsilon + m_e(A).$$

Definition: A set $L \subseteq \mathbb{R}^d$ is **measurable** when it satisfies the above equivalent conditions.

Note: If L is a bounded set, contained in a rectangle K, then $\mu(K) - m_e(K - L)$ deserves the name of interior measure of L, so that condition $\mu(K) = m_e(L \cap K) + m_e(L^c \cap K)$ states that the interior and exterior measures of L coincide.

Theorem: *The Lebesgue measurable sets in $\mathbb{R}^d$ form a σ-algebra containing the rectangles (hence all the Borel sets) and any set of null measure, and $m = m_e$ defines a measure (the **Lebesgue measure**) invariant under translations.*

Proof: If L is measurable, clearly so is L^c, and $\emptyset$ and $\mathbb{R}^d$ are obviously measurable.

Any set N of null measure is measurable, since $m_e(N\cap I)+m_e(N^c\cap I) = m_e(N^c\cap I) \leq m_e(I)$.

Any rectangle J is measurable because, for any rectangle I, we have that $J^c \cap I$ is a finite union of disjoint rectangles, $J^c \cap I = I_1 \amalg \ldots \amalg I_n$, and by the lemma

$$m_e(J \cap I) + m_e(J^c \cap I) \leq \mu(J \cap I) + \sum_{i=1}^{n} m_e(I_i) = \mu(J \cap I) + \sum_{i=1}^{n} \mu(I_i) = \mu(I).$$

1. *If L_1 and L_2 are measurable, then so is $L_1 \cup L_2$* (hence so is $L_1 \cap L_2 = (L_1^c \cup L_2^c)^c$):

   $m_e\big((L_1 \cup L_2) \cap A\big) + m_e\big((L_1 \cup L_2)^c \cap A\big) =$
   $= m_e\big(L_1 \cap (L_1 \cup L_2) \cap A\big) + m_e\big(L_1^c \cap (L_1 \cup L_2) \cap A\big) + m_e\big((L_1 \cup L_2)^c \cap A\big)$
   $= m_e(L_1\cap A)+m_e(L_2\cap L_1^c\cap A)+m_e(L_2^c\cap L_1^c\cap A) = m_e(L_1\cap A)+m_e(L_1^c\cap A) = m_e(A).$

2. *Countable disjoint unions of measurable sets are measurable, and $m\big(\amalg_n L_n\big) = \sum_n m(L_n)$:*

   $m_e\big((\bigcap_{n=1}^{k-1} L_n^c) \cap A\big) = m_e\big((\bigcap_{n=1}^{k-1} L_n^c) \cap L_k^c \cap A\big) + m_e\big((\bigcap_{n=1}^{k-1} L_n^c) \cap L_k \cap A\big) = m_e\big((\bigcap_{n=1}^{k} L_n^c) \cap A\big) + m_e(L_k \cap A)$. Hence

   $$m_e(A) = m_e(L_1^c \cap A) + m_e(L_1 \cap A) = m_e(L_1^c \cap L_2^c \cap A) + \sum_{n=1}^{2} m_e(L_n \cap A) = \ldots$$
   $$= m_e\big((\textstyle\bigcap_{n=1}^{k} L_n^c) \cap A\big) + \sum_{n=1}^{k} m_e(L_n \cap A) \geq m_e(L^c \cap A) + \sum_{n=1}^{k} m_e(L_n \cap A),$$

   where $L = \amalg_n L_n$. Considering the limit as $k \to \infty$, we see that L is measurable:

   $$m_e(A) \geq m_e(L^c \cap A) + \sum_{n=1}^{\infty} m_e(L_n \cap A) \geq$$
   $$\geq m_e(L^c \cap A) + m_e\big((\textstyle\bigcup_n L_n) \cap A\big) = m_e(L^c \cap A) + m_e(L \cap A)\ .$$

   When $A = L$, we have $m(L) \geq \sum_n m(L_n \cap L) = \sum_n m(L_n)$; hence $m(L) = \sum_n m(L_n)$.

3. *Countable unions of measurable sets are measurable* (hence countable intersections):

   In fact $L = \bigcup_n L_n$ is the disjoint union of the measurable sets $L_1^c \cap \ldots \cap L_{n-1}^c \cap L_n$.

4. *If L is measurable, so is any translation $a + L$:*

   We have $(a + L) \cap I = a + \big(L \cap (I - a)\big)$, $(a + L)^c \cap I = a + \big(L^c \cap (I - a)\big)$ and the exterior measure m_e is invariant under translations.

Proposition: *A set $L \subseteq \mathbb{R}^d$ is measurable if and only if $L = B \cup N$ with B a Borel set and $m(N) = 0$; hence if and only if L is contained in a Borel set B such that $m(B - L) = 0$.*

Proof: If $m(L) < \infty$, there is a Borel set $B \supseteq L$ (in fact a countable intersection of countable covers of L by rectangles) such that $m(L) = m(B)$; hence $m(B - L) = 0$.

When $m(L) = \infty$, we put $\mathbb{R}^d = \amalg_n I_n$ as a countable disjoint union of rectangles. We have Borel sets $B_n \supseteq (L \cap I_n)$ with $m\big(B_n - (L \cap I_n)\big) = 0$; hence $L \subseteq B = \bigcup_n B_n$ and $m(B - L) = 0$.

Finally, if B is a Borel set containing L^c and $m(B - L^c) = 0$, then $L = B^c \cup (B \cap L)$, and $m(B \cap L) = m(B - L^c) = 0$.

Proposition: *If $L_1 \subseteq \mathbb{R}^{n_1}$, $L_2 \subseteq \mathbb{R}^{n_2}$ are measurable, so is $L_1 \times L_2 \subseteq \mathbb{R}^{n_1+n_2}$ and*

$$m(L_1 \times L_2) = m_1(L_1)m_2(L_2) \ , \quad \text{(where } 0 \cdot \infty = 0).$$

Proof: Let $I \subset \mathbb{R}^{n_2}$ be a rectangle. Since $\mu(B) = m(B \times I)$ is an invariant measure on $\mathcal{B}(\mathbb{R}^{n_1})$, we have $\mu = m_2(I)m_1$; hence $m(B \times I) = m_1(B)m_2(I)$ for any Borel set $B \subseteq \mathbb{R}^{n_1}$.

Now, if $m_1(B_1) < \infty$, then $\mu(B) = m(B_1 \times B)$ is an invariant measure on $\mathcal{B}(\mathbb{R}^{n_2})$.

It follows that $\mu = m_1(B_1)m_2$; hence $m(B_1 \times B_2) = m_1(B_1)m_2(B_2)$.

If $m_1(B_1) = m_2(B_2) = \infty$, there is a rectangle $I \subset \mathbb{R}^{n_2}$ such that $0 < m_2(B_2 \cap I) < \infty$.

Hence $\infty = m\big(B_1 \times (B_2 \cap I)\big) \leq m(B_1 \times B_2)$, and $m(B_1 \times B_2) = \infty$.

Finally, in general $L_i = B_i \cup N_i$, with B_i a Borel set and $m_i(N_i) = 0$, so that

$$L_1 \times L_2 = (B_1 \times B_2) \cup (N_1 \times L_2) \cup (L_1 \times N_2)$$

is measurable, because $B_1 \times B_2$ is a Borel set and $m(N_1 \times \mathbb{R}^m) = m(\mathbb{R}^n \times N_2) = 0$, since $m(N_1 \times I) = 0$ for any interval I.

Moreover, $m(L_1 \times L_2) = m(B_1 \times B_2) = m_1(B_1)m_2(B_2) = m_1(L_1)m_2(L_2)$.

Lemma: *There is a unique measure m on $\mathcal{B}(\mathbb{R}^d)$ such that $m(I) = \mu(I)$ for any rectangle I.*

Proof: Let $\bar{m}$ be a measure such that $\bar{m}(I) = \mu(I)$ for any rectangle I, and let B be a Borel set.

If $B \subseteq \bigcup_n I_n$, then $\bar{m}(B) \leq \sum_n \bar{m}(I_n) = \sum_n \mu(I_n)$; hence $\bar{m}(B) \leq m(B)$.

Now, if B is contained in a rectangle I, then $\bar{m}(B) \leq \bar{m}(I) = \mu(I) < \infty$ and

$$\mu(I) - \bar{m}(B) = \bar{m}(I) - \bar{m}(B) = \bar{m}(I - B) \leq m(I - B) = \mu(I) - m(B) \ ,$$

so that $m(B) \leq \bar{m}(B)$. When B is unbounded, we consider a partition $\mathbb{R}^d = \amalg_n I_n$ as a disjoint union of rectangles I_n, and we conclude:

$$\bar{m}(B) = \textstyle\sum_n \bar{m}(I_n \cap B) = \sum_n m(I_n \cap B) = m(B) \ .$$

Theorem: *If a measure $\tilde{m}$ on $\mathcal{B}(\mathbb{R}^d)$ is invariant under translations and $\tilde{m}\big([0,1)^d\big) = c < \infty$, then we have $\tilde{m} = cm$.*

Proof: If $c = 0$, then $\bar{m}(\mathbb{R}^d) = 0$, because $\mathbb{R}^d = \amalg_{n_1,\dots,n_d \in \mathbb{Z}}\big((n_1,\dots,n_d) + [0,1)^d\big)$; hence $\bar{m} = 0$.

If $c > 0$, replacing $\tilde{m}$ by $\frac{1}{c}\tilde{m}$, we may assume that $c = 1$.

For any rectangle $I = \prod_i [a_i, b_i)$, with $b_i - a_i = n_i \in \mathbb{N}$, we have $\tilde{m}(I) = m(I)$, because I is a disjoint union of $n_1 \cdots n_d$ translations of $[0,1)^d$.

Hence, for any rectangle $J = \prod_i [a_i, b_i)$, with $b_i - a_i = \frac{p_i}{q_i} \in \mathbb{Q}$, we have $\tilde{m}(J) = m(J)$, because the disjoint union of $q_1 \cdots q_d$ translations of J is a rectangle I with integer edges.

Now, for any $\varepsilon > 0$ and any open rectangle K, there are rectangles J_1, J_2 with rational edges such that $J_1 \subseteq K \subseteq J_2$ and $m(J_2) - m(J_1) < \varepsilon$; hence $\tilde{m}(K) = m(K)$.

Finally, if L is a degenerated rectangle, for any $\varepsilon > 0$ there is an open rectangle $K \supset L$ such that $\tilde{m}(K) = m(K) < \varepsilon$; hence $\tilde{m}(L) = m(L) = 0$.

We conclude by the previous lemma.

Corollary: *The Lebesgue measure is invariant under rigid motions, and any linear subvariety of $\mathbb{R}^n$ of dimension $< n$ has null measure.*

Proof: Let φ is a rigid motion. The measure $\mu(B) = m\big(\varphi(B)\big)$ is invariant under translations, and $m\big(\varphi(D)\big) = m(D) > 0$ when D is a ball, because $\varphi(D)$ is a translation of D.

Finally, if X is a linear subvariety of dimension $d < n$, then $\varphi(X) = \mathbb{R}^d \times 0$ for some rigid motion φ, and $\mathbb{R}^d \times 0$ has null measure.

Theorem: *If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and $B \in \mathcal{B}(\mathbb{R}^n)$, then $m\big(T(B)\big) = |\det T| m(B)$.*

Proof: When $\det T = 0$, the theorem holds because $\operatorname{Im} T$ is contained in a vector subspace of dimension $< n$, and if the theorem holds for two endomorphisms S, T, it also holds for $S \circ T$.

Now, any linear automorphism T is (see p. 60) $T = S_1 \circ H \circ S_2$, where S_1 and S_2 are isometries and $H(x_1, \ldots, x_n) = (a_1 x_1, \ldots, a_n x_n)$, with $a_1, \ldots, a_n \in \mathbb{R}$, and the theorem clearly holds for H.

Corollary: *Given vectors $e_1 = (a_{11}, \ldots, a_{n1}), \ldots, e_n = (a_{1n}, \ldots, a_{nn})$ in $\mathbb{R}^n$, we have*

$$|\det(a_{ij})| = m\big(\{x_1 e_1 + \ldots + x_n e_n;\ 0 \leq x_1, \ldots, x_n \leq 1\}\big).$$

Proof: Just apply the theorem to $B = [0,1]^n$ and $T(x_1, \ldots, x_n) = x_1 e_1 + \ldots + x_n e_n$.

4.4 Integral Calculus

Let $(X, \mathcal{A})$ be a measurable space, and recall that a function $f\colon X \to [0, \infty]$ is measurable when $f^{-1}(B) \in \mathcal{A}$, $\forall B \in \mathcal{B}\big([0,\infty]\big)$. In fact, it is enough that $\{x \in X\colon f(x) > c\} \in \mathcal{A}$, $\forall c \in \mathbb{R}_+$ (and we may replace the sign $>$ by $<$, $\geq$ or $\leq$) because $(c, \infty]$ generate $\mathcal{B}\big([0,\infty]\big)$.

If $f, h\colon X \to [0,\infty]$ are measurable, so is $f \times h\colon X \times X \to [0,\infty] \times [0,\infty]$, because any open set in $[0,\infty] \times [0,\infty]$ is a countable union of open sets $U_1 \times U_2$; hence $f + h$ is measurable, the set $\{x \in X\colon f(x) = h(x)\}$ is in $\mathcal{A}$, etc.

Definition: A measurable function $s\colon X \to [0, \infty)$ is **simple** when $s = a_1 I_{A_1} + \ldots + a_n I_{A_n}$ for some **partition** $X = \amalg_i A_i$, with $A_i \in \mathcal{A}$, $a_i < \infty$.

Lemma: *If some functions $f_n\colon X \to [0,\infty]$ are measurable, then so are $h(x) = \sup f_n(x)$ and $g(x) = \inf f_n(x)$, and the function $f(x) = \lim f_n(x)$ when it exists.*

Proof: $\{x \in X\colon \sup f_n(x) > c\} = \bigcup_n \{x \in X\colon f_n(x) > c\} \in \mathcal{A}$,
$\{x \in X\colon \inf f_n(x) \geq c\} = \bigcap_n \{x \in X\colon f_n(x) \geq c\} \in \mathcal{A}$,
$\lim f_n = \inf_{n \geq 1} \big(\sup_{i \geq n} f_i\big)$.

Theorem: *Any measurable function $f\colon X \to [0,\infty]$ is the limit $f = \lim s_n$ of an increasing sequence of simple functions s_n.*

Proof: Let $Q_n = \big\{\frac{m}{2^n};\ 0 \leq m \leq 2^n n\big\}$ be the set of all rationals in $[0,n]$ with denominator 2^n.

Now put $s_n(x) = \max\{q \in Q_n\colon q \leq f(x)\} \in Q_n$, and s_n is a measurable function because $\{x \in X\colon s_n(x) \geq q\} = \{x \in X\colon f(x) \geq q\}$, $\forall q \in Q_n$.

Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space. The integral of a simple function s is

$$\int_X \Big(\sum_i a_i I_{A_i}\Big) \mathrm{d}\mu = \sum_i a_i \mu(A_i)\ ; \quad s = \sum_i a_i I_{A_i},\ X = \amalg_i A_i,$$

(where $0 \cdot \infty = 0$) and it does not depend on the decomposition:

If $\sum_i a_i I_{A_i} = \sum_j b_j I_{B_j}$, with $X = \amalg_j B_j$, then $a_i = b_j$ whenever $A_i \cap B_j \neq \emptyset$, and

$$\textstyle\sum_i a_i \mu(A_i) = \sum_i \sum_j a_i \mu(A_i \cap B_j) = \sum_j \sum_i b_j \mu(A_i \cap B_j) = \sum_j b_j \mu(B_j).$$

The **integral** of a measurable function $f\colon X \to [0,\infty]$ is defined to be

$$\int_X f\mathrm{d}\mu = \sup\left\{\int_X s\mathrm{d}\mu\,;\ s \text{ is simple and } 0 \le s \le f\right\} \in [0,\infty].$$

For any set $A \in \mathcal{A}$, the functions $f|_A$ and $I_A f$ also are measurable, and we have

$$\int_A (f|_A)\mathrm{d}\mu = \int_X I_A f\mathrm{d}\mu.$$

1. *When f is simple, this definition coincides with the former one*:
   If $s = \sum_j b_j I_{B_j} \le f = \sum a_i I_{A_i}$, $X = \amalg_i A_i = \amalg_j B_j$, then $b_j \le a_i$ when $A_i \cap B_j \ne \emptyset$, and
   $$\textstyle\sum_j b_j\mu(B_j) = \sum_j\sum_i b_j\mu(A_i\cap B_j) \le \sum_i\sum_j a_i\mu(A_i\cap B_j) = \sum_i a_i\mu(A_i).$$
2. *If f is measurable, then* $\displaystyle\int cf\mathrm{d}\mu = c\int f\mathrm{d}\mu$ *for any* $c \in [0,\infty)$.
3. *If f, h are measurable and $f \le h$, then* $\displaystyle\int f\mathrm{d}\mu \le \int h\mathrm{d}\mu$.
4. *If $A, B \in \mathcal{A}$ and $A \subseteq B$, then* $\displaystyle\int_A f\mathrm{d}\mu \le \int_B f\mathrm{d}\mu$.
5. *If $A \in \mathcal{A}$ and $\mu(A) = 0$, then* $\displaystyle\int_A f\mathrm{d}\mu = 0$.
   We have $\int_A s\mathrm{d}\mu = 0$ when s is simple; hence $\int_A f\mathrm{d}\mu = \sup\{\int_A s\mathrm{d}\mu;\ s \le f\} = 0$.
6. **Monotone Convergence Theorem:** *If f_n is an increasing sequence of measurable functions, then*
   $$\lim\int f_n\mathrm{d}\mu = \int(\lim f_n)\mathrm{d}\mu.$$
   The sequence (f_n) converges to a function f because (f_n) is increasing, and $f = \lim f_n$ is measurable; hence $\int f_n\mathrm{d}\mu \le \int f\mathrm{d}\mu$, so that $\lim\int f_n\mathrm{d}\mu \le \int f\mathrm{d}\mu$.
   Now, given a simple function $s = \sum_i a_i I_{A_i} \le f$, we fix $r < 1$ and we consider
   $$B_n = \{x \in X\colon rs(x) \le f_n(x)\}.$$
   We have $B_n \subseteq B_{n+1}$ because (f_n) is increasing, and $X = \bigcup_n B_n$ because $f_n(x) \to f(x)$ and $rs(x) < f(x)$ (since $r < 1$ and $s(x) < \infty$ by the definition of simple function).
   $$r\int_{B_n} s\mathrm{d}\mu \le \int_{B_n} f_n\mathrm{d}\mu \le \int_X f_n\mathrm{d}\mu \le \lim\int_X f_n\mathrm{d}\mu.$$
   Now, $r\int_{B_n} s\mathrm{d}\mu \to r\int_X s\mathrm{d}\mu$ when $n \to \infty$, because $\mu(B_n \cap A_i) \to \mu(A_i)$.
   When $r \to 1$, we see that $\int_X s\mathrm{d}\mu \le \lim\int_X f_n\mathrm{d}\mu$; hence $\int_X f\mathrm{d}\mu \le \lim\int_X f_n\mathrm{d}\mu$.
7. *If f and h are measurable, then* $\displaystyle\int(f+h)\mathrm{d}\mu = \int f\mathrm{d}\mu + \int h\mathrm{d}\mu$.
   If $f = \sum a_i I_{A_i}$ and $h = \sum b_j I_{B_j}$, then $f + h = \sum_{i,j}(a_i + b_j)I_{A_i\cap B_j}$ and
   $$\sum_{i,j}(a_i+b_j)\mu(A_i\cap B_j) = \sum_{i,j} a_i\mu(A_i\cap B_j) + \sum_{i,j} b_j\mu(A_i\cap B_j) = \sum_i a_i\mu(A_i) + \sum_j b_j\mu(B_j).$$
   In the general case, $f = \lim s_n$ and $h = \lim t_n$, where (s_n) and (t_n) are increasing sequences of simple functions, so that $f + h = \lim(s_n + t_n)$ and we conclude by the monotone convergence theorem.

8. *If f_n are measurable functions, then* $\displaystyle\sum_{n=1}^{\infty}\int f_n \mathrm{d}\mu = \int\left(\sum_{n=1}^{\infty} f_n\right)\mathrm{d}\mu.$

9. *If $A, B \in \mathcal{A}$ and $\mu(A\cap B) = 0$, then* $\displaystyle\int_{A\cup B} f\mathrm{d}\mu = \int_A f\mathrm{d}\mu + \int_B f\mathrm{d}\mu.$

   In fact $I_A + I_B = I_{A\cup B} + I_{A\cap B}$, and $\int_{A\cap B} f\mathrm{d}m = 0$ when $\mu(A\cap B) = 0$.

10. *If f and h are measurable and $f = h$ almost everywhere, then* $\displaystyle\int f\mathrm{d}\mu = \int h\mathrm{d}\mu.$

11. *If f is measurable and $\displaystyle\int f\mathrm{d}\mu = 0$, then $f = 0$ almost everywhere.*

$$\mu(\{f(x)\geq\varepsilon\}) = \int_{\{f(x)\geq\varepsilon\}}\mathrm{d}\mu \leq \int \varepsilon^{-1} f\mathrm{d}\mu = 0.$$

12. *If f is measurable, then $\mu_f(A) := \displaystyle\int_A f\mathrm{d}\mu$ is a measure on $\mathcal{A}$.*

    If $A = \amalg_n A_n$, then $\mu_f(A) = \int\sum_n I_{A_n} f\mathrm{d}\mu = \sum_n\int I_{A_n} f\mathrm{d}\mu = \sum_n \mu_f(A_n)$.

Definition: If $f\colon X\to\mathbb{R}$ is measurable, so are $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$, and we have $f = f_+ - f_-$. We say that f is **integrable** when $\int f_+\mathrm{d}\mu$ and $\int f_-\mathrm{d}\mu$ are finite, and we put

$$\int_X f\mathrm{d}\mu = \int_X f_+\mathrm{d}\mu - \int_X f_-\mathrm{d}\mu \in \mathbb{R}.$$

1. *If f is integrable, so is cf, $\forall c\in\mathbb{R}$, and* $\displaystyle\int cf\mathrm{d}\mu = c\int f\mathrm{d}\mu.$

2. *If f and h are integrable, so is $f+h$ and* $\displaystyle\int(f+h)\mathrm{d}\mu = \int f\mathrm{d}\mu + \int h\mathrm{d}\mu.$

   We have $(f+h)_+ - (f+h)_- = f + h = f_+ - f_- + h_+ - h_-$.

   Hence $(f+h)_+ + f_- + h_- = (f+h)_- + f_+ + h_+$, and integrating we conclude.

3. *If f and h are integrable and $f\leq h$, then* $\displaystyle\int f\mathrm{d}\mu \leq \int h\mathrm{d}\mu.$

4. *If f is measurable and $|f|\leq h$ for some integrable function h, then f is integrable and*

$$\left|\int f\mathrm{d}\mu\right| \leq \int |f|\mathrm{d}\mu.$$

   $|f| = f_+ + f_-$, and $\int(f_+ + f_-)\mathrm{d}\mu$ is finite if and only if so are $\int f_+\mathrm{d}\mu$ and $\int f_-\mathrm{d}\mu$.

5. *If f_n is an increasing (or decreasing) sequence of integrable functions with integrable limit,*

$$\lim\int f_n\mathrm{d}\mu = \int(\lim f_n)\mathrm{d}\mu.$$

   If (f_n) is increasing, then $0\leq f_n - f_1 \leq f - f_1$, and we conclude by the monotone convergence theorem. If (f_n) is decreasing, then $(-f_n)$ is increasing, and we conclude.

Dominated Convergence Theorem: *Let $f_n, f\colon X\to\mathbb{R}$ be measurable functions such that $f = \lim f_n$. If there is an integrable function h such that $|f_n|\leq h$ for any index n, then*

$$\lim\int f_n\mathrm{d}\mu = \int f\mathrm{d}\mu.$$

Proof: Since $|f_n| \to |f|$, we also have $|f| \leq h$; hence f is integrable. Moreover, $m_n = \inf_{k\geq n} f_k$ is an increasing sequence, and $M_n = \sup_{k\geq n} f_k$ is a decreasing sequence, both with limit f.

Since m_n, M_n are integrable, because $|m_n|, |M_n| \leq h$, and $m_n \leq f_n \leq M_n$, we conclude:

$$\int m_n \mathrm{d}\mu \leq \int f_n \mathrm{d}\mu \leq \int M_n \mathrm{d}\mu,$$

$$\lim \int m_n \mathrm{d}\mu = \int (\lim m_n)\mathrm{d}\mu = \int f \mathrm{d}\mu = \int (\lim M_n)\mathrm{d}\mu = \lim \int M_n \mathrm{d}\mu.$$

Differentiation under the Integral Sign: *Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f\colon U \times X \to \mathbb{R}$ be a function such that $f_t(x) = f(t,x)$ is measurable, $\forall t \in U$, and $f_x(t) = f(t,x)$ is of class $\mathcal{C}^m$, $0 \leq m \leq \infty$, for all $x \in X$. If for any multi-index $|\alpha| \leq m$ we have an integrable function g_α such that $|\partial_\alpha f(t,x)| \leq g_\alpha(x)$, $\forall (t,x) \in U \times X$, then $F(t) = \int_X f(t,x)\mathrm{d}\mu$ is of class $\mathcal{C}^m$ and*

$$\frac{\partial^{|\alpha|}}{\partial t^\alpha} \int_X f(t_1,\dots,t_n,x)\mathrm{d}\mu = \int_X \frac{\partial^{|\alpha|} f}{\partial t^\alpha}(t_1,\dots,t_n,x)\mathrm{d}\mu \ , \quad |\alpha| \leq m.$$

Proof: Fix $a \in U$. Given any sequence $t_n \to a$, we have

$$F(t_n) = \int f(t_n,x)\mathrm{d}\mu \to \int \left(\lim_{t_n\to a} f(t_n,x)\right)\mathrm{d}\mu = \int f(a,x)\mathrm{d}\mu = F(a)$$

by the dominated convergence theorem. Hence $F(t)$ is continuous. Moreover,

$$\frac{F(a+h_i) - F(a)}{h} = \int \frac{f(a+h_i,x) - f(a,x)}{h}\mathrm{d}\mu \ , \quad h_i = (0,\dots,h,\dots,0),$$

and the integrand may be extended to a continuous function on a neighborhood of $h=0$, where the value is $\frac{\partial f}{\partial t_i}(a,x)$. By the mean value theorem this extension is absolutely bounded by g_i.

Hence the first term also may be extended to a continuous function on a neighborhood of $h=0$, so that $\frac{\partial F}{\partial t_i}$ exists and $\frac{\partial F}{\partial t_i} = \int \frac{\partial f}{\partial t_i}\mathrm{d}\mu$. In particular, $\frac{\partial F}{\partial t_i}$ is continuous, because so is $\frac{\partial f}{\partial t_i}$.

4.4.1 Change of Variables Formula

Now we consider $\mathbb{R}^n$ with the σ-algebra of Borel sets and the Lebesgue measure m; hence a function f is measurable when $f^{-1}(B)$ is a Borel set for any Borel set $B \subseteq \mathbb{R}$, and we put

$$\int_{\mathbb{R}^n} f(x_1,\dots,x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_{\mathbb{R}^n} f\mathrm{d}m.$$

Theorem: *If $f\colon \mathbb{R}^n \to [0,\infty]$ is measurable and $\Gamma_f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}\colon 0 \leq y < f(x)\}$, then*

$$\int_{\mathbb{R}^n} f\mathrm{d}m = m(\Gamma_f).$$

Proof: It holds when $f = \sum_i a_i I_{B_i}$ is a simple function, because $m\big(B_i \times [0,a_i)\big) = a_i m(B_i)$.

In general, $f = \lim s_n$ is a limit of an increasing sequence of simple functions, so that $\Gamma_f = \bigcup_s \Gamma_{s_n}$ and we conclude by the monotone convergence theorem:

$$m(\Gamma_f) = \lim m(\Gamma_{s_n}) = \lim \int s_n \mathrm{d}m = \int (\lim s_n)\mathrm{d}m = \int f\mathrm{d}m.$$

Monotone Class Theorem: *The σ-algebra generated by an algebra $\mathcal{A}$ is the least family $\mathcal{M} \supseteq \mathcal{A}$ closed under countable increasing unions and countable decreasing intersections.*

Proof: The family $\mathcal{B} = \{B \in \mathcal{M}\colon B \cap A \in \mathcal{M}, \forall A \in \mathcal{A}\}$ contains $\mathcal{A}$, and it is closed under countable increasing unions, because $(\bigcup_n B_n) \cap A = \bigcup_n (B_n \cap A)$, and countable decreasing intersections, because $(\bigcap_n B_n) \cap A = \bigcap_n (B_n \cap A)$; hence $\mathcal{B} = \mathcal{M}$.

Now $\mathcal{C} = \{B \in \mathcal{M}\colon B \cap A \in \mathcal{M}, \forall A \in \mathcal{M}\}$ contains $\mathcal{A}$, and it is closed under countable increasing unions and decreasing intersections; hence $\mathcal{C} = \mathcal{M}$ is closed under finite intersections. Analogously $\mathcal{D} = \{B \in \mathcal{M}\colon B^c \in \mathcal{M}\}$ contains $\mathcal{A}$, and it is closed under countable increasing unions and decreasing intersections; hence $\mathcal{B} = \mathcal{M}$ is closed under complements.

Since $\mathcal{M}$ is closed under complements and finite intersections, it is closed under finite unions; hence under countable unions $\bigcup A_n = \bigcup_n (A_1 \cup \ldots \cup A_n)$, and $\mathcal{M}$ is a σ-algebra.

Lemma: *If $B \subseteq \mathbb{R}^{n+m}$ is a Borel set, and we put $B_y = \{x \in \mathbb{R}^n\colon (x,y) \in B\}$, where $y \in \mathbb{R}^m$, then the function $h_B(y) = m(B_y)$ is measurable, and*

$$m(B) = \int_{\mathbb{R}^m} m(B_y)\mathrm{d}y_1 \ldots \mathrm{d}y_m.$$

Proof: Replace $\mathbb{R}^{n+m}$ by $I \times \mathbb{R}^m$, where I is a rectangle, and let $\mathcal{M}$ be the family of all Borel sets $B \subseteq I \times \mathbb{R}^m$ such that h_B is measurable.

If h_B is measurable, so is $h_{B^c} = m(I) - h_B$, and if we have a countable increasing union $B = \bigcup_n B_n$ of Borel sets $B_n \in \mathcal{M}$, then $B \in \mathcal{M}$, because $h_B = \sup h_{B_n}$; hence $\mathcal{M}$ also is closed under countable decreasing intersections.

Clearly $\mathcal{M}$ contains the finite disjoint unions of (eventually unbounded) rectangles in $I \times \mathbb{R}^m$, and such family is an algebra (if J, J' are rectangles, then $J \cap J'$ is a rectangle, and J^c is a finite disjoint union of rectangles): By the monotone class theorem, $\mathcal{M} = \mathcal{B}(I \times \mathbb{R}^m)$.

In general, $h_B = \sum_i h_{I_i \cap B}$ is measurable, where $\mathbb{R}^n = \amalg I_i$ is a countable partition.

Now, $\mu(B) = \int_{\mathbb{R}^m} h_B \mathrm{d}m$ is a measure on $\mathcal{B}(\mathbb{R}^{n+m})$:

$$\mu\big(\coprod_n B_n\big) = \int_{\mathbb{R}^m} \big(\sum_n h_{B_n}\big)\mathrm{d}m = \sum_n \int_{\mathbb{R}^m} h_{B_n}\mathrm{d}m = \sum_n \mu(B_n),$$

and it is invariant. Since $\mu\big([0,1)^{n+m}\big) = 1$, we conclude that $m(B) = \int_{\mathbb{R}^m} h_B \mathrm{d}m$.

Cavalieri's Principle: *If $m(B_y) = m(\bar{B}_y)$, $\forall y \in \mathbb{R}^m$, then $m(B) = m(\bar{B})$.*

Fubini's Theorem: *If $f\colon \mathbb{R}^{n+m} \to [0,\infty]$ is measurable, so is $h_f(y) = \int_{\mathbb{R}^n} f(x,y)\mathrm{d}m$ and*

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y)\mathrm{d}x_1 \ldots \mathrm{d}x_n \mathrm{d}y_1 \ldots \mathrm{d}y_m = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\mathrm{d}x_1 \ldots \mathrm{d}x_n \right) \mathrm{d}y_1 \ldots \mathrm{d}y_m.$$

Proof: By the lemma, it holds for any simple function.

In general, we put $f = \lim s_n$ where (s_n) is an increasing sequence of simple functions.

By the monotone convergence theorem, $h_f = \lim h_{s_n}$ and we conclude:

$$\int_{\mathbb{R}^m} h_f \mathrm{d}m = \lim \int_{\mathbb{R}^m} h_{s_n}\mathrm{d}m = \lim \int_{\mathbb{R}^{n+m}} s_n \mathrm{d}m = \int_{\mathbb{R}^{n+m}} f\mathrm{d}m.$$

Corollary: *If $f\colon \mathbb{R}^{n+m} \to \mathbb{R}$ is integrable, and $f_y(x) = f(x,y)$ is integrable for any $y \in \mathbb{R}^m$, then $h_f(y) = \int_{\mathbb{R}^n} f(x,y)\mathrm{d}m$ is integrable and Fubini's formula holds for f.*

Proof: By the theorem, $h_+(y) = \int_{\mathbb{R}^n} f_+ \mathrm{d}m$ and $h_-(y) = \int_{\mathbb{R}^n} f_- \mathrm{d}m$ have finite integral; hence $h_f = h_+ - h_-$ is integrable, and $\int h_f \mathrm{d}m = \int h_+ \mathrm{d}m - \int h_- \mathrm{d}m = \int f_+ \mathrm{d}m - \int f_- \mathrm{d}m = \int f\mathrm{d}m$.

Definition: Let $\varphi\colon U \to V$, $\varphi(x_1,\dots,x_n) = \big(y_1(x_1,\dots,x_n),\dots,y_n(x_1,\dots,x_n)\big)$, be a map of class $\mathcal{C}^1$, where $U, V \subseteq \mathbb{R}^n$ are open sets. The **jacobian** $J_\varphi(p)$ at a point $p \in U$ is

$$J_\varphi(p) = \det \varphi_{*,p} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} (p).$$

Change of Variables Formula: *Let $\varphi\colon U \to V$ be a $\mathcal{C}^1$-diffeomorphism. If $f\colon V \to [0,\infty]$ is measurable, then so is $(f\circ\varphi)|J_\varphi|$ and*

$$\int_V f(y_1,\dots,y_n)\mathrm{d}y_1\dots\mathrm{d}y_n = \int_U f\big(y_1(x),\dots,y_n(x)\big)\left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right| \mathrm{d}x_1\dots\mathrm{d}x_n.$$

Proof: If f is measurable, so is $f\circ\varphi$ because φ is a homeomorphism; hence $(f\circ\varphi)|J_\varphi|$ because $|J_\varphi|$ is continuous. And we only have to show that $\int_V f\mathrm{d}m \le \int_U (f\circ\varphi)|J_\varphi|\mathrm{d}m$, since the reverse inequality follows applying this inequality to $\varphi^{-1}\colon V \to U$.

If we prove this inequality when f is the indicator function of a Borel set, the monotone convergence theorem shows that it holds when f is measurable.

1. *Given $\varepsilon > 0$, each point $p \in U$ has a neighborhood U_p such that*

   $$\varphi(x + C) \subseteq \varphi(x) + (1+\varepsilon)\varphi_{*,p}(C)$$

   for any cube[4] $x + C \subseteq U_p$. *In particular,* $m\big(\varphi(x+C)\big) \le (1+\varepsilon)^n |J_\varphi(p)| m(C)$.

   By the key lemma of infinitesimals along the diagonal, in a neighborhood of p we have $\varphi(y) - \varphi(x) = F(x,y)(y-x)$, where $F(x,y)$ is a continuous family of endomorphisms of $\mathbb{R}^n$ such that $F(x,x) = \varphi_{*,x}$; hence

   $$\varphi(y) - \varphi(x) = \varphi_{*,p}(y-x) + H(x,y)(y-x),$$

   where $H(x,y) = F(x,y) - \varphi_{*,p}$ is a continuous family such that $H(p,p) = 0$.

   Since $\varphi_{*,p}$ is an isomorphism, there exists $k > 0$ such that $C \subseteq k\varphi_{*,p}(C)$ for any cube C centered at the origin, and we may fix a neighborhood U_p where $H(x,y)(C) \subseteq \frac{\varepsilon}{k}C$, $\forall x, y \in U_p$. We conclude:

   $$\varphi(x+C) \subseteq \varphi(x) + \varphi_{*,p}(C) + \tfrac{\varepsilon}{k}C \subseteq \varphi(x) + \varphi_{*,p}(C) + \varepsilon\varphi_{*,p}(C) = \varphi(x) + (1+\varepsilon)\varphi_{*,p}(C).$$

2. *Given $\alpha > 1$, each point $p \in U$ has a neighborhood U_p such that $m(\varphi B) \le \alpha \int_B |J_\varphi|\mathrm{d}m$ for any Borel set $B \subseteq U_p$.*

   In a neighborhood U_p we have $m\big(\varphi(x+C)\big) < \sqrt{\alpha}|J_\varphi(p)|m(C)$ for any cube $x + C \subseteq U_p$, and $|J_\varphi(p)| < \sqrt{\alpha}J$, where J is the infimum of $|J_\varphi|$ on U_p; so that $m\big(\varphi(x+C)\big) < \alpha J m(C)$.

   We may assume that U_p is a rectangle, so that any Borel set $B \subseteq U_p$ admits a countable cover by cubes $x_n + C_n \subseteq U_p$ such that $\sum_n m(C_n) < \alpha m(B)$, because any rectangle is a countable disjoint union of cubes; hence

   $$m(\varphi B) \le \sum_n m\big(\varphi(x_n + C_n)\big) \le \alpha J \sum_n m(C_n) \le \alpha^2 J m(B) \le \alpha^2 \int_B |J_\varphi|\mathrm{d}m.$$

Now, given $\alpha > 1$, fix a countable open cover $U = \bigcup_i U_i$ such that $m(\varphi B_i) \le \alpha \int_{B_i} |J_\varphi|\mathrm{d}m$ for any Borel set $B_i \subseteq U_i$.

[4] That is to say, C is a rectangle with center at the origin and sides of equal length.

Any Borel set $B \subseteq U$ admits a partition $B = \amalg_i B_i$, where B_i is a Borel set in U_i, hence

$$m(\varphi B) = \sum_i m(\varphi B_i) \leq \alpha \int_{B_i} |J_\varphi| \mathrm{d}m = \alpha \int_B |J_\varphi| \mathrm{d}m.$$

Since it holds for any $\alpha > 1$, we conclude that $m(\varphi B) \leq \int_B |J_\varphi| \mathrm{d}m$.

Corollary: *If $N \subset U$ has null measure, so does $\varphi(N)$. If $L \subseteq U$ is measurable, so is $\varphi(L)$.*

Proof: Take a Borel set $N \subseteq B$ with $m(B) = 0$. Then $m(\varphi N) \leq m(\varphi B) = \int_B |J_\varphi| \mathrm{d}m = 0$.
Now, any measurable set is $L = B \cup N$, where B is a Borel set and $m(N) = 0$.
Hence $\varphi(L) = \varphi(B) \cup \varphi(N)$ is measurable, since $\varphi(B)$ is a Borel set and $m(\varphi N) = 0$.

Corollary: *If $f\colon V \to \mathbb{R}$ is integrable, then so is $(f \circ \varphi)|J_\varphi|$ and the change of variables formula holds for f.*

Proof: If f_+, f_- have finite integrals, so have $(f_+ \circ \varphi)|J_\varphi| = \big((f \circ \varphi)|J_\varphi|\big)_+$ and $(f_- \circ \varphi)|J_\varphi| = \big((f \circ \varphi)|J_\varphi|\big)_-$; hence $(f \circ \varphi)|J_\varphi|$ is integrable.

Definition: Let $U \subseteq \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}^d$ a map of class $\mathcal{C}^1$. We say that $x \in U$ is a **critical point** of f when $\mathrm{rk}\, f_{*,x} < d$, and the **critical set** Δ_f is the set of critical points.

Sard's Theorem: *Let $U \subseteq \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}^d$ a map of class $\mathcal{C}^m$, $m < \frac{n}{d}$. The set of critical values $f(\Delta_f)$ has null measure in $\mathbb{R}^d$.*

Proof: Let us consider the equations $y_i = y_i(x_1, \ldots, x_n)$ of f, and let Δ_f^k be the set of points $x \in U$ where all the partial derivatives of the functions $y_i(x_1, \ldots, x_n)$ up to order k vanish at x. The case $n = 0$ is trivial, and we proceed by induction on n.

1. *The set $f(\Delta_f - \Delta_f^1)$ has null measure*: Let us show that any point $x \in \Delta_f - \Delta_f^1$ has an open neighborhood V such that $f\big(V \cap (\Delta_f - \Delta_f^1)\big)$ has null measure.

   Re-labelling the coordinates if necessary, we may assume that $(\partial_1 y_1)(x) \neq 0$. Hence, replacing U by a smaller neighborhood, and composing f with a diffeomorphism, we may assume that $y_1(x_1, \ldots, x_n) = x_1$. Now, for any $t \in \mathbb{R}$ we have a map

$$f_t\colon U_t = \{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\colon (t, x_2, \ldots, x_n) \in U\} \longrightarrow \mathbb{R}^{d-1}$$
$$f_t(x_2, \ldots, x_n) = f(t, x_2, \ldots, x_n),$$
$$f(\Delta_f - \Delta_f^1) = \textstyle\bigcup_t \{t\} \times f_t\big(\Delta_{f_t}\big).$$

   By induction $f_t\big(\Delta_{f_t}\big)$ has null measure in $\mathbb{R}^{d-1}$, so that $f(\Delta_f - \Delta_f^1)$ has null measure in $\mathbb{R}^d$ by Cavalieri's principle.

2. *The set $f(\Delta_f^k - \Delta_f^{k+1})$ has null measure for all $k \geq 1$*: Let us show that any point $x \in \Delta_f^k - \Delta_f^{k+1}$ has an open neighborhood V such that $f\big(V \cap (\Delta_f^k - \Delta_f^{k+1})\big)$ has null measure.

   Replacing U by a smaller neighborhood, we may assume that $\Delta_f^{k+1} = \emptyset$, and composing f with a diffeomorphism, we may assume that Δ_f^k is contained in the hyperplane $x_1 = 0$. Now we have a map

$$f_0\colon U_0 = \{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\colon (0, x_2, \ldots, x_n) \in U\} \longrightarrow \mathbb{R}^{d-1}$$
$$f_0(x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n),$$
$$f(\Delta_f^k) = f_0\big(\Delta_{f_0}^k\big),$$

   and by induction $f_0\big(\Delta_{f_0}^k\big)$ has null measure in $\mathbb{R}^{d-1}$.

3. *The set $f(\Delta_f^k)$ has null measure when $k > \frac{n}{m}$*: Let us show that $f(K \cap \Delta_f^k)$ has null measure for any compact set $K \subset U$.

   By the key lemma along the diagonal, there is a constant $c_0 > 0$ and a finite cover of $K \cap \Delta_f^k$ by open sets V such that $\|f(x_2) - f(x_1)\| \leq c\|x_2 - x_1\|^k$ whenever $x_1 \in K \cap \Delta_f^k$, $x_2 \in V$. Hence, we have a constant $c > 0$ such that if $C \subset V$ is a cube with edge r which intersects $K \cap \Delta_f^k$, then the exterior measure is

   $$m_e\big(f(C)\big) \leq cr^{mk} = c \cdot m(C)^{mk/n}.$$

   Cover $K \cap \Delta_f^k$ with a finite number of such cubes $\{C_j\}$, with disjoint interiors. Given a natural number N, subdivide each cube C_j into N^n cubes $\{C_j^i\}$ of equal sizes. If a cube C_j^i intersects $K \cap \Delta_f^k$, then $m_e\big(f(C_j^i)\big) \leq= c \cdot m(C_j^i)^{mk/n} = cN^{-mk}m(C_j)$. Hence,

   $$m_e\big(f(C_j \cap \Delta_f^k)\big) \leq \textstyle\sum_i m_e\big(f(C_j^i \cap \Delta_f^k)\big) \leq N^{n-mk}m(C_j).$$

   Since $n - mk < 0$, when $N \to \infty$, we obtain that $m_e\big(f(C_j \cap \Delta_f^k)\big) = 0$, and we conclude.

4.5 Convergence in $\mathcal{C}^m(U)$

Definition: Let X, Y be topological spaces and let $\mathcal{C}(X, Y)$ be the set of all continuous maps. The **compact-open** topology on $\mathcal{C}(X, Y)$ is generated by the subsets

$$\langle K, U\rangle = \{f \in \mathcal{C}(X, Y) \colon f(K) \subseteq U\},$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

A base of open sets are just the finite intersections $\langle K_1, U_1\rangle \cap \ldots \cap \langle K_n, U_n\rangle$.

If $U \subseteq V$, then $\langle K, U\rangle \subseteq \langle K, V\rangle$. If $K \subseteq L$, then $\langle L, U\rangle \subseteq \langle K, U\rangle$. Hence, if X is σ-compact and Y has a countable base of open sets, then $\mathcal{C}(X, Y)$ also has a countable base.

Definition: Put $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A **topological vector space** is a $\mathbb{K}$-vector space E with a **linear topology**, a topology such that the structural maps $E \times E \to E$ and $\mathbb{K} \times E \to E$ are continuous, where $E \times E$ and $\mathbb{K} \times E$ are endowed with the direct product topology, so that also are continuous the natural maps

$$\mathbb{K}^n \times E^n \longrightarrow E\ , \quad (\lambda_1, \ldots, \lambda_n, e_1, \ldots, e_n) \mapsto \lambda_1 e_1 + \ldots + \lambda_n e_n.$$

The topological $\mathbb{K}$-vector spaces, with the continuous $\mathbb{K}$-linear maps, define a category.

Since the translations are continuous, they are homeomorphisms, and we see that any linear topology is determined by the family $\{U_i\}$ of neighborhoods of 0, since $\{p+U_i\}$ is just the family of neighborhoods of any point $p \in E$.

Definition: A **seminorm** on a real vector space E is a map $q \colon E \to \mathbb{R}_{\geq 0}$ such that

1. $q(e + v) \leq q(e) + q(v)$; $\forall e, v \in E$.

2. $q(\lambda e) = |\lambda| \cdot q(e)$; $\forall \lambda \in \mathbb{K}$, $e \in E$.

and it is a **norm** when $q(e) = 0 \Rightarrow e = 0$. Sometime we put $q(e) = \|e\|$.

A seminorm defines a **pseudometric**[5] $\mathrm{d}(x,y) = q(y-x)$, and it is a metric when q is a norm. Hence, a seminorm defines a topology on E. A base of neighborhoods of $x \in E$ is defined by the balls

$$B_r(x) = B(x,r) = B_{\mathrm{d}}(x,r) := \{y \in E \colon \mathrm{d}(x,y) < r\} = \{y \in E \colon \|y-x\| < r\} \ , \quad r \in \mathbb{R}_+,$$

and this topology is a linear topology because

$$B_r(x) + B_s(y) \subseteq B_{r+s}(x+y) \quad , \quad (\lambda - \varepsilon, \lambda + \varepsilon)B_r(x) \subseteq B_{(|\lambda|+|\varepsilon|)r}(\lambda x).$$

In general, any family of seminorms $\{q_i\}$ on E defines a linear topology, and a topological vector space is **locally convex** when the topology is defined by a family of seminorms. Then a base of neighborhoods of $x \in E$ is defined by the finite intersections of balls

$$\{y \in E \colon q_{i_1}(y-x) < r_1\} \cap \ldots \cap \{y \in E \colon q_{i_n}(y-x) < r_n\}.$$

When the topology of E is separated and may be defined by a countable family of seminorms $\{q_n\}$, then E is metrizable, the topology being defined by the translation-invariant metric $\mathrm{d}(x,y) = \sum_n 2^{-n} \min\{1, q_n(y-x)\}$. Now, a sequence (e_n) is a **Cauchy sequence** for d when for any neighborhood U of 0, there is an index k such that $e_n - e_m \in U$, $\forall m,n \geq k$. Hence Cauchy sequences do not depend on the fixed family $\{q_n\}$, only on the linear topology of E, and we say that E is **complete** if any Cauchy sequence converges.

Examples: Any scalar product on a real vector space E defines a norm $q(e) = \sqrt{e \cdot e}$.

If $B(X)$ is the ring of all bounded functions on a set X, then $q(f) = \sup_{x \in X} |f(x)|$ is a norm.

If $(X, \mathcal{A})$ is a measurable space, then $q(f) = \int_X |f| \mathrm{d}\mu$ is a seminorm on the vector space of all integrable functions.

Let q be a seminorm on a vector space F. If $f \colon E \to F$ is a linear map, then $(f^*q)(e) := q\big(f(e)\big)$ is a seminorm on E, and the corresponding topology is just the initial topology. In particular, if $V \subset F$ is a vector subspace, then the restriction $q|_V$ is a seminorm, and the corresponding topology on V is the induced topology.

If q_1, q_2 are two seminorms on a vector space E, then $q_1 + q_2$ and $\sup\{q_1, q_2\}$ also are seminorms, defining the same topology as the pair of seminorms $\{q_1, q_2\}$.

Definition: Any compact set K in a topological space X defines a seminorm $\| \ \|_K$ on the ring of real (or complex) valued continuous functions $\mathcal{C}(X)$,

$$\|f\|_K = \sup_{p \in K} |f(p)| \quad , \quad \mathrm{d}_K(f,h) = \|h - f\|_K \ ,$$

and the seminorms $\| \ \|_K$ define the **compact convergence** topology on $\mathcal{C}(X)$.

When X is σ-compact, $X = \bigcup_n K_n$, $K_n \subseteq \overset{\circ}{K}_{n+1}$, this linear topology is defined by the seminorms $\| \ \|_{K_n}$, because $K \subseteq K_n$ for some index n.

Proposition: *The topology of the compact convergence on $\mathcal{C}(X)$ coincides with the compact-open topology. In particular, $\mathcal{C}(X)$ admits a countable base of open sets when X is σ-compact.*

Proof: If $f \in \langle K, U \rangle$ and ε is smaller than the distance of $f(K)$ to $X - U$, then $B_{\mathrm{d}_K}(f, \varepsilon) \subseteq \langle K, U \rangle$, and we see that $\langle K, U \rangle$ is open in the topology of the compact convergence.

[5] A map $\mathrm{d} \colon X \times X \to \mathbb{R}_{\geq 0}$ such that $\mathrm{d}(x,x) = 0$, $\mathrm{d}(x,y) = \mathrm{d}(y,x)$, and $\mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(x,z)$. Any family of pseudometrics $\{\mathrm{d}_i\}$ induces a topology on X, a base of neighborhoods of $x \in X$ being defined by the finite intersections of balls $\bigcap_i B_{\mathrm{d}_i}(x, r_i)$, $r_i > 0$, where $B_{\mathrm{d}}(x,r) = \{y \in X \colon \mathrm{d}(x,y) < r\}$. This topology is separated when $x \neq y \Rightarrow \mathrm{d}_i(x,y) \neq 0$ for some index i.

Now, given a ball $B_{\mathrm{d}_K}(f, 2\varepsilon)$, any point $p \in K$ has a compact neighborhood $\bar{U}_p$ in K such that $f(\bar{U}_p) \subseteq \big(f(p) - \varepsilon, f(p) + \varepsilon\big)$, so that $\mathrm{d}_{\bar{U}_p}(h - f) < 2\varepsilon$ when $h \in \langle \bar{U}_p, \big(f(p) - \varepsilon, f(p) + \varepsilon\big) \rangle$.

Since K is compact, it admits a finite cover $K = \bar{U}_{p_1} \cup \ldots \cup \bar{U}_{p_n}$, and we see that $B_{\mathrm{d}_K}(f, 2\varepsilon)$ is open in the compact-open topology because

$$\langle \bar{U}_{p_1}, \big(f(p_1) - \varepsilon, f(p_1) + \varepsilon\big) \rangle \cap \ldots \cap \langle \bar{U}_{p_n}, \big(f(p_n) - \varepsilon, f(p_n) + \varepsilon\big) \rangle \subseteq B_{\mathrm{d}_K}(f, 2\varepsilon).$$

Theorem: *If X is a σ-compact space, then $\mathcal{C}(X)$ is metrizable and complete.*

Proof: Any Cauchy sequence converges (p. 38) to a function continuous on any compact set.

Definition: Let $U \subseteq \mathbb{R}^d$ be an open set. Any compact set $K \subset U$ defines seminorms $\| \ \|_{K,r}$ on $\mathcal{C}^m(X)$, $1 \leq m \leq \infty$, (see p. 104 for notations)

$$\|f\|_{K,r} = \sup_{|\alpha| \leq r} \left(\tfrac{1}{\alpha!} \ \|\partial_\alpha f\|_K \right) \ ; \quad r \leq m, \ r < \infty,$$

where the factorials are introduced so that $\|fg\|_{K,r} \leq \|f\|_{K,r} \|g\|_{K,r}$. In fact

$$\partial_\alpha (fg) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (\partial_\beta f)(\partial_{\alpha - \beta} g).$$

These seminorms define a linear topology on $\mathcal{C}^m(U)$, the topology of **compact convergence of derivatives**. We have $\lim f_n = f$ when (f_n) uniformly converges to f, and $(\partial_\alpha f_n)$, $|\alpha| \leq m$, uniformly converges to $\partial_\alpha f$, on any compact set $K \subset U$.

If we put $U = \bigcup_n K_n$, $K_n \subseteq \overset{\circ}{K}_{n+1}$, we see that it is defined by the countable family of seminorms $\| \ \|_{K_n,r}$, because $K \subseteq K_n$ for some index n.

Moreover, $\mathcal{C}^m(U)$ admits a countable base of open sets, because this topology is just the initial topology of the maps $\partial_\alpha \colon \mathcal{C}^m(U) \to \mathcal{C}(U)$, $|\alpha| \leq m$, and $\mathcal{C}(U)$ has a countable base.

Theorem: *$\mathcal{C}^m(U)$ is metrizable and complete, $1 \leq m \leq \infty$.*

Proof: Any Cauchy sequence (f_n) uniformly converges to a continuous function g, and the partial derivatives $\partial_\alpha f$, $|\alpha| \leq m$, uniformly converge to certain continuous functions g_α, and we have to prove that $g_\alpha = \partial_\alpha g$, so that $g \in \mathcal{C}^m(U)$.

We may use that the derivative preserves uniform limits (p. 38) or we may argue directly.

If $p = (a_1, \ldots, a_d) \in U$, then

$$f_n(a_1, \ldots, x_i, \ldots, a_d) = f_n(a_1, \ldots, a_i, \ldots, a_d) + \int_{a_i}^{x_i} \frac{\partial f_n}{\partial x_i}(a_1, \ldots, t_i, \ldots, a_d) \, \mathrm{d}t_i,$$
$$g(a_1, \ldots, x_i, \ldots, a_d) = g(a_1, \ldots, a_i, \ldots, a_d) + \int_{a_i}^{x_i} g_i(a_1, \ldots, t_i, \ldots, a_d) \, \mathrm{d}t_i,$$

and we see that $g_1(p) = \frac{\partial g}{\partial x_1}(p)$. Iterating, we conclude.

Chapter 5

Algebra II

5.1 Actions of a Group

Definitions: A (left[1]) **action** of a group G on a set X, or a G-**set**, is a map $G \times X \dot{\to} X$ such that[2]

1. $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$, $x \in X$.
2. $1 \cdot x = x$, for all $x \in X$.

It defines on X an equivalence relation, $x \equiv x'$ when $x' = gx$ for some $g \in G$, and the **orbit** of $x \in X$ is the equivalence class $Gx = \{gx \colon g \in G\}$. The **isotropy subgroup** is

$$I_x = \{g \in G \colon gx = x\}.$$
$$I_{gx} = g I_x g^{-1}. \tag{5.1}$$

An action is **transitive** if there is a unique orbit, $G = Gx$ for some $x \in X$.

A point $x \in X$ is a **fixed** point when $I_x = G$, and X^G denotes the set of fixed points.

A map $f \colon X \to Y$ is a **morphism** of G-sets if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$, $x \in X$; and the set of all G-morphisms $X \to Y$ is denoted by $\mathrm{Hom}_G(X, Y)$.

Isomorphisms of G-sets are bijective morphisms of G-sets.

If $H \subseteq G$ is a subgroup, G/H is a G-set with the action $g_1 \cdot [g_2] = [g_1 g_2]$.

Theorem: $\mathrm{Hom}_G(G/H, X) = X^H$, $f \mapsto f([1])$, *for any subgroup* $H \subseteq G$.

Proof: The map $f \colon G/H \to X$, $f(gH) = gx$, is well defined if and only if $x \in X^H$.

Theorem: $G/I_x = Gx$, $[g] \mapsto gx$.

Proof: If $g_1 x = g_2 x$, then $g_1^{-1} g_2 x = x$, $g_1^{-1} g_2 \in I_x$, $[g_1] = [g_2]$.

Class Formula: *If a finite group G acts on a finite set X, then there are non-fixed points $x_i \in X$ and divisors $d_i > 1$ of the order of G such that*

$$|X| = |X^G| + \textstyle\sum_{x_i} [G : I_{x_i}] = |X^G| + \sum_i d_i.$$

Proof: X is the disjoint union of all the orbits, $|X^G|$ is the number of one point orbits, and the cardinals $d_i = [G : I_{x_i}]$ of the other orbits divide $|G|$ by Lagrange's theorem.

[1] Right actions being left actions of the **opposite group** G^{op}, whose product is $a * b := ba$.

[2] An action is just a group morphism $\rho \colon G \to \mathrm{Aut}(X)$, $(\rho g)(x) := g \cdot x$, where $\mathrm{Aut}(X)$ is the group of all bijections $X \to X$.

Definition: A p**-group** is a group of order some power of a prime number p.

Lemma: *If G is a p-group, and X a finite G-set, then* $|X| \equiv |X^G|$ (mod. p).

Proof: In the class formula, any divisor d_i is a power p^{n_i}, $n_i \geq 1$.

Theorem: *The* ***center*** *of any p-group $G \neq 1$ is non trivial,*

$$Z(G) = \{a \in G \colon ag = ga, \ \forall g \in G\} \neq 1.$$

Proof: G acts on G by conjugation, and the set of fixed points is $Z(G)$.
By the lemma, $|Z(G)| \equiv |G| \equiv 0$ (mod. p), and $|Z(G)| \neq 1$.

Cauchy's Theorem: *If a prime p divides $|G|$, then G has some subgroup of order p.*

Proof: The group $\mathbb{Z}/p\mathbb{Z}$ acts on $X = \{(g_1, \ldots, g_p) \in G^p \colon \ g_1 \ldots g_p = 1\}$ by means of a cyclic permutation,

$$g_1 \ldots g_p = 1, \ g_1 \ldots g_{p-1} = g_p^{-1}, \ g_p g_1 \ldots g_{p-1} = 1.$$

Since $|X| = |G|^{p-1}$ is a multiple of p, there is a fixed point $(g, \ldots, g) \neq (1, \ldots, 1)$.
Hence, $g^p = 1$, $g \neq 1$, and the order of (g) is p.

Definition: Let p be a prime number. The **Sylow p-subgroups** of a finite group G are the subgroups of order the greatest power p^n dividing $|G| = p^n m$.

Lemma: *Any subgroup H of order p^i, $i < n$, is contained in a subgroup H' of order p^{i+1} such that $H \lhd H'$* (i.e., H is a normal subgroup of H').

Proof: The p-group H acts on G/H, the set of fixed points being $N(H)/H$, where the subgroup $N(H) = \{g \in G \colon gHg^{-1} = H\}$ is the **normalizer** of H in G.
Since $|G/H|$ is a multiple of p, so is $|N(H)/H|$.
By Cauchy's theorem, $N(H)/H$ has some subgroup $\bar{H}$ of order p.
Now $H' = \pi^{-1}(\bar{H})$ is a subgroup of $N(H)$ of order p^{i+1} and $H \lhd H'$.

Corollary: *Any p-group G admits a sequence $1 = H_0 \lhd H_1 \lhd \ldots \lhd H_{n-1} \lhd H_n = G$, where H_i is a subgroup of order p^i.*

Corollary (First Sylow Theorem): *There exist Sylow p-subgroups of G.*

Second Sylow Theorem: *All Sylow p-subgroups of G are conjugate.*

Proof: Let P', P be Sylow p-subgroups. P' acts on G/P and p does not divide $|G/P|$.
There is a fixed point $\bar{g} \in G/P$, $P' \subseteq gPg^{-1}$, and $P' = gPg^{-1}$ since both have order p^n.

Third Sylow Theorem: *Put $|G| = p^n m$. The number of Sylow p-subgroups of G divides the common index m and it is congruent to 1 modulo p.*

Proof: G acts transitively, by conjugation, on the set X of all Sylow p-subgroups of G.
$N(P)$ is the isotropy subgroup of P; hence $|X| = [G : N(P)]$ divides $[G : P] = m$.
Let us show that P is the unique fixed point under the action of P.
If $gP'g^{-1} = P'$, $\forall g \in P$, then $P \subset N(P')$, and P, P' are Sylow p-subgroups of $N(P)$.
Hence $P' = P$ by the second Sylow theorem. Now $|X| \equiv |X^P| = 1$ (mod. p).

Definition: A finite group G is **solvable** if there are subgroups $1 = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$ with abelian quotients H_i/H_{i-1}.

Theorem: *Let G be a finite group. If G is solvable, so is any subgroup H.*
If $H \lhd G$, then G is solvable if and only if so are H and G/H.

Proof: If $1 = H_0 \lhd H_1 \ldots \lhd H_n = G$ is a resolution, we put $H'_i = H_i \cap H$.
Then $H'_{i-1} \lhd H'_i$, and $H'_i/H'_{i-1} \hookrightarrow H_i/H_{i-1}$; hence H'_i/H'_{i-1} is abelian, and H is solvable.
If $H \lhd G$, and $\pi\colon G \to G/H$ is the canonical projection, we put $\bar{H}_i = \pi(H_i)$. Then $\bar{H}_{i-1} \lhd \bar{H}_i$ and $\bar{H}_i/\bar{H}_{i-1}$ is a quotient of H_i/H_{i-1}; hence it is abelian, and G/H is solvable.
Conversely, if we have resolutions of H and G/H,

$$
\begin{aligned}
1 = H_0 \ &\lhd \ H_1 \ \lhd \ \ldots \ \lhd \ H_{n-1} \ \lhd \ H_n = H, \\
1 = \bar{H}_0 \ &\lhd \ \bar{H}_1 \ \lhd \ \ldots \ \lhd \ \bar{H}_{d-1} \ \lhd \ \bar{H}_d = G/H,
\end{aligned}
$$

then $1 \lhd H_1 \ldots \lhd H_n = \pi^{-1}(\bar{H}_0) \lhd \pi^{-1}(\bar{H}_1) \ldots \lhd \pi^{-1}(\bar{H}_d) = G$ is a resolution of G.

5.2 Modules

Definitions: Let A be a ring (commutative with unity). A structure of A**-module** on an abelian group M is given by a product $A \times M \xrightarrow{\cdot} M$ such that[3]

1. $a(m_1 + m_2) = am_1 + am_2$; $\forall m_1, m_2 \in M$.
2. $(a + b)m = am + bm$; $\forall a, b \in A$, $m \in M$.
3. $(ab)m = a(bm)$; $\forall a, b \in A$, $m \in M$.
4. $1 \cdot m = m$; $\forall m \in M$

and a subgroup $N \subseteq M$ is a **submodule** if $a \in A$, $m \in N \Rightarrow am \in N$.
A group morphism $f\colon M \to M'$ is a **morphism** of A-modules (an **isomorphism** of A-modules if moreover it is bijective) when it is A-linear,

$$f(am) = a \cdot f(m) \ ; \quad \forall a \in A, m \in M.$$

Examples: Modules over a field k are just k-vector spaces, submodules are just vector subspaces and morphisms of k-modules are just k-linear maps.
Any abelian group admits a unique structure of $\mathbb{Z}$-module, submodules are just subgroups and morphisms of $\mathbb{Z}$-modules are just group morphisms.
Sums and intersections of submodules are submodules.
If I is an ideal, $IM = \{a_1m_1 + \ldots + a_nm_n \colon a_i \in I,\ m_i \in M\}$ is a submodule of M.
The submodules of A are just the ideals of A.
The set $\mathrm{Hom}_A(M, N)$ of all A-linear maps $M \to N$ is an A-module with the obvious structure, $(f + h)(m) = f(m) + h(m)$ and $(af)(m) = a(f(m))$.
Direct products $\prod_{i\in I} M_i$ and **direct sums** $\bigoplus_{i\in I} M_i$ (formed by all sequences $\{m_i\}_{i\in I}$ with a finite number of non-zero terms) of A-modules are A-modules.
Any family $\{m_i\}_{i\in I}$ of elements of M defines a morphism of A-modules

$$f\colon \ \bigoplus_I A \longrightarrow M,\ \ f((a_i)_{i\in I}) = \sum_i a_i m_i,$$

[3]That is to say a ring morphism $\rho\colon A \to \mathrm{End}(M)$, $(\rho a)(m) := a \cdot m$, where $\mathrm{End}(M)$ is the (non commutative) ring of all group morphisms $M \to M$.

and the image $\sum_i Am_i$ is the submodule **generated** by $\{m_i\}_{i\in I}$. If f is an isomorphism, we say that M is a **free** module of base $\{m_i\}_{i\in I}$.

M is said to be a **finitely generated** A-module if $M = Am_1 + \ldots + Am_n$.

The proofs given in the case of vector spaces (pp. 50–52) show that

1. If N is a submodule of an A-module M, the quotient group M/N admits a unique structure of A-module such that the canonical projection $\pi\colon M \to M/N$ is a morphism of A-modules, and M/N has the corresponding universal property.

2. If $f\colon M \to N$ is a morphism of A-modules, then $\operatorname{Ker} f$ and $\operatorname{Im} f$ are submodules, and the isomorphism theorem $M/\operatorname{Ker} f \simeq \operatorname{Im} f$ holds.

3. If $p\colon M \to \bar{M}$ is an A-linear surjective map, we have a lattice isomorphism (where a submodule $\bar{P} \subseteq \bar{M}$ corresponds to the kernel $P = \pi^{-1}(\bar{P})$ of $M \to \bar{M} \to \bar{M}/\bar{P}$, so that $M/P = \bar{M}/\bar{P}$)

$$(N := \operatorname{Ker} p) \qquad \begin{bmatrix}\text{Submodules}\\ \text{of } \bar{M}\end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix}\text{Submodules of } M\\ \text{containing } N\end{bmatrix}$$

4. Modules of finite length are defined to be modules admitting some flag, and the common length of all flags is the **length** $l(M)$ of the module (but simple A-modules are **residue fields** $A/\mathfrak{m}$ of maximal ideals, so that typically A has not finite length).

5. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then $l(M) = l(M') + l(M'')$.

Theorem: *Any ring $A \neq 0$ has some maximal ideal.*

Proof: Let X be the ordered set of all ideals $I \neq A$. If $\{I_j\}$ is a chain in X, then $I = \bigcup_j I_j$ is an ideal $\neq A$ (p. 54) containing any ideal I_j. By Zorn's lemma, X has a maximal element.

Corollary: *Any ideal $I \neq A$ is contained in some maximal ideal.*

Proof: Any maximal ideal of A/I defines a maximal ideal of A containing I (p. 71).

Corollary: *An element $f \in A$ is invertible if and only if it is in no maximal ideal of A.*

Proof: If f is not a unit, then $fA \neq A$, and some maximal ideal contains fA.

Corollary: *Let $A \neq 0$. If there is a surjective morphism $A^n \to A^m$, then $n \geq m$. Hence, any base of A^n has n elements.*

Proof: Let $\mathfrak{m}$ be a maximal ideal of A, and put $k = A/\mathfrak{m}$.

If a morphism $A^n \to A^m$ is surjective, so is the k-linear map $A^n/\mathfrak{m}A^n \to A^m/\mathfrak{m}A^m$.

Since $A^n/\mathfrak{m}A^n = A^n/\mathfrak{m}^n = (A/\mathfrak{m})^n = k^n$, we have $n \geq m$. q.e.d.

Morphisms of A-modules $f\colon M' \to M$ naturally induce morphisms of A-modules

$$f_*\colon \operatorname{Hom}_A(N, M') \longrightarrow \operatorname{Hom}_A(N, M),\ \ f_*(g) = f \circ g,$$
$$f^*\colon \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M', N),\ \ f^*(g) = g \circ f.$$

Theorem: *A sequence of morphisms of A-modules* $M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ *is exact if and only if for any A-module N the following sequence is exact*

$$0 \longrightarrow \mathrm{Hom}_A(M'', N) \xrightarrow{p^*} \mathrm{Hom}_A(M, N) \xrightarrow{i^*} \mathrm{Hom}_A(M', N)$$

Proof: Assume that $\mathrm{Im}\, i = \mathrm{Ker}\, p$ and p is surjective.

If $f\colon M'' \to N$ vanishes on $\mathrm{Im}\, p = M''$, then $f = 0$; hence p^* is injective.

Since $pi = 0$, we have $0 = (pi)^* = i^*p^*$; hence $\mathrm{Im}\, p^* \subseteq \mathrm{Ker}\, i^*$.

Finally, if $f\colon M \to N$ vanishes on $\mathrm{Im}\, i = \mathrm{Ker}\, p$, by the universal property f factors through $p\colon M \to M/\mathrm{Ker}\, p \simeq M''$, so that $f \in \mathrm{Im}\, p^*$.

Conversely, since p^* is injective when $N = M''/\mathrm{Im}\, p$, the canonical map $M'' \to N$ is zero; hence $\mathrm{Im}\, p = M''$, and p is surjective.

$\mathrm{Im}\, i \subseteq \mathrm{Ker}\, p$, because $pi = (pi)^*(\mathrm{Id}_{M''}) = i^*p^*(\mathrm{Id}_{M''}) = 0$.

$\pi\colon M \to N = M/\mathrm{Im}\, i$. Since $i^*(\pi) = 0$, there is a morphism $f\colon M'' \to N$ such that $\pi = p^*(f) = fp$; hence $\mathrm{Ker}\, p \subseteq \mathrm{Ker}\, \pi = \mathrm{Im}\, i$,

$$\begin{array}{ccccccc} M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \\ & & {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle f} & & & \\ & & M/\mathrm{Im}\, i & & & & \end{array}$$

Theorem: *A sequence of morphisms of A-modules* $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M''$ *is exact if and only if for any A-module N the following sequence is exact*

$$0 \longrightarrow \mathrm{Hom}_A(N, M') \xrightarrow{i_*} \mathrm{Hom}_A(N, M) \xrightarrow{p_*} \mathrm{Hom}_A(N, M'')$$

Proof: The direct statement is easy to check. To prove the converse, just take $N = A$, since we have natural isomorphisms $\mathrm{Hom}_A(A, M) = M$, $f \mapsto f(1)$.

Theorem: *Let* $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ *be an exact sequence of A-modules. The following conditions are equivalent (and we say that the sequence* ***splits****),*

1. *There is an A-linear* ***section*** $s\colon M'' \to M$; $ps = \mathrm{Id}_{M''}$.
2. *There is an A-linear* ***retract*** $r\colon M \to M'$; $ri = \mathrm{Id}_{M'}$.
3. $p_*\colon \mathrm{Hom}_A(N, M) \longrightarrow \mathrm{Hom}_A(N, M'')$ *is surjective for any A-module N.*
4. $i^*\colon \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_A(M', N)$ *is surjective for any A-module N.*
5. *There is an isomorphism* $M' \oplus M'' \simeq M$ *such that the following diagram commutes,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{i_1} & M' \oplus M'' & \xrightarrow{\pi_2} & M'' & \longrightarrow & 0 \\ & & \| & & \wr| & & \| & & \\ 0 & \longrightarrow & M' & \xrightarrow{i} & M & \xrightarrow{p} & M'' & \longrightarrow & 0 \end{array}$$

Proof: ($1 \Rightarrow 3$) Because $p_*s_* = (ps)_* = \mathrm{Id}$.

($3 \Rightarrow 1$) Just take $N = M''$, and consider the identity of M''.

($2 \Rightarrow 4$) Because $i^*r^* = (ri)^* = \mathrm{Id}$.

($4 \Rightarrow 2$) Just take $N = M'$, and consider the identity of M'.

$(1 \Rightarrow 5)$ The required isomorphism is $i + s\colon M' \oplus M'' \to M$ (see p. 54).

$(2 \Rightarrow 5)$ The morphism $(r, p)\colon M \to M' \oplus M''$ clearly gives a commutative diagram. Let us prove that it is an isomorphism:
If $r(m) = 0$ and $p(m) = 0$, then $m = i(m')$ and $m' = ri(m') = r(m) = 0$; hence $m = 0$.
Given $m' \in M'$ and $m'' = p(m) \in M''$, there exists $x \in m'$ such that $r(m + i(x)) = m'$.

Finally, $(5 \Rightarrow 1)$ and $(5 \Rightarrow 2)$ are obvious, since π_2 admits the section $i_2(m'') = (0, m'')$ and i_1 admits the retract $\pi_1(m', m'') = m'$.

5.2.1 Injective and Projective Modules

Definitions: An A-module P is **projective** if $\mathrm{Hom}_A(P, -)$ preserves exact sequences; i.e., if a morphism of A-modules $p\colon M \to M''$ is surjective, so is the morphism

$$p_*\colon \mathrm{Hom}_A(P, M) \longrightarrow \mathrm{Hom}_A(P, M'').$$

Dually, an A-module Q is **injective** if $\mathrm{Hom}_A(-, Q)$ preserves exact sequences; i.e., if a morphism of A-modules $i\colon M' \to M$ is injective, then the following morphism is surjective

$$i^*\colon \mathrm{Hom}_A(M', Q) \longrightarrow \mathrm{Hom}_A(M, Q).$$

Theorem: *If P is projective, any exact sequence $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} P \longrightarrow 0$ splits.*
Dually, if Q is injective, any exact sequence $0 \longrightarrow Q \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ splits.

Proof: If P is projective, then $p_*\colon \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, P)$ is surjective, and there is a morphism $s\colon P \to M$ such that $\mathrm{Id}_P = p_*(s) = ps$.

If Q is injective, then $i^*\colon \mathrm{Hom}_A(M, Q) \to \mathrm{Hom}_A(Q, Q)$ is surjective, and there is a morphism $r\colon M \to Q$ such that $\mathrm{Id}_Q = i^*(r) = ri$.

Theorem: *$\oplus_i P_i$ is projective if and only if so are the modules P_i.*
Dually, $\prod_i Q_i$ is injective if and only if so are the modules Q_i.

Proof: It follows from the universal properties of the direct sum and the direct product,

$$\mathrm{Hom}_A(\oplus_i P_i, M) = \textstyle\prod_i \mathrm{Hom}_A(P_i, M)$$
$$\mathrm{Hom}_A(M, \textstyle\prod_i Q_i) = \textstyle\prod_i \mathrm{Hom}_A(M, Q_i)$$

Theorem: *Any free module is projective. Any module is a quotient of a projective module.*

Proof: A is projective because $\mathrm{Hom}_A(A, M) = M$; hence $\oplus_I A$ is projective.

Moreover, any generating system $\{m_i\}_{i \in I}$ of M defines an epimorphism $\oplus_I A \to M$.

Ideal Criterion: *If the restriction morphism $\mathrm{Hom}_A(A, Q) \to \mathrm{Hom}_A(I, Q)$ is surjective for any ideal I of A, then Q is an injective A-module.*

Proof: Given a submodule N of M and a morphism $f\colon N \to Q$, take a morphism $\rho\colon A \to M$ which does not factor through N, and consider the morphism $\phi\colon I = \rho^{-1}(N) \to Q$, $\phi = f \circ \rho$.

By hypothesis it may be extended to a morphism $\phi'\colon A \to Q$.

Since ϕ' vanishes on the kernel of $A \to Am = \mathrm{Im}\,\rho$, it induces a morphism $\phi'\colon Am \to Q$ coinciding with f on $N \cap Am$. Now the exact sequence

$$0 \longrightarrow N \cap Am \xrightarrow{j} N \oplus Am \xrightarrow{s} N + Am \longrightarrow 0$$

where $j(n) = (n, -n)$, shows that $f + \phi' \colon N \oplus Am \to Q$ defines a morphism $N + Am \to Q$ coinciding with f on N. Now, applying Zorn's lemma to the pairs (M', f'), where M' is a submodule of M containing N and $f' \colon M' \to Q$ extends f, with the order

$$(M_1', f_1') \ \leq \ (M_2', f_2') \ \Leftrightarrow \ M_1' \subseteq M_2', \text{ and } f_2' \text{ extends } f_1',$$

we see that there exists (M', f') maximal, and $M' = M$ since we may not extend f'.

Definition: An A-module Q is **divisible** if the morphisms $Q \xrightarrow{a\cdot} Q$, $a \neq 0$, are surjective.

Corollary: *Divisible modules over a principal ideal domain are injective.*

Proof: $Q \xrightarrow{a\cdot} Q$ is just the morphism $\mathrm{Hom}_A(A, Q) \to \mathrm{Hom}_A(aA, Q) \simeq \mathrm{Hom}_A(A, Q)$, $a \neq 0$.

Corollary: *If p is an irreducible element of a principal ideal domain A, then $B = A/p^nA$ is an injective B-module.*

Proof: Given an ideal p^rA/p^nA of B and a morphism of B-modules $\phi \colon p^rA/p^nA \to B$, we have $0 = \phi(\bar{p}^n) = \bar{p}^{n-r}\phi(\bar{p}^r)$; hence $\phi(\bar{p}^r) = b\bar{p}^r$ for some $b \in B$.

The required extension $\phi' \colon B \to B$ is just $\phi'(x) = bx$.

Theorem: *Any A-module M is a submodule of an injective A-module.*

Proof: Any cyclic group is a subgroup of the injective $\mathbb{Z}$-module $Q = \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$, so that for any non zero $m \in M$ we have a group morphism $\omega \colon M \to Q$ not vanishing on m.

If we put $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, Q)$, the natural morphism of A-modules $M \to M^{**}$ is injective, and it is enough to embed M^{**} into an injective A-module.

Now, we have a natural isomorphism of A-modules

$$\mathrm{Hom}_{\mathbb{Z}}(M, Q) = \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, Q)) \quad , \quad \mathrm{Hom}_{\mathbb{Z}}(M, Q) = \mathrm{Hom}_A(M, A^*) \ ,$$

so that A^* is an injective A-module, because Q is an injective $\mathbb{Z}$-module. Considering an epimorphism $\oplus_I A \to M^*$, we obtain an injective morphism $M^{**} \to \prod_I A^*$ into an injective A-module.

5.2.2 Localization

Definition: The **localization** M_S of an A-module M by a multiplicative system S of A is the quotient of $M \times S$ by the equivalence relation

$$(m, s) \equiv (n, t) \ \Leftrightarrow \ \text{ there are } u, v \in S \ \text{ such that } \ mu = nv, \ su = tv,$$

and it is an A_S-module with the operations

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st} \quad , \quad \frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st}$$

where $\frac{m}{s}$ is the class of (m, s). Therefore $\frac{m}{s} = 0$ if and only if $um = 0$ for some $u \in S$.

Any morphism of A-modules $f \colon M \to N$ defines a morphism of A_S-modules

$$f_S \colon M_S \longrightarrow N_S, \ f_S\left(\tfrac{m}{s}\right) = \tfrac{f(m)}{s}.$$

We have a canonical **localization morphism** of A-modules $M \to M_S$, $m \mapsto \frac{m}{1}$.

Universal Property: *Let N be an A_S-module. Any morphism of A-modules $f\colon M \to N$ uniquely factors through a morphism of A_S-modules $\phi\colon M_S \to N$, $\phi(\frac{m}{1}) = f(m)$.*

$$\mathrm{Hom}_{A_S}(M_S, N) = \mathrm{Hom}_A(M, N).$$

Proof: The unique possible morphism, $\phi(\frac{m}{s}) = s^{-1}f(m)$, is well defined,

$$\phi\big(\tfrac{um}{us}\big) = (us)^{-1}f(um) = s^{-1}u^{-1}uf(m) = s^{-1}f(m).$$

Notation: M_f denotes the localization by $S = \{1, f, \ldots, f^n, \ldots\}$.
If $\mathfrak{p}$ is a prime ideal, $M_\mathfrak{p}$ denotes the localization by $S = A - \mathfrak{p}$.

Theorem: *If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence, so is the sequence*

$$M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S.$$

Proof: If $\frac{m}{s} \in \mathrm{Ker}\, g_S$, then $\frac{g(m)}{s} = 0$, and $0 = tg(m) = g(tm)$ for some $t \in S$.
Hence $tm = f(m')$, and $\frac{m}{s} = \frac{tm}{ts} = \frac{f(m')}{ts} = f_S\Big(\frac{m'}{ts}\Big) \in \mathrm{Im}\, f_S$. q.e.d.

If N is a submodule of M, then N_S may be identified with a submodule of M_S, and

1. $(N + N')_S = N_S + N'_S$.
2. $(N \cap N')_S = N_S \cap N'_S$.
3. $(M \oplus M')_S = M_S \oplus M'_S$.
4. $(M/N)_S = M_S/N_S$.
5. $(\mathrm{Ker}\, f)_S = \mathrm{Ker}\, f_S$.
6. $(\mathrm{Im}\, f)_S = \mathrm{Im}\, f_S$.

Proof: The non obvious equalities follow localizing the exact sequences

$$0 \longrightarrow M' \longrightarrow M' \oplus M \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow N \cap N' \longrightarrow N \oplus N' \longrightarrow N + N' \longrightarrow 0$$
$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Ker}\, f \longrightarrow M \longrightarrow N$$

5.2.3 Tensor Product

Definitions: An ordered set I is **filtered** if for any pair $i, j \in I$ there is $k \in I$ such that $i, j \leq k$.
An **inductive system** of sets is a family of sets $\{X_i\}_{i \in I}$ and maps $\phi_{ji}\colon X_i \to X_j$, $i \leq j$, such that $\phi_{ii} = \mathrm{Id}_{X_i}$ and, whenever $i \leq j \leq k$,

$$\begin{array}{ccc} X_i & \longrightarrow & X_j \\ & \searrow \;\; \swarrow & \\ & X_k & \end{array} \qquad\qquad \phi_{ki} = \phi_{kj}\phi_{ji}$$

The **inductive limit** $\varinjlim X_i$ is the quotient set $\big(\amalg_i X_i\big)/\equiv$ of the disjoint union, where

$$x_i \equiv x_j \ \text{ when } x_i = x_j \text{ in } X_k, \text{ for some } k \geq i, j,$$

(if $x_i = x_j$ in X_k, and $x_j = x_l$ in $X_{k'}$, then $x_i = x_j = x_l$ in X_r for any index $r \geq k, k'$.)

We have canonical maps $\phi_j \colon X_j \to \varinjlim X_i$, $\phi_j(x_j) = [x_j]$, such that

$$\begin{array}{ccc} X_i & \longrightarrow & X_j \\ & \searrow \quad \swarrow & \\ & \varinjlim X_i & \end{array} \qquad \phi_i = \phi_j \phi_{ji}$$

Definitions: A **projective system** of sets is a family of sets $\{X_i\}_{i \in I}$ and maps $\phi_{ij} \colon X_j \to X_i$, $i \leq j$, such that $\phi_{ii} = \mathrm{Id}_{X_i}$ and, whenever $i \leq j \leq k$,

$$\begin{array}{ccc} & X_k & \\ & \swarrow \quad \searrow & \\ X_j & \longrightarrow & X_i \end{array} \qquad \phi_{ik} = \phi_{ij} \phi_{jk}$$

The subset of $\prod_i X_i$ formed by all congruent sequences (x_i), $\phi_{ij}(x_j) = x_i$ when $i \leq j$, is the **projective limit** $\varprojlim X_i$. (If I is the empty set, $\varprojlim X_i := \{p\}$.)

We have canonical maps $\phi_j \colon \varprojlim X_i \to X_j$, $\phi_j(x_i) = x_j$, such that

$$\begin{array}{ccc} & \varprojlim X_i & \\ & \swarrow \quad \searrow & \\ X_j & \longrightarrow & X_i \end{array} \qquad \phi_i = \phi_{ij} \phi_j$$

If the maps ϕ_{ji} are group (ring,...) morphisms, then the limits inherit a group (ring,...) structure and the canonical maps ϕ_i are group (ring,...) morphisms.

Examples: The ring morphisms $K[x]/(x^n) \to K[x]/(x^m)$, $m \leq n$, form a projective system of rings, and the projective limit is the ring of formal series $K[[x]]$.

Any module M is the inductive limit of its submodules $M_i \subseteq M$, and the projective limit of its quotients $M \to N_i$, since $\varinjlim X_i = X_k$ and $\varprojlim X_i = X_k$ when I has a final element k.

Universal Property: *Let (X_i, ϕ_{ji}) be an inductive system. Given maps $f_i \colon X_i \to Y$ such that $f_i = f_j \phi_{ji}$ when $i \leq j$, there exists a unique map $f \colon \varinjlim X_i \to Y$ such that $f_i = f \phi_i$,*

$$\mathrm{Hom}(\varinjlim X_i, Y) = \varprojlim \mathrm{Hom}(X_i, Y), \ f \mapsto (f \phi_i).$$

Dually, let (X_i, ϕ_{ij}) be a projective system. Given maps $f_i \colon Y \to X_i$ such that $f_i = \phi_{ij} f_j$ when $i \leq j$, there exists a unique map $f \colon Y \to \varprojlim X_i$ such that $f_i = \phi_i f$,

$$\mathrm{Hom}(Y, \varprojlim X_i) = \varprojlim \mathrm{Hom}(Y, X_i), \ f \mapsto (\phi_i f).$$

Proof: In the first case, the unique possible map is $f([x_i]) = f_i(x_i)$, and in the second case, $f(y) = (f_i(y))$. Both are well defined since $f_i = f_j \phi_j^i$, and $f_i = \phi_i^j f_j$. q.e.d.

Let M, N, P be three A-modules. The A-bilinear maps $M \times N \to P$ form an A-module $F(P) = \mathrm{Bil}_A(M, N; P)$. We shall construct a bilinear map $\xi \colon M \times N \to M \otimes_A N$ such that any bilinear map $g \colon M \times N \to P$ uniquely factors through a morphism $f \colon M \otimes_A N \to P$,

$$\mathrm{Hom}_A(M \otimes_A N, P) = \mathrm{Bil}_A(M, N; P), \ f \mapsto f \circ \xi.$$

Remark that f is surjective if and only if the image of g spans P; i.e., g does not factor through a strict submodule of P. Since $M \otimes_A N$ must be the projective limit of its quotients, we consider **pairs** P_g, where P is an A-module and $g\colon M \times N \to P$ is A-bilinear, and **morphisms of pairs** $f\colon P'_{g'} \to P_g$ are defined to be A-linear maps $f\colon P' \to P$ such that $g = f \circ g'$.

Definition: A pair Q_ξ is **minimal** if any injective morphism of pairs $P_g \to Q_\xi$ is an isomorphism (so that Q_ξ is a quotient of $M \otimes_A N$, and we should have $M \otimes_A N = \varprojlim Q_\xi$).

Lemma: *If* $0 \longrightarrow P' \xrightarrow{i} P \xrightarrow{p} P''$ *is an exact sequence, then so is the sequence*

$$0 \longrightarrow F(P') = \mathrm{Bil}_A(M, N; P') \xrightarrow{i_*} F(P) = \mathrm{Bil}_A(M, N; P) \xrightarrow{p_*} F(P'') = \mathrm{Bil}_A(M, N; P'')$$

and F *preserves direct products and projective limits,*

$$\begin{aligned} \mathrm{Bil}_A(M, N; P \times P') &= \mathrm{Bil}_A(M, N; P) \times \mathrm{Bil}_A(M, N; P'), \\ \mathrm{Bil}_A(M, N; \varprojlim P_i) &= \varprojlim \mathrm{Bil}_A(M, N; P_i). \end{aligned}$$

Proof: Straightforward. q.e.d.

Lemma: *Any pair* P_g *is dominated by a minimal pair: there exists a morphism of pairs* $Q_\xi \to P_g$*, where* Q_ξ *is minimal.*

Proof: Just take the submodule Q generated by the image of $g\colon M \times N \to P$, with the bilinear map $\xi\colon M \times N \to Q$, $\xi(m, n) = g(m, n)$. q.e.d.

Lemma: If Q_ξ is a minimal pair, any two morphisms of pairs $f, h\colon Q_\xi \to P_g$ coincide.

Proof: Since $0 \to \mathrm{Ker}\,(h - f) \xrightarrow{i} Q \xrightarrow{h-f} P$ is exact, we have an exact sequence

$$\begin{gathered} 0 \longrightarrow F(\mathrm{Ker}\,(h - f)) \xrightarrow{i_*} F(Q) \xrightarrow{h_* - f_*} F(P) \\ (h_* - f_*)(\xi) = h\xi - f\xi = g - g = 0 \end{gathered}$$

and there is $\xi' \in F(\mathrm{Ker}\,(h - f))$ such that $i\colon \mathrm{Ker}\,(h - f)_{\xi'} \to Q_\xi$ is a morphism of pairs.

Since Q_ξ is minimal, $\mathrm{Ker}\,(h - f) = Q$, and $f = h$. q.e.d.

Now we order minimal pairs (identifying isomorphic pairs): $Q'_{\xi'} \geq Q_\xi$ if there is a morphism of pairs $f'\colon Q'_{\xi'} \to Q_\xi$. If also $Q_\xi \geq Q'_{\xi'}$, there is a morphism $f\colon Q_\xi \to Q'_{\xi'}$, and we have morphisms $f'f\colon Q_\xi \to Q_\xi$, $ff'\colon Q'_{\xi'} \to Q'_{\xi'}$; hence $f'f = \mathrm{Id}_Q$, $ff' = \mathrm{Id}_{Q'}$, and $Q_\xi = Q'_{\xi'}$.

This is a filtered order, because F preserves direct products:

If Q_ξ and $Q'_{\xi'}$ are minimal pairs, then $\xi \times \xi' \in F(Q) \times F(Q') = F(Q \times Q')$. We have morphisms of pairs

$$(Q \times Q')_{\xi \times \xi'} \rightrightarrows \begin{matrix} Q_\xi \\ Q'_{\xi'} \end{matrix}$$

and any minimal pair dominating $(Q \times Q')_{\xi \times \xi'}$ dominates Q_ξ and $Q'_{\xi'}$.

Hence the minimal pairs form a projective system $\{(Q_i)_{\xi_i}\}$, and the projective limit $M \otimes_A N$ is a pair with $\xi = (\xi_i) \in \varprojlim F(Q_i) = F(M \otimes_A N)$.

A minimal pair dominating $(M \otimes_A N)_\xi$ dominates any other minimal pair; hence there is a final minimal pair, obviously the projective limit $(M \otimes_A N)_\xi$, and for every pair P_g we have a unique morphism of pairs $(M \otimes_A N)_\xi \to P_g$. We see that $\xi\colon M \times N \to M \otimes_A N$ is the universal bilinear map and, if we put $m \otimes n := \xi(m,n)$,

$$(am + a'm') \otimes n = a(m \otimes n) + a'(m' \otimes n),$$
$$m \otimes (an + a'n') = a(m \otimes n) + a'(m \otimes n').$$

Universal Property: *Any bilinear map $g\colon M \times N \longrightarrow P$ uniquely factors through a morphism of A-modules $f\colon M \otimes_A N \to P$, $f(m \otimes n) = g(m,n)$,*

$$\mathrm{Hom}_A(M \otimes_A N, P) = \mathrm{Bil}_A(M, N; P).$$

If $f\colon M \to M'$, $h\colon N \to N'$ are morphisms of A-modules, then $M \times N \to M' \otimes_A N'$, $(m,n) \mapsto f(m) \otimes h(n)$, is A-bilinear, and it induces a morphism of A-modules

$$f \otimes h\colon M \otimes_A N' \to M' \otimes_A N', \ (f \otimes h)(m \otimes n) = f(m) \otimes h(n).$$

Lemma: $\mathrm{Bil}_A(M, N; P) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$.

Proof: The morphism $f\colon M \to \mathrm{Hom}_A(N,P)$ corresponds to $g(m,n) = f(m)(n)$.

Theorem: *If $M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ is an exact sequence, so is exact the sequence*

$$M' \otimes_A N \xrightarrow{i \otimes 1} M \otimes_A N \xrightarrow{p \otimes 1} M'' \otimes_A N \longrightarrow 0$$

Proof: If $\mathbf{E}$ is the exact sequence $M' \to M \to M'' \to 0$, for any A-module P the sequence $\mathrm{Hom}_A(\mathbf{E}, \mathrm{Hom}_A(N,P)) = \mathrm{Hom}_A(\mathbf{E} \otimes_A N, P)$ is exact; hence $\mathbf{E} \otimes_A N$ is exact. q.e.d.

1. $(M \otimes_A N) \otimes_A P = M \otimes_A (N \otimes_A P)$, where $(m \otimes n) \otimes p = m \otimes (n \otimes p)$.
2. $M \otimes_A N = N \otimes_A M$, where $m \otimes n = n \otimes m$.
3. $M \otimes_A (\oplus_i N_i) = \oplus_i (M \otimes_A N_i)$, where $m \otimes (\sum_i n_i) = \sum_i m \otimes n_i$.
4. $M \otimes_A (\varinjlim N_i) = \varinjlim (M \otimes_A N_i)$.
5. $A \otimes_A M = M$, where $a \otimes m = am$.
6. $(A/I) \otimes_A M = M/IM$, where $\bar{a} \otimes m = [am]$.

Proof of the Properties of the Tensor Product of Modules:

(1) $\mathrm{Hom}((M \otimes N) \otimes P, X) = \mathrm{Hom}(M \otimes N, \mathrm{Hom}(P, X)) = \mathrm{Hom}(M, \mathrm{Hom}(N, \mathrm{Hom}(P, X)))$
$$= \mathrm{Hom}(M, \mathrm{Hom}(N \otimes P, X)) = \mathrm{Hom}(M \otimes (N \otimes P), X).$$

(2) $\mathrm{Hom}(M \otimes N, X) = \mathrm{Bil}_A(M, N; X) = \mathrm{Bil}_A(N, M; X) = \mathrm{Hom}(N \otimes M, X)$.

(3) $\mathrm{Hom}(M \otimes (\oplus_i N_i), X) = \mathrm{Hom}(M, \mathrm{Hom}(\oplus_i N_i, X)) = \mathrm{Hom}(M, \prod_i \mathrm{Hom}(N_i, X))$
$$= \prod_i \mathrm{Hom}(M, \mathrm{Hom}(N_i, X)) = \prod_i \mathrm{Hom}(M \otimes N_i, X) = \mathrm{Hom}(\oplus_i (M \otimes N_i), X).$$

(4) $\mathrm{Hom}(M \otimes (\varinjlim N_i), X) = \mathrm{Hom}(M, \mathrm{Hom}(\varinjlim N_i, X)) = \mathrm{Hom}(M, \varprojlim \mathrm{Hom}(N_i, X))$
$$= \varprojlim \mathrm{Hom}(M, \mathrm{Hom}(N_i, X)) = \varprojlim \mathrm{Hom}(M \otimes N_i, X) = \mathrm{Hom}(\varinjlim (M \otimes N_i), X).$$

(5) $\mathrm{Hom}(A \otimes M, X) = \mathrm{Hom}(A, \mathrm{Hom}(M, X)) = \mathrm{Hom}(M, X)$.

(6) The sequence $I \otimes_A M \longrightarrow A \otimes_A M = M \longrightarrow (A/I) \otimes_A M \longrightarrow 0$ is exact.

Definition: An A-module P is **flat** if $(-) \otimes_A P$ preserves exact sequences; i.e., if $M' \to M$ is an injective morphism of A-modules, so is $M' \otimes_A P \to M \otimes_A P$.

Any free (or projective) module is flat.

If $i\colon E' \to E$ is an injective linear map, then $i \otimes 1\colon E' \otimes_k F \to E \otimes_k F$ is also injective, since i admits a retract (p. 54). Any k-vector space is a flat k-module.

Definition: Let $A \to B$ be a ring morphism. The morphisms $1 \otimes b\colon M \otimes_A B \to M \otimes_A B$ define a structure of B-module on $M \otimes_A B$, $b\big(\sum_i m_i \otimes b_i\big) = \sum_i m_i \otimes (bb_i)$. We denote it by $M_B = M \otimes_A B$, and we say that it is the **base change** of M.

The morphism of A-modules $M \to M_B$, $m \mapsto m \otimes 1$, is the **base change morphism**.

Any morphism of A-modules $f\colon M \to M'$ induces a morphism of B-modules

$$f_B = f \otimes 1\colon M_B \longrightarrow M'_B,\ \ f_B\big(\textstyle\sum_i m_i \otimes b_i\big) = \sum_i f(m_i) \otimes b_i.$$

Universal Property: *Let N be a B-module. For any morphism of A-modules $f\colon M \to N$ there exists a unique morphism of B-modules $\phi\colon M_B \to N$ such that $\phi(m \otimes 1) = f(m)$,*

$$\mathrm{Hom}_A(M, N) = \mathrm{Hom}_B(M_B, N).$$

Proof: By the universal property of the tensor product, there exists a unique morphism of A-modules $\phi\colon M \otimes_A B \to N$, $\phi(m \otimes b) = bf(m)$, and it is a morphism of B-modules.

Corollary: $M_S = M \otimes_A A_S$.

Proof: $\mathrm{Hom}_{A_S}(M_S, N) = \mathrm{Hom}_A(M, N) = \mathrm{Hom}_{A_S}(M \otimes_A A_S, N)$.

Theorem: $(M \otimes_A B) \otimes_B N = M \otimes_A N,\ (m \otimes b) \otimes n = m \otimes (bn),$ *where N is a B-module.*

Proof: In the isomorphism $\mathrm{Hom}_A(M \otimes_A N, X) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, X))$ it is easy to check that $\mathrm{Hom}_B(M \otimes_A N, X)$ corresponds to $\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, X))$; hence

$$\begin{aligned}\mathrm{Hom}_B(M_B \otimes N, X) &= \mathrm{Hom}_B(M_B, \mathrm{Hom}_B(N, X))\\ &= \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, X)) = \mathrm{Hom}_B(M \otimes_A N, X).\end{aligned}$$

Corollary: $(M_B)_C = M_C$, where $(m \otimes b) \otimes c = m \otimes (bc)$.

Corollary: $(M_B) \otimes_B (M'_B) = (M \otimes_A M')_B$, where $(m \otimes b_1) \otimes (m' \otimes b_2) = (m \otimes m') \otimes b_1 b_2$.

Proof: $(M \otimes_A B) \otimes_B (M'_B) = M \otimes_A (M'_B) = (M \otimes_A M') \otimes_A B$.

Definition: We fix a ring k (a field in this course) and we name k-**algebra** any ring morphism $k \to A$, and **morphisms** of k-algebras are ring morphisms $A \to B$ such that the following triangle is commutative

$$\begin{array}{ccc} & k & \\ \swarrow & & \searrow \\ A & \longrightarrow & B\end{array}$$

If A, B are k-algebras, the morphism of k-modules

$$A \otimes_k B \otimes_k A \otimes_k B \longrightarrow A \otimes_k B,\ \ a_1 \otimes b_1 \otimes a_2 \otimes b_2 \mapsto a_1 a_2 \otimes b_1 b_2,$$

induces a k-bilinear map $(A \otimes_k B) \times (A \otimes_k B) \to A \otimes_k B$. This product

$$\left(\sum_i a_i \otimes b_i\right) \cdot \left(\sum_j a_j \otimes b_j\right) = \sum_{i,j} a_i a_j \otimes b_i b_j$$

defines a ring structure on $A \otimes_k B$, and it is also a k-algebra with the natural morphism

$$k \longrightarrow A \otimes_k B, \ \lambda \mapsto \lambda \otimes 1 = 1 \otimes \lambda.$$

Universal Property: *If $f\colon A \to C$, $h\colon B \to C$ are morphisms of k-algebras, there is a unique morphism of k-algebras $\phi\colon A \otimes_k B \to C$ such that $\phi(a \otimes 1) = f(a)$, $\phi(1 \otimes b) = h(b)$.*

$$\mathrm{Hom}_{k\text{-alg}}(A \otimes_k B, C) = \mathrm{Hom}_{k\text{-alg}}(A, C) \times \mathrm{Hom}_{k\text{-alg}}(B, C).$$

Proof: By the universal property of the tensor product, there exists a unique morphism of k-modules $\phi\colon A \otimes_k B \to C$, $\phi(a \otimes b) = f(a)h(b)$, and it is a morphism of k-algebras.

Corollary: $k[x_1, \ldots, x_n] \otimes_k L = L[x_1, \ldots, x_n]$.
$(k[x]/(P)) \otimes_k L = L[x]/(P)$.

5.3 Categories and Functors

Definition: A **category C** is given by a family of **objects**, a set $\mathrm{Hom}_{\mathbf{C}}(M, N)$ for each pair M, N of objects (the **morphisms** $M \to N$), a morphism $\mathrm{Id}_M\colon M \to M$ for each object M (the **identity** of M) and a map (the **composition** of morphisms)

$$\mathrm{Hom}_{\mathbf{C}}(M, N) \times \mathrm{Hom}_{\mathbf{C}}(N, P) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(M, P), \ (f, g) \mapsto g \circ f.$$

for any three objects M, N, P; satisfying the following conditions:

1. Composition of morphisms is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.
2. We have $f \circ \mathrm{Id}_M = f$ and $\mathrm{Id}_N \circ f = f$ for any morphism $f\colon M \to N$.

A morphism $f\colon M \to N$ is an **isomorphism** (**automorphism** when $M = N$) when $fg = \mathrm{id}_N$, $gf = \mathrm{id}_M$ for some morphism $g\colon N \to M$ (obviously unique and we put $g = f^{-1}$).

We have the category of sets, of groups, of rings, of A-modules, of topological spaces,...

Definitions: A **covariant functor** $F\colon \mathbf{C} \rightsquigarrow \mathbf{C}'$ between two categories associates an object $F(M)$ of $\mathbf{C}'$ to any object M of $\mathbf{C}$, and a morphism $F(f)\colon F(M) \to F(N)$ of $\mathbf{C}'$ to any morphism $f\colon M \to N$ of $\mathbf{C}$, so that

1. $F(\mathrm{Id}_M) = \mathrm{Id}_{F(M)}$ for any object M of $\mathbf{C}$.
2. $F(f \circ g) = F(f) \circ F(g)$ for any pair of morphisms $M \xrightarrow{g} N \xrightarrow{f} P$ of $\mathbf{C}$.

Analogously, **contravariant functors** associate a morphism $F(f)\colon F(N) \to F(M)$ to any morphism $f\colon M \to N$, so that $F(f \circ g) = F(g) \circ F(f)$. Any functor may be assumed to be covariant, since contravariant functors $\mathbf{C} \rightsquigarrow \mathbf{C}'$ are just covariant functors $\mathbf{C}^{\mathbf{op}} \rightsquigarrow \mathbf{C}'$, where the **opposite** category $\mathbf{C}^{\mathbf{op}}$ of $\mathbf{C}$ has the same objects, but $\mathrm{Hom}_{\mathbf{C}^{\mathbf{op}}}(M, N) = \mathrm{Hom}_{\mathbf{C}}(N, M)$, the composition $f \circ g$ of two morphisms in $\mathbf{C}^{\mathbf{op}}$ being the composition $g \circ f$ in $\mathbf{C}$.

Given two (covariant) functors $F, G\colon \mathbf{C} \rightsquigarrow \mathbf{C}'$, to give a **natural transformation** or **morphism of functors** $\theta\colon F \to G$ is just to give a morphism $\theta_M\colon F(M) \to G(M)$ in **C**' for any object M in **C**, such that for any morphism $f\colon M \to N$ in **C** the following square commutes

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_N} \\ G(M) & \xrightarrow{G(f)} & G(N) \end{array} \qquad \theta_N \circ F(f) = G(f) \circ \theta_M$$

and it is an **isomorphism of functors** if θ_M is an isomorphism in **C**' for any object M, so that $\theta^{-1}\colon G \to F$, $(\theta^{-1})_M := (\theta_M)^{-1}$, also is a natural isomorphism and $\theta_N \circ F(f) \circ \theta_M^{-1} = G(f)$.

An **equivalence**[4] of categories is a pair of covariant functors $F\colon \mathbf{C} \rightsquigarrow \mathbf{C}'$, $G\colon \mathbf{C}' \rightsquigarrow \mathbf{C}$ and a pair of isomorphisms of functors $\theta\colon G \circ F \xrightarrow{\sim} \mathrm{Id}_{\mathbf{C}}$, $\theta'\colon F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathbf{C}'}$.

In such case, for any two objects M, N of **C**, we have natural maps

$$\mathrm{Hom}_{\mathbf{C}}(M,N) \xrightarrow{F} \mathrm{Hom}_{\mathbf{C}'}(F(M),F(N)) \xrightarrow{G} \mathrm{Hom}_{\mathbf{C}}(GF(M),GF(N)) \overset{\theta}{\simeq} \mathrm{Hom}_{\mathbf{C}}(M,N)$$

and the composition is the identity, $\theta_N \circ G(F(f)) \circ \theta_M^{-1} = f$, because $\theta\colon G \circ F \to \mathrm{Id}_{\mathbf{C}}$ is functorial.

Therefore F is injective (hence so is G) and we see that F induces bijections[5]

$$F\colon \mathrm{Hom}_{\mathbf{C}}(M,N) \longrightarrow \mathrm{Hom}_{\mathbf{C}'}(F(M),F(N)) \ , \quad f \mapsto F(f).$$

Definitions: Let X be an object of a category **C**. We say that $X^\bullet(T) = \mathrm{Hom}_{\mathbf{C}}(T,X)$ is the set of points of X parameterized by T, or just T-**points** of X.

Any morphism $t\colon S \to T$ in **C** induces a natural map

$$X^\bullet(T) = \mathrm{Hom}_{\mathbf{C}}(T,X) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(S,X) = X^\bullet(S), \ x \mapsto x|_t = x \circ t,$$

so that $X^\bullet\colon \mathbf{C} \rightsquigarrow \mathbf{Sets}$ is a contravariant functor, named **functor of points** of X, and we say that $x|_t$ is the **specialization** of the T-point x at the point t of the parameter space T. A point of X is **generic** if any other point of X is a specialization. So, the identity $x_g\colon X \to X$ is a generic point of X, since any point $x\colon T \to X$ is a specialization, $x = \mathrm{Id}_X \circ x = x_g|_x$.

Any morphism $f\colon X \to Y$ naturally induces a morphism $f = f_*\colon X^\bullet \to Y^\bullet$, $f(x) := f \circ x$.

Yoneda's Lemma: *For any contravariant functor* $F\colon \mathbf{C} \rightsquigarrow \mathbf{Sets}$ *we have a canonical bijection*

$$\mathrm{Hom}_{\mathrm{nat}}(X^\bullet, F) = F(X), \quad \theta \mapsto \theta(\mathrm{Id}_X).$$

In particular, $\mathrm{Hom}_{\mathbf{C}}(X,Y) = \mathrm{Hom}_{\mathrm{nat}}(X^\bullet, Y^\bullet)$ *for any two objects* X, Y *of* ***C***.

Proof: Any morphism of functors $X^\bullet \to F$ preserves specializations, hence it is fully determined by the image of a generic point, and the considered map is injective.

Finally, any element $\xi \in F(X)$ defines a morphism $\theta\colon X^\bullet \to F$, $\theta(x) = F(x)(\xi) \in F(T)$ for any point $x\colon T \to X$. Now $\theta(\mathrm{Id}) = F(\mathrm{Id})(\xi) = \xi$, so that the map is surjective.

[4] an **antiequivalence** when the involved functors are contravariant.

[5] We have isomorphisms $F(\theta_M), \theta'_{F(M)}\colon FGF(M) = F(M)$, $G(\theta'_{M'}), \theta_{G(M')}\colon GFG(M') = G(M')$, and it would be sensible to assume that $F(\theta_M) = \theta'_{F(M)}$, $G(\theta'_{M'}) = \theta_{G(M')}$. Given the isomorphism of functors θ, just consider the isomorphisms $\theta'_{M'} := G^{-1}(\theta_{G(M')})$, since $\theta_{GF(M)} = GF(\theta_M)$ by the naturalness of θ in the morphism $\theta_M\colon GF(M) \to M$.

Definitions: The **direct product** of two objects X, Y is a pair of morphisms

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & Y \end{array}$$

with the universal property $(\pi_1, \pi_2)\colon \mathrm{Hom}_{\mathbf{C}}(T, X \times Y) = \mathrm{Hom}_{\mathbf{C}}(T, X) \times \mathrm{Hom}_{\mathbf{C}}(T, Y)$; i.e. the points of the direct product are the direct product of points, $(X \times Y)^\bullet = X^\bullet \times Y^\bullet$.

The **final** object pt is defined to be an object with a unique T-point $T \to \mathrm{pt}$ for any object T; i.e. with the universal property $\mathrm{Hom}_{\mathbf{C}}(T, \mathrm{pt}) = *$.

Dually we define the **direct sum** $X \oplus Y$ (or **coproduct** $X \coprod Y$) as a pair of morphisms

$$\begin{array}{ccc} & X \oplus Y & \\ {}^{j_1}\nearrow & & \nwarrow^{j_2} \\ X & & Y \end{array}$$

with the universal property $(j_1^*, j_2^*)\colon \mathrm{Hom}_{\mathbf{C}}(X \oplus Y, T) = \mathrm{Hom}_{\mathbf{C}}(X, T) \times \mathrm{Hom}_{\mathbf{C}}(Y, T)$, and the **initial** object $\emptyset$ is defined by the universal property $\mathrm{Hom}_{\mathbf{C}}(\emptyset, T) = *$.

In general, we define the product $\prod_i X_i$ and coproduct $\bigoplus_i X_i$ of any set of objects $\{X_i\}_{i \in I}$ (so that $\prod_i X_i = pt$ and $\bigoplus_i X_i = \emptyset$ when I is empty) by the universal properties

$$\mathrm{Hom}(T, \textstyle\prod_i X_i) = \prod_i \mathrm{Hom}(T, X_i) \quad , \quad \mathrm{Hom}(\bigoplus_i X_i, T) = \prod_i \mathrm{Hom}(X_i, T).$$

Projective limits and **inductive limits** are defined by the universal properties

$$\mathrm{Hom}(T, \varprojlim X_i) = \varprojlim \mathrm{Hom}(T, X_i) \quad , \quad \mathrm{Hom}(\varinjlim X_i, T) = \varprojlim \mathrm{Hom}(X_i, T).$$

Definition: Fix an object S in $\mathbf{C}$. The morphisms $X \to S$, named S-objects, with the sets $\mathrm{Hom}_S(X, Y)$ of S-morphisms, i.e. of morphisms $f\colon X \to Y$ fitting in a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

form a category $\mathbf{C}|_S$, and the direct product in $\mathbf{C}|_S$ of two S-objects $f\colon X \to S$, $g\colon Y \to S$ is said to be the **fibred product** of the morphisms f and g. It is a S-object $X \times_S Y \to S$ endowed with S-morphisms $\pi_1\colon X \times_S Y \to X$, $\pi_2\colon X \times_S Y \to Y$ such that, for any S-object T, we have a bijection

$$(\pi_1, \pi_2)\colon \mathrm{Hom}_S(T, X \times_S Y) = \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y).$$

When $T = X$, and we fix the identity morphism $T \to X$, we obtain the very important

Graph Formula: $\mathrm{Hom}_S(X, Y) = \mathrm{Hom}_X(X, X \times_S Y)$, $f \mapsto \mathrm{Id}_X \times f$.

1. $F(E) = \Lambda^p E$, $F(X) = X/G$, $F(X) = X^G$, $F(N) = \mathrm{Hom}_A(M, N)$, $F(M) = M_S$, $F(P) = \mathrm{Bil}_A(M, N; P)$, $F(N) = M \otimes_A N$, $F(M) = M_B$ are covariant functors.

2. $F(E) = E^*$, $F(E) = T_p E$, $F(N) = \mathrm{Hom}_A(N, M)$ are contravariant functors.

3. The Reflexivity Theorem shows that the functor $E \rightsquigarrow E^*$ induces an antiequivalence of the category of finite dimensional k-vector spaces with itself, the inverse functor being F itself.

4. The Fundamental Theorem of Projective Geometry points that the lattice of linear subvarieties defines an equivalence of the category of projective spaces of dimension $n \geq 2$ and projectivizations of semilinear transformations with a certain a certain category of ordered sets and isomorphisms, but the construction of the inverse functor requires additional work (see [2]).

5. Given a group G, the categories of left G-sets and right G-sets are equivalent, a left action corresponding to the right action $x \cdot g = g^{-1} \cdot x$.

6. In the category of sets (and topological spaces), the direct sum is the disjoint union, the initial object is the empty set and the final object is the one point set.

7. In the category **Sets** of sets, the fibred product of two maps $f\colon X \to S$, $g\colon Y \to S$ is the set $X \times_S Y = \{(x,y) \in X \times Y\colon f(x) = g(y)\}$, with the projections $\pi_1\colon X \times_S Y \to X$, $\pi_1(x,y) = x$, and $\pi_2\colon X \times_S Y \to Y$, $\pi_2(x,y) = y$. In particular, $X \times_{\mathrm{pt}} Y = X \times Y$, $X \times_S y = f^{-1}(y)$ when y is a point of S, and $X \times_S Y = X \cap Y$ when X and Y are subsets of S.

8. In an arbitrary category, the universal property of $X \times_S Y$ shows that for any object T we have a natural bijection $(\pi_1, \pi_2)\colon \mathrm{Hom}(T, X \times_S Y) = \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y)$; i.e. the points of the fibred product are the fibred product of points, $(X \times_S Y)^\bullet = X^\bullet \times_{S^\bullet} Y^\bullet$.

9. In the opposite category of rings, the direct sum is $A \times B$ (that we also denote $A \oplus B$), k-algebras are just k-objects, and the fibred product is $A \otimes_k B$. Hence, the graph formula states that $\mathrm{Hom}_{k\text{-alg}}(A, B) = \mathrm{Hom}_{B\text{-alg}}(A \otimes_k B, B)$.

5.3.1 Grothendieck's Representability Theorem

Definition: A contravariant functor $F\colon \mathbf{C} \rightsquigarrow \mathbf{Sets}$ is **representable** when it is isomorphic to the functor of points $Q^\bullet$ of some object Q. By Yoneda's lemma, any functorial isomorphism $F \simeq Q^\bullet$ is defined by a unique element $\xi \in F(Q)$, and we say that the pair Q_ξ **represents** F: for any pair T_η, where $\eta \in F(T)$, there is a unique morphism of pairs $f\colon T_\eta \to Q_\xi$; i.e. $F(f)(\xi) = \eta$,

$$\mathrm{Hom}_{\mathbf{C}}(T, Q) \ = \ F(T), \ f \mapsto F(f)(\xi).$$

Dually, a covariant functor $F\colon \mathbf{C} \rightsquigarrow \mathbf{Sets}$ is representable when for some pair Q_ξ, where $\xi \in F(Q)$, the following natural maps are bijective,

$$\mathrm{Hom}_{\mathbf{C}}(Q, M) \longrightarrow F(M), \ f \mapsto F(f)(\xi).$$

Theorem: *If it exists, the representant of a functor is unique, up to canonical isomorphisms: If two pairs Q_ξ and $Q'_{\xi'}$ represent a covariant (resp. contravariant) functor F, then there is a unique isomorphism $f\colon Q \to Q'$ such that $F(f)(\xi) = \xi'$ (resp. $F(f)(\xi') = \xi$).*

Proof: There is a unique morphism $f\colon Q \to Q'$ such that $F(f)(\xi) = \xi'$, and a unique morphism $f'\colon Q' \to Q$ such that $F(f')(\xi') = \xi$.

Hence $F(f'f)(\xi) = \xi$, $F(ff')(\xi') = \xi'$, so that $f'f = \mathrm{Id}_Q$, $ff' = \mathrm{Id}_{Q'}$.

Examples: All universal properties state that certain pair Q_ξ represents a functor F (in quotients, ξ is the canonical projection, in localizations it is the localization morphism, in base changes it is the base change morphism, etc.). Hence any universal property determines a well defined (up to a canonical isomorphism) object, if it exists.

Definitions: In the category of sets, a sequence of maps

$$X \xrightarrow{i} Y \overset{f,g}{\rightrightarrows} Z$$

is **exact** when i is injective with image $\operatorname{Im} i = \{y \in Y : f(y) = g(y)\}$.

In an arbitrary category $\mathbf{C}$, a sequence of morphisms

$$M \xrightarrow{i} N \overset{f,g}{\rightrightarrows} P$$

is exact (i is the **kernel** of f and g) when, for any object X, so is the sequence of maps

$$\operatorname{Hom}_{\mathbf{C}}(X, M) \xrightarrow{i_*} \operatorname{Hom}_{\mathbf{C}}(X, N) \overset{f_*,g_*}{\rightrightarrows} \operatorname{Hom}_{\mathbf{C}}(X, P)$$

and a covariant functor is **left exact** when it preserves finite direct products (hence the final object), in the sense that the natural morphisms $F(\prod_i M_i) \to \prod_i F(M_i)$ are isomorphisms, and kernels. Any representable covariant functor is left exact and preserves projective limits, and Grothendieck's representability theorem states that the converse is true under a very mild condition. Dually a sequence of morphisms

$$M \overset{f,g}{\rightrightarrows} N \xrightarrow{p} P$$

is exact (p is the **cokernel** of f and g) when so are the sequences of maps

$$\operatorname{Hom}_{\mathbf{C}}(P, X) \xrightarrow{p^*} \operatorname{Hom}_{\mathbf{C}}(N, X) \overset{f^*,g^*}{\rightrightarrows} \operatorname{Hom}_{\mathbf{C}}(M, X)$$

and a covariant functor is **right exact** when it preserves finite coproducts (hence the initial object) and cokernels, and it is **exact** when it is left and right exact. Finally, a contravariant functor $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$ is left or right exact when so is the covariant functor $F : \mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{C}'$.

Examples: In the category of A-modules, the first condition states that the sequence of A-linear maps $0 \to M \xrightarrow{i} N \xrightarrow{f-g} P$ is exact, and the second one states that so is $M \xrightarrow{f-g} N \xrightarrow{p} P \to 0$.

In the category of sets, the cokernel is the quotient of N by the coarse equivalence relation such that $f(m) \equiv g(m)$. In the category of topological spaces, it is the same quotient endowed with the quotient topology (the finest topology such that p is continuous), and in the category of G-sets with the obvious action of G.

In an arbitrary category $\mathbf{C}$, the action of a finite group $G = \{g_1, \dots, g_n\}$ on an object X is defined to be a group morphism $G \to \operatorname{Aut}_{\mathbf{C}}(X)$, so that we have morphisms $g_i : X \to X$. Now the objects X^G and X/G are defined to be the kernel and cokernel respectively of these morphisms $g_1, \dots, g_n : X \to X$.

Representability Theorem: *Let $F : A$-**mod** $\rightsquigarrow$ **Sets** be a covariant (resp. contravariant) left exact functor.*

If any pair is dominated by a minimal pair, then F is an inductive limit of representable subfunctors,

$$F(M) = \varinjlim \operatorname{Hom}_A(Q_i, M) \qquad \Big(\text{resp. } F(M) = \varinjlim \operatorname{Hom}_A(M, Q_i)\Big).$$

*If the natural maps $F(\varprojlim M_i) \to \varprojlim F(M_i)$ (*resp. $F(\varinjlim M_i) \to \varprojlim F(M_i)$*) are bijective, then F is representable,*

$$F(M) = \operatorname{Hom}_A(Q, M) \qquad \Big(\text{resp. } F(M) = \operatorname{Hom}_A(Q, M)\Big).$$

Proof: In the covariant case, the proof given in p. 130 holds (pairs exist since $F(0) = * \neq \emptyset$!) replacing the exact sequence $0 \to \operatorname{Ker}(h - f) \to Q \to P$ by the exact sequence

$$\mathrm{Ker}(h-f) \xrightarrow{i} Q \overset{f,h}{\rightrightarrows} P,$$

because any pair P_ξ is dominated by a minimal pair: consider the submodules $M_i \subseteq P$ such that $\xi \in F(M_i)$ (since $F(M_i) \to F(P)$ is injective, we identify $F(M_i)$ with a subset of $F(P)$) and put $M = \bigcap_i M_i$. Since we have $\xi \in \bigcap_i F(M_i) = \varprojlim F(M_i) = F(\varprojlim M_i) = F(\bigcap_i M_i) = F(M)$, it is clear that M_ξ is a minimal pair dominating P_ξ.

In the contravariant case the proof also holds with the following changes:

A pair Q_ξ is minimal if any surjective morphism of pairs $Q_\xi \to P_g$ is an isomorphism.

A pair Q_ξ dominates a pair P_g if there is a morphism of pairs $P_g \to Q_\xi$.

Minimal pairs are just submodules of the representant; hence, if it exists, it is the inductive limits of all minimal pairs. In this case, the coincidence of all morphisms $f, h\colon P_g \to Q_\xi$, when Q_ξ is minimal (so that minimal pairs are just representable subfunctors $\xi\colon Q^\bullet \hookrightarrow F$), follows from the exact sequence $P \rightrightarrows Q \to Q/\mathrm{Im}(h-f) \to 0$; and, in order to show that any pair P_ξ is dominated by a minimal pair, we consider the submodules $M_i \subseteq P$ such that $\xi \in F(P/M_i) \subseteq F(P)$ and we put $M = \bigcup_i M_i$, since we have $\xi \in \bigcap_i F(P/M_i) = \varprojlim F(P/M_i) = F(\varinjlim P/M_i) = F(P/\bigcup_i M_i) = F(P/M)$, so that $(P/M)_\xi$ is a minimal pair dominating P_ξ.

Definition: A covariant functor $F\colon A$-**mod** $\rightsquigarrow A$-**mod** is A-**linear** when so are the maps $F\colon \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(F(M),F(N))$, and F is (linearly) **representable** if we have an isomorphism of functors $F \simeq \mathrm{Hom}_A(Q,-)$ for some A-module Q (we have natural A-linear isomorphisms $F(M) = \mathrm{Hom}_A(Q,M)$, not just natural bijections, so that, when a pair Q_ξ represents F, then $F(Q) = \mathrm{Hom}_A(Q,Q)$ has a structure of (eventually non commutative) A-algebra with unity $\xi \in F(Q)$).

We have analogous definitions for contravariant functors $F\colon A$-**mod** $\rightsquigarrow A$-**mod**.

Corollary: *A covariant A-linear functor $F\colon A$-**mod** $\rightsquigarrow A$-**mod** is representable if and only if it preserves projective limits and for any exact sequence of A-modules $0 \to M' \to M \to M''$, so is the sequence $0 \to F(M') \to F(M) \to F(M'')$.*

*A contravariant A-linear functor $F\colon A$-**mod** $\rightsquigarrow A$-**mod** is representable if and only if it transforms inductive limits into projective limits and for any exact sequence of A-modules $M' \to M \to M'' \to 0$, so is the sequence $0 \to F(M'') \to F(M) \to F(M')$.*

Proof: Any covariant $\mathbb{Z}$-linear functor F preserves split exact sequences, hence $F(M \times N) = F(M) \times F(N)$ and $F(0) = 0$, and it is left exact if and only if $0 \to F(M') \to F(M) \to F(M'')$ is exact whenever so is $0 \to M' \to M \to M''$, because $F(f-g) = F(f) - F(g)$.

Finally, the bijection $\mathrm{Hom}_A(Q,M) = F(M)$, $f \mapsto F(f)(\xi)$, is A-linear when so is F.

An analogous argument holds in the contravariant case.

Set Theoretic Questions: Rigorously, in the Representability theorem we must prove that minimal pairs (up to isomorphisms of pairs) form a set. In the contravariant case, any minimal pair M_ξ is fully determined (up to isomorphisms) by the subset $M^\bullet(A) \subseteq F(A)$, because a submodule $M \hookrightarrow M'$ is fully determined by the morphisms $A \to M'$ factoring through M, and any two minimal pairs are submodules of a third minimal pair.

In the covariant case, if Q is an A-module such that for any A-module N and any $0 \neq n \in N$ there is a morphism $f\colon N \to Q$ such that $f(n) \neq 0$, then any minimal pair M_ξ is fully determined (up to isomorphisms) by the subset $\mathrm{Hom}_A(M,Q) \subseteq F(Q)$. Since $An \simeq A/\mathfrak{a}$, just consider for any ideal $\mathfrak{a} \subset A$ an injective A-module $Q_\mathfrak{a}$ containing $A/\mathfrak{a}$, and put $Q = \prod_\mathfrak{a} Q_\mathfrak{a}$.

For a representability theorem in abstract categories see exercises 64-68 in pp. 504-505.

5.4 The Spectrum of a Ring

Definitions: The **spectrum** of a ring A is the set $\operatorname{Spec} A$ of all prime ideals, and any element $f \in A$ is said to be a **function** on $\operatorname{Spec} A$, where the **value** $f(x)$ of f at a **point** $x \in \operatorname{Spec} A$, defined by a prime ideal $\mathfrak{p}$, is just the class of f into the **residue field** $\kappa(x)$

$$f(x) = [f] \in \kappa(x) := (A/\mathfrak{p})_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}.$$

Although the residue field $\kappa(x)$ changes with the point, *the zero value is absolute.*

The ideal of all functions vanishing at x is just $\mathfrak{p}$, and **zeros** of ideals

$$(I)_0 = \begin{bmatrix}\text{Points of Spec}\, A \text{ where} \\ \text{all functions in } I \text{ vanish}\end{bmatrix} = \bigcap_{f\in I} (f)_0 = \begin{bmatrix}\text{Prime ideals of} \\ A \text{ containing } I\end{bmatrix} = \operatorname{Spec}(A/I)$$

are closed sets of a topology on $\operatorname{Spec} A$, named **Zariski topology**,

$$(A)_0 = \emptyset \quad , \quad (0)_0 = \operatorname{Spec} A$$
$$\left(\textstyle\sum_j I_j\right)_0 = \textstyle\bigcap_j (I_j)_0$$
$$(IJ)_0 = (I \cap J)_0 = (I)_0 \cup (J)_0$$

and only the last equality requires a proof: if $f_1 \in I, f_2 \in J$ do not vanish at a point x, then neither does $f_1 f_2$, and $f_1 f_2 \in IJ \subseteq I \cap J$.

The **basic open sets** $U_f = \operatorname{Spec} A - (f)_0$ define a base of this topology since closed sets are intersections of zeros of functions: *Given a closed set Y and an exterior point x, there is a function $f \in A$ vanishing on Y such that $f(x) \neq 0$.* Hence, if I_Y is the ideal of all the functions $f \in A$ vanishing on a given closed set Y, we have $Y = (I_Y)_0$.

Recall (p. 124) that $\operatorname{Spec} \neq \emptyset$ when $A \neq 0$, so that $I = A$ when $(I)_0 = \emptyset$. Hence

Chinese Remainder Theorem: *If $(I)_0 \cap (J)_0 = \emptyset$, then $I \cap J = IJ$ and $A/IJ = (A/I) \times /A/J)$.*

Corollary: $A/(\mathfrak{m}_1 \ldots \mathfrak{m}_n) = (A/\mathfrak{m}_1) \times \ldots \times (A/\mathfrak{m}_n)$ *when $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are maximal ideals.*

Proof: By induction on n, since $(\mathfrak{m}_1)_0 \cap (\mathfrak{m}_2 \ldots \mathfrak{m}_n)_0 = (\mathfrak{m}_1)_0 \cap \{(\mathfrak{m}_2)_0 \cup \ldots \cup (\mathfrak{m}_n)_0\} = \emptyset$.

Proposition: $\overline{x} = (\mathfrak{p})_0$, *and we say that x is the **generic point** of its closure $\overline{x}$.*

Hence $\operatorname{Spec} A$ *is T_0 and closed points correspond to maximal ideals of A.*

Proof: A point $x \in \operatorname{Spec} A$ is in a closed set $(I)_0$ when $I \subseteq \mathfrak{p}$, so that $(\mathfrak{p})_0 \subseteq (I)_0$.

Hence $(\mathfrak{p})_0$ is the minimal closed set containing x.

Theorem: $\operatorname{Spec} A$ *is a compact topological space.*

Proof: Given a family of closed sets with empty intersection, $\emptyset = \bigcap_j (I_j)_0 = \left(\sum_j I_j\right)_0$, then $\sum_j I_j = A$, since any ideal $I \neq A$ is contained in some prime ideal.

Hence $1 = f_1 + \ldots + f_n$, where $f_i \in I_{j_i}$, and a finite family has empty intersection,

$$(I_{j_1})_0 \cap \ldots \cap (I_{j_n})_0 = (I_{j_1} + \ldots + I_{j_n})_0 = (A)_0 = \emptyset.$$

Definitions: A topological space X is **irreducible** if for any finite closed cover $X = \bigcup_{i\in I} Y_i$ we have $X = Y_i$ for some index $i \in I$ (when $I = \emptyset$, we see that $X \neq \emptyset$), and an **irreducible component** of X is a maximal irreducible subspace.

The closure of any irreducible subspace is irreducible; hence the irreducible components are closed. By Zorn's lemma, any irreducible subspace of X is in some irreducible component; hence X is the union of its irreducible components.

When C is an irreducible closed set in $\operatorname{Spec} A$, the ideal I_C is prime (if $C \subseteq (fh)_0 = (f)_0 \cup (h)_0$ then $C \subseteq (f)_0$ or $C \subseteq (h)_0$) and $C = (I_C)_0$; hence C is the closure of a (unique) point:

Proposition: *The irreducible components of* $\operatorname{Spec} A$ *correspond to the minimal prime ideals of* A*, and any prime ideal contains some minimal prime ideal. In fact, we have a lattice anti-isomorphism*:

$$\begin{bmatrix}\text{Prime ideals}\\ \text{of } A\end{bmatrix} \xrightarrow{\sim} \begin{bmatrix}\text{Irreducible closed}\\ \text{subspaces of Spec } A\end{bmatrix}, \ \mathfrak{p} \mapsto (\mathfrak{p})_0$$

1. The spectrum $\operatorname{Spec} k$ of a field has a unique point.

2. $\operatorname{Spec} \mathbb{Z}$ has a closed point for every prime number p, with residue field $\mathbb{F}_p$, and a generic point with residue field $\mathbb{Q}$.

3. $\operatorname{Spec} k[x]$ has a closed point for any unitary irreducible polynomial P, with residue field $k[x]/(P)$, and a generic point with residue field $k(x)$.

4. The bijection $(I)_0 = \operatorname{Spec}(A/I)$ is a homeomorphism since $(f)_0 \cap (I)_0 = (\bar{f}\,)_0$. Hence,
$$\operatorname{Spec}(A_1 \oplus A_2) = (\operatorname{Spec} A_1) \amalg (\operatorname{Spec} A_2).$$
In fact, $A = A_1 \oplus A_2 = I_1 + I_2$, and $I_1 \cap I_2 = 0$, where $I_1 = 0 \oplus A_2$, and $I_2 = A_1 \oplus 0$, so that we have $\operatorname{Spec} A = (I_1)_0 \cup (I_2)_0$, $(I_1)_0 \cap (I_2)_0 = \emptyset$, and $(I_i)_0 = \operatorname{Spec} A/I_i = \operatorname{Spec} A_i$.

Definition: A ring morphism $j\colon A \to B$ induces a map $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$, where $x = \phi(y)$ when $\mathfrak{p}_x = \mathfrak{p}_y \cap A := \{a \in A\colon j(a) \in \mathfrak{p}_y\}$, so that $\kappa(x) \hookrightarrow \kappa(y)$.

By definition $(jf)(y) = f(x) = f(\phi(y))$; hence $\phi^{-1}(f)_0 = (fB)_0$, and ϕ is continuous:

$$\phi^{-1}\big((I)_0\big) = \phi^{-1}\big(\bigcap_{f\in I}(f)_0\big) = \bigcap_{f\in I}\phi^{-1}(f)_0 = \bigcap_{f\in I}(fB)_0 = (IB)_0.$$

Theorem: *The canonical projection* $A \to A/I$ *induces a homeomorphism* $\operatorname{Spec} A/I = (I)_0$.

Proof: We know (p. 71) that it is a continuous bijection, and it is a homeomorphism because we have $(J)_0 \cap (\operatorname{Spec} A/I) = \big(\pi(J)\big)_0$ for any ideal J of A.

Theorem: *Let* S *be a multiplicative system of* A*. The canonical morphism* $A \to A_S$ *induces a homeomorphism of* $\operatorname{Spec} A_S$ *onto the subspace of* $\operatorname{Spec} A$ *defined by the prime ideals not intersecting* S*.*

Proof: If $\mathfrak{q}$ is a prime ideal of A_S, then $\mathfrak{p} = A \cap \mathfrak{q}$ is disjoint from S and (p. 88) $\mathfrak{q} = \mathfrak{p}A_S$, so that $\operatorname{Spec} A_S \to \operatorname{Spec} A$ is injective.

If a prime ideal $\mathfrak{p}$ of A is disjoint from S, then $A \cap \mathfrak{p}A_S = \mathfrak{p}$:

$$a/1 = b/s,\ b \in \mathfrak{p} \ \Rightarrow\ au = bv \in \mathfrak{p} \text{ for some } \ u, v \in S \ \Rightarrow\ a \in \mathfrak{p}.$$

Hence $\mathfrak{p}A_S \neq A_S$, and it is a prime ideal: $\frac{a}{s} \cdot \frac{b}{t} \in \mathfrak{p}A_S \ \Rightarrow\ ab \in A \cap \mathfrak{p}A_S = \mathfrak{p} \ \Rightarrow\ a$ or $b \in \mathfrak{p}$.

This continuous map defines a homeomorphism of $\operatorname{Spec} A_S$ onto the image because we have $(I)_0 \cap (\operatorname{Spec} A_S) = (IA_S)_0$ for any ideal I of A.

Corollary: *Prime ideals of* $A_{\mathfrak{p}}$ *correspond to prime ideals of* A *contained in* $\mathfrak{p}$*; hence* $A_{\mathfrak{p}}$ *has a unique maximal ideal,* $\mathfrak{p}A_{\mathfrak{p}}$*.*

Corollary: $\operatorname{Spec} A_f = U_f = \operatorname{Spec} A - (f)_0$.

Definition: The **radical** of a ring A is the set $\operatorname{rad} A = \{a \in A\colon a^n = 0, \text{ for some } n \geq 1\}$ of all nilpotent elements, and a ring is **reduced** if $\operatorname{rad} A = 0$.

Corollary: *Nilpotent functions are just functions vanishing at any point of the spectrum. (The radical of a ring is the intersection of all prime ideals).*

Proof: If $f^n = 0$, then $(f)_0 = (f^n)_0 = \operatorname{Spec} A$.

If $(f)_0 = \operatorname{Spec} A$, then $\operatorname{Spec} A_f = \emptyset$; hence $A_f = 0$, and f is nilpotent.

Proposition: *If a ring morphism $A \to B$ is injective,* $\operatorname{Spec} B \to \operatorname{Spec} A$ *has dense image.*

Proof: Let $x \in \operatorname{Spec} A$ be the point of a minimal prime. Now, $B_x \neq 0$ since $A_x \to B_x$ is injective; hence $x = \operatorname{Spec} A_x$ is in the image, and it is dense. q.e.d.

In fact, the spectrum of a ring A must be considered to be the pair $(A, \operatorname{Spec} A)$, and morphisms $X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ are defined to be ring morphisms $A \to B$, i.e. $\operatorname{Hom}(Y, X) = \operatorname{Hom}_{\text{rings}}(A, B)$, so that any morphism induces a continuous map, which does not determine it. Hence, if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then a point $\phi\colon Y \to X$ of X parameterized by Y is just a ring morphism $A \to B$, and

$$X \times_{\operatorname{Spec} k} Y = \operatorname{Spec}(A \otimes_k B) \quad , \quad X \amalg Y = \operatorname{Spec}(A \oplus B).$$

Now we shall calculate the fibre $\phi^{-1}(x)$ over the point $x\colon \operatorname{Spec} \kappa(x) \to X = \operatorname{Spec} A$ defined by a prime ideal $\mathfrak{p}$ of A:

Fibre Formula: $\phi^{-1}(x) = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) = \operatorname{Spec}\big(B \otimes_A \kappa(x)\big) = Y \times_X x$.

Proof: If $\mathfrak{p} = \mathfrak{m}$ is a maximal ideal, the fibre $(\mathfrak{m}B)_0$ is the spectrum of $B/\mathfrak{m}B = B \otimes_A \kappa(x)$.

In general, since $\phi^{-1}(x) \subseteq \operatorname{Spec} B_{\mathfrak{p}}$, it is the fibre of $\operatorname{Spec} B_{\mathfrak{p}} \to \operatorname{Spec} A_{\mathfrak{p}}$ over the unique closed point of $\operatorname{Spec} A_{\mathfrak{p}}$, defined by the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Corollary: *If $x \in \operatorname{Spec} A$ is defined by a maximal ideal $\mathfrak{m}$, then $\phi^{-1}(x) = \operatorname{Spec} B/\mathfrak{m}B$.*

If $x \in \operatorname{Spec} A$ is defined by a minimal prime ideal $\mathfrak{p}$, then $\phi^{-1}(x) = \operatorname{Spec} B_{\mathfrak{p}}$.

1. The fibre of the projection $\mathbb{A}_{2,\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y] \to \operatorname{Spec} \mathbb{C}[x] = \mathbb{A}_{1,\mathbb{C}}$ over the point $x = a$ is $\operatorname{Spec} \mathbb{C}[x,y]/(x-a)$, the points being defined by the ideal $(x-a)$ and the maximal ideals $(x-a, y-b)$. The fibre over the generic point is $\operatorname{Spec} \mathbb{C}(x)[y]$, the points being defined by the ideal 0 and the ideals (P), where $P(x,y)$ is irreducible of non zero degree in y. The complex plane $\operatorname{Spec} \mathbb{C}[x,y]$ is formed by the closed points $x = a$, $y = b$, the generic points of the irreducible curves $P(x,y) = 0$, and the generic point of the plane.

2. The fibre of the projection $\mathbb{A}_{1,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x] \to \operatorname{Spec} \mathbb{Z}$ over a prime number p is $\operatorname{Spec} \mathbb{F}_p[x]$, the points being defined by the ideal (p) and the maximal ideals (p, Q), where the reduction $\bar{Q}$ modulo p is irreducible. The fibre over the generic point is $\operatorname{Spec} \mathbb{Q}[x]$, the points being defined by the ideal 0 and the ideals (P), where P is irreducible in $\mathbb{Q}[x]$.

3. Let us consider $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$. The fibre of $\operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$ over a prime p is

$$\operatorname{Spec} \mathbb{F}_p[x]/(x^2+1) = \begin{cases} \text{one point } x = 1, \text{ if } p = 2 \\ \text{one point, if } -1 \text{ is not a quadratic residue mod. } p \\ 2 \text{ points } x = \pm a, \text{ if } -1 \text{ is a quadratic residue mod. } p \end{cases}$$

Hence (p. 73) the maximal ideals of $\mathbb{Z}[i]$ are $(2, 1+i) = (1+i)$, the ideals (p) when $p \equiv 3$ (mod. 4), and the ideals $(p, i \pm a)$, with $a^2 \equiv -1$ (mod. p), when $p \equiv 1$ (mod. 4).

Definition: The localization of an A-module M at the prime ideal of a point $x \in \operatorname{Spec} A$ is denoted by M_x. The **support** of an element $m \in M$ is

$$\operatorname{supp}(m) = \{x \in \operatorname{Spec} A\colon\ m_x \neq 0\},$$

(where $m_x := m/1$) and $\operatorname{supp}(M) = \{x \in \operatorname{Spec} A\colon\ M_x \neq 0\} = \bigcup_{m \in M} \operatorname{supp}(m)$ is the support of M.

Lemma: $(\operatorname{Ann}(m))_0 = \operatorname{supp}(m)$. *Therefore,* $m = 0$ *if and only if* $m_x = 0,\ \forall x \in \operatorname{Spec} A$.

Proof: Condition $m_x = 0$ states that $fm = 0$ for some function f not vanishing at x; i.e., x is not in the zeros of the ideal $\operatorname{Ann}(m) = \{f \in A\colon fm = 0\}$.

Now, if $m_x = 0$ at any point, then $(\operatorname{Ann}(m))_0 = \emptyset$; hence $\operatorname{Ann}(m) = A$, and $m = 0$.

Corollary: $M = 0$ *if and only if* $M_x = 0$ *at any point* $x \in \operatorname{Spec} A$.

Theorem: *A sequence* $M' \xrightarrow{f} M \xrightarrow{g} M''$ *of morphisms of* A*-modules is exact if and only if so is the localization* $M'_x \xrightarrow{f_x} M_x \xrightarrow{g_x} M''_x$ *at any point* $x \in \operatorname{Spec} A$.

Proof: If it is exact at any point, then $(\operatorname{Im} gf)_x = \operatorname{Im}(gf)_x = \operatorname{Im}(g_x f_x) = 0$.

Hence $\operatorname{Im} gf = 0$, and $\operatorname{Im} f \subseteq \operatorname{Ker} g$.

Now, localizing $\operatorname{Ker} g / \operatorname{Im} f$ we see that it is 0,

$$(\operatorname{Ker} g / \operatorname{Im} f)_x = (\operatorname{Ker} g)_x / (\operatorname{Im} f)_x = (\operatorname{Ker} g_x)/(\operatorname{Im} f_x) = 0.$$

Definition: A ring $\mathcal{O}$ is **local** if it has a unique maximal ideal $\mathfrak{m}$. For example, A_x.

Nakayama's Lemma: *Let* $\mathcal{O}$ *be a local ring of maximal ideal* $\mathfrak{m}$*. If* M *is a finite* $\mathcal{O}$*-module and* $\mathfrak{m}M = M$*, then* $M = 0$.

Proof: If $M \neq 0$, we consider a minimal generating system $m_1, \ldots, m_n$.

$$M = \mathfrak{m}M = \mathfrak{m}(\mathcal{O}m_1 + \ldots + \mathcal{O}m_n) = \mathfrak{m}m_1 + \ldots + \mathfrak{m}m_n,$$

and $m_1 = f_1 m_1 + f_2 m_2 + \ldots + f_n m_n$ for some functions $f_1, \ldots, f_n \in \mathfrak{m}$.

Hence $1 - f_1$ is invertible (it is not in $\mathfrak{m}$) and

$$(1 - f_1) m_1 = f_2 m_2 + \ldots + f_n m_n;$$

$m_1 \in \mathcal{O}m_2 + \ldots + \mathcal{O}m_n$, and $m_2, \ldots, m_n$ generate M. Absurd.

Corollary: $M = \mathcal{O}m_1 + \ldots + \mathcal{O}m_n$ *if and only if* $M/\mathfrak{m}M = k\bar{m}_1 + \ldots + k\bar{m}_n$; $(k = \mathcal{O}/\mathfrak{m})$.

Proof: Put $M' = M/(\mathcal{O}m_1 + \ldots + \mathcal{O}m_n)$.

The exact sequence $\mathcal{O}^n \to M \to M' \to 0$ induces an exact sequence

$$k^n \longrightarrow M \otimes_{\mathcal{O}} k \longrightarrow M' \otimes_{\mathcal{O}} k \longrightarrow 0$$

and $\bar{m}_1, \ldots, \bar{m}_n$ span $M \otimes_{\mathcal{O}} k = M/\mathfrak{m}$ if and only if $0 = M'/\mathfrak{m}M'$; i.e., $M' = 0$.

5.5 Differential Calculus

Definition: In this section k will be a ring (not a field, as in the rest of these notes). A **k-derivation** of a k-algebra A into an A-module M is a group morphism $D\colon A \to M$ vanishing on constants ($D\lambda = 0$ for all $\lambda \in k$) such that

$$D(ab) = (Da)b + a(Db);\ \ a, b \in A.$$

Derivations $A \to M$ are k-linear and they form an A-module $\mathrm{Der}_k(A, M)$ with the operations

$$(D + D')a = Da + D'a \quad , \quad (bD)a = b(Da).$$

Any morphism of A-modules $f\colon M \to M'$ induces a morphism of A-modules

$$f_*\colon \mathrm{Der}_k(A, M) \longrightarrow \mathrm{Der}_k(A, M'),\ \ f_*D = f \circ D,$$

and any morphism of k-algebras $j\colon A \to B$ induces B-linear morphisms (N a B-module)

$$j^*\colon \mathrm{Der}_k(B, N) \longrightarrow \mathrm{Der}_k(A, N),\ \ j^*D = D \circ j.$$

Example: Any derivation $D\colon k[x_1, \ldots, x_n] \to M$ is determined by the derivatives Dx_i, so that $D = \sum_i (Dx_i)\frac{\partial}{\partial x_i}$, and $\mathrm{Der}_k(k[x_1, \ldots, x_n], M) = M\,\frac{\partial}{\partial x_1} \oplus \ldots \oplus M\,\frac{\partial}{\partial x_n}$.

Derivations First Exact Sequence: *If $A \to B$ is a morphism of k-algebras, for any B-module N we have an exact sequence of B-modules*

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_k(B, N) \longrightarrow \mathrm{Der}_k(A, N)$$

Derivations Second Exact Sequence: *When $B = A/I$, we have an exact sequence*

$$0 \longrightarrow \mathrm{Der}_k(B, N) \longrightarrow \mathrm{Der}_k(A, N) \longrightarrow \mathrm{Hom}_A(I, N) = \mathrm{Hom}_B(I/I^2, N)$$

Proof: The restriction of a derivation $D\colon A \to N$ to I is A-linear since $f \in I$ annihilates any B-module: $D(af) = a(Df) + f(Da) = a(Df)$.

Finally, a derivation $D\colon A \to N$ factors through B if and only if it vanishes on I.

Example: If $k = A/\mathfrak{m}$ is a field, then $\mathrm{Der}_k(A, k) \xrightarrow{\sim} \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. In fact, this morphism is injective by the second exact sequence, and to show that it is surjective, we introduce the **differential** $\mathrm{d}_p\colon A \to \mathfrak{m}/\mathfrak{m}^2$, $\mathrm{d}_p f = [\Delta f] = [f - f(p)]$, at the point p defined by $\mathfrak{m}$,

$$\begin{aligned} fg &= (f(p) + \Delta f)(g(p) + \Delta g), \\ \Delta(fg) &= f(p)(\Delta g) + g(p)(\Delta f) + (\Delta f)(\Delta g), \\ \mathrm{d}_p(fg) &= f(p)\mathrm{d}_p f + g(p)\mathrm{d}_p f. \end{aligned}$$

Now, if $\omega\colon \mathfrak{m}/\mathfrak{m}^2 \to k$ is k-linear, the derivation $Df = \omega(\mathrm{d}_p f)$ coincides with ω on $\mathfrak{m}$.

Definitions: The existence of a universal derivation follows from the representability theorem; but we shall give a direct construction. The **diagonal ideal** is the kernel Δ of the morphism $\mu\colon A \otimes_k A \to A$, $\mu(a \otimes b) = ab$, and the **module of differentials** is $\Omega_{A/k} = \Delta/\Delta^2$ (both sides define the same A-module structure, since it is annihilated by $a \otimes 1 - 1 \otimes a \in \Delta$).

The **differential** is the k-derivation $\mathrm{d}\colon A \to \Omega_{A/k}$, $\mathrm{d}a = [a \otimes 1 - 1 \otimes a]$,

$$\mathrm{d}(ab) = (b \otimes 1)[a \otimes 1 - 1 \otimes a] + (1 \otimes a)[b \otimes 1 - 1 \otimes b] = b(\mathrm{d}a) + a(\mathrm{d}b).$$

Lemma: $\Omega_{A/k}$ *is generated by the image of the differential* $\mathrm{d}\colon A \to \Omega_{A/k}$.

Proof: If $\sum_i a_i \otimes b_i \in \Delta$, then $\sum_i a_i b_i = 0$; hence $\sum_i 1 \otimes a_i b_i = 0$, and

$$\sum_i a_i \otimes b_i = \sum_i a_i \otimes b_i - \sum_i 1 \otimes a_i b_i = \sum_i b_i(a_i \otimes 1 - 1 \otimes a_i).$$

Universal Property: *For any k-derivation $D\colon A \to M$ there exists a unique morphism of A-modules $f\colon \Omega_{A/k} \to M$ such that $f(\mathrm{d}a) = Da$,*

$$\mathrm{Der}_k(A, M) = \mathrm{Hom}_A(\Omega_{A/k}, M).$$

Proof: The A-linear morphism $\phi\colon A \otimes_k A \to M$, $\phi(a \otimes b) = b(Da)$, vanishes on Δ^2,

$$\phi((a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)) = D(ab) - a(Db) - b(Da) + ab(D1) = 0,$$

and it induces an A-linear map $f\colon \Delta/\Delta^2 \to M$, $f(\mathrm{d}a) = \phi(a \otimes 1 - 1 \otimes a) = Da$.
The uniqueness follows from the above lemma.

Corollary: $\Omega_{A/k} = A\mathrm{d}x_1 \oplus \ldots \oplus A\mathrm{d}x_n$, *when* $A = k[x_1, \ldots, x_n]$.

Proof: $\mathrm{Der}_k(k[x_1, \ldots, x_n], M) = M\,\frac{\partial}{\partial x_1} \oplus \ldots \oplus M\,\frac{\partial}{\partial x_n}$.

Theorem: $\Omega_{A_S/k} = \left(\Omega_{A/k}\right)_S$.

Proof: If M is an A_S-module, then $\mathrm{Der}_k(A_S, M) = \mathrm{Der}_k(A, M)$, since a derivation $D\colon A \to M$ only may be defined by the derivation

$$\bar{D}\colon A_S \longrightarrow M,\ \bar{D}\left(\frac{a}{s}\right) = \frac{sDa - aDs}{s^2}$$

and it is well defined: if $\frac{a}{s} = \frac{b}{t}$, then $rat = rbs$, $r \in S$. Deriving and dividing by rst

$$\frac{Da}{s} + \frac{aDt}{st} = \frac{Db}{t} + \frac{bDs}{st}$$
$$\frac{sDa - aDs}{s^2} = \frac{tDb - bDt}{t^2}$$

$$\mathrm{Hom}_{A_S}(\Omega_{A_S/k}, M) = \mathrm{Der}_k(A_S, M) = \mathrm{Der}_k(A, M) = \mathrm{Hom}_A(\Omega_{A/k}, M) = \mathrm{Hom}_{A_S}((\Omega_{A/k})_S, M).$$

Corollary: $\Omega_{(A_1 \oplus A_2)/k} = \Omega_{A_1/k} \oplus \Omega_{A_2/k}$.

Proof: We have $A = A_1 \oplus A_2 = A_{f_1} \oplus A_{f_2}$, where $f_1 = (1, 0)$, $f_2 = (0, 1)$.
Hence any A-module M is $M = M_{f_1} \oplus M_{f_2}$.
Now, $(\Omega_{A/k})_{f_i} = \Omega_{A_{f_i}/k} = \Omega_{A_i/k}$.

Theorem: *The module of differentials is stable under base changes,* $\Omega_{A_K/K} = \Omega_{A/k} \otimes_k K$.

Proof: If M is an A_K-module, any k-derivation $D\colon A \to M$ uniquely factors through a K-derivation $D \otimes 1\colon A_K \to M$, and

$$\begin{aligned}\mathrm{Hom}_{A_K}(\Omega_{A_K/K}, M) &= \mathrm{Der}_K(A_K, M) = \mathrm{Der}_k(A, M) = \mathrm{Hom}_A(\Omega_{A/k}, M)\\ &= \mathrm{Hom}_{A_K}(\Omega_{A/k} \otimes_A A_K, M) = \mathrm{Hom}_{A_K}(\Omega_{A/k} \otimes_k K, M).\end{aligned}$$

Differentials First Exact Sequence: *If $A \to B$ is a morphism of k-algebras, we have an exact sequence of B-modules*

$$\Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow \Omega_{B/A} \longrightarrow 0$$

Proof: For any B-module N, we have an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_B(\Omega_{B/A}, N) & \longrightarrow & \mathrm{Hom}_B(\Omega_{B/k}, N) & \longrightarrow & \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, N) \\ & & \| & & \| & & \| \\ & & \mathrm{Der}_A(B, N) & & \mathrm{Der}_k(B, N) & & \mathrm{Der}_k(A, N) \end{array}$$

Differentials Second Exact Sequence: *When $B = A/I$, we have an exact sequence*

$$I/I^2 \xrightarrow{\mathrm{d}\otimes 1} \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow 0$$

Proof: For any B-module N we have an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_B(\Omega_{B/k}, N) & \longrightarrow & \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, N) & \longrightarrow & \mathrm{Hom}_B(I/I^2, N) \\ & & \| & & \| & & \| \\ & & \mathrm{Der}_k(B, N) & & \mathrm{Der}_k(A, N) & & \mathrm{Hom}_A(I, N) \end{array}$$

Corollary: $\Omega_{A/k} = (k[x]/(P, P'))\mathrm{d}x$, *where* $A = k[x]/(P)$.

Example: If $k = A/\mathfrak{m}$ is a field, then $\mathrm{d}\otimes 1\colon \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \Omega_{A/k} \otimes_A k$ is an isomorphism, since so is (p. 143) the transpose $\mathrm{Der}_k(A, k) = \mathrm{Hom}_A(\Omega_{A/k}, k) = \mathrm{Hom}_k(\Omega_{A/k} \otimes_A k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$.

5.6 Finite Algebras over a Field

Lemma: *Any prime ideal of a finite k-algebra A is a maximal ideal.*

Proof: If A is integral, the linear map $h_a\colon A \to A$, $h_a(x) = ax$, is injective when $a \neq 0$.
Hence it is surjective, and $1 = h_a(b) = ab$ for some $b \in A$, so that A is a field.

Lemma: *The number of maximal ideals of any finite k-algebra A is $\leq [A : k]$.*

Proof: We have $n \leq [(A/\mathfrak{m}_1) \times \ldots \times (A/\mathfrak{m}_n) : k] = [A/\mathfrak{m}_1 \ldots \mathfrak{m}_n : k] \leq [A : k]$ (p. 139).

Theorem: *The spectrum of a finite k-algebra is a finite discrete space,* $\mathrm{Spec}\, A = \{x_1, \ldots, x_n\}$, *and A decomposes as a direct sum of local algebras (finite extensions if A is reduced)*

$$A = A_{x_1} \oplus \ldots \oplus A_{x_n}.$$

Proof: $\mathrm{Spec}\, A$ is a finite space and any point is closed, hence it is discrete.
The natural morphism $A \to A_{x_1} \oplus \ldots \oplus A_{x_n}$ is an isomorphism at any point $y \in \mathrm{Spec}\, A$,

$$(A_{x_1} \oplus \ldots \oplus A_{x_n})_y = (A_{x_1})_y \oplus \ldots \oplus (A_{x_n})_y = A_y$$

because $(A_x)_y = 0$ when $x \neq y$, since prime ideals of $(A_x)_y$ correspond to prime ideals of A contained in $\mathfrak{m}_x$ and $\mathfrak{m}_y$, and there are no such ideals.
If A is reduced, A_{x_i} is a reduced local algebra; hence it is a field.

Theorem: *If A, B are finite k-algebras, any injective morphism $A \to B$ induces a surjective continuous map* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$.

Proof: If $x \in \operatorname{Spec} A$, the morphism $A_x \to B_x$ is injective.

Hence $B_x \neq 0$, and any point of $\operatorname{Spec} B_x$ is in the fibre over $x = \operatorname{Spec} A_x$.

Definitions: The **points** of a k-algebra A with values in an extension L of k, or L-points, are the morphisms of k-algebras $A \to L$.

If A is a k-algebra, a point $x \in \operatorname{Spec} A$ is **rational** when $k = \kappa(x)$.

A finite k-algebra is **rational** when so is any point of $\operatorname{Spec} A$.

1. L-valued points of $A = k[x]/(P)$ are just roots of $P(x)$ in L.

2. When $A = L$ is a finite extension, L-points of L are just automorphisms of L over k.

3. Any k-morphism $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ preserves rational points since $\kappa(y)$ is an extension of $\kappa(x)$ when $x = \phi(y)$. Hence subalgebras of a rational algebra are also rational (by the above theorem) and it is clear that quotients, direct sums and tensor products of rational algebras are also rational.

4. By the Chinese remainder theorem, $k[x]/(P) = \bigoplus_i k[x]/(P_i^{n_i})$, where $P = P_1^{n_1} \dots P_s^{n_s}$ is the irreducible factor decomposition. The points of $\operatorname{Spec} k[x]/(P)$ correspond to the irreducible factors P_i, and the residue field is $k[x]/(P_i)$, rational points correspond to roots of P in k, and $k[x]/(P)$ is rational if and only if all the roots of P are in k.

Points Formula: $\operatorname{Hom}_{k\text{-alg}}(A, L) = \begin{bmatrix} \text{Points of } \operatorname{Spec} A_L \\ \text{of residue field } L \end{bmatrix}$

Proof: When $L = k$, any morphism $A \to k$ is surjective and it is fully determined by the kernel, which defines a rational point of $\operatorname{Spec} A$. In fact, two morphisms with equal kernel differ in an automorphism of k, which is the identity.

The general case now follows from the graph formula,

$$\operatorname{Hom}_{k\text{-alg}}(A, L) = \operatorname{Hom}_{L\text{-alg}}(A \otimes_k L, L).$$

Theorem: *The concept of rational local algebra is **geometric** (stable under base changes $k \to K$; if A is a finite local rational k-algebra, then A_K is a finite local rational K-algebra).*

Proof: Let $\mathfrak{m}$ be the unique maximal ideal of a finite rational local k-algebra $\mathcal{O}$.

Any element of $\mathfrak{m}$ is nilpotent, and $k = \mathcal{O}/\mathfrak{m}$. Now the exact sequence

$$0 \longrightarrow \mathfrak{m} \otimes_k K \longrightarrow \mathcal{O} \otimes_k K \longrightarrow k \otimes_k K = K \longrightarrow 0$$

shows that $\mathfrak{m}_K$ is a maximal ideal of $\mathcal{O}_K$, and $\mathcal{O}_K/\mathfrak{m}_K = K$.

Since $\mathfrak{m} \otimes_k K$ is generated by nilpotent elements, $\mathcal{O}_K$ is a rational local K-algebra.

Kronecker's Theorem: *Let A be a finite k-algebra. There is a finite extension $k \to L$ such that A_L is a rational L-algebra.*

Proof: By induction on $[A : k]$. When $[A : k] = 1$, then $A = k$ is rational.

When $[A : k] > 1$, if A has a rational point, $A = A_1 \oplus B$, and by induction, B is rational over a finite extension L, and $A_L = (A_1)_L \oplus B_L$ is rational.

If A has no rational point, we consider a residue field $A \to A/\mathfrak{m} = K$. The points formula shows that A_K has some rational point; hence there is a finite extension $K \to L$ such that $(A_K)_L = A_L$ is a rational L-algebra. q.e.d.

In fact the proof shows that A is rational over a quotient L of an iterated tensor product $A^{\otimes n}$. With this additional condition, L is unique up to (non canonical) isomorphisms and it is the **decomposition field** of A.

In fact, if A is rational over another quotient L' of $A^{\otimes m}$, then so are $A^{\otimes n}$ and its quotient L, and there is a morphism $L \to L'$ by the points formula. Hence $[L:k] \leq [L':k]$, and analogously $[L':k] \leq [L:k]$, so that $L \to L'$ is an isomorphism.

The condition of $A = k[x]/(P)$ being rational over L states that P has all its roots $\alpha_1, \ldots, \alpha_n$ in L, and that of being a quotient of $A^{\otimes n}$ states that L is generated by some roots of P, so that the decomposition field of P (i.e. of $k[x]/(P)$) over k is just $L = k(\alpha_1, \ldots, \alpha_n)$, see p. 80.

Definitions: Given a commutative square of extensions

$$\begin{array}{ccc} k & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ K_2 & \longrightarrow & L \end{array}$$

the field of fractions of the image of $K_1 \otimes_k K_2 \to L$ is the **composite** K_1K_2, formed by sums, products and quotients of elements of K_1 and K_2.

The composites of K_1 and K_2 are just the residue fields of Spec $(K_1 \otimes_k K_2)$; hence they exist, and when $k \to K_1$ is a finite extension, any composite is a quotient of $K_1 \otimes_k K_2$.

Definitions: A field $\bar{k}$ is **algebraically closed** if any non constant polynomial with coefficients in $\bar{k}$ has a root in $\bar{k}$ (any finite extension $\bar{k} \to K$ is trivial).

An algebraic extension $k \to \bar{k}$ is an **algebraic closure** of k if $\bar{k}$ is algebraically closed.

For example, $\mathbb{C}$ is algebraically closed, so that it is an algebraic closure of $\mathbb{R}$, and the field $\bar{\mathbb{Q}}$ of all algebraic complex numbers is an algebraic closure of $\mathbb{Q}$.

Theorem: *Any field has an algebraic closure, unique up to non canonical isomorphisms.*

Proof: Let L_P be a decomposition field of $P \in k[x]$ and let us consider the k-algebra

$$A = \varinjlim L_{P_1} \otimes_k \ldots \otimes_k L_{P_n}$$

and the residue field $\bar{k}$ of a maximal ideal $\mathfrak{m}$ of A, which is an algebraic extension of k since it is generated by the images of the morphisms $L_P \to A \to \bar{k}$.

These morphisms also show that any polynomial $P \in k[x]$ has all the roots in $\bar{k}$, hence $\bar{k}$ is algebraically closed (p. 81).

If $k \to k'$ is another algebraic closure, we consider a composite $k'\bar{k}$, and we have $\bar{k} \xrightarrow{\sim} k'\bar{k}$, since k' is algebraic over k. Analogously $k' \xrightarrow{\sim} k'\bar{k}$, and $\bar{k} \simeq k'$.

5.6.1 Trivial and Separable Algebras

Definition: A finite k-algebra A is **trivial** when $A = k \oplus \ldots \oplus k = \oplus_X k = \mathrm{Hom}(X, k)$; i.e., when the number of points of Spec A coincides with the degree $[A:k]$. In such a case, $A = \oplus k$, $a \mapsto (p_1(a), \ldots, p_n(a))$, where $\mathrm{Hom}_{k\text{-alg}}(A, k) = \{p_1, \ldots, p_n\}$.

If A, B are trivial k-algebras, clearly so are $A \oplus B$, $A \otimes_k B$ and any quotient A/I; and also any subalgebra $C \hookrightarrow A$, since trivial algebras are just reduced rational algebras.

Finally, given an extension $k \to L$, by the points formula:

1. *The number of L-points of a finite k-algebra A is bounded by $[A:k]$, and the equality holds if and only if A_L is trivial, $A \otimes_k L = \oplus L$.* In such a case $A_L = \oplus L$, $a \otimes \lambda \mapsto (p_1(a)\lambda, \ldots, p_n(a)\lambda)$, where $\mathrm{Hom}_{k\text{-alg}}(A, L) = \{p_1, \ldots, p_n\}$.

2. *The number of automorphisms of a finite extension $k \to L$ is bounded by $[L : k]$, and the equality holds if and only if $L \otimes_k L = \oplus L$.* In such case $L \otimes_k L = \oplus L$, $\mu \otimes \lambda \mapsto (g_1(\mu)\lambda, \dots, g_n(\mu)\lambda)$, where $\mathrm{Aut}_{k\text{-alg}}(L) = \{g_1, \dots, g_n\}$.

Galois Theorem (Trivial Case $G = 1$)**:** *The functors $F(A) = \mathrm{Spec}\, A = \mathrm{Hom}_{k\text{-alg}}(A, k)$ and $R(X) = \oplus_X k = \mathrm{Hom}(X, k)$ define an equivalence of categories*

$$\big[\text{Trivial } k\text{-algebras}\big]^{\mathrm{op}} \;\leftrightsquigarrow\; \big[\text{Finite Sets}\big].$$

Proof: The natural morphism $A \to \mathrm{Hom}(\mathrm{Spec}\, A, k)$ is injective because $A = \oplus k$ is reduced; hence it is an isomorphism, since both algebras have equal degree (the cardinal of $\mathrm{Spec}\, A$).

The natural map $X \to \mathrm{Spec}\big(\mathrm{Hom}(X, k)\big)$, $x \mapsto \mathfrak{m}_x = \{f \colon X \to k \colon f(x) = 0\}$, is injective; hence it is bijective, since the number of points of the spectrum is bounded by the degree.

Theorem: *A finite k-algebra A is* ***separable*** *if the following equivalent conditions hold*

1. *A is* ***locally***[6] *trivial: $A_L = \oplus L$ for some finite extension $k \to L$.*
2. *A is geometrically reduced: A_K is reduced for any extension $k \to K$.*
3. *$A \otimes_k A$ is reduced.*
4. $\Omega_{A/k} = 0$.

Proof: $(1 \Rightarrow 2)$ $0 \to A_K \to A_{KL} = (A_L)_{KL} = (\oplus L)_{KL} = \oplus KL$; hence A_K is reduced.

$(2 \Rightarrow 3)$ A is reduced, $A = \oplus_i K_i$; hence $A \otimes_k A = \oplus_i (A \otimes_k K_i)$ is reduced.

$(3 \Rightarrow 4)$ If $A \otimes_k A = \oplus_i K_i$, any ideal I of $A \otimes_k A$ is a direct sum of components. Hence $I = I^2$, and $\Omega_{A/k} = \Delta/\Delta^2 = 0$.

$(4 \Rightarrow 1)$ If $A_L = B \oplus \dots$, then $0 = (\Omega_{A/k})_L = \Omega_{A_L/L} = \Omega_{B/L} \oplus \dots$; hence $\Omega_{B/L} = 0$.

By Kronecker's theorem, there is a finite extension $k \to L$ such that A_L is a rational L-algebra.

Now (see p. 145) for any maximal ideal $\mathfrak{m}$ of A_L we have $\mathfrak{m}/\mathfrak{m}^2 = \Omega_{B/L} \otimes_B L = 0$.

Hence $\mathfrak{m} = 0$ by Nakayama's lemma, so that $B = L$ and A_L is a trivial L-algebra.

Corollary: *Separability is a local and geometric concept. Subalgebras, quotients, direct sums, and tensor products of separable algebras are also separable. Moreover,*

$$\mathrm{Hom}_{k\text{-alg}}(A, K) = \begin{bmatrix} \text{Components of} \\ A \otimes_k K \text{ equal to } K \end{bmatrix} \qquad (A \text{ separable finite } k\text{-algebra})$$

$$\mathrm{Aut}_{k\text{-alg}}(L) = \begin{bmatrix} \text{Components of} \\ L \otimes_k L \text{ equal to } L \end{bmatrix} \qquad (L \text{ separable finite extension})$$

Primitive Element Theorem: *If the field k is infinite, any finite separable k-algebra is generated by some element, $A = k[a]$.*

Proof: A is locally trivial, $0 \to A \to A_L = L \oplus \overset{n}{\dots} \oplus L$, where $n = [A : k]$.

Elements $(\lambda_1, \dots, \lambda_n) \in A$ such that $\lambda_i = \lambda_j$ form a proper vector subspace $V_{ij} \subset A$, since A contains a base of $A_L = L^n$. When k is infinite[7], $\bigcup V_{ij} \neq A$ and there is $a \in A$ such that the n morphisms $k[a] \to A_L \to L$ are different. The degree of $k[a]$ is $\geq n$, and $A = k[a]$.

[6] If $k \to L$ is a finite extension, $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$ should be viewed as an open cover of $\mathrm{Spec}\, k$ in a convenient "Grothendieck topology".

[7] Including V_{ij} into an hyperplane, in the dual projective space $\mathbb{P}(A^*)$ we have a finite number of points, and there exists an exterior hyperplane; just project from a point and use induction on dimension.

Definitions: A polynomial $P \in k[x]$ is **separable** when so is the k-algebra $k[x]/(P)$; i.e., when any root of P is simple, g.c.d.$(P, P') = 1$.

An element a of a finite k-algebra A is **separable** when so is the k-algebra $k[a] \simeq k[x]/(P_a)$; i.e., when so is the annihilator polynomial $P_a(x)$.

Proposition: *$A = k[a_1, \ldots, a_n]$ is separable if and only if so are $a_1, \ldots, a_n$.*

Proof: If A is separable, so are the subalgebras $k[a_i]$.

If $k[a_1], \ldots, k[a_n]$ are separable algebras, so is $k[a_1] \otimes_k \ldots \otimes_k k[a_n]$.

Hence A, since we have an epimorphism $k[a_1] \otimes_k \ldots \otimes_k k[a_n] \to A$.

Definition: A field k is **perfect** if any finite extension is separable; i.e., any irreducible polynomial in $k[x]$ has simple roots.

Fields of null characteristic and the finite fields $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ are perfect (pp. 86, 87) and the example of p. 87 shows that the field $\mathbb{F}_2(t)$ is not perfect.

Proposition: *A field k of characteristic $p > 0$ is perfect if and only if the **Frobënius morphism** $F \colon k \to k$, $F(a) = a^p$, is surjective. Therefore, finite fields are perfect.*

Proof: If k is perfect, then $k \to k(\sqrt[p]{a}\,)$ is separable; hence $\sqrt[p]{a} \in k$.

Conversely, if F is surjective, and a polynomial $Q \in k[x]$ has a multiple root, then $Q' = 0$ (p. 86) and Q is not irreducible,

$$Q = a_0 + a_p x^p + a_{2p} x^{2p} + \ldots = (b_0^p + b_p^p x + b_{2p}^p x^2 + \ldots) = (b_0 + b_p x + b_{2p} x^2 + \ldots)^p.$$

Definition: The **trace** of $a \in A$ is the trace of the endomorphism $h_a \colon A \to A$, $h_a(x) = ax$.

So we get a linear form $\mathrm{tr} \colon A \to k$, $\mathrm{tr}(a) = \mathrm{tr}\, h_a$, and a **trace metric**,

$$\mathrm{Tr}(a, b) = \mathrm{tr}(ab).$$

The trace of a nilpotent endomorphism T is null since (p. 180) the characteristic polynomial is $c_T(x) = x^n$; hence $\mathrm{rad}\, A \subseteq \mathrm{rad}\, \mathrm{Tr}$.

Theorem: *The trace metric is stable under base changes $k \to K$.*

Proof: The matrix of h_a in a base $e_1, \ldots, e_n$ of A is just the matrix of $h_{a \otimes 1} \colon A_K \to A_K$ in the base $e_1 \otimes 1, \ldots, e_n \otimes 1$ of A_K, so that $\mathrm{tr}(a) = \mathrm{tr}(a \otimes 1)$.

Hence the trace's metric in A_K is just $\mathrm{Tr} \otimes 1 \colon A_K \otimes_K A_K = (A \otimes_k A) \otimes_k K \to K$.

Corollary: $\begin{bmatrix} \text{Radical of the} \\ \text{trace metric in } A \end{bmatrix} \otimes_k K = \begin{bmatrix} \text{Radical of the} \\ \text{trace metric in } A_K \end{bmatrix}$

Proof: Let us consider the polarity $\phi \colon A \to A^*$, $\phi(a)(b) = \mathrm{Tr}(a, b) = \mathrm{tr}(ab)$.

$$0 \longrightarrow \mathrm{rad}\, \mathrm{Tr} \longrightarrow A \xrightarrow{\ \phi\ } \mathrm{Hom}_k(A, k)$$

$$0 \longrightarrow (\mathrm{rad}\, \mathrm{Tr})_K \longrightarrow A_K \xrightarrow{\ \phi \otimes 1\ } \mathrm{Hom}_k(A, k) \otimes_k K = \mathrm{Hom}_{A_K}(A_K, K)$$

Since $\phi \otimes 1$ is the polarity of the trace metric in A_K, we see that $(\mathrm{rad}\, \mathrm{Tr})_K$ is just the radical of the trace metric in A_K.

Theorem: *A finite k-algebra is separable if and only if the trace metric is non singular.*

Proof: The trace metric of a trivial algebra is non singular (the matrix in the usual base is I).

Now, if A is locally trivial, then $(\mathrm{rad}\,\mathrm{Tr})_L = 0$, and $\mathrm{rad}\,\mathrm{Tr} = 0$.

If $\mathrm{rad}\,\mathrm{Tr} = 0$, then $\mathrm{rad}\, A_K \subseteq (\mathrm{rad}\,\mathrm{Tr})_K = 0$, and A is geometrically reduced.

Corollary: *A finite extension $k \to K$ is separable if and only if the trace* $\mathrm{tr}\colon K \to k$ *is non null.*

Proof: The radical $\mathrm{rad}\,\mathrm{Tr}$ is an ideal of the field K; hence it is null or K.

5.6.2 Galois Theory

Theorem: *The sequence $k \to A \rightrightarrows A \otimes_k A$ is exact for any k-algebra $A \neq 0$.*

Proof: Take a k-linear retract $\omega\colon A \to k$ (p. 55). If $a \otimes 1 = 1 \otimes a$, then

$$a = (1 \otimes \omega)(a \otimes 1) = (1 \otimes \omega)(1 \otimes a) = \omega(a) \in k.$$

Definition: A finite extension $k \to L$ is a **Galois** extension when $L \otimes_k L$ is a trivial L-algebra

$$L \otimes_k L = \oplus_G L = \mathrm{Hom}(G, L)\ , \quad a \otimes b = \big(\tau(a)b\big)_{\tau \in G},$$

where $G = \mathrm{Aut}(L/k) := \mathrm{Aut}_{k\text{-alg}}(L) = \mathrm{Hom}_{k\text{-alg}}(L, L)$ is said to be the **Galois group**.

In such case the exact sequence $k \to L \rightrightarrows L \otimes_k L = \oplus_G L$ shows that $k = L^G$.

1. The decomposition field L of a separable k-algebra A is a quotient of $A^{\otimes m}$. Since A_L is a trivial L-algebra, so is $L \otimes_k L$, and $k \to L$ is a Galois extension: the **Galois envelope** of A over k. Hence the decomposition field $k(\alpha_1, \ldots, \alpha_d)$ of any separable polynomial $P \in k[x]$ is a Galois extension of k, and the Galois group G is a permutation group of the roots $\alpha_1, \ldots, \alpha_d$ of $P(x)$.

2. Any Galois extension $k \to L$ also is a Galois extension of any intermediate field K, since we have an epimorphism $\oplus L = L \otimes_k L \to L \otimes_K L$.

   Moreover, any composite of L with L is $\simeq L$, so that any two morphisms $f_1, f_2\colon L \to E$ into another extension always have equal image. Hence, if an intermediate field K is also a Galois extension of k, then any automorphism $\tau \in G$ preserves it, $\tau(K) = K$, and we have a restriction morphism $G = \mathrm{Aut}(L/k) \to \mathrm{Aut}(K/k)$.

3. Since a Galois extension $k \to L$ is trivial over L, so is any intermediate field K; hence a k-algebra is trivial over L if and only if it is (isomorphic to) a direct sum $K_1 \oplus \ldots \oplus K_n$ of intermediate fields (i.e. subrings of L containing k).

Theorem: *If a finite group G acts on a k-algebra A, then the algebra of invariants is stable under base changes $k \to B$,*

$$(A \otimes_k B)^G = A^G \otimes_k B.$$

Proof: The functor $(-) \otimes_k B$ being exact, it preserves invariants under a finite group.

That is to say, put $G = \{\tau_1, \ldots, \tau_n\}$. The exact sequence $A^G \to A \xrightarrow{f} A \oplus \overset{n}{\ldots} \oplus A$ of k-linear maps, where $f(a) = (\tau_1(a) - a, \ldots, \tau_n(a) - a)$, induces an exact sequence

$$(A^G)_B \longrightarrow A_B \xrightarrow{f \otimes 1} A_B \oplus \overset{n}{\ldots} \oplus A_B.$$

Artin's Theorem: *Let G be a group of automorphisms of a finite extension $k \to L$. If $k = L^G$, then L is a Galois extension of k and G is the Galois group, $G = \mathrm{Aut}(L/k)$.*

Proof: By the points formula, each element of $G = \{\tau_1 = \mathrm{Id}, \dots, \tau_n\}$ defines a rational component A_i of $L \otimes_k L$; hence $L \otimes_k L = (A_1 \oplus \dots \oplus A_n) \oplus (B_1 \oplus \dots)$ and

$$L = L^G \otimes_k L = (L \otimes_k L)^G = (A_1 \oplus \dots \oplus A_n)^G \oplus (B_1 \oplus \dots)^G.$$

Since L is a local ring, we have $B_1 \oplus \dots = 0$.

Moreover $A_i = L$, since any nilpotent element $a \in A_1$ defines a nilpotent in L,

$$(a, \dots, \tau_i(a), \dots) \in (A_1 \oplus \dots \oplus A_n)^H = L.$$

Galois Correspondence: Let $k \to L$ be a Galois extension of group G, and let A be a finite k-algebra trivial over L,

$$A \otimes_k L = \oplus_{F(A)} L = \mathrm{Hom}(F(A), L) \ , \quad a \otimes \lambda = \big(p(a)\lambda\big)_{p \in F(A)},$$

where $F(A) := \mathrm{Hom}_{k\text{-alg}}(A, L) = \mathrm{Spec}\, A_L$ is a finite G-set ($\tau p = \tau \circ p$ for any point $p\colon A \to L$) fully determining A:

$$A = A \otimes_k L^G = (A \otimes_k L)^G = \mathrm{Hom}(F(A), L)^G = \mathrm{Hom}_G(F(A), L),$$

where the action of G on $A \otimes_k L = \oplus_{F(A)} L = \mathrm{Hom}(F(A), L)$ is just $\tau \cdot f = \tau \circ f \circ \tau^{-1}$.

Hence we define the **associated covering** of a finite G-set Δ to be

$$R(\Delta) = \mathrm{Hom}(\Delta, L)^G = \mathrm{Hom}_G(\Delta, L) \subseteq \oplus_\Delta L,$$

so that the natural morphism $A \to RF(A) = \mathrm{Hom}_G(F(A), L)$ is an isomorphism for any finite k-algebra A trivial over L.

Now, any point $x \in \Delta$ defines a morphism of k-algebras $R(\Delta) \hookrightarrow \mathrm{Hom}(\Delta, L) \xrightarrow{\delta_x} L$, and let us see that this natural morphism $\Delta \to FR(\Delta)$ also is an isomorphism:

Galois Theorem: *Let $k \to L$ be a Galois extension of group $G = \mathrm{Aut}(L/k)$. The above functors F and R define an equivalence of categories*

$$\begin{bmatrix}\text{Finite } k\text{-algebras} \\ \text{trivial over } L\end{bmatrix}^{\mathrm{op}} \rightsquigarrow \begin{bmatrix}\text{Finite} \\ G\text{-sets}\end{bmatrix}, \qquad \begin{matrix} RF(A) = A \\ FR(\Delta) = \Delta \end{matrix}$$

Proof: We determine the k-morphisms $R(\Delta) \to L$ with the points formula:

$$R(\Delta) \otimes_k L = \mathrm{Hom}(\Delta, L)^G \otimes_k L = \mathrm{Hom}(\Delta, L \otimes_k L)^G = \mathrm{Hom}_G(\Delta, \oplus_G L),$$

where the action of G on the left factor of the L-algebra $L \otimes_k L = \oplus_G L$ corresponds to the natural action $(\tau f)(g) = f(g\tau)$ on $\oplus_G L = \mathrm{Hom}(G, L)$. Hence, since G acts via L-morphisms,

$$\mathrm{Hom}_G(\Delta, \oplus_G L) = \mathrm{Hom}_G(\Delta \times G, L) = \mathrm{Hom}((\Delta \times G)/G, L) = \mathrm{Hom}(\Delta, L),$$

and we conclude that $\Delta = \mathrm{Hom}_{L\text{-alg}}(R(\Delta) \otimes_k L, L) = \mathrm{Hom}_{k\text{-alg}}(R(\Delta), L) = FR(\Delta)$.

Corollary: $\mathrm{Hom}_{k\text{-alg}}(A, B) = \mathrm{Hom}_G(F(B), F(A))$,
$\mathrm{Hom}_G(\Delta, \Delta') = \mathrm{Hom}_{k\text{-alg}}(R(\Delta'), R(\Delta))$.

Proof: See page 134.

Corollary: *If $H \subseteq G$ is a subgroup, then $R(G/H) = L^H$ and $F(L^H) = G/H$.*

1. $\begin{bmatrix}\text{Intermediate fields} \\ \text{between } k \text{ and } L\end{bmatrix} = \begin{bmatrix}\text{Subgroups} \\ \text{of } G\end{bmatrix}, \quad \begin{matrix} K \longmapsto \mathrm{Aut}(L/K) \\ L^H \longleftarrow H \end{matrix}$ *, is a lattice anti-isomorphism.*

2. $\mathrm{Hom}_{k\text{-alg}}(L^H, L^{H'}) = \{\bar\tau \in G/H \colon H' \subseteq \tau H \tau^{-1}\}$.

3. $L^H \simeq L^{H'}$ *if and only if* H' *and* H *are conjugate subgroups.*

4. $\mathrm{Aut}(L^H/k) = N(H)/H$.

5. L^H *is a Galois extension of* k *if and only if* H *is a normal subgroup of* G*, and in such a case the Galois group is* $\mathrm{Aut}(L^H/k) = G/H$.

Proof: We have $R(G/H) = \mathrm{Hom}_G(G/H, L) = L^H$, $F(L^H) = FR(G/H) = G/H$.

(1) Let $i \colon L^H \to L$ be the inclusion. The fibre H of the restriction map

$$G = F(L) \longrightarrow G/H = F(L^H)$$

over $[\mathrm{Id}] = i$ is just $\mathrm{Hom}_{L^H\text{-alg}}(L, L) = \mathrm{Aut}(L/L^H)$; hence $H = \mathrm{Aut}(L/L^H)$.

Moreover, $K = L^{\mathrm{Aut}(L/K)}$, since L is a Galois extension of K.

The remaining statements readily follow from the equality

$$\mathrm{Hom}_{k\text{-alg}}(L^H, L^{H'}) = \mathrm{Hom}_G(G/H', G/H) = (G/H)^{H'} = \{\bar g \in G/H \colon H' \subseteq gHg^{-1}\}.$$

Lagrange's Theorem: *Let* $k \to L$ *be a Galois extension of group* G*. Any composite* LE *is a Galois extension of* E*, and the Galois group is* $\mathrm{Aut}(L/L \cap E)$.

Proof: LE is a quotient of $L \otimes_k E$; hence $LE \otimes_E LE = \oplus LE$, being a quotient of

$$L \otimes_k E \otimes_E LE = L \otimes_K LE = L \otimes_K L \otimes_L LE = (\oplus L) \otimes_L LE = \oplus LE.$$

Moreover $\mathrm{Aut}(LE/E) \hookrightarrow \mathrm{Aut}(L/k)$, and $L^{\mathrm{Aut}(LE/E)} = L \cap (LE)^{\mathrm{Aut}(LE/E)} = L \cap E$.

Corollary: *If* G *is the Galois group of a separable polynomial* $P(x)$*, then the roots of any irreducible factor of* P *form an orbit under the action of* G.

Proof: Let L be a decomposition field of $P(x)$. If $P = P_1 \dots P_r$, where P_i is irreducible, then $A = k[x]/(P) = K_1 \oplus \dots \oplus K_r$, where $K_i = k[x]/(P_i)$.

Hence $P(A) = P(K_1) \oplus \dots \oplus P(K_r)$, where $P(K_i) = \mathrm{Hom}_{k\text{-alg}}(k[x]/(P_i), L)$ is an orbit, formed by the roots of P_i in L.

5.6.3 The Frobënius Automorphism

Let L be a finite field of characteristic p, so that L is an extension of $\mathbb{F}_p$.

If $n = [L \colon \mathbb{F}_p]$, we have a $\mathbb{F}_p$-linear isomorphism $L \simeq \mathbb{F}_p^n$; hence L has $q = p^n$ elements.

Theorem: *Let* $q = p^n$ *be a power of a prime number. There exists a unique field with* q *elements, namely the decomposition field* $\mathbb{F}_q$ *of* $x^q - x$ *over* $\mathbb{F}_p$.

Proof: The automorphism $F \colon \mathbb{F}_q \to \mathbb{F}_q$, $F(a) = a^q$, fixes any root of $x^q - x$; hence it is the identity, and any element of $\mathbb{F}_q$ is a root of $Q = x^q - x$.

Since Q is separable, because $Q' = -1$, it has q different roots, and $\mathbb{F}_q$ has q elements.

If L is another field with q elements, then the non null elements of L form a multiplicative group of order $q - 1$, so that they are roots of $x^{q-1} - 1$.

Hence all the roots of $x^q - x$ are in L, and L is the decomposition field of $x^q - x$ over $\mathbb{F}_p$.

Theorem: *The extension $\mathbb{F}_p \to \mathbb{F}_q$ is a Galois extension of cyclic Galois group, generated by the* ***Frobënius automorphism*** *$F(\alpha) = \alpha^p$.*

Proof: Let $H = (F)$. Since $\mathbb{F}_q^H = \mathbb{F}_p$, Artin's theorem let us conclude.

Corollary: *Let $Q \in \mathbb{F}_p[x]$ be separable, product of irreducible polynomials of degrees $n_1, \ldots, n_r$. The Galois group of Q over $\mathbb{F}_p$ is generated by a permutation of form $n_1, \ldots, n_r$.*

Lemma: *Any finite subgroup H of the multiplicative group k^* of a field k is cyclic.*

Proof: If d is the annihilator of H, then all the elements of H are roots of $x^d - 1$.
Hence $d \geq |H|$, and the classification of abelian groups shows that H is cyclic.

Corollary: *In $\mathbb{F}_p[x]$ there are irreducible polynomials of arbitrary degree.*

Proof: Let $q = p^n$. Since $\mathbb{F}_q^*$ is a cyclic group, $\mathbb{F}_q = \mathbb{F}_p(\theta)$.
The irreducible polynomial of θ over $\mathbb{F}_p$ has degree n.

Reduction Theorem: *Let G be the Galois group of $Q = x^n + c_1x^{n-1} + \ldots + c_n \in \mathbb{Z}[x]$ over $\mathbb{Q}$. If $\bar{G}$ is the Galois group over $\mathbb{F}_p$ of the reduction $\bar{Q} \in \mathbb{F}_p[x]$, there is a subgroup $H \subseteq G$ and an epimorphism $\varphi\colon H \to \bar{G}$. If $\bar{Q}$ is separable, then φ is an isomorphism, and $\tau \in H$ and $\varphi(\tau) \in \bar{G}$ have equal form as permutations of the roots.*

Proof: Let $\alpha_1, \ldots, \alpha_n$ be the complex roots of $Q(x)$.

1. $A = \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ is a finite $\mathbb{Z}$-module, because $\alpha_i^n = -c_1\alpha_i^{n-1} - \ldots - c_n$.

2. $A^G = \mathbb{Z}$. If $\frac{a}{b} \in A^G = A \cap \mathbb{Q}$, then $\mathbb{Z}[\frac{a}{b}] \subseteq A$ is a finite $\mathbb{Z}$-module (p. 176); hence its elements have bounded denominator, so that $\frac{a}{b} \in \mathbb{Z}$.

3. A/pA is a finite $\mathbb{F}_p$-algebra, hence $\operatorname{Spec}(A/pA) = \{x_1, \ldots, x_d\}$ is finite. Let $\mathfrak{m}_i$ be the maximal ideal of A defined by x_i, and put $K_i = A/\mathfrak{m}_i$. Since

$$(\mathfrak{m}_1 + (\mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_d))_0 = \{x_1\} \cap \{x_2, \ldots, x_d\} = \emptyset,$$

by the Chinese remainder theorem we have epimorphisms

$$A \longrightarrow A/(\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_d) = K_1 \oplus \ldots \oplus K_d \longrightarrow K_1 = \mathbb{F}_p[\bar{\alpha}_1, \ldots, \bar{\alpha}_n].$$

Since $Q(x) = \prod_i (x - \alpha_i)$, then $\bar{Q}(x) = \prod_i (x - \bar{\alpha}_i)$; hence K_1 is the decomposition field of $\bar{Q}$ and $\bar{G} = \operatorname{Aut}(K_1/\mathbb{F}_p)$. Now the subgroup $H = \{\tau \in G \colon \tau(\mathfrak{m}_1) = \mathfrak{m}_1\}$ acts on $K_1 = A/\mathfrak{m}_1$ and we have a natural morphism $\varphi\colon H \longrightarrow \bar{G}$, $\varphi(\tau)(\bar{a}) = [\tau(a)]$.

4. φ is surjective. We put $K_1 = \mathbb{F}_p(\bar{\theta})$, and we fix $\theta \in A$ with value $\bar{\theta}$ at x_1 and vanishing at $x_2, \ldots, x_d$. Now all the coefficients of $R(x) = \prod_{\tau \in G}(x - \tau\theta)$ are in $A^G = \mathbb{Z}$, and $\bar{\tau}(\bar{\theta})$ is a root of $\bar{R}(x)$, $\forall \bar{\tau} \in \bar{G}$. Hence $\bar{\tau}(\bar{\theta}) = [\tau\theta]$ for some $\tau \in G$.

   Since $\tau(\theta)$ does not vanish at x_1, we conclude that $\tau \in H$, and $\varphi(\tau) = \bar{\tau}$.

5. If $\bar{Q}$ is separable, then $\tau \in H$ defines a permutation of the roots α_i, and $\varphi(\tau)$ defines the same permutation of the roots $\bar{\alpha}_i$. Hence φ is injective.

Definition: If $\bar{Q}$ is separable, the **Frobënius automorphism** of $Q(x)$ at a prime number p is the unique element $F_p \in H$ such that $\varphi(F_p)$ is the Frobënius automorphism of K_1,

$$F_p(a) \equiv a^p \text{ (mod. } \mathfrak{m}_1), \ a \in A.$$

F_p depends on the maximal ideal $\mathfrak{m}_1$; but *it is well defined up to conjugation* since it is $\sigma F_p \sigma^{-1}$ when we fix the maximal ideal $\mathfrak{m}_i = \sigma(\mathfrak{m}_1)$, and G acts transitively on $\operatorname{Spec}(A/pA)$:

If $\operatorname{Spec}(A/pA)$ has two orbits, we fix a function $f \in A$ only vanishing at the orbit of x_1 (it exists since $A \to K_1 \oplus \ldots \oplus K_d$ is surjective). Now $N(f) = \prod_{\tau \in G} \tau(f) \in A^G = \mathbb{Z}$ only vanishes at the orbit of x_1. Absurd, $N(f) \in \mathfrak{m}_1 \cap \mathbb{Z} = p\mathbb{Z} \subseteq \mathfrak{m}_i$. q.e.d.

1. *There are polynomials of arbitrary degree n with Galois group S_n over $\mathbb{Q}$.*

   Let $Q_2 \in \mathbb{F}_2[x]$ irreducible of degree n; $Q_3 \in \mathbb{F}_3[x]$ with an irreducible factor of degree $n-1$ and a root in $\mathbb{F}_3$; and $Q_p \in \mathbb{F}_p[x]$ with an irreducible factor of degree 2 and $n-2$ different roots in $\mathbb{F}_p$, $p \neq 2, 3$. Since $\mathbb{Z}/6p\mathbb{Z} = \mathbb{F}_2 \oplus \mathbb{F}_3 \oplus \mathbb{F}_p$, there is $Q \in \mathbb{Z}[x]$ with reductions Q_2, Q_3, Q_p modulo 2, 3, p. The Galois group of Q is a transitive subgroup $G \subseteq S_n$ with a $(n-1)$-cycle and a transposition: $(2, \ldots, n)$, $(ij) \in G$.

   Since G is transitive, it contains a transposition $(1k)$. Conjugating $(1k)$ with $(2, \ldots, n)$ we see that $(12), (13), \ldots, (1n) \in G$ and these transpositions generate $S_n = G$.

2. If G does not contain n-cycles, then the reduction $\bar{Q}$ never is irreducible. Any quartic of group $\{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$, for example x^4+1, is irreducible; but it has reducible reduction at any prime p, since it is inseparable, or it has 4 roots in $\mathbb{F}_p$ (when $F_p = \mathrm{Id}$) or it is a product of two quadratic factors (when F_p has form 2,2).

3. If any automorphism $\tau \in G$ fixes some root of Q, then $\bar{Q}$ has a root in $\mathbb{F}_p$ (if $\bar{Q}$ is separable). Since any automorphism of $\mathbb{Q}(\sqrt{2}, i)$ fixes $\sqrt{2}$, $i\sqrt{2}$ or $1+i$, it fixes a root of $x^8 - 2^4$; hence $x^8 - 2^4$ has a root in $\mathbb{F}_p$ when $p \neq 2$, and we see that 2^4 is a 8-th power at any prime.

The polynomial $x^n - 1$ is separable when $p = \operatorname{car} k$ does not divide n, and its roots in a decomposition field L form a subgroup μ_n of L^*; hence a cyclic group

$$\mu_n = \{\varepsilon_n, \varepsilon_n^2, \ldots, \varepsilon_n^n = 1\}.$$

Theorem: *$k \to k(\varepsilon_n)$ is an abelian extension of group $G \subseteq (\mathbb{Z}/n\mathbb{Z})^*$.*

Proof: $k(\varepsilon_n)$ is a Galois extension because it is the decomposition field of a separable polynomial, and $\tau \in G$ induces an automorphism of μ_n; hence $\tau(\varepsilon_n) = \varepsilon_n^i$, $i \in (\mathbb{Z}/n\mathbb{Z})^*$.

Now, the morphism $G \to (\mathbb{Z}/n\mathbb{Z})^*$, $\tau \mapsto i$, is injective since τ is determined by $\tau(\varepsilon_n)$.

Theorem: *The Frobënius automorphism of $x^n - 1$ at any prime p not dividing n is*

$$F_p = [p] \in (\mathbb{Z}/n\mathbb{Z})^*.$$

Proof: If the reduction of $x^n - 1$ is separable, $\bar{\varepsilon}_n^i \neq \bar{\varepsilon}_n^j$, $i \neq j$; so that $\varepsilon_n^i \not\equiv \varepsilon_n^j$ (mod. $\mathfrak{m}_1$).

Since $F_p(\varepsilon_n) \equiv \varepsilon_n^p$ (mod. $\mathfrak{m}_1$), we see that $F_p(\varepsilon_n) = \varepsilon_n^p$.

Corollary: $\mathbb{Q} \to \mathbb{Q}\big(e^{\frac{2\pi i}{n}}\big)$ *is an abelian extension of group $(\mathbb{Z}/n\mathbb{Z})^*$, hence of degree $\phi(n)$, and the cyclotomic polynomial $\Phi_n(x)$ is irreducible.*

Proof: G contains any prime not dividing a n, and these primes generate $(\mathbb{Z}/n\mathbb{Z})^*$.

Lemma: *If n is odd, the discriminant of $Q(x) = x^n - 1$ is $\Delta = (-1)^{\frac{n-1}{2}} n^n$.*

Proof: $\Delta = (-1)^{\frac{n-1}{2}} \prod_i Q'(\alpha_i) = (-1)^{\frac{n-1}{2}} n^n \prod_i \alpha_i^{n-1} = (-1)^{\frac{n-1}{2}} n^n$.

Quadratic Reciprocity Law: $\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right)$, *where p, q are odd primes.*

Proof: Let Δ be the discriminant of $x^q - 1$. The unique extension of degree 2 contained in $\mathbb{Q}\big(e^{\frac{2\pi i}{q}}\big)$ is $K = \mathbb{Q}(\sqrt{\Delta}\,)$, and the Frobënius automorphism F_p of $x^q - 1$ is the identity on K whenever $[p]$ is in the unique subgroup of index 2 of $(\mathbb{Z}/q\mathbb{Z})^*$, formed by all the quadratic residues.

Since $\mathbb{Z}[\sqrt{\Delta}] \subset \mathbb{Z}[e^{\frac{2\pi i}{q}}]$, the restriction of F_p to K is the Frobënius automorphism of $x^2 - \Delta$ at p, which is the identity when Δ is a quadratic residue modulo p. Hence (p. 73)

$$\left(\frac{p}{q}\right) = \left(\frac{\Delta}{p}\right) = \left(\frac{-1}{p}\right)^{\frac{q-1}{2}} \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right).$$

5.6.4 Radical Extensions

Lemma: *If $\operatorname{car} k \neq 2$, any extension $k \to L$ of degree 2 is $L = k(\sqrt{a}\,)$ for some $a \in k$.*

Proof: If $\beta \in L$ is not in k, it is a root of some polynomial $x^2 + bx + c \in k[x]$.
Hence $2\beta = -b \pm \sqrt{b^2 - 4c}$, and $L = k(\beta) = k(\sqrt{b^2 - 4c}\,)$.

Proposition: *All the complex roots of a polynomial $P \in \mathbb{Q}[x]$ are quadratic irrationals if and only if the Galois group G of $P(x)$ over $\mathbb{Q}$ is a 2-group.*

Proof: If all the roots of P are quadratic irrationals, then the decomposition field L is contained in an extension by quadratic radicals, and $|G| = [L : \mathbb{Q}] = 2^d$ (p. 81).

If G is a 2-group, there exist subgroups $1 \subset H_1 \subset \ldots \subset H_d = G$, $|H_i| = 2^i$, (p. 122).

Now we see that L is an extension by quadratic radicals,

$$\mathbb{Q} = L^{H_d} \xrightarrow{\;2\;} L^{H_{d-1}} \xrightarrow{\;2\;} \ldots \xrightarrow{\;2\;} L^{H_1} \xrightarrow{\;2\;} L$$

Corollary: *An algebraic number $\alpha \in \mathbb{C}$ is a quadratic irrational if and only if the Galois group G over $\mathbb{Q}$ of the irreducible polynomial $P_\alpha(x)$ is a 2-group.*

Proof: If G is a 2-group, any root of P_α (hence α) is a quadratic irrational.

Conversely, if α is a quadratic irrational, we have to show that so is any complex root β of P_α, and by definition $\alpha \in \mathbb{Q}(\alpha_1, \ldots, \alpha_r)$, where $\alpha_i^2 \in \mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})$.

Let $L \subset \mathbb{C}$ be the Galois envelope of $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$.

Since P_α is irreducible, $\mathbb{Q}(\alpha) \simeq \mathbb{Q}(\beta)$, and $\tau(\alpha) = \beta$ for some automorphism $\tau \colon L \to L$; hence $\beta \in \mathbb{Q}(\tau\alpha_1, \ldots, \tau\alpha_r)$ and $(\tau\alpha_i)^2 \in \mathbb{Q}(\tau\alpha_1, \ldots, \tau\alpha_{i-1})$. q.e.d.

1. $\mathbb{Q} \to \mathbb{Q}\big(e^{\frac{2\pi i}{n}}\big)$ is a Galois extension of degree $\phi(n)$; hence $e^{\frac{2\pi i}{n}}$ *is a quadratic irrational if and only if $\phi(n)$ is a power of 2.*

2. If $p = 2^k + 1$ is prime, then $e^{\frac{2\pi i}{p}}$ is a quadratic irrational. *The regular polygons of* 17, 257, *and* 65537 *sides are ruler-and-compass constructible.*

3. Other proof of **D'Alembert's theorem**: We have to show that any finite extension $\mathbb{C} \to L$ is trivial, and we may assume that L is a Galois extension of $\mathbb{R}$. Let G be the Galois group, and let H be a Sylow 2-subgroup, so that $[L^H : \mathbb{R}]$ is odd. Then $\operatorname{gr} P_\alpha = [\mathbb{R}(\alpha) : \mathbb{R}]$ is odd for all $\alpha \in L^H$, and P_α has a real root by Bolzano's theorem. Hence $\alpha \in \mathbb{R}$, $L^H = \mathbb{R}$, $H = G$, and L is an extension of $\mathbb{C}$ by quadratic radicals; $L = \mathbb{C}$.

Independence Theorem: *Let A be a k-algebra and L an extension of k. Any family of morphisms of k-algebras $p_i \colon A \to L$ is L-linearly independent.*

Proof: When $L = k$, the morphisms $p_i \colon A \to k$ are surjective, so that $\mathfrak{m}_i = \operatorname{Ker} p_i$ is a maximal ideal, and we conclude since $A/(\mathfrak{m}_1 \dots \mathfrak{m}_r) = (A/m_1) \oplus \dots \oplus (A/\mathfrak{m}_r) = k^r$ (p. 139).

In the general case, just consider the rational points $p_i \otimes 1 \colon A_L \to L$.

Definitions: A finite extension $k \to K$ is a **radical extension** if

$$K = k(\alpha_1, \dots, \alpha_r) \ , \quad \alpha_i^{n_i} \in k(\alpha_1, \dots, \alpha_{i-1}).$$

A polynomial $P \in k[x]$ is **solvable by radicals** if the decomposition field L over k admits a finite extension $L \to K$ such that K is a radical extension of k.

Theorem: *If k contains all the n-th roots of unity, and* $\operatorname{char} k$ *does not divide n, then*

1. *Any extension $k(\alpha)$, $\alpha^n = a \in k$, is cyclic, of degree a divisor d of n, and $\alpha^d \in k$.*
2. *If $k \to L$ is a cyclic extension of degree n, then $L = k(\sqrt[n]{a}\,)$ for some $a \in k$.*

Proof: (1) The decomposition field of the separable polynomial $x^n - a$ is $k(\alpha)$, hence it is a Galois extension. If $\tau \in G$, then $(\tau\alpha)^n = \tau(\alpha^n) = a$, and $\tau(\alpha) = u\alpha$, $u \in \mu_n$.

So we get an injective morphism $G \hookrightarrow \mu_n$.

Since μ_n is a cyclic group of order n, then G is a cyclic group of order a divisor d of n.

If $G = (\sigma)$, then $\sigma(\alpha) = v\alpha$, where $v^d = 1$. Hence $\alpha^d \in k$, since $\sigma(\alpha^d) = (\sigma\alpha)^d = (v\alpha)^d = \alpha^d$.

(2) Let $G = (\sigma)$ be the Galois group.

Since $\sigma^n = \mathrm{Id}$, the annihilator polynomial of σ is $x^n - 1$ by the independence theorem.

Hence ε_n is an eigenvalue of σ and there exists $0 \neq \alpha \in L$ such that $\sigma(\alpha) = \varepsilon_n\alpha$.

Now $\sigma(\alpha^n) = (\varepsilon_n\alpha)^n = \alpha^n$, and $\alpha^n \in k$. Moreover, $L = k(\alpha)$ since $\sigma^i(\alpha) = \varepsilon_n^i\alpha \neq \alpha$, $i < n$.

Lemma: *Let $k \to L$ be a Galois extension of group G. If $k \to E$ is an abelian extension, then* $\operatorname{Aut}(EL/E) = H \lhd G$*, and G/H is abelian.*

Proof: $E \cap L$ is an abelian extension of k, hence it defines a subgroup $H \lhd G$ such that G/H is abelian, and Lagrange's theorem states that $H = \operatorname{Aut}(EL/E)$.

Theorem: *Let k be a field of characteristic 0. A polynomial $P \in k[x]$ is solvable by radicals if and only if the Galois group G is solvable.*

Proof: Let L be the decomposition field of P, and $G = \operatorname{Aut}(L/k)$.

If $L \to k(\alpha_1, \dots, \alpha_r)$, where $\alpha_i^{n_i} \in k(\alpha_1, \dots, \alpha_{i-1})$, we put $E_i = k(\varepsilon_n, \alpha_1, \dots, \alpha_i)$, where $n = n_1 \dots n_r$. Now $E_0 = k(\varepsilon_n)$ is an abelian extension of k (p. 154), and E_i is a cyclic extension of E_{i-1}. If H_i is the Galois group of E_iL over E_i,

$$\begin{array}{ccccccccc}
L & \longrightarrow & E_0L & \longrightarrow & E_1L & \longrightarrow & \dots & \longrightarrow & E_rL \\
\uparrow{\scriptstyle G} & & \uparrow{\scriptstyle H_0} & & \uparrow{\scriptstyle H_1} & & & & \uparrow{\scriptstyle H_r} \\
k & \longrightarrow & E_0 & \longrightarrow & E_1 & \longrightarrow & \dots & \longrightarrow & E_r
\end{array}$$

the above lemma states that $H_r \lhd \ldots \lhd H_0 \lhd G$ and the successive quotients are abelian.

Since $H_r = 1$, because $L \subset E_r$, we see that G is solvable.

Conversely, if G is solvable of order n, by Lagrange's theorem the Galois group of $L(\varepsilon_n)$ over $k(\varepsilon_n)$ is a subgroup H of G; hence it is solvable: we have subgroups $1 = H_0 \lhd H_1 \lhd \ldots \lhd H_r = H$, where H_i/H_{i-1} is cyclic, of order a divisor of n.

Hence $L(\varepsilon_n)$ is a radical extension of $k(\varepsilon_n)$, and P is solvable by radicals. q.e.d.

1. $1 \lhd A_3 \lhd S_3$ is a resolution of S_3, and a resolution of S_4 is $1 \lhd V \lhd A_4 \lhd S_4$, where V is the Klein group $\{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$. But S_n is not solvable when $n \geq 5$ because any 3-cycle is the **commutator** of two 3-cycles,
$$(ijk) = \sigma\tau\sigma^{-1}\tau^{-1};\ \sigma = (ijl),\ \tau = (ikm),$$
so that a resolution of S_n, since H_{i-1} contains any commutator of elements of H_i, would lead to an absurd: $H_0 = 1$ contains any 3-cycle. Equations of degree 2, 3 and 4 are solvable by radicals; but equations of group S_5 (they exist, p. 154) are not solvable by radicals.

2. If the Galois group is solvable of order $p_1 p_2 \ldots p_r$, where p_i is prime, the above proof shows that the equation is solved with r radicals $\sqrt[p_1]{\ }, \ldots, \sqrt[p_r]{\ }$ and p_i-th roots of unity. Since $|S_3| = 2 \cdot 3$, cubics are solved with a square root and a cubic root, and since $|S_4| = 2^3 3$, quartics are solved with three square roots and a cubic root.

5.6.5 Inseparable Algebras

Definition: The separable elements of a finite k-algebra A form a subalgebra $\pi_0^k(A)$, and the **separability degree** of A over k is $[A:k]_s = [\pi_0^k(A):k]$.

Theorem: *The maximal separable subalgebra is stable under base changes $k \to K$,*
$$\pi_0^k(A) \otimes_k K \ = \ \pi_0^K(A \otimes_k K).$$

Proof: The kernel A_1 of the differential $A \to \Omega_{A/k}$ contains $\pi_0^k(A)$, hence $\pi_0^k(A) \subseteq \bigcap_n A_n$, where $A_{n+1} = (A_n)_1$. When $A_{n+1} = A_n$, the algebra A_n is separable (p. 148) and $A_n \subseteq \pi_0^k(A)$.

Hence $\pi_0^k(A) = \bigcap_n A_n$, and we conclude because the subalgebra $\bigcap_n A_n$ is stable under base changes, since so is the module of differentials (p. 144).

Corollary: $|\mathrm{Hom}_{k\text{-alg}}(A,K)| \leq [A:k]_s$, *and the equality holds if and only if A_K is rational.*

$|\mathrm{Aut}(L/k)| \leq [L:k]_s$, *and the equality holds if and only if $L \otimes_k L$ is a rational L-algebra* (L is a **normal** extension of k).

Proof: By the above theorem we may assume that $k = K$.

Let $B = A_1 \oplus \ldots \oplus A_n$ be the rational components of A. Since separable algebras are reduced, $\pi_0^k(B) \hookrightarrow B/\mathrm{rad}\, B = k^n$, and since $k^n \subseteq \pi_0^k(B)$, we see that $\pi_0^k(B) = k^n$.

Hence $|\mathrm{Hom}_{k\text{-alg}}(A,k)| \leq [A:k]_s$, and the equality holds if and only if A is rational.

Corollary: $[L:k]_s = [L:K]_s [K:k]_s$.

Proof: The extensions L and K are rational over some extension E, so that
$$[L:k]_s = |\mathrm{Hom}_{k\text{-alg}}(L,E)| \quad , \quad [K:k]_s = |\mathrm{Hom}_{k\text{-alg}}(K,E)|.$$

The map $\mathrm{Hom}_{k\text{-alg}}(L,E) \to \mathrm{Hom}_{k\text{-alg}}(K,E)$ is surjective by the points formula (p. 145), and the fibre over any point is $\mathrm{Hom}_{K\text{-alg}}(L,E)$, hence of cardinal $[L:K]_s$ since $L \otimes_K E$ is a rational E-algebra (it is a quotient of $L \otimes_k E$).

Corollary: *The functor π_0^k preserves epimorphisms.*

Proof: After a base change (p. 146) we may assume that A is a rational local k-algebra.

Corollary: *A finite k-algebra A is said to be* ***purely inseparable*** *if the following equivalent conditions hold*

1. *A is geometrically local (A_K is local for any extension $k \to K$).*

2. *A_L is a rational local L-algebra for some extension $k \to L$.*

3. *$\pi_0^k(A) = k$ (that is to say, $[A : k]_s = 1$).*

Proof: ($1 \Rightarrow 2$) When A_L is rational (p. 146) it is local and rational.

($2 \Rightarrow 3$) If A_L is local and rational, then $\pi_0^k(A)_L = \pi_0^L(A_L) = L$; hence $\pi_0^k(A) = k$.

($3 \Rightarrow 1$) $\pi_0^K(A_K) = \pi_0^k(A)_K = K$, and we conclude by the following lemma.

Lemma: *A finite k-algebra A is local if and only if $\pi_0^k(A)$ is a field.*

Proof: $\pi_0^k(A)$ is local if so is A (p. 145) and, being reduced, it is a field.
If $A = A_1 \oplus A_2 \oplus \ldots$ is not local, then $\pi_0^k(A) = \pi_0^k(A_1) \oplus \pi_0^k(A_2) \oplus \ldots$ is not integral.

Corollary: *If G is the group of automorphisms of a normal extension $k \to L$, then $k \to L^G$ is a purely inseparable extension (and $L^G \to L$ is a Galois extension by Artin's theorem).*

Proof: Since $[L : k]_s = |G| = [L : L^G]$, it follows that $[L^G : k]_s = 1$.

Examples: The decomposition field L of a finite k-algebra A is a quotient of $A^{\otimes m}$ and, A_L being rational, so is $L \otimes_k L$; hence L is a normal extension of k.

A polynomial Q is purely inseparable when so is the k-algebra $k[x]/(Q)$; i.e., when all the roots coincide. So, $x^{p^n} - a$ is purely inseparable when $\operatorname{char} k = p$. In fact, if α is a root, then $\alpha^{p^n} = a$, and $x^{p^n} - a = x^{p^n} - \alpha^{p^n} = (x - \alpha)^{p^n}$.

Corollary: *Let $p = \operatorname{car} k$. If $k \to L$ is a purely inseparable extension, then any element $\alpha \in L$ is a root of a polynomial $x^{p^n} - a \in k[x]$.*

Proof: Let Q_α be the irreducible polynomial of α over k, and $q = p^n$ the greatest power such that $Q_\alpha(x) = Q(x^q)$ for some polynomial Q (separable, it is irreducible and $Q' \neq 0$).

Now $a = \alpha^q \in \pi_0^k(L) = k$, and α is a root of $x^q - a$ (in fact $Q_\alpha(x) = x^q - a$, because $\deg Q_\alpha(x) = \deg Q(x^q) \geq q$).

Chapter 6

Projective Geometry

6.1 Projective Spaces

Definitions: The **projective space** $\mathbb{P}(E)$ of a vector space E is the set of all 1-dimensional vector subspaces. We have a projection (defined when $e \neq 0$)

$$\pi \colon E \longrightarrow \mathbb{P}(E), \ \pi(e) = \langle e \rangle.$$

The **dimension** of $\mathbb{P}(E)$ is $n = \dim E - 1$, and we put $\mathbb{P}_n = \mathbb{P}(E)$.
Linear subvarieties are subsets $X = \pi(V) = \mathbb{P}(V)$, where V is a vector subspace.
The dimension of X is $\dim V - 1$, and the unique subvariety of dimension -1 is $\emptyset = \pi(0)$.
The **codimension** of X is $\operatorname{codim} X = \dim \mathbb{P}(E) - \dim X = \dim E - \dim V$.
Hyperplanes are linear subvarieties of codimension 1.

Since $V = \bigcup_{e \in V} \langle e \rangle$, we have a lattice isomorphism

$$\begin{bmatrix} \text{Vector sub-} \\ \text{spaces of } E \end{bmatrix} \xrightarrow{\ \pi\ } \begin{bmatrix} \text{Linear sub-} \\ \text{varieties of } \mathbb{P}(E) \end{bmatrix}, \ V \mapsto \pi(V).$$

so that the maximum of $X = \pi(V)$, $Y = \pi(W)$ is $X + Y = \pi(V + W)$; and (p. 52)

$$\dim (X + Y) = \dim X + \dim Y - \dim (X \cap Y).$$

For example, two different lines in $\mathbb{P}_2$ cut one another in a unique point.
In general, since $\dim (X + Y) \leq \dim \mathbb{P}(E)$, we have

$$\operatorname{codim} (X \cap Y) \leq \operatorname{codim} X + \operatorname{codim} Y.$$

Theorem: *Let $X = \pi(V)$. Linear subvarieties of dimension $1 + \dim X$ containing X correspond to points of $\mathbb{P}(E/V)$ and we have a lattice isomorphism*

$$\begin{bmatrix} \text{Linear subvarieties} \\ \text{of } \mathbb{P}(E) \text{ containing } X \end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix} \text{Linear subvarieties} \\ \text{of } \mathbb{P}(E/V) \end{bmatrix}, \ \pi(W) \mapsto \pi(W/V).$$

Proof: The isomorphism of p. 51, and the formula $\dim W/V = \dim W - \dim V$.

Theorem: *Hyperplanes of $\mathbb{P}(E)$ correspond to points of the **dual** space $\mathbb{P}(E^*)$, and we have a lattice anti-isomorphism (where linear subvarieties of codimension d correspond to linear subvarieties of dimension $d - 1$)*

$$\begin{bmatrix} \text{Linear sub-} \\ \text{varieties of } \mathbb{P}(E) \end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix} \text{Linear sub-} \\ \text{varieties of } \mathbb{P}(E^*) \end{bmatrix}, \ \pi(V) \mapsto \pi(V^{\circ}).$$

Proof: The isomorphism of p. 55, and the formula $\dim V^{\circ} = \dim E - \dim V$.

Duality Principle: Any ordered set $(X, \leq)$ defines a **dual** order $X^* = (X, \leq^*)$, where we put $x \leq^* y$ whenever $y \leq x$. Since $\dim \mathbb{P}(E^*) = \dim \mathbb{P}(E)$, the dual of a n-dimensional projective lattice (i.e. isomorphic to the lattice of linear subvarieties of a projective space of dimension n) also is a n-dimensional projective lattice. Hence, any statement on projective lattices has a dual equivalent statement. The statement *"any two different points in a plane may be joined by a unique line"*, is dual to *"any two different lines in a plane intersect at a unique point"*, and the dual figure of a triangle in a plane (three non collinear points) is again a triangle (three non concurrent lines).

Desargues Theorem: *If the three straight lines joining corresponding vertices of two triangles ABC and $A'B'C'$ all meet in a point P, then the three intersection points L, M, N of pairs of corresponding sides lie on a straight line.*

Proof: If p represents P, there are representants a, b, c, a', b', c' of A, B, C, A', B', C' such that $p = a + a' = b + b' = c + c'$. Therefore

$$\begin{aligned} l = a - b = b' - a' &\text{ represents } L, \\ m = a - c = c' - a' &\text{ represents } M, \\ n = b - c = c' - b' &\text{ represents } N, \end{aligned}$$

so that $l - m + n = 0$, and the points L, M, N are collinear.

Lemma: *Given $n+2$ points $(P_0, \ldots, P_n; U)$ in $\mathbb{P}_n$, no hyperplane containing $n+1$, there is a base $e_0, \ldots, e_n$ of E, unique up to a common factor, such that*

$$P_0 = \pi(e_0),\ \ldots,\ P_n = \pi(e_n),\ U = \pi(e_0 + \ldots + e_n).$$

Proof: To prove the existence, we put $P_i = \pi(v_i)$.

Now $v_0, \ldots, v_n$ span E because $P_0 + \ldots + P_n = \mathbb{P}_n$. If

$$U = \pi(\lambda_0 v_0 + \ldots + \lambda_n v_n),$$

then $\lambda_i \neq 0$, because $P_0 + \ldots + \widehat{P_i} + \ldots + U = \mathbb{P}_n$, and we put $e_i = \lambda_i v_i$.

If we have another base $e'_0, \ldots, e'_n$, and

$$e'_0 = \lambda_0 e_0,\ \ldots,\ e'_n = \lambda_n e_n,\ e'_0 + \ldots + e'_n = \mu(e_0 + \ldots + e_n),$$

then $\lambda_0 = \mu, \ldots, \lambda_n = \mu$, and the factors λ_i coincide.

Definitions: A **projective reference system** in $\mathbb{P}_n$ is a sequence $(P_0, \ldots, P_n; U)$ of $n+2$ points, no hyperplane containing $n+1$. If $P = \pi(e)$, and $e = x_0 e_0 + \ldots + x_n e_n$ in the normalized base of the lemma, $(x_0, \ldots, x_n)$ are the **homogeneous coordinates** of P.

They are well defined up to a common factor, and may be not all zero.

The **cross-ratio** of four (different) collinear points is

$$(P_1, P_2; P_3, P_4) = \frac{x_0}{x_1} \in k,$$

where (x_0, x_1) are the coordinates of P_4 in the reference system $(P_1, P_2; P_3)$.

The cross-ratio defines a bijection $\mathbb{P}_1 - \{P_1, P_2, P_3\} \to k - \{0, 1\}$. The **projective parameter** of $P = (x_0, x_1)$ is $\theta = \frac{x_0}{x_1}$, so defining a bijection $\theta \colon \mathbb{P}_1 \to k \cup \{\infty\}$.

Definition: A map $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E')$ is a **projectivity** if $\tau = \pi(T)$ for some isomorphism $T\colon E \to E'$, in the sense that the following square commutes

$$\begin{array}{ccc} E & \xrightarrow{T} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}(E) & \xrightarrow{\tau} & \mathbb{P}(E') \end{array}$$

so that τ induces a lattice isomorphism between linear subvarieties of $\mathbb{P}(E)$ and $\mathbb{P}(E')$, and any projective statement is invariant under projectivities.

Projectivities $\mathbb{P}(E) \to \mathbb{P}(E)$ form a group $PGl(E)$, and **homographies** are projectivities $\mathbb{P}_1 \to \mathbb{P}_1$. The equations of a projectivity $\mathbb{P}_n \to \mathbb{P}_n$ are $X' = AX$, where A is an invertible matrix of $n+1$ rows and columns. In particular, homographies are

$$\begin{cases} x_0' = ax_0 + bx_1 \\ x_1' = cx_0 + dx_1 \end{cases} \quad , \quad \theta' = \frac{a\theta + b}{c\theta + d} \quad ; \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Lemma: *The linear representant $T\colon E \to E'$ of a projectivity $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E')$ is well defined, up to a non null factor.*

Proof: If $\pi(T) = \pi(\bar{T})$, and we put $S = T^{-1}\bar{T}$, then $\pi(S)\colon \mathbb{P}(E) \to \mathbb{P}(E)$ is the identity.

The former lemma states that $S = \lambda \mathrm{Id}$, so that $\bar{T} = \lambda T$.

Theorem: *If $(P_0, \ldots, P_n; U)$, $(P_0', \ldots, P_n'; U')$ are two projective reference systems in $\mathbb{P}_n$, there is a unique projectivity $\mathbb{P}_n \xrightarrow{\tau} \mathbb{P}_n$ such that $P_i' = \tau P_i$, $U' = \tau U$.*

Proof: Any two bases of E differ in a unique linear automorphism $T\colon E \to E$.

Corollary: *The cross-ratio classifies, up to projectivities, quadruples of collinear points.*

Proof: If $(P_1, P_2; P_3, P_4) = (P_1', P_2'; P_3', P_4')$, and we take a homography τ transforming P_1, P_2, P_3 into P_1', P_2', P_3', we have that $P_4' = \tau P_4$ since

$$(P_1', P_2'; P_3', P_4') = (P_1, P_2; P_3, P_4) = (\tau P_1, \tau P_2; \tau P_3, \tau P_4) = (P_1', P_2'; P_3', \tau P_4).$$

Corollary: $(P_1, P_2; P_3, P_4) = \dfrac{(\theta_1 - \theta_3)(\theta_2 - \theta_4)}{(\theta_1 - \theta_4)(\theta_2 - \theta_3)}$ *where θ_i is the projective parameter of P_i.*

Proof: The homography transforming P_1, P_2, P_3 into $\infty, 0, 1$ is $\tau(\theta) = \dfrac{(\theta_1 - \theta_3)(\theta_2 - \theta)}{(\theta_1 - \theta)(\theta_2 - \theta_3)}$.

Corollary: $(P_1, P_2; P_3, P_4) = (P_2, P_1; P_4, P_3) = (P_3, P_4; P_1, P_2) = (P_4, P_3; P_2, P_1)$.

Definition: Four collinear points have 24 orderings. Since the cross-ratio is invariant under the **Klein group** $V = \{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$, there are at most 6 different values

$$(P_1, P_2; P_3, P_4) = \lambda \qquad (P_1, P_3; P_2, P_4) = 1 - \lambda \qquad (P_2, P_1; P_3, P_4) = \frac{1}{\lambda}$$
$$(P_2, P_3; P_1, P_4) = 1 - \frac{1}{\lambda} \qquad (P_3, P_1; P_2, P_4) = \frac{1}{1 - \lambda} \qquad (P_3, P_2; P_1, P_4) = \frac{\lambda}{\lambda - 1}$$

and, when $(P_1, P_2; P_3, P_4) = -1$, we say that P_4 is the **harmonic conjugate** of P_3 with respect to P_1, P_2. In such a case we also have $(P_2, P_1; P_3, P_4) = (P_1, P_2; P_4, P_3) = -1$.

6.1.1 Affine Spaces

Definitions: An **affine space** of dimension n over a field k is a set $\mathbb{A}_n$ (the set of **points**) with a free transitive action of a n-dimensional k-vector space V (the space of **free vectors**): if $p, q \in \mathbb{A}_n$, then $q = p + v$ for a unique vector $v \in V$.

An **affine reference** is a sequence $(p_0, p_0 + v_1, \ldots, p_0 + v_n)$ of $n+1$ points such $v_1, \ldots, v_n$ is a base of V, so that any point is $q = p_0 + y_1 v_1 + \ldots + y_n v_n$, and $(y_1, \ldots, y_n)$ are the **affine coordinates** of q in this reference.

The **linear subvariety** of $\mathbb{A}_n$ passing through a point p with **direction** a vector subspace $W \subseteq V$ is $p + W$, and it is an affine space with space of free vectors W.

Two linear subvarieties S, S' are **parallel**, $S \parallel S'$, when both have equal direction.

We agree that the empty subset also is a linear subvariety.

Let V and V' be vector spaces over some fields k, k' respectively. A **semilinear morphism** $(\sigma, T)\colon (k, V) \to (k', V')$ is a ring morphism[1] $\sigma\colon k \to k'$ and group morphism $T\colon V \to V'$ such that $T(\lambda v) = \sigma(\lambda) T(v)$, $\forall \lambda \in k, v \in V$. The map T fully determines σ when $T \neq 0$.

A **semiaffine map** $(\sigma, \vec{\phi}, \phi)\colon (k, V, \mathbb{A}_n) \to (k', V', \mathbb{A}'_m)$ between affine spaces is a semilinear morphism $(\sigma, \vec{\phi})\colon (k, V) \to (k', V')$ and a map $\phi\colon \mathbb{A}_n \to \mathbb{A}'_m$ such that $\phi(p + v) = \phi(p) + \vec{\phi}(v)$, $\forall p \in \mathbb{A}_n, v \in V$. The map ϕ fully determines $\vec{\phi}$; hence also σ when ϕ is not constant.

When $n \neq 0$, semiaffine isomorphisms (named **semiaffinities**) are bijective semiaffine maps: since $\vec{\phi}$ and σ are bijective, both are isomorphisms, and $(\sigma^{-1}, \vec{\phi}^{-1}, \phi^{-1})$ is a semiaffine map.

When $k = k'$ and σ is the identity, we eliminate the prefix *semi*.

Examples: Vector spaces are affine space with the obvious additive action, and any point p of an affine space $\mathbb{A}_n$ defines an affine isomorphism $V \simeq \mathbb{A}_n$, $v \mapsto p + v$.

Lemma: *Given a reference $p_0, \ldots, p_n$ in $\mathbb{A}_n$, a ring morphism $\sigma\colon k \to k'$ and points $p'_0, \ldots, p'_n$ in $\mathbb{A}'_m$, there is a unique σ-semiaffine map $\phi\colon \mathbb{A}_n \to \mathbb{A}'_m$ such that $\phi(p_0) = p'_0, \ldots, \phi(p_n) = p'_n$.*

Proof: Put $p_i = p_0 + v_i$, $p'_i = p'_0 + v'_i$. Then $\vec{\phi}\colon V \to V'$, $\vec{\phi}(\sum_i \lambda_i v_i) = \sum_i \sigma(\lambda_i) v'_i$ is a σ-semilinear map, and $\phi\colon \mathbb{A}_n \to \mathbb{A}'_m$, $\phi(p_0 + v) = p'_0 + \vec{\phi}(v)$, is a σ-semiaffine morphism (clearly the unique with the required property):

$$\phi(p_0 + e + v) = p'_0 + \vec{\phi}(e + v) = p'_0 + \vec{\phi}(e) + \vec{\phi}(v) = \phi(p_0 + e) + \vec{\phi}(v).$$

Definition: A map $\phi\colon \mathbb{A}_n \to \mathbb{A}'_m$ between affine spaces is a **parallel morphism** when it preserves the parallelism of lines, in the sense that for any points $a, b, c, d \in \mathbb{A}_n$ we have

$$ab \parallel cd \ \Rightarrow\ \phi(a)\phi(b) \parallel \phi(c)\phi(d)\ ,$$

where pq is the line joining the points p and q, or just the point p when $p = q$.

In particular, if $ab \parallel cd$ and $\phi(a) = \phi(b)$, then we also have $\phi(c) = \phi(d)$.

Semiaffine maps clearly are parallel morphisms.

Lemma: *Let $T, T'\colon V \to V'$ be additive maps such that $T'(v) \in \langle T(v) \rangle$, $\forall v \in V$. If the image of T is not contained in a 1-dimensional vector subspace, then $T' = \lambda T$ for some $\lambda \in k'$.*

Proof: When $T(v) \neq 0$, we have $T'(v) = \lambda_v T(v)$ for a unique scalar $\lambda_v \in k'$. Now, if $T(e)$ and $T(v)$ are linearly independent, then $T(e + v) = T(e) + T(v) \neq 0$; and $\lambda_e = \lambda_v$ because

$$\lambda_{e+v} T(e) + \lambda_{e+v} T(v) = \lambda_{e+v} T(e + v) = T'(e + v) = T'(e) + T'(v) = \lambda_e T(e) + \lambda_v T(v).$$

[1] When $k = k' = \mathbb{R}$, semilinear maps are linear because the unique ring morphism $\sigma\colon \mathbb{R} \to \mathbb{R}$ is the identity. In fact, σ is the identity in $\mathbb{Q}$ and it is continuous because it preserves the order: $\sigma(\mathbb{R}_+) = \sigma(\mathbb{R}^{*2}) \subseteq \mathbb{R}^{*2} = \mathbb{R}_+$.

When $T(e)$ and $T(v)$ are linearly dependent, by hypothesis there is $w \in V$ such that $T(w)$ is linearly independent from $T(e)$ and $T(v)$, so that $\lambda_e = \lambda_w = \lambda_v$.

Theorem: *Let $\phi\colon \mathbb{A}_n \to \mathbb{A}'_m$ be a parallel morphism, $m \geq 2$. If the image of ϕ is not contained in a line, then ϕ is a semiaffine map.*

Proof: Once we fix points $p \in \mathbb{A}_n$, $\phi(p) \in \mathbb{A}'_m$, we have to prove that any parallel morphism $T\colon V \to V'$ such that $T(0) = 0$ is a semilinear morphism.

1. *The map T is additive*, $T(e+v) = T(e) + T(v)$.

   Since T transforms the parallelogram with vertices $0, e, v, e+v$ into a parallelogram with vertices $0, e', v', (e+v)'$, there are $\alpha, \beta \in k'$ such that
   $$(e+v)' = \alpha e' + v' = e' + \beta v'.$$
   When $v' \notin \langle e' \rangle$ we see that $(e+v)' = e' + v'$, because either $e' = 0$ (so that $\alpha e' = e'$) or e', v' are linearly independent (so that $\alpha = \beta = 1$). The case $e' \notin \langle v' \rangle$ is analogous.

   Otherwise we have $\langle e' \rangle = \langle v' \rangle$, and by hypothesis there is $w \in V$ such that $w' \notin \langle v' \rangle$; hence $(v+w)' = v' + w' \notin \langle v' \rangle$, and $w' \notin \langle (e+v)' \rangle = \langle e' + \beta v' \rangle \subseteq \langle v' \rangle$.

   We conclude that $(e+v)' = e' + v'$ since, by the former case, we have
   $$e' + v' + w' = e' + (v+w)' = (e+v+w)' = (e+v)' + w'.$$

2. *There is a map $\sigma\colon k \to k'$ such that $T(\lambda v) = \sigma(\lambda) T(v)$, $\forall \lambda \in k;\ v \in V$.*

   Fix $\lambda \in k$ and put $T'(v) := T(\lambda v)$, so that the map T' is additive. Since T is parallel and $T(0) = 0$, we have that $T(\lambda v) \in \langle T(v) \rangle$, $\forall v \in V$, and we conclude by the lemma.

3. *The map $\sigma\colon k \to k'$ is a morphism of rings.*

   Take $v \in V$ such that $T(v) \neq 0$. We conclude by the following equalities:
   $$\sigma(\lambda+\mu)T(v) = T((\lambda+\mu)v) = T(\lambda v) + T(\mu v) = \sigma(\lambda)T(v) + \sigma(\mu)T(v) = (\sigma(\lambda)+\sigma(\mu))T(v),$$
   $$\sigma(\lambda\mu)T(v) = T(\lambda\mu v) = \sigma(\lambda)T(\mu v) = \sigma(\lambda)\sigma(\mu)T(v).$$

Definition: A bijective map $\tau\colon \mathbb{A}_n \to \mathbb{A}'_m$ is a **collineation** if it induces an isomorphism between the lattices of linear subvarieties (in particular $n = m$).

A subset W of a vector space V is a vector subsapce if $0 \in W$ and W contains $ke + kv$ whenever $e, v \in W$. When $k \neq \mathbb{F}_2$, it is enough that $0 \in W$ and W contains the line
$$pq = \{(1-t)p + tq\}_{t \in k}$$
joining any two points $p, q \in W$, since $ke + kv = (1-t)ke + tkv$ when $t \neq 0, 1$. Therefore a subset S of an affine space is a linear subvariety whenever it contains the line joining any two points of S (the plane spanned by any three points of S when $k = \mathbb{F}_2$). It follows that collineations are just bijections preserving lines (planes if $k = \mathbb{F}_2$); hence the name.

Semiaffinities, hence parallel isomorphisms, obviously are collineations.

Fundamental Theorem of Affine Geometry: *A bijection $\tau\colon \mathbb{A}_n \to \mathbb{A}'_n$ between affine spaces of dimension $n \geq 2$ is a semiaffinity if and only if it is a collineation.*

Proof: Since parallel lines are just coplanar non-intersecting (or coincident) lines, collineations are parallel morphisms; hence semiaffinities in dimension ≥ 2 by the above theorem.

Definitions: The affine functions $f\colon \mathbb{A}_n \to k$ form a vector space F of dimension $n+1$.

The linear map $V \hookrightarrow E = F^*$, $v(f) = \vec{f}(v)$, identifies V with $\{\omega \in E\colon \omega(1) = 0\}$, and the **vector extension** $\mathbb{A}_n \hookrightarrow E$, $x(f) = f(x)$, is an affine map identifying $\mathbb{A}_n$ with the linear subvariety $\{\omega \in E\colon \omega(1) = 1\}$ of direction V.

Hence we have a map $\mathbb{A}_n \hookrightarrow \mathbb{P}_n = \mathbb{P}(E)$ identifying $\mathbb{A}_n$ with the complement in $\mathbb{P}_n$ of $H = \pi(V)$, the **hyperplane at infinity** (the hyperplane of improper points).

Conversely, given a projective space $\mathbb{P}_n = \mathbb{P}(E)$ and an hyperplane $H = \pi(V)$, the complement $\mathbb{A}_n = \mathbb{P}_n - H$ is an affine space with space of free vectors $V^{\mathrm{o}} \otimes_k V = (E/V)^* \otimes_k V$, the action being $\pi(e) + \omega \otimes v = \pi(e + \omega(e)v)$. Hence, once we fix $0 \neq \omega_0 \in V^{\mathrm{o}}$, the space of free vectors is V and we have a vector extension $\mathbb{A}_n \to E$, $p \mapsto e$, where $p = \pi(e)$ and $\omega_0(e) = 1$.

Universal Property: *Let $j\colon \mathbb{A}_n \hookrightarrow E$ be an injective affine morphism into a vector space E of dimension $n+1$ such that $0 \notin j(\mathbb{A}_n)$. Any σ-semiaffine morphism $\phi\colon \mathbb{A}_n \to F$ to a k'-vector space F admits a unique extension to a σ-semilinear morphism $T\colon E \to F$.*

Proof: We may see $\mathbb{A}_n$ as a subset of E. Any affine reference $(p_0, p_1 = p_0 + v_1, \dots, p_n = p_0 + v_n)$ of $\mathbb{A}_n$ defines a base of E because otherwise $p_0 \in \langle v_1, \dots, v_n\rangle$ and $0 \in \mathbb{A}_n$.

If ϕ is a σ-semiaffine morphism, then $T(\sum_i \lambda_i p_i) = \sum_i \sigma(\lambda_i)\phi(p_i)$ is the unique possible semilinear extension, and in fact $T|_{\mathbb{A}_n} = \phi$ because both are σ-semiaffine morphisms and coincide on an affine reference system.

Definitions: A projective reference $(P_0, \dots, P_n; U)$ is **affine** when $P_1 + \ldots + P_n = H$. The **origin** is P_0, the **axis** are lines the $P_0 + P_i$, and U is the **unit point**. In the corresponding homogeneous coordinates $(x_0, \dots, x_n)$, the equation of the hyperplane at infinity is $x_0 = 0$, and any proper point has well defined **affine coordinates** $y_1 = \frac{x_1}{x_0}, \dots, y_n = \frac{x_n}{x_0}$.

Linear subvarieties of $\mathbb{A}_n$ are just affine parts of linear subvarieties of $\mathbb{P}_n$ and, except $\emptyset$, they correspond to linear subvarieties of $\mathbb{P}_n$ not contained in H. Two linear subvarieties of $\mathbb{A}_n$ are parallel when the infinity zones coincide. The **simple ratio** (A, B, C) of three collinear points in $\mathbb{A}_n$ is the cross-ratio $(A, B; C, P)$ with the improper point P of the line ABC, and C is the **middle point** of A and B when $(A, B, C) = -1$.

Theorem[2]**:** *The group of affinities of $\mathbb{A}_n$ is canonically isomorphic to the group of projectivities of $\mathbb{P}(E)$ leaving invariant the hyperplane at infinity H.*

Proof: Let $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E)$ be a projectivity such that $\tau(H) = H$. Then τ admits a unique linear representant $T\colon E \to E$ inducing the identity on E/V, so that $T(\mathbb{A}_n) = \mathbb{A}_n$ and we see that τ induces an affinity of $\mathbb{A}_n$.

If τ is the identity on $\mathbb{A}_n$, then it fixes a projective reference system, so that $\tau = \mathrm{Id}$.

Finally, any affinity $\phi\colon \mathbb{A}_n \to \mathbb{A}_n$ induces an isomorphism $T\colon E \to E$ by the universal property, and $T(V) = V$. Hence $\tau = \pi(T)$ is a projectivity extending ϕ and $\tau(H) = H$.

Definitions: Affinities fixing any point at infinity are **homotheties** and **translations**, according to they fix or not a proper point.

Definition: A bijective map $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E')$ is a **collineation** if it induces an isomorphism between the lattices of linear subvarieties (in particular $\dim \mathbb{P}(E) = \dim \mathbb{P}(E')$.)

[2] F. Klein, in the *Erlangen Program*, views Geometry as the action of a group G on a set: concepts are invariants of the group action, statements are relations between invariants, and theorems are true relations. Projective Geometry is defined by the action of the group of projectivities on $\mathbb{P}_n$, and Affine Geometry by the action of the group of affinities: *Affine geometry is the projective geometry of a fixed hyperplane.*

Since a subset $X \subseteq \mathbb{P}(E)$ is a linear subvariety when it contains the line joining any two points of X, a bijective map τ is a collineation whenever it preserves lines.

Any semilinear isomorphism $T\colon E \to E'$ induces a collineation $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E')$.

Fundamental Theorem of Projective Geometry: *Any collineation $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E')$ between projective spaces of dimension ≥ 2 is defined by a semilinear isomorphism $T\colon E \to E'$.*

Proof: Fix an hyperplane H in $\mathbb{P}(E)$, so that $H' = \tau(H)$ is an hyperplane in $\mathbb{P}(E')$, and consider the corresponding affine spaces $\mathbb{A}_n$ and $\mathbb{A}'_n$. Then τ induces a collineation $\tau\colon \mathbb{A}_n \to \mathbb{A}'_n$, hence a semiaffinity by the fundamental theorem of Affine Geometry, and it extends to a semilinear isomorphism $T\colon E \to E'$ by the above universal property.

Now $\tau_1 = \pi(T)$ is a collineation coinciding with τ out of H; hence $\bar{\tau} = \tau_1^{-1}\tau$ is the identity out of H. If $P \in H$, we consider two lines R_1, R_2 intersecting H at P. Since each line has two points out of H, they are invariant under $\bar{\tau}$, and we see that $\bar{\tau}(P) = P$. Hence, $\tau = \tau_1$.

6.2 Metrics

Let E be a vector space of finite dimension over a field k of characteristic $\neq 2$.

Definitions: A (symmetric) **metric** is a 2-covariant symmetric tensor S, and we put

$$e \cdot v := S(e, v) = \phi(e)(v),$$

where $\phi\colon E \to E^*$, $\phi(e) = i_e S = S(e, -)$, is the **polarity** of S. A **metric vector space** is a vector space endowed with a metric. A linear map $f\colon (E, S) \to (\bar{E}, \bar{S})$ is a **metric morphism** when $e \cdot v = f(e) \cdot f(v)$, and an **isometry** is a metric isomorphism.

Two vectors are **orthogonal** if $e \cdot v = 0$. Orthogonal vectors to a vector subspace V form a vector subspace $V^\perp$, the **orthogonal** of V, and $V \subseteq V^{\perp\perp}$.

The **radical** is the kernel of the polarity, $\operatorname{rad} E = E^\perp$ (hence $\operatorname{rad} V = V^\perp \cap V$), and the **rank** is $\dim(E/\operatorname{rad} E)$. The space E is **non singular** when the polarity is an isomorphism, $\operatorname{rad} E = 0$, and **totally isotropic** when the polarity is null, $\operatorname{rad} E = E$.

A non null vector $e \in E$ is **isotropic** if $e \cdot e = 0$.

A space is **elliptic** if there are no isotropic vectors (in particular it is non singular).

The **orthogonal sum** $E \perp E'$ of two metric vector spaces is the direct sum $E \oplus E'$, endowed with the following natural metric, so that $\operatorname{rad}(E \perp E') = (\operatorname{rad} E) \perp (\operatorname{rad} E')$,

$$(e_1 + e'_1) \cdot (e_2 + e'_2) = e_1 \cdot e_2 + e'_1 \cdot e'_2.$$

A metric S on E is **projectable** by an epimorphism $p\colon E \to \bar{E}$ if there exists a metric $\bar{S}$ in $\bar{E}$, clearly unique, such that p is a metric morphism.

Theorem: *A metric is projectable if and only if* $\operatorname{Ker} p \subseteq \operatorname{rad} E$.

Proof: If it is projectable and $p(e) = 0$, then $e \cdot v = p(e) \cdot p(v) = 0$; hence $e \in \operatorname{rad} E$.

Conversely, if $\operatorname{Ker} p \subseteq \operatorname{rad} E$, the metric $p(e) \cdot p(v) = e \cdot v$ is well defined: if $e' = e + u$, $u \in \operatorname{rad} E$, then $e' \cdot v = e \cdot v + u \cdot v = e \cdot v$.

Corollary: *Any metric projects onto $E/\operatorname{rad} E$, and the projection is non singular.*

Theorem: *Any metric vector space E decomposes, uniquely up to isometries, as an orthogonal sum of a totally isotropic space and a non singular space,*

$$E = (\operatorname{rad} E) \perp (E/\operatorname{rad} E).$$

Proof: If $E = (\mathrm{rad}\, E) \oplus V$, then $E = (\mathrm{rad}\, E) \perp V$, and V is non singular.

Moreover, if $E = T \perp F$, where T is totally isotropic and F is non singular, then

$$\mathrm{rad}\, E = (\mathrm{rad}\, T) \perp (\mathrm{rad}\, F) = T$$

and the metric morphism $F \to E/\mathrm{rad}\, E = E/T$ is an isometry.

Lemma: *If E is non singular, then $\dim V^\perp = \dim E - \dim V$, and $V = V^{\perp\perp}$.*
If moreover V is non singular, $E = V \perp V^\perp$.

Proof: Composing the polarity $E \xrightarrow{\sim} E^*$ with the epimorphism $E^* \to V^*$ we get an exact sequence $0 \longrightarrow V^\perp \longrightarrow E \longrightarrow V^* \longrightarrow 0$ and we conclude.

Corollary: *Let E be non singular. If V is a non singular subspace, there is an isometry $E \to E$, the* ***symmetry*** *with respect to V, which is the identity on V and $-\mathrm{Id}$ on $V^\perp$.*

Definitions: A plane is **hyperbolic** if it is non singular and has some isotropic vector. A **hyperbolic space** is an orthogonal sum of hyperbolic planes. Two isotropic vectors e, e' with $e \cdot e' = 1$ form a **hyperbolic pair**, so that they span a hyperbolic plane $\langle e, e' \rangle$.

Lemma: *Let E be non singular. Any base $e_1, \ldots, e_i$ of a totally isotropic subspace T may be completed so as to obtain i mutually orthogonal hyperbolic pairs; hence T is contained in a hyperbolic space $\langle e_1, e_1' \rangle \perp \ldots \perp \langle e_i, e_i' \rangle$.*

Proof: By induction on i. By the above lemma, there is a vector v orthogonal to $e_2, \ldots, e_i$, and such that $e_1 \cdot v = 1$. If we put $e_1' = v - \frac{v \cdot v}{2} e_1$, we obtain a hyperbolic pair e_1, e_1'.

By induction, in $\langle e_1, e_1' \rangle^\perp$ we have vectors $e_2', \ldots, e_i'$ such that the pairs e_j, e_j' are hyperbolic and mutually orthogonal.

Corollary: *Any hyperbolic plane has some hyperbolic pair, and all hyperbolic spaces of equal dimension are isometric.*

Witt's Theorem: *Let E be non singular. Any isometry $\sigma\colon V \to V'$ between vector subspaces may be extended to an isometry of E.*

Proof: We put $V = \mathrm{rad}\, V \perp F$, and we complete a base of $\mathrm{rad}\, V$ in $F^\perp$ so as to obtain mutually orthogonal hyperbolic pairs, which span a hyperbolic space $H \subseteq F^\perp$.

Now $V' = \mathrm{rad}\, V' \perp F'$, with $F' = \sigma F$, and analogously completing a base of $\mathrm{rad}\, V'$ in $F'^\perp$, so as to obtain a hyperbolic space $H' \subseteq F'^\perp$, we see that we may extend σ to an isometry $H \perp F \to H' \perp F'$; and we may assume that V is non singular.

When $V = V'$, we may extend σ by the identity on $V^\perp$.

If $V = \langle e \rangle$, then $V' = \langle e' \rangle$, where $e' = \sigma(e)$. Since $V + V'$ is a plane not totally isotropic, and the vectors $e' + e$, $e' - e$ are non null and orthogonal, someone is not isotropic.

If so is $e' - e$, the symmetry with respect to $\langle e' - e \rangle^\perp$ transforms e into e', and it extends σ.

If so is $e' + e$, the symmetry with respect to $\langle e' + e \rangle$ extends σ.

When $\dim V > 1$, we decomposes it as an orthogonal sum of two non singular subspaces, $V = F \perp G$, so that $V' = F' \perp G'$. By induction, we may extend $\sigma\colon F \to F'$ to an isometry σ_1 of E; we put $G_1 = \sigma_1(G)$. Since G_1 and G' are contained in $F'^\perp$, the isometry $\sigma\sigma_1^{-1}\colon G_1 \to G'$ may be extended to an isometry of $F'^\perp$ which, extended by the identity on F', defines an isometry σ_2 of E such that the isometry $\sigma_2\sigma_1\colon E \to E$ extends σ.

Corollary: *If E is non singular, all maximal totally isotropic subspaces have equal dimension, named **index** of the metric.*

Corollary: *Let E be non singular. If $E \perp F \simeq E \perp F'$, then $F \simeq F'$.*

Proof: Since $\mathrm{rad}\, F = \mathrm{rad}\,(E \perp F) \simeq \mathrm{rad}\,(E \perp F') = \mathrm{rad}\, F'$, we have that

$$E \perp (F/\mathrm{rad}\, F) \simeq E \perp (F'/\mathrm{rad}\, F'),$$

and, by Witt's theorem, $F/\mathrm{rad}\, F \simeq F'/\mathrm{rad}\, F'$, and $F \simeq F'$.

Theorem: *Any metric vector space E decomposes, uniquely up to isometries, as an orthogonal sum of a totally isotropic space, a hyperbolic space and an elliptic space.*

Proof: If E is non singular, any maximal totally isotropic subspace T is contained in a hyperbolic space H, and we have $E = H \perp H^\perp$, where $H^\perp$ is elliptic, since T is maximal.

The uniqueness follows from the above corollary.

6.2.1 Classification

Theorem: *The dimension and rank classify metrics when k is algebraically closed.*

Proof: The elliptic part has dimension 0 or 1 (according to the rank being even or odd) because if it would have two linearly independent vectors e, v, we may fix $\lambda \in k$ so that

$$(\lambda e + v)^2 = (e^2)\lambda^2 + 2(e \cdot v)\lambda + v^2 = 0.$$

Moreover, all elliptic spaces of dimension 1 are isometric since they have some vector of square $e^2 = 1$: just divide any vector $e \neq 0$ by $\sqrt{e \cdot e}$.

Definition: The **matrix** of a metric S in a base $e_1, \ldots, e_n$ of E is the matrix $A = (a_{ij})$, where $a_{ij} = e_i \cdot e_j$, so that $e \cdot v = X^t AY$, where X and Y are the coordinates of e and v respectively. Hence the matrix in another base is just $B^t AB$, where B is the base change matrix. In particular, in the real case $k = \mathbb{R}$, the sign of the determinant $|A|$ is an invariant.

Remark that the rank of S is just the rank of the matrix A.

Proposition: *In the real case, a metric is a scalar product if and only if all the principal minors are positive,*

$$a_{11} > 0 \ , \ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \ , \ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0 \ , \ldots, \ \begin{vmatrix} a_{11} & \ldots & a_{1r} \\ \ldots & \ldots & \ldots \\ a_{r1} & \ldots & a_{rr} \end{vmatrix} > 0 \ , \ldots, \ |A| > 0.$$

Proof: If S is a scalar product, any principal minor, being the determinant of the matrix of S in a base of $\mathbb{R}e_1 + \ldots + \mathbb{R}e_r$, is positive.

Conversely, if any principal minor is positive, by induction on the dimension we have that S is a scalar product on the hyperplane $H = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{n-1}$, so that H admits an orthonormal base $u_1, \ldots, u_{n-1}$. Now, there is a non null vector $e \in E$ such that $H \cdot e = 0$, because the linear map $E \to \mathbb{R}^{n-1}$, $e \mapsto (u_1 \cdot e, \ldots, u_{n-1} \cdot e)$ is not injective, and $e \notin H$ since S is positive-definite on H. The matrix of S in the base $u_1, \ldots, u_{n-1}, e$ is just $\mathrm{diag}(1, \ldots, 1, a)$, and $a > 0$ because $|A| > 0$. We conclude that S is positive-definite on E.

Lemma: *When $k = \mathbb{R}$, any elliptic space is positive definite or negative definite.*

Proof: If $e^2 > 0$ and $v^2 < 0$, then $(\lambda e + v)^2 = (e^2)\lambda^2 + 2(e \cdot v)\lambda + v^2 = 0$ for some $\lambda \in \mathbb{R}$, since the discriminant of the polynomial is $4(e \cdot v)^2 - 4(e^2)(v^2) > 0$.

Theorem: *If $k = \mathbb{R}$, the dimension, rank, index and sign of the elliptic part classify metrics.*

Proof: The existence of orthonormal bases shows that the dimension classifies elliptic spaces of positive sign; hence also those of negative sign. q.e.d.

In the real case, to calculate the rank, index and sign of a metric S, we fix a scalar product $e \cdot v$ in E, and we put $e * v = S(e, v)$. The scalar product defines an isomorphism $E \simeq E^*$, and the polarity $T\colon E \to E^* = E$ of S is an endomorphism such that $e * v = (Te) \cdot v$, so that T is a symmetric, $(Te) \cdot v = e \cdot (Tv)$, and, according to the Spectral theorem:

The endomorphism T diagonalizes in some orthonormal base of E. Hence all the roots of the characteristic polynomial $c_T(x)$ are real. If n is the degree of $c_T(x)$, r_+ and r_- are the number of positive and negative roots (counted with multiplicities), and r_0 is the multiplicity of the null root, then the rank r, index i, and sign s of S are

$$r = n - r_0, \;\; i = \min(r_+, r_-), \;\; s = \operatorname{sgn}(r_+ - r_-).$$

In fact, let $\alpha_1, \ldots, \alpha_n$ be the roots of c_T and let $e_1, \ldots, e_n$ be an orthonormal base of eigenvectors, $T(e_i) = \alpha_i e_i$. Then $e_i * e_j = (Te_i) \cdot e_j = \alpha_i \delta_{ij}$ and, dividing e_i by $\sqrt{|\alpha_i|}$, when $\alpha_i \neq 0$, the matrix of S is $\operatorname{diag}(0, \overset{r_0}{\ldots}, 0, 1, \overset{r_+}{\ldots}, 1, -1, \overset{r_-}{\ldots}, -1)$. q.e.d.

If A is the matrix of S in a base $e_1, \ldots, e_n$, and we consider the scalar product $e_i \cdot e_j = \delta_{ij}$, then the matrix of the endomorphism T is A, since $e_i \cdot (Te_j) = e_i * e_j = a_{ij}$. The characteristic polynomial $c_T(x) = |xI - A|$ is not an invariant of S; but so are the numbers r_+, r_-, r_0.

Definition: Any non-singular metric on a 1-dimensional vector space is obviously classified by the square $e \cdot e$ of any non null vector, well defined in the group k^*/k^{*2}. Now a non-singular metric S on a vector space E of dimension n defines an isomorphism $E \to E^*$; hence an isomorphism $\Lambda^n E \to \Lambda^n E^* = (\Lambda^n E)^*$, and a non singular metric on $\Lambda^n E$, classified by an element $\operatorname{disc} S \in k^*/k^{*2}$ named **discriminant** of S.

If A is the matrix of S in a base of E, then $\operatorname{disc} S = |A| \in k^*/k^{*2}$.

In general, the discriminant of a metric is defined to be that of the non singular part.

It is clear that $\operatorname{disc}(E \perp E') = (\operatorname{disc} E)(\operatorname{disc} E')$.

Lemma: *Let $\mathbb{F}_q$ be a finite field of characteristic $\neq 2$. If $a, b \in \mathbb{F}_q$ are non null, then the equation $ax^2 + by^2 = 1$ has some solution in $\mathbb{F}_q$.*

Proof: Since $\mathbb{F}_q^2$ has $\frac{q+1}{2}$ elements (p. 73), so does the image of the map $f\colon \mathbb{F}_q \to \mathbb{F}_q$, $f(x) = b^{-1}(1 - ax^2)$. Hence they intersect, and we conclude.

Theorem: *The dimension, rank and discriminant classify metrics when k is a finite field.*

Proof: By induction on the rank r.

When $r = 1$, the discriminant classifies non singular metrics of dimension 1.

If $r \geq 2$, by the above lemma $e \cdot e = 1$ for some $e \in E$.

Hence $E = \langle e \rangle \perp \langle e \rangle^\perp$, and we conclude by induction since $\operatorname{disc} E = \operatorname{disc} \langle e \rangle^\perp$.

Definition: An **alternate** metric is a 2-form $\Omega \in \Lambda^2 E^*$. The above proofs remain valid for alternate metrics; but, since any vector is isotropic, non singular spaces are hyperbolic.

Theorem: *The rank classifies alternate metrics, and it is always even. Any alternate metric of rank* $2r$*, in some base* $\omega_1, \ldots, \omega_n$ *of* E^**, is* $\Omega = \omega_1 \wedge \omega_2 + \ldots + \omega_{2r-1} \wedge \omega_{2r}$.

Definition: Let $\sigma\colon K \to K$ be an involution ($\sigma^2 = \mathrm{Id}$, $\sigma \neq \mathrm{Id}$) of a field K. Then the equality $\alpha = \frac{\alpha+\sigma\alpha}{2} + \frac{\alpha-\sigma\alpha}{2}$ shows that K is an extension of degree 2 of $k := \{\alpha \in K\colon \sigma\alpha = \alpha\}$, and that $K = k \oplus kj$, where $\sigma(j) = -j$. We put $\bar{\alpha} = \sigma\alpha$, and $a = -j^2 \in k$. An **hermitian** metric on a finite-dimensional K-vector space E is a right-linear, left-semilinear (of automorphism σ) map $H\colon E \times E \to K$, such that $H(v,e) = \overline{H(e,v)}$.

The orthogonal of V is $V^\perp = \{e \in E\colon H(e,v) = 0, \forall v \in V\}$, and the radical is $\operatorname{rad} H = E^\perp$.

H defines a symmetric metric S and an alternate metric Ω on the k-vector space E,

$$H(e,v) = S(e,v) + \Omega(e,v)j,$$

Moreover, we have $S(e,v) = \Omega(e,jv)$ and $\Omega(e,v) = a^{-1}S(je,v)$. Hence,

1. The orthogonal $V^\perp$ of a K-vector subspace $V \subseteq E$ is the same for H and S. In particular, $\operatorname{rad} H = \operatorname{rad} S$, and $\operatorname{rk} S = 2\operatorname{rk} H$.

2. $H(e,e) = S(e,e)$, and the isotropic vectors of H are those of S.

3. Any maximal totally isotropic subspace T of H so is for S.

   If $T \oplus ke$ would be totally isotropic for S, then $e \in T^\perp$, and $H(e,e) = S(e,e) = 0$; and $T \oplus Ke$ would be totally isotropic for H.

Theorem: *Two hermitian metrics* H, H' *are equivalent if and only if the associated symmetric metrics* S, S' *are equivalent.*

Proof: If $\tau\colon (E,H) \to (E',H')$ is a K-linear isometry, then $\tau\colon (E,S) \to (E',S')$ is a k-linear isometry, $S(e,e) = H(e,e) = H'(\tau e, \tau e) = S'(\tau e, \tau e)$.

Conversely, if $\tau\colon (E,S) \to (E',S')$ is a k-linear isometry, take $e \in E$ such that $S(e,e) \neq 0$ and put $e' = \tau(e)$, so that $H'(e',e') = S'(e',e') = S(e,e) = H(e,e)$, and $(Ke,H) \simeq (Ke',H')$; hence $(Ke,S) \simeq (Ke',S')$.

(Ke,S) being non singular, we have $((Ke)^\perp, S) \simeq ((Ke')^\perp, S')$ by Witt's theorem.

Now $((Ke)^\perp, H) \simeq ((Ke')^\perp, H')$ by induction on the dimension, and $(E,H) \simeq (E',H')$.

6.2.2 Quadrics

Definition: A map $q\colon E \to k$ is said to be a **quadratic form** when, in the coordinates $(x_1, \ldots, x_n)$ of a base $e_0, \ldots, e_n$ of E (hence of any base) it is defined by some homogeneous polynomial of degree 2; i.e. $q(x_0e_0 + \ldots + x_ne_n) = \sum\limits_{0 \leq i \leq j \leq n} c_{ij}x_ix_j$.

The quadratic form of a metric S is the map $q\colon E \to k$, $q(e) = e^2 = S(e,e)$.

It vanishes on every isotropic vector, and it fully determines the metric,

$$e \cdot v = \tfrac{1}{2}\big(q(e+v) - q(e) - q(v)\big).$$

If $(a_{ij}) = (e_i \cdot e_j)$ is the matrix of S in a base, the quadratic form is just

$$q(x_0, \ldots, x_n) = \sum_{i,j} a_{ij}x_ix_j = \sum_i a_{ii}x_i^2 + \sum_{i<j} 2a_{ij}x_ix_j$$

so that any quadratic form is associated to a unique metric (recall that $\operatorname{car} k \neq 2$).

Definitions: A **quadric** in $\mathbb{P}(E)$ is a non null metric on E, up to a factor, $\mathcal{Q} = \langle S \rangle$. That is to say, quadrics in $\mathbb{P}(E)$ are just points of the projective space $\mathbb{P}(S_2E)$.

Two quadrics $\langle S \rangle$, $\langle S' \rangle$ in $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are **projectively equivalent** if some projectivity $\pi(T)\colon \mathbb{P}(E) \to \mathbb{P}(E')$ transforms one into the other: $S'(Te, Tv) = \lambda S(e, v)$.

Two points are **conjugate** with respect to a quadric if they are represented by orthogonal vectors, and **points** of the quadric are represented by isotropic vectors.

The points in the **vertex** $\pi(\operatorname{rad} S)$ are the **singular** points of the quadric. If $X = \pi(V)$ is not contained in the vertex, then $\mathcal{Q} \cap X$ denotes the quadric defined by the restriction of S to V. The **directrix** of a quadric $\mathcal{Q}$ is the non singular quadric $\mathcal{Q} \cap X$, where X is a linear subvariety defined by some supplement of $\operatorname{rad} E$ in E.

The **polar variety** of $X = \pi(V)$ is $X^\perp = \pi(V^\perp)$, and if a point P is not in the vertex, then $P^\perp$ is the **polar** hyperplane of P.

The **tangent** hyperplane at a non singular point P of the quadric is just its polar hyperplane, and a linear subvariety X is said to be tangent at P when $P \in X \subseteq P^\perp$.

If $\mathcal{Q}$ is non singular, then $\phi\colon E \to E^*$ is an isomorphism, and we have the **dual** quadric $\langle S^* \rangle$ in the dual space $\mathbb{P}(E^*)$, where $S^*(\phi e, \phi v) = S(e, v)$, and points of the dual quadric are tangent hyperplanes of $\mathcal{Q}$.

When we multiply a metric by a non null factor λ, the rank and index do not change, but in the real case the sign changes when λ is negative. Therefore,

Theorem: *If the field k is algebraically closed, the dimension and rank (n, r) projectively classify quadrics, and $1 \leq r \leq n + 1$.*

Conics in $\mathbb{P}_{2,\mathbb{C}}$	Locus	Quadrics in $\mathbb{P}_{3,\mathbb{C}}$	Locus
$x_0^2 = 0$	double line	$x_0^2 = 0$	double plane
$x_0^2 + x_1^2 = 0$	pair of lines	$x_0^2 + x_1^2 = 0$	pair of planes
$x_0^2 + x_1^2 + x_2^2 = 0$	non singular conic	$x_0^2 + x_1^2 + x_2^2 = 0$	cone
		$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$	non singular quadric

Theorem: *If $k = \mathbb{R}$, the dimension, rank and index (n, r, i) projectively classify quadrics, and we have $0 \leq 2i \leq r \leq n + 1$.*

Special Theory of Relativity: This theory assumes that spacetime is a 4-dimensional real affine space and the light cone is a cone, defined by a quadratic form of index 1 (well-defined up to a factor: A **Minkowski spacetime** is a real affine space $\mathbb{A}_4$ whose space of free vectors V is endowed with a 1-dimensional vector space $\mathcal{Q}$ of symmetric metrics of index 1. Let us fix a **Lorentz metric** $g \in \mathcal{Q}$ (i.e. a metric of type $(+, -, -, -)$, named **time metric**, the **proper time** interval between two events p and $q = p + v$ being $\sqrt{g(v, v)}$, whenever $g(v, v) > 0$), and let us fix a metric $h \in \mathcal{Q}$ of type $(-, +, +, +)$, named **space metric**[3], so that we have $h = -c^2 g$ for some positive constant c, the speed of light. Isotropic vectors are **light vectors**.

The other hypothesis is that all inertial observers are equivalent, so that the equation of the light cone must be $\sqrt{x^2 + y^2 + z^2} = ct$ in any inertial reference frame. Hence an **inertial** reference system is defined to be an event $p \in \mathbb{A}_4$ (the origin) and a base e_0, e_1, e_2, e_3 of V such that the matrix of g is $\operatorname{diag}(1, -c^{-2}, -c^{-2}, -c^{-2})$; i.e. the matrix of h is $\operatorname{diag}(-c^{-2}, 1, 1, 1)$. Now any event is $q = p + te_0 + xe_1 + ye_2 + ze_3$, where t is the observed time and the spatial vector $\vec{r} = xe_1 + ye_2 + ze_3$ is the position of q (both relative to the inertial observer), and we say that

[3]The metrics g and h are well defined up to a positive factor. To replace g by $\lambda^2 g$ modifies time intervals by a factor λ, so that to fix the time metric $g \in \mathcal{Q}$ is just to fix the time unit, and to replace h by $\lambda^2 h$ modifies lengths by a factor λ, so that to fix the space metric $h \in \mathcal{Q}$ is just to fix the length unit.

(t, x, y, z) are the **inertial coordinates** of q in the considered inertial reference. The straight line $p + \mathbb{R}e_0$ is the trajectory of the inertial observer (e_0 being the 4-velocity, the spacetime displacement per time unit) and e_1, e_2, e_3 is an orthonormal base of the Euclidean structure that the space metric $h = -c^2 g$ defines on the subspace $(\mathbb{R}e_0)^{\perp} = \langle e_1, e_2, e_3 \rangle$.

Inertial trajectories are straight lines where g is positive-definite, velocities of inertial observers being vectors[4] u such that $g(u, u) = 1$, so that h defines a Euclidean structure on $(\mathbb{R}u)^{\perp}$, and vectors in $(\mathbb{R}u)^{\perp}$ are **spatial** (or simultaneity) vectors for inertial observers of velocity u.

In presence of electromagnetic forces, the theory of relativity assumes the existence of an endomorphism $\tilde{F}\colon V \to V$ such that $\tilde{F}(u)$ is the force acting on a unit charge at rest, measured by an inertial observer of 4-velocity u. Let us consider the 2-covariant tensor $F(e, v) = h\big(\tilde{F}(e), v\big)$. Since any force always is a spatial vector, we have $F(u, u) = 0$, so that F is an alternate tensor, and an **electromagnetic field** is defined to be a 2-form F on a Minkowski spacetime. Once we fix an inertial reference e_0, e_1, e_2, e_3, we have

$$F = E_1 \mathrm{d}t \wedge \mathrm{d}x + E_2 \mathrm{d}t \wedge \mathrm{d}y + E_3 \mathrm{d}t \wedge \mathrm{d}z - \tfrac{1}{c}(B_3 \mathrm{d}x \wedge \mathrm{d}y + B_2 \mathrm{d}z \wedge \mathrm{d}x + B_1 \mathrm{d}y \wedge \mathrm{d}z),$$

where the spatial vectors $\vec{E} = E_1 e_1 + E_2 e_2 + E_3 e_3$ and $\vec{B} = B_1 e_1 + B_2 e_2 + B_3 e_3$ depend on the inertial observer. However, the scalars $\|\vec{E}\|^2 - \|\vec{B}\|^2$ and $\langle \vec{E} | \vec{B} \rangle^2$ do not depend on the inertial reference since the characteristic polynomial of the endomorphism $\tilde{F}$ is

$$x^4 - \tfrac{1}{c^2}(\|\vec{E}\|^2 - \|\vec{B}\|^2)x^2 - \tfrac{1}{c^4}\langle \vec{E} | \vec{B} \rangle^2.$$

Affine Elements of Quadrics: A quadric in an affine space $(\mathbb{P}(E), H)$ is **parabolic** when the infinity H is a tangent hyperplane.

A **center** of a quadric is a point conjugate to any point at infinity.

A line passing through a center is a **diameter**, and it is an **asymptote** if it is tangent.

Two quadrics are **affinely equivalent** if some affinity transforms one into the other.

Lemma: *If (r, i) are the rank and index of a quadric $\mathcal{Q}$, and (r', i') are those of the intersection with the hyperplane at infinity $H = \pi(V)$, there are three possible cases,*

1. *V does not contain $\operatorname{rad} E$. In this case $\operatorname{rad} V = V \cap \operatorname{rad} E$, and $r' = r$, $i' = i$.*

2. *$\operatorname{rad} E = \operatorname{rad} V$. In this case $r' = r - 1$, and $i' = i, i - 1$ (according to V contains or not a maximal totally isotropic subspace of E).*

3. *$\operatorname{rad} E$ is a hyperplane of $\operatorname{rad} V$. In this case $r' = r - 2$, $i' = i - 1$.*

Proof: If the hyperplane V does not contain $\operatorname{rad} E$, then $E = V \perp \langle e \rangle$, $e \in \operatorname{rad} E$; so that $\operatorname{rad} V = V \cap \operatorname{rad} E$, since any vector in V is orthogonal to e, and we have an isometry $V/\operatorname{rad} V = E/\operatorname{rad} E$; hence $r' = r$, $i' = i$. This is case 1.

If $\operatorname{rad} V = \operatorname{rad} E$, then $V/\operatorname{rad} V$ is a non singular hyperplane of $E/\operatorname{rad} E$; hence $r' = r - 1$, and $i' = i$, $i - 1$, according to $V/\operatorname{rad} V$ contains or not a maximal totally isotropic subspace of $E/\operatorname{rad} E$. This is case 2.

If $\operatorname{rad} E \subset \operatorname{rad} V$, projecting to $E/\operatorname{rad} E$ we may assume $\operatorname{rad} E = 0$. Then $\dim V^{\perp} = 1$.

Since $\operatorname{rad} V = V^{\perp} \cap V \neq 0$, we have $\dim \operatorname{rad} V = 1$ (hence $r' = r - 2$), and $V = V^{\perp\perp}$ is the orthogonal of an isotropic vector: $i' = i - 1$. This is case 3.

[4] In fact, the Special Theory of Relativity always assume that time is oriented; that the 4-velocity u of any punctual body always is in a fixed connected component of the hyperboloid $g(u, u) = 1$.

Lemma: *If the intersections $\mathcal{Q}\cap H_1$, $\mathcal{Q}\cap H_2$ of a quadric with two hyperplanes are projectively equivalent, there is a projectivity τ such that $\tau(\mathcal{Q})=\mathcal{Q}$, $\tau(H_1)=H_2$.*

Proof: We put $H_i=\pi(V_i)$, and we distinguish the three possible cases.

1. We put $\operatorname{rad} E=\operatorname{rad} V_i\perp T_i$, and $V_i=(\operatorname{rad} V_i)\perp F_i$. We have

$$E=\operatorname{rad} V_1\perp T_1\perp F_1=\operatorname{rad} V_2\perp T_2\perp F_2.$$

By hypothesis we have an isometry $(V_1,\lambda S)\to(V_2,S)$ and, T_i being totally isotropic, it may be extended to an isometry $T\colon (E,\lambda S)\to(E,S)$, and $T(V_1)=V_2$.

2. We fix a supplement $\bar{E}$ of $\operatorname{rad} E$, and we put $F_i=V_i\cap\bar{E}$, $\bar{E}=F_i\perp\langle e_i\rangle$. We have that $V_i=\operatorname{rad} E\perp F_i$, and

$$E=\operatorname{rad} E\perp F_1\perp\langle e_1\rangle=\operatorname{rad} E\perp F_2\perp\langle e_2\rangle$$

where $a_1=e_1\cdot e_1$, $a_2=e_2\cdot e_2$ are non null. By hypothesis $(F_1,\lambda S)$ and (F_2,S) are isometric; hence, if $m=\dim F_1$, in the group k^*/k^{*2} we have

$$a_1\operatorname{disc} F_1=\operatorname{disc}\bar{E}=a_2\operatorname{disc} F_2=a_2\lambda^m\operatorname{disc} F_1$$

and $a_1=\lambda^m a_2$ in k^*/k^{*2}. If m is even, we have an isometry $\langle e_1\rangle\simeq\langle e_2\rangle$ and, by Witt's theorem, it may be extended to an isometry $\bar{T}\colon\bar{E}\to\bar{E}$ transforming $\langle e_1\rangle^{\perp}=F_1$ into $\langle e_2\rangle^{\perp}=F_2$, and it may be extended by the identity $T\colon\operatorname{rad} E\perp\bar{E}\to\operatorname{rad} E\perp\bar{E}$ so as to obtain an isometry transforming V_1 into V_2.

If m is odd, $a_2=\lambda a_1$ in k^*/k^{*2}, and $(\langle e_1\rangle,\lambda S)\simeq(\langle e_2\rangle,S)$. Since $(F_1,\lambda S)\simeq(F_2,S)$, we obtain an isometry $T\colon(E,\lambda S)\to(E,S)$, and $T(V_1)=V_2$.

3. We put $\operatorname{rad} V_i=\operatorname{rad} E\perp\langle e_i\rangle$, and let $\bar{E}_i$ be a supplement of $\operatorname{rad} E$ containing $\langle e_i\rangle$. The orthogonal of $\langle e_i\rangle$ in $\bar{E}_i$ is just $V_i\cap\bar{E}_i$. Now, $\bar{E}_1$ and $\bar{E}_2$ are isometric and non singular, hence by Witt's theorem the isometry $\langle e_1\rangle\simeq\langle e_2\rangle$ may be extended to an isometry $\bar{E}_1\to\bar{E}_2$, transforming $V_1\cap\bar{E}_1$ into $V_2\cap\bar{E}_2$. We obtain an isometry $T\colon\operatorname{rad} E\perp\bar{E}_1\to\operatorname{rad} E\perp\bar{E}_2$, and $T(V_1)=V_2$.

Theorem: *Two quadrics are affinely equivalent if and only if both, and their intersections with the hyperplane at infinity, are projectively equivalent.*

Proof: If $\mathcal{Q}$ and $\mathcal{Q}'$ are projectively equivalent, $\mathcal{Q}'=\tau(\mathcal{Q})$ for some projectivity τ, then $\mathcal{Q}\cap H$ and $\mathcal{Q}'\cap\tau(H)$ are projectively equivalent. If moreover $\mathcal{Q}\cap H$ and $\mathcal{Q}'\cap H$ are projectively equivalent, so are $\mathcal{Q}'\cap\tau(H)$ and $\mathcal{Q}'\cap H$. By the above lemma, $\sigma(\mathcal{Q}')=\mathcal{Q}'$, and $\sigma(\tau H)=H$ for some projectivity σ. Hence $\mathcal{Q}'=\sigma\tau\mathcal{Q}$, and $\sigma\tau(H)=H$.

Euclidean Geometry: An Euclidean structure on a real affine space $\mathbb{A}_n$ is given by a scalar product Ω on the space V of free vectors, well defined up to a positive factor; i.e. a non singular quadric $\langle\Omega\rangle$ of index 0 at infinity $H=\pi(V)$, named **absolute**: *classical Euclidean geometry is the projective geometry of a fixed hyperplane with a fixed imaginary non-singular quadric.* An affine reference system $(P_0,\ldots,P_n;U)$ is said to be **Euclidean** if there is a normalized base $e_0,\ldots,e_n$ such that $e_1,\ldots,e_n$ is an orthonormal base, so that the absolute is $x_1^2+\ldots+x_n^2=0$.

An affinity $\tau\colon\mathbb{P}_n\to\mathbb{P}_n$ is a **similarity** if the absolute is invariant, $\tau\Omega=\rho^2\Omega$ (where τ denotes the normalized linear representant, inducing the identity on E/V) for some positive real number ρ, the **similarity ratio** of τ. **Motions** are similarities of ratio 1.

Lemma: *Let Q_1, Q_2 be the points where a line $P_1 + P_2$ intersects a quadric $\mathcal{Q} = \langle S \rangle$. If ϕ is a complex number*[5] *such that $\cos\phi = \frac{e_1 \cdot e_2}{\sqrt{e_1^2 e_2^2}}$, where $P_i = \pi(e_i)$, then $(P_1, P_2; Q_1, Q_2) = e^{2\phi i}$.*

Proof: We have $Q_i = \pi(\alpha_i e_1 + e_2)$, where α_1, α_2 are the roots of $0 = e_1^2 t^2 + 2(e_1 \cdot e_2)t + e_2^2$.

$$(P_1, P_2; Q_1, Q_2) = \frac{\alpha_2}{\alpha_1} = \frac{e_1 \cdot e_2 + \sqrt{(e_1 \cdot e_2)^2 - e_1^2 e_2^2}}{e_1 \cdot e_2 - \sqrt{(e_1 \cdot e_2)^2 - e_1^2 e_2^2}} = \frac{\frac{e_1 \cdot e_2}{\sqrt{e_1^2 e_2^2}} + \sqrt{\frac{(e_1 \cdot e_2)^2}{e_1^2 e_2^2} - 1}}{\frac{e_1 \cdot e_2}{\sqrt{e_1^2 e_2^2}} - \sqrt{\frac{(e_1 \cdot e_2)^2}{e_1^2 e_2^2} - 1}}$$
$$= \frac{\cos\phi + \sqrt{\cos^2\phi - 1}}{\cos\phi - \sqrt{\cos^2\phi - 1}} = \frac{\cos\phi + i\sin\phi}{\cos\phi - i\sin\phi} = \frac{e^{\phi i}}{e^{-\phi i}} = e^{2\phi i}.$$

Definition: The point at infinity of a line being represented by the direction of the line, the **angle** determined by two intersecting lines r_1, r_2 may be defined[6] to be $\phi = |\frac{1}{2i} \ln (r_1, r_2; i, j)|$, where i, j are the self-conjugate lines with respect to the absolute in the pencil determined by r_1 and r_2. The lines are **perpendicular** when $(r_1, r_2; i, j) = -1$; i.e., $\phi = \pi/2$.

Now let $\mathcal{Q}$ be a quadric of rank r in an Euclidean space.

Quadrics differing in a motion are considered to be equal.

Since any symmetric endomorphism diagonalizes in an orthonormal base (p. 168) there are Euclidean references where the intersection $\mathcal{Q} \cap H$ with the hyperplane at infinity is

$$a_1 x_1^2 + \ldots + a_{r'} x_{r'}^2 = 0,$$

and we shall place the origin P_0 so that the equation of the quadric becomes simple.

We distinguish the three possible cases (p. 171) and we assume that $\operatorname{rad} E = 0$ (if $\operatorname{rad} E \neq 0$, the equation is the same, just put $E = \bar{E} \perp \operatorname{rad} E$):

1. The vertex has some proper point P_0, and the quadric is $a_1 y_1^2 + \ldots + a_r y_r^2 = 0$.
2. The pole of the infinity is a center P_0, and the quadric is $a_1 y_1^2 + \ldots + a_{r-1} y_{r-1}^2 = 1$.
3. The quadric is a paraboloid tangent to H at a point, say P_1 (hence $a_1 = 0$). The polar variety of $P_2 + \ldots + P_n$ is a line that passing through P_1 and intersecting the quadric at a proper point P_0. The quadric is $y_1 = a_2 y_2^2 + \ldots + a_{r-1} y_{r-1}^2$.

Metric Elements of Conics: Given a non singular conic in an Euclidean plane, an **axis** is a line conjugate to the perpendicular lines, and the **vertices** are the intersection points with the axes. A **focus** is any point where conjugate lines are just perpendicular lines (and the corresponding polar line is said to be a **directrix**), so that the focuses of a conic $\langle S \rangle$ are defined by the singular conics of the focal series $\langle S^* + \lambda \Omega^* \rangle$, where $\langle S^* \rangle$ is the dual conic and $\langle \Omega^* \rangle$ is the absolute, viewed as a pair of imaginary lines in the dual plane.

Non Euclidean Geometries: In the Euclidean geometry, the absolute defines a singular imaginary quadric $\langle \Omega^* \rangle$ in the dual space $\mathbb{P}_n^*$, with vertex at a point, the hyperplane at infinity. Perpendicularity is just conjugation with respect to $\langle \Omega^* \rangle$, proper points are points defining hyperplanes in $\mathbb{P}_n^*$ not intersecting $\langle \Omega^* \rangle$ at real points, and proper lines are lines passing through

[5] Well-defined up to a sign and an integer multiple of π, and ϕ is real when Q_1, Q_2 are conjugate complex points, $(e_1 \cdot e_2)^2 < e_1^2 e_2^2$. Since $\cos di = \cosh d = \frac{1}{2}(e^d + e^{-d})$, it may be fixed to be a pure imaginary $\phi = di$ when Q_1, Q_2 are real points, $(e_1 \cdot e_2)^2 \geq e_1^2 e_2^2$, not separating P_1 and P_2 (i.e., e_1^2 and e_2^2 have equal sign).

[6] It is well defined up to an integer multiple of π, and a sign, since we may interchange i and j. Hence, in an Euclidean plane, up to an integer multiple of π if we fix an imaginary point I of the absolute.

some proper point. Non Euclidean geometries are obtained by fixing the absolute to be an arbitrary non singular quadric in $\mathbb{P}_n$ (non ruled, so that proper points exist).

Hyperbolic Geometry: The absolute $\langle\Omega^*\rangle$ is a real quadric of index 1, hence the dual of a real quadric $\mathcal{Q} = \langle\Omega\rangle$ in $\mathbb{P}_n$. Proper points of this geometry are interior points (without real tangents) of $\mathcal{Q}$, and points at infinity are those of $\mathcal{Q}$. Proper lines are lines intersecting $\mathcal{Q}$ at two points, and the angle determined by two intersecting lines r_1, r_2 is $\phi = |\frac{1}{2i}\ln(r_1, r_2; i, j)|$, where i, j are the self-conjugate lines in the pencil determined by r_1 and r_2, which are imaginary conjugate lines. In this geometry *the distance is an absolute concept*!

$$\mathrm{d}(P_1, P_2) = \left|\tfrac{1}{2}\ln(P_1, P_2; Q_1, Q_2)\right| = \operatorname{arc\,cosh}\left(\frac{|e_1 \cdot e_2|}{\sqrt{e_1^2 e_2^2}}\right)$$

where $P_i = \pi(e_i)$ and the line $P_1 + P_2$ intersects $\mathcal{Q}$ at the points Q_1, Q_2 (the cross-ratio being positive because Q_1, Q_2 do not separate P_1, P_2). Straight lines have infinite length; but this geometry violates Euclid's fifth postulate: by an exterior point pass infinite parallel lines.

Hence, *the fifth postulate is independent of the other four postulates.*

Elliptic Geometry: The absolute $\langle\Omega^*\rangle$ is an imaginary non singular quadric, dual to an imaginary quadric $\mathcal{Q} = \langle\Omega\rangle$ in $\mathbb{P}_n$. All points are proper, hence there are no parallel lines.

Angles are defined to be $\phi = |\frac{1}{2i}\ln(r_1, r_2; i, j)|$, and we also have an absolute distance (but now Q_1, Q_2 are imaginary conjugate points and straight lines have finite length π):

$$\mathrm{d}(P_1, P_2) = \left|\tfrac{1}{2i}\ln(P_1, P_2; Q_1, Q_2)\right| = \arccos\left(\frac{|e_1 \cdot e_2|}{|e_1| \cdot |e_2|}\right).$$

EUCLIDEAN, AFFINE AND PROJECTIVE CLASSIFICATION OF CONICS

(r',i') \ (r,i)	(1,0)	(2,0)	(2,1)	(3,0)	(3,1)
(0,0)	$\emptyset$		line $y = 0$		
(1,0)	double line $y^2 = 0$	pair of parallel imaginary lines $y^2 = -a^2$ $a > 0$	pair of parallel real lines $y^2 = a^2$ $a > 0$		parabola $y = \dfrac{x^2}{a}$ $a > 0$
(2,0)		pair of intersecting imaginary lines $x^2 + a^2y^2 = 0$ $1 \geq a > 0$		imaginary ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1$ $a \geq b > 0$	ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $a \geq b > 0$
(2,1)			pair of intersecting real lines $x^2 - a^2y^2 = 0$ $1 \geq a > 0$		hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $a > 0, b > 0$

Conics in the same column are projectively equivalent.
Conics in the same cell are affinely equivalent.
Parameter values in a cell determine an euclidean equivalence class.

EUCLIDEAN, AFFINE AND PROJECTIVE CLASSIFICATION OF CUADRICS

(r',i') \ (r,i)	(1,0)	(2,0)	(2,1)	(3,0)	(3,1)	(4,0)	(4,1)	(4,2)
(0,0)	$\emptyset$		**plane** $z=0$					
(1,0)	**double plane** $z^2=0$	**pair of parallel imaginary planes** $z^2=-a^2$ $a>0$	**pair of parallel real planes** $z^2=a^2$ $a>0$		**parabolic cilinder** $y=\frac{x^2}{a}$ $a>0$			
(2,0)		**pair of intersecting imaginary planes** $x^2+a^2y^2=0$ $1\geq a>0$		**imaginary cilinder** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$ $a\geq b>0$	**elliptic cilinder** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $a\geq b>0$		**non-ruled paraboloid** $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ $a\geq b>0$	
(2,1)			**pair of intersecting real planes** $x^2-a^2y^2=0$ $1\geq a>0$		**hyperbolic cilinder** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $a>0, b>0$			**ruled paraboloid** $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ $a>0, b>0$
(3,0)				**imaginary cone** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2=0$ $1\geq a\geq b>0$		**imaginary ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ $a\geq b\geq c>0$	**elipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $a\geq b\geq c>0$	
(3,1)					**cone** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=0$ $a\geq b>0$		**non-ruled hyperboloid** $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $a>0, b\geq c>0$	**ruled hyperboloid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ $a\geq b>0, c>0$

Cuadrics in the same column are projectively equivalent.
Cuadrics in the same cell are affinely equivalent.
Parameter values in a cell determine an euclidean equivalence class.

6.3 Modules over a Principal Ideal Domain

Let Σ be the field of fractions of a principal ideal domain A.

Let M be an A-module, and $M_\Sigma = M \otimes_A \Sigma$ the localization of M by $S = A - \{0\}$.

Definition: The **rank** of M is $\operatorname{rk} M = \dim_\Sigma M_\Sigma$. In particular $\operatorname{rk} A^r = r$.

If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\operatorname{rk} M = \operatorname{rk} M' + \operatorname{rk} M''$.

Lemma: *Any submodule N of a free A-module L of rank r is free, of rank $\leq r$.*

Proof: By induction on r. If $r = 1$, then $N \simeq aA \simeq A$ or 0.

If $r > 1$, then L is a direct sum of two non null free modules, $L = L' \oplus L''$, and we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L & \xrightarrow{\pi} & L'' & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & N' = N \cap L' & \longrightarrow & N & \xrightarrow{\pi} & N'' = \pi(N) & \longrightarrow & 0 \end{array}$$

By induction, N' and N'' are free, and the sum of ranks is $\leq r$.

Since N'' is free, the bottom exact sequence splits and $N \simeq N' \oplus N''$ is free, of rank $\leq r$.

Corollary: *Any submodule of a finitely generated A-module is finitely generated.*

Proof: Any finitely generated module is a quotient of a free module A^r.

Definitions: The kernel of the localization morphism $M \to M_\Sigma = M \otimes_A \Sigma$ is the **torsion submodule** $T(M) = \{m \in M : am = 0, \text{ for some non null } a \in A\}$.

M is a **torsion** module if $M = T(M)$, and M is **torsion free** if $T(M) = 0$.

Lemma: *Any finitely generated torsion free A-module M is free.*

Proof: Let $m_1, \ldots, m_n$ be a generating system of M, where $m_1, \ldots, m_r$ are linearly independent and $a_i m_i \in Am_1 + \ldots + Am_r$, with $a_i \neq 0$. Let $b = a_{r+1} \ldots a_n \neq 0$.

Since bM is a submodule of the free module $Am_1 + \ldots + Am_r$, it is free.

If M is torsion free, $M \to bM$, $m \mapsto bm$, is an isomorphism, and we conclude.

Theorem: *Any finitely generated A-module decomposes, uniquely up to isomorphisms, as a direct sum of a free module and a torsion module*

$$M \simeq A^r \oplus T(M), \ \ r = \operatorname{rk} M.$$

Proof: $M/T(M)$ is torsion free; hence it is free and the following exact sequence splits,

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow M/T(M) \longrightarrow 0$$
$$M \simeq (M/T(M)) \oplus T(M).$$

Now, if $M \simeq L \oplus T$, where L is free and T is torsion free, localizing we have that $M_\Sigma \simeq L_\Sigma$, and L is free of rank $r = \operatorname{rk} M$. Moreover,

$$T(M) \simeq T(L) \oplus T(T) = 0 \oplus T = T.$$

Definition: The **annihilator ideal** of an A-module M is $\operatorname{Ann} M = \{b \in A : \ bM = 0\}$.

When A is a principal ideal domain, the generator a is the **annihilator** of M, and it is well defined up to invertible factors. We put $\operatorname{Ker} b = \{m \in M : bm = 0\}$, so that $\operatorname{Ker} a = M$.

Torsion finitely generated A-modules have non null annihilator a, and finite length, since $l(A/aA) < \infty$. Moreover, $\operatorname{Ann}(M_1 \oplus M_2) = \operatorname{Ann}(M_1) \cap \operatorname{Ann}(M_2)$.

Lemma: $\operatorname{Ker} pq = \operatorname{Ker} p \oplus \operatorname{Ker} q$, *when p and q are coprime.*

Proof: (See p. 82). By Bézout's identity, $1 = \lambda p + \mu q$, and for all $m \in M$,

$$m = \lambda pm + \mu qm.$$

If $m \in \operatorname{Ker} pq$, then $\lambda pm \in \operatorname{Ker} q$, and $\mu qm \in \operatorname{Ker} p$; hence $\operatorname{Ker} pq = \operatorname{Ker} p + \operatorname{Ker} q$.
If $m \in \operatorname{Ker} p \cap \operatorname{Ker} q$, then $m = \lambda pm + \mu qm = 0 + 0 = 0$.

First Decomposition Theorem: *Let $p_1^{n_1} \cdots p_s^{n_s}$ be the irreducible factor decomposition of the annihilator of a finitely generated torsion A-module M. Then M decomposes uniquely as a direct sum of submodules annihilated by $p_i^{n_i}$,*

$$M = \operatorname{Ker} p_1^{n_1} \oplus \ldots \oplus \operatorname{Ker} p_s^{n_s}.$$

Proof: Since $M = \operatorname{Ker}(p_1^{n_1} \cdots p_s^{n_s})$, the existence follows from the above lemma.
Now, if $M = M_1 \oplus \ldots \oplus M_s$, where $p_i^{n_i} M_i = 0$, then $M_i \subseteq \operatorname{Ker} p_i^{n_i}$.
If some inclusion is strict, so is $M = \oplus_i M_i \subset \oplus_i \operatorname{Ker} p_i^{n_i} = M$. Absurd.

Definition: A finitely generated A-module is **primary** if it is annihilated by some power of an irreducible element $p \in A$. Modules isomorphic to A/p^nA are named primary monogenous.

Second Decomposition Theorem: *Any primary finitely generated A-module M decomposes, uniquely up to isomorphisms, as a direct sum of primary monogenous modules*

$$M \simeq (A/p^{n_1}A) \oplus \ldots \oplus (A/p^{n_s}A), \; n_1 \geq \cdots \geq n_s.$$

Proof: We consider a minimal generating system $m_1, \ldots, m_s$ of M, and we proceed by induction on s, since theorem is obvious when M is monogenous.

If the annihilator of M is p^n, some generator, say m_1, is not annihilated by p^{n-1}, and we have an exact sequence of modules over the ring $B = A/p^nA$,

$$0 \longrightarrow B = Am_1 \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0.$$

It splits, since B is an injective B-module (p. 127); hence $M \simeq A/p^nA \oplus \bar{M}$, where $\bar{M}$ is generated by $\bar{m}_2, \ldots, \bar{m}_s$, and we obtain the existence by induction.

Finally, put $k = A/pA$. We have $(p^iA/p^jA) \otimes_A k \simeq k$ when $0 \leq i < j$, so that the number ν_j of terms A/p^jA in a decomposition of M does not depend on the decomposition

$$\begin{aligned} &\dim_k(M \otimes k) = \nu_1 + \ldots + \nu_n, \\ &\dim_k((pM) \otimes_A k) = \nu_2 + \ldots + \nu_n, \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\dim_k((p^{n-1}M) \otimes_A k) = \nu_n. \end{aligned}$$

6.3.1 Classification

Let M be a finitely generated A-module. By the above theorem, there are irreducible elements $p_i \in A$ and powers $p_i^{n_{ij}}$, well defined up to units and named **elementary divisors** of M, such that M decomposes as a direct sum

$$M \simeq (A \oplus \overset{r}{\dots} \oplus A) \oplus \Big(\bigoplus_{i,j} A/p_i^{n_{ij}} A \Big) , \qquad r = \operatorname{rk} M.$$

Classification Theorem: *The rank and elementary divisors classify finitely generated modules over a principal ideal domain A.*

Definitions: If M is a module over an arbitrary ring A, the **tensor algebra** of M

$$T^\bullet_A M = \bigoplus\nolimits_n M^{\otimes n} = A \oplus M \oplus (M \otimes_A M) \oplus \dots$$

is a (non commutative) algebra with a canonical morphism $M \to T^\bullet M$ such that any morphism of A-modules $M \to B$ into a (not necessarily commutative) A-algebra B uniquely factors trough a morphism of A-algebras $T^\bullet M \to B$.

$$\operatorname{Hom}_A(M, B) = \operatorname{Hom}_{A\text{-alg}}(T^\bullet_A M, B).$$

The **symmetric algebra** $S^\bullet_A M$ of M is the quotient of $T^\bullet_A M$ by the two sided ideal generated by the elements $m \otimes m' - m' \otimes m$, while the **exterior algebra** $\Lambda^\bullet_A M$ of M is the quotient of $T^\bullet_A M$ by the two sided ideal generated by the elements $m \otimes m$,

$$S^\bullet_A M = \bigoplus\nolimits_n S^n M = (T^\bullet_A M)/(m \otimes m' - m' \otimes m),$$
$$\Lambda^\bullet_A M = \bigoplus\nolimits_n \Lambda^n M = (T^\bullet M)/(m \otimes m).$$

Both commute with base changes, $(S^\bullet M) \otimes_A B = S^\bullet(M_B)$ and $(\Lambda^\bullet M) \otimes_A B = \Lambda^\bullet(M_B)$, since so do tensor products; hence they commute with localizations.

Moreover, $\Lambda^0 M = A$ and $\Lambda^1 M = M$ since the ideal $(m \otimes m)$ has no element of degree ≤ 1, and $S^\bullet_A M$ is a commutative algebra while $\Lambda^\bullet_A M$ is an **anticommutative** algebra,

$$a_n b_m = (-1)^{nm} b_m a_n \ ; \quad a_n \in \Lambda^n M,\, b_m \in \Lambda^m M.$$

We have a canonical morphism $M \to S^\bullet M$ such that any A-linear map $M \to B$ into a commutative A-algebra B uniquely factors trough a morphism of A-algebras $S^\bullet M \to B$,

$$\operatorname{Hom}_A(M, B) = \operatorname{Hom}_{A\text{-alg}}(S^\bullet_A M, B).$$

Proposition: $\Lambda^\bullet(M \oplus N) = (\Lambda^\bullet M) \otimes_A (\Lambda^\bullet N)$, *where the algebra structure of the right term is defined by the product* $(a_n \otimes b_m)(a_r \otimes b_s) = (-1)^{mr} a_n a_r \otimes b_m b_s$.

Proof: The natural morphism $M \oplus N \to (\Lambda^\bullet M) \otimes (\Lambda^\bullet N)$ induces a morphism of algebras $T^\bullet(M \oplus N) \to (\Lambda^\bullet M) \otimes (\Lambda^\bullet N)$, and it factors through $\Lambda^\bullet(M \oplus N)$.

On the other hand, the natural morphisms $\Lambda^\bullet M \to \Lambda^\bullet(M \oplus N)$, $\Lambda^\bullet N \to \Lambda^\bullet(M \oplus N)$ induce the inverse morphism $(\Lambda^\bullet M) \otimes (\Lambda^\bullet N) \to \Lambda^\bullet(M \oplus N)$.

Corollary: *If L is a free module of rank n, then $\Lambda^p L$ is a free module of rank $\binom{n}{p}$. Hence, if M is monogenous, $\Lambda^p M = 0$ for all $p > 1$.*

Definition: The **invariant factors** of a finitely generated module M over a principal ideal domain A are the annihilators ϕ_j (well defined up to units) of $\Lambda^j M$, $j \geq 1$.

Theorem: *The invariant factors classify finitely generated modules over a principal ideal domain A. If r is the rank of M, and $p_i^{n_{ij}}$ are the elementary divisors ($n_{i1} \geq n_{i2} \geq \ldots$), then*

$$\phi_1 = \ldots = \phi_r = 0,\ \phi_{r+j} = p_1^{n_{1j}} p_2^{n_{2j}} \ldots$$
$$M \simeq A/\phi_1 A \oplus \ldots \oplus A/\phi_d A$$

Proof: $\Lambda^\bullet(N_1 \oplus N_2) = (\Lambda^\bullet N_1) \otimes_A (\Lambda^\bullet N_2)$, and $\Lambda^\bullet N = A \oplus N$ when N is monogenous.

Since $A/I \otimes_A A/J = A/(I+J)$, the annihilator $\phi_j A$ of $\Lambda^j M$ is the intersection of all sums of j ideals of the family $0, \overset{r}{\ldots}, 0, (p_i^{n_{ij}})$, so that $\phi_1 = \ldots = \phi_r = 0$, and $\phi_{r+j} A = \prod_i p_i^{n_{ij}} A$.

The chinese remainder theorem shows that $M \simeq A/\phi_1 A \oplus \ldots \oplus A/\phi_d A$.

Definition: Let us consider a presentation $L'_n \xrightarrow{f} L_m \xrightarrow{\pi} M \longrightarrow 0$, where L'_n and L_m are free modules of ranks n and m. The image of the morphism

$$\Lambda^{m-i} L' \otimes (\Lambda^{m-i} L)^* \xrightarrow{\Lambda f \otimes 1} \Lambda^{m-i} L \otimes (\Lambda^{m-i} L)^* \longrightarrow A$$

is the i-th **Fitting ideal** $F_i(M) = c_i A$. It is generated by all minors of order $m-i$ of the matrix of f (we agree that $F_i = 0$ when $m - i > n$, and $F_i = A$ when $m - i < 1$).

Proposition: $c_i = \phi_{i+1} \ldots \phi_d$, *so that* $\phi_i = \frac{c_{i-1}}{c_i}$.

Proof: Localizing at a point of $\operatorname{Spec} A$ we may assume that A is a local ring.

Let $k = A/pA$ be the residue field. Now a decomposition

$$M = A/\phi_1 A \oplus \ldots \oplus A/\phi_d A = A \oplus \overset{r}{\ldots} \oplus A \oplus A/\phi_{r+1} \oplus \ldots \oplus A/\phi_d A$$

defines generators $m_1, \ldots, m_d$ of M, and we consider the exact sequence

$$L'_n \otimes_A k \xrightarrow{f \otimes 1} L_m \otimes_A k \xrightarrow{\pi \otimes 1} M \otimes_A k = k^d \longrightarrow 0.$$

By Nakayama's lemma, there is a base $e_1, \ldots, e_m$ of L such that $\pi(e_i) = m_i$, $i \leq d$.

If $\pi(e_j) = \sum_i a_{ij} m_i$, $j > d$, replacing e_j by $e_j - \sum_i a_{ij} e_i$ we may assume that $\pi(e_j) = 0$, so that $\phi_{r+1} e_{r+1}, \ldots, \phi_d e_d, e_{d+1}, \ldots, e_m$ is base of $\operatorname{Ker} \pi$.

Using the same argument with L', we obtain bases where any coefficient of the matrix of f is null, except some diagonal coefficients $\phi_{r+1}, \ldots, \phi_d, 1, \overset{m-d}{\ldots}, 1$.

Now the theorem is obvious.

Classification of Abelian Groups: *The invariant factors classify finitely generated abelian groups. Any finitely generated abelian group is a direct sum of infinite cyclic groups and cyclic groups of order some powers of prime numbers.*

Corollary: *Any finite abelian group of order n has subgroups of any order dividing n.*

Proof: In fact, $p^{n-i}\mathbb{Z}/p^n\mathbb{Z}$ is a subgroup of $\mathbb{Z}/p^n\mathbb{Z}$ of order p^i.

Corollary: *A system of diophantine linear equations $AX = B$ is compatible if and only if A and $(A|B)$ have equal rank r, and equal greatest common divisor of the minors of order r.*

Proof: Let us consider the morphism $f\colon \mathbb{Z}^n \to \mathbb{Z}^m$, $f(X) = AX$, and put $r = \operatorname{rk} A$.

The rank of $M = \mathbb{Z}^m/\operatorname{Im} f$ is $m - r$, and the order of $T(M)$ is $c_{m-r}(A)$.

If the system is compatible, then $M = M/\mathbb{Z}B$, and $c_i(A) = c_i(A|B)$ for any index i.

If it has no integer solution, then $B \neq 0$ in M.

If $B \notin T(M)$, then $\mathrm{rk}\,(M/\mathbb{Z}B) = \mathrm{rk}\,(M) - 1$, and $\mathrm{rk}\,(A|B) = r + 1$.
Otherwise $T(M/\mathbb{Z}B) \subset T(M)$, and $c_{m-r}(A|B) < c_{m-r}(A)$.

Definition: Let T be an endomorphism of a k-vector space E of finite dimension n. We put

$$p(x) \cdot e = p(T)(e)$$

where $p(x) \in k[x]$, $e \in E$. This $k[x]$-module E_T is a finite torsion $k[x]$-module since $\dim_k E < \infty$, and submodules $V \subseteq E$ are just invariant vector subspaces, $T(V) \subseteq V$.

Two endomorphisms T, T' of E are **equivalent** if the corresponding modules are isomorphic, $E_T \simeq E_{T'}$; i.e., if there is a linear automorphism τ of E such that

$$T' = \tau \circ T \circ \tau^{-1}.$$

The invariant factors of T are defined to be those of the $k[x]$-module E_T, and we may calculate them with a matrix A of T in a base $e_1, \ldots, e_n$ of E. In fact the exact sequence

$$0 \longrightarrow k[x,y] \xrightarrow{(x-y)\cdot} k[x,y] \longrightarrow k[x] \longrightarrow 0$$
$$0 \longrightarrow k[x] \otimes_k k[x] \xrightarrow{x\otimes 1 - 1\otimes x} k[x] \otimes_k k[x] \longrightarrow k[x] \longrightarrow 0$$

splits because $k[x]$ is free, and applying $(-) \otimes_{k[x]} E_T$ we obtain an exact sequence

$$0 \longrightarrow k[x] \otimes_k E \xrightarrow{x\otimes 1 - 1\otimes T} k[x] \otimes_k E \longrightarrow E_T \longrightarrow 0$$

where $k[x] \otimes_k E$ is a free $k[x]$-module of base $(1 \otimes e_1, \ldots, 1 \otimes e_n)$, and the matrix of $x \otimes 1 - 1 \otimes T$ is just the characteristic matrix $xI - A$ (and the **characteristic polynomial** of T may be intrisically defined to be the determinant of the endomorphism $x \otimes 1 - 1 \otimes T$ of the free $k[x]$-module $k[x] \otimes_k E$ of rank n).

Classification of Endomorphisms: *If c_i is the greatest common divisor of all minors of order $n - i$ of $xI - A$, then the invariant factors of T are $\phi_i = \frac{c_{i-1}}{c_i}$.*

Corollary: *The characteristic polynomial $c_0 = |xI - A|$ is the product of the invariant factors, $c_0 = \phi_1 \ldots \phi_d$. Hence, the characteristic polynomial is a multiple of the annihilator polynomial ϕ_1, and both have the same irreducible factors.*

Lemma: $1, x - \alpha, \ldots, (x - \alpha)^{n-1}$ *form a base of* $k[x]/((x - \alpha)^n)$.

Proof: A base of $k[y]/(y^n)$ is $(1, y, \ldots, y^{n-1})$; where $y = x - \alpha$. q.e.d.

When $E_T \simeq k[x]/((x - \alpha)^n)$, the matrix of T in the base $\{e_j = (x - \alpha)^{j-1}\}$ is

$$T(e_j) = x \cdot (x - \alpha)^{j-1} = (x - \alpha)(x - \alpha)^{j-1} + \alpha(x - \alpha)^{j-1} = e_{j+1} + \alpha e_j,$$

$$\begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 1 & \alpha & & \vdots \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots 0 & 1 & \alpha \end{pmatrix} \tag{6.1}$$

If the elementary divisors of T are $(x-\alpha_i)^{n_{ij}}$, in some base the matrix of T is a **Jordan matrix** (B_{ij} is a matrix $n_{ij}\times n_{ij}$ of the form 6.1, with α_i in the diagonal),

$$\begin{pmatrix} \ddots & & \\ & B_{ij} & \\ & & \ddots \end{pmatrix} \tag{6.2}$$

so obtaining reduced equations of endomorphisms if k is algebraically closed.

If $k=\mathbb{R}$, we consider a complex number $\alpha = a+bi$, $b\neq 0$, and the irreducible polynomial

$$(x-\alpha)(x-\bar\alpha) = x^2 - 2ax + (a^2+b^2).$$

Lemma: $\mathbb{R}[x]/(x-\alpha)^n(x-\bar\alpha)^n = \mathbb{C}[x]/(x-\alpha)^n$, *isomorphism of* $\mathbb{R}[x]$*-modules.*

Proof: The annihilator of $\mathbb{C}[x]/((x-\alpha)^n)$ is a multiple of $(x-\alpha)^n(x-\bar\alpha)^n$, since it has real coefficients. We conclude because $\dim_{\mathbb{R}}\mathbb{C}[x]/((x-\alpha)^n) = 2n$. q.e.d.

The matrix of T in the base $\{e_j = (x-\alpha)^{j-1},\ e'_j = i(x-\alpha)^{j-1}\}$ is

$$\begin{aligned} T(e_j) &= x\cdot(x-\alpha)^{j-1} = (x-\alpha)^j + \alpha(x-\alpha)^{j-1} \\ &= (x-\alpha)^j + a(x-\alpha)^{j-1} + bi(x-\alpha)^{j-1} = e_{j+1} + ae_j + be'_j \\ T(e'_j) &= x\cdot i(x-\alpha)^{j-1} = i(x-\alpha)^j + i\alpha(x-\alpha)^{j-1} \\ &= i(x-\alpha)^j + ai(x-\alpha)^{j-1} - b(x-\alpha)^{j-1} = e'_{j+1} + ae'_j - be_j \end{aligned}$$

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ I & A & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots 0 & I & A \end{pmatrix}, \quad A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.3}$$

In general, in some base of E the matrix of T is a Jordan matrix 6.2, where the matrices B_{ij} have the form 6.1 or 6.3.

Definition: Two projectivities $\tau,\tau'\colon \mathbb{P}(E)\to\mathbb{P}(E)$ are **equivalent** if $\tau' = \sigma\tau\sigma^{-1}$ for some projectivity σ (if T and T' are linear representants of τ and τ', this condition states that the endomorphisms T' and λT are equivalent for some non null constant $\lambda\in k$).

Lemma: *If* $\phi_i(x)$ *are the invariant factors of* T*, then those of* λT *are* $\phi_i(x/\lambda)$*.*

Proof: If $E = k[x]/(\phi_1(x))\oplus\ldots\oplus k[x]/(\phi_n(x))$ and we put $y=\lambda x$, then

$$E \simeq k[y]/\big(\phi_1(\tfrac{y}{\lambda})\big)\oplus\ldots\oplus k[y]/\big(\phi_n(\tfrac{y}{\lambda})\big)$$

where the structure of $k[y]$-module of E is defined by the endomorphism λT.

This isomorphism shows that the invariant factors of λT are just $\phi_i(\frac{y}{\lambda})$.

Classification of Projectivities: *Two projectivities are equivalent if and only if the invariant factors of the linear representants have proportional (non null) roots,* $\phi'_i(x) = \phi_i(\lambda x)$*.*

$PGl(2,\mathbb{C})$

Invariant Factors	Equation	Fixed Points
$\phi_1 = (t-1)(t-a)$	$\theta' = a\theta$	2 points
$\phi_1 = (t-1)^2$	$\theta' = \theta + 1$	1 point
$\phi_1 = \phi_2 = t-1$	$\theta' = \theta$	all

$PGl(3,\mathbb{C})$

Invariant Factors	Equations	Fixed Points
$\phi_1 = (t-1)(t-a)(t-b)$	$x' = ax,\ y' = by$	3 points
$\phi_1 = (t-1)^2(t-a)$	$x' = x+1,\ y' = ay$	2 points
$\phi_1 = (t-1)^3$	$x' = x+1,\ y' = x+y$	1 point
$\phi_1 = (t-1)(t-a)$, $\phi_2 = t-1$	$x' = x,\ y' = ay$	1 line, 1 point
$\phi_1 = (t-1)^2$, $\phi_2 = t-1$	$x' = x+1,\ y' = y$	1 line
$\phi_1 = \phi_2 = \phi_3 = t-1$	$x' = x,\ y' = y$	all

$PGl(4,\mathbb{C})$

Invariant Factors	Equations	Fixed Points
$\phi_1 = (t-1)(t-a)(t-b)(t-c)$	$x' = ax,\ y' = by,\ z' = cz$	4 points
$\phi_1 = (t-1)^2(t-a)(t-b)$	$x' = x+1,\ y' = ay,\ z' = bz$	3 points
$\phi_1 = (x-1)^2(t-a)^2$	$x' = x+1,\ y' = ay,\ z' = y+az$	2 points
$\phi_1 = (t-1)^3(t-a)$	$x' = x+1,\ y' = x+y,\ z' = az$	2 points
$\phi_1 = (t-1)^4$	$x' = x+1,\ y' = x+y,\ z' = y+z$	1 point
$\phi_1 = (t-1)(t-a)(t-b)$, $\phi_2 = t-1$	$x' = x,\ y' = ay,\ z' = bz$	1 line, 2 points
$\phi_1 = (t-1)^2(t-a)$, $\phi_2 = t-1$	$x' = x+1,\ y' = y,\ z' = az$	1 line, 1 point
$\phi_1 = (t-1)^2(t-a)$, $\phi_2 = t-a$	$x' = x+1,\ y' = ay,\ z' = az$	1 line, 1 point
$\phi_1 = (t-1)^3$, $\phi_2 = t-1$	$x' = x+1,\ y' = x+y,\ z' = z$	1 line
$\phi_1 = \phi_2 = (t-1)(t-a)$	$x' = x,\ y' = ay,\ z' = az$	2 lines
$\phi_1 = (t-1)(t-a)$, $\phi_2 = \phi_3 = t-1$	$x' = x,\ y' = y,\ z' = az$	1 plane, 1 point
$\phi_1 = (t-1)^2$, $\phi_2 = \phi_3 = t-1$	$x' = x+1,\ y' = y,\ z' = z$	1 plane
$\phi_1 = \phi_2 = (t-1)^2$	$x' = x+1,\ y' = y,\ z' = y+z$	1 line
$\phi_1 = \phi_2 = \phi_3 = \phi_4 = t-1$	$x' = x,\ y' = y,\ z' = z$	all

6.3.2 Grothendieck K-Group

Definitions: An **additive** function on the category $\mathcal{C}$ of finitely generated A-modules, with values in an abelian group G, assigns to any finitely generated A-module M an element $\chi(M) \in G$ so that $\chi(M) = \chi(M') + \chi(M'')$ for any exact sequence

$$(*) \qquad 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

An additive function $\xi : \mathcal{C} \to K$ is **universal** if any other additive function $\chi : \mathcal{C} \to G$ uniquely factors trough ξ; i.e., there is a unique group morphism $f : K \to G$ such that $\chi = f \circ \xi$. If $\xi_1 : \mathcal{C} \to K_1$, $\xi_2 : \mathcal{C} \to K_2$ are universal additive functions, then there is a unique group isomorphism $f : K_1 \to K_2$ such that $\xi_2 = f \circ \xi_1$.

(We also have a K-group of finite length A-modules, of finite projective A-modules, etc.)

The existence of a universal additive function follows from the representability theorem; but we shall give a direct construction. Let us consider the free abelian group on isomorphism classes of finitely generated A-modules, and the quotient $K(A)$ by the subgroup generated by elements $M - M' - M''$, one for each exact sequence $(*)$. If $[M]$ denotes the class of M in $K(A)$, then $[M] = [M'] + [M'']$ and, for any additive function χ there is a unique group morphism $f : K(A) \to G$ such that $\chi(M) = f([M])$. Remark that $[M \oplus N] = [M] + [N]$, so that any element of $K(A)$ is a difference $[M_1] - [M_2]$, but not uniquely.

Theorem: *If A is a principal ideal domain, then* $\mathrm{rg}\colon K(A) \to \mathbb{Z}$ *is an isomorphism.*

Proof: The decomposition theorems show that in $K(A)$ we have $[M] = r[A] + \sum_{ij}[A/p_i^{n_{ij}}A]$. Moreover, the exact sequence $0 \to A \xrightarrow{f} A \to A/fA \to 0$, where $f \neq 0$, shows that $[A/fA] = [A] - [A] = 0$. Hence $[M] = r[A]$, and we conclude.

Definition: The 0-**cycles group** of $X = \operatorname{Spec} A$ is the free abelian group $Z_0(X)$ on the closed points of X. The 0-cycle $z(M)$ of an A-module of finite length M is

$$z(M) = \sum_x l(M_x) \cdot x,$$

where x runs over the closed points of X and M_x is the localization of M at x. It is an additive function, because localization preserves exact sequences and lengths are additive functions.

Examples: Let A be a principal ideal domain. Any **fractional ideal** (a non null finitely generated submodule of the field of fractions) is $\frac{a_1}{b}A + \ldots + \frac{a_n}{b}A = \frac{1}{b}(a_1A + \ldots + a_nA) = \frac{a}{b}A = \mathfrak{m}_1^{e_1} \ldots \mathfrak{m}_r^{e_r}$ for some eventually negative exponents. Hence, the 0-cycles group $Z_0(\operatorname{Spec} A)$ is isomorphic to the multiplicative group of fractional ideals, the cycle $z(M)$ of a torsion finitely generated A-module M corresponding to the ideal generated by the product of the elementary divisors $c_0 = \prod_{ij} p_i^{n_{ij}} = \prod_i \phi_i$.

In $k(x)$, any monic rational function is $\frac{x^m + a_1x^{m-1} + \ldots}{x^n + b_1x^{n-1} + \ldots} = p_1^{e_1} \ldots p_r^{e_r}$, where $p_1, \ldots, p_r$ are irreducible monic polynomials. Hence, the 0-cycles group $Z_0(\mathbb{A}_{1,k})$ is just the multiplicative group of monic rational functions, the cycle $z(E_T)$ of the $k[x]$-module defined by an endomorphism $T\colon E \to E$ corresponding to the characteristic polynomial.

Any positive rational number is $p_1^{e_1} \ldots p_r^{e_r}$ for some prime numbers $p_1, \ldots, p_r$. Hence $Z_0(\operatorname{Spec} \mathbb{Z}) = \mathbb{Q}_+^*$, the cycle $z(G)$ of a finite abelian group G corresponding to the order $|G|$.

Theorem: *The morphism $z\colon K_{\mathrm{fl}}(A) \to Z_0(\operatorname{Spec} A)$ is an isomorphism.*

Proof: This morphism is surjective because $x = z(A/\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of the closed point x.

Moreover, any A-module of finite length admits a flag, and the successive quotients are isomorphic to $A/\mathfrak{m}$. Hence the classes $[A/\mathfrak{m}]$ generate the group $K_{\mathrm{fl}}(A)$. Since these classes are linearly independent in the group $Z_0(A)$, we conclude that the considered morphism is injective.

Corollary: *The order is the universal additive function on finite abelian groups.*

The characteristic polynomial is the universal additive function on endomorphisms of finite dimensional vector spaces.

6.4 Classification of Pairs of Metrics

Let S, S' be symmetric metrics, where S is non singular, on a k-vector space E of finite dimension, $\operatorname{char} k \neq 2$. We put $e \cdot v = S(e, v)$.

S being non singular, we have $S'(e,v)=(Te)\cdot v$ for a unique endomorphism $T\colon E\to E$, and T defines a structure of $k[x]$-module on E. Since S and S' are symmetric, we have

$$(Te)\cdot v = e\cdot (Tv),$$

and the polarity $\phi\colon E \xrightarrow{\sim} E^*$ of S is an isomorphism of $k[x]$-modules.

If $\tilde{T}\colon F\to F$ is the endomorphism of another pair of metrics $\tilde{S},\tilde{S}'$, where $\tilde{S}$ is non singular, then a k-linear isomorphism $F\xrightarrow{\sim} E$ transforms the pair (S,S') into $(\tilde{S},\tilde{S}')$ if and only if it is an isomorphism of $k[x]$-modules transforming S into $\tilde{S}$.

Let $A=k[x]/(\phi)$, where $\phi(x)$ is the annihilator polynomial of T.

By the representability theorem, the functor $M\rightsquigarrow M^*$ is representable, so that there is an A-module D such that

$$M^* = \mathrm{Hom}_A(M,D).$$

When $M=A$, we see that $D=A^*$; hence $E^*=\mathrm{Hom}_A(E,A^*)$.

Now, since the annihilator of the $k[x]$-module $A^*=\mathrm{Hom}_k(A,k)$ is also $\phi(x)$, we have $A^*=A\xi$ for some linear form $\xi\colon A\to k$, and

$$E^*\simeq \mathrm{Hom}_A(E,A),\ \omega=\omega^A \text{ when } \omega(e)=\xi\big[\omega^A(e)\big].$$

Therefore, ϕ induces an isomorphism of A-modules

$$\phi\colon E \xrightarrow{\ \sim\ } E^*\simeq \mathrm{Hom}_A(E,A)$$

and there is a unique A-bilinear map $S^A\colon E\times E\to A$ such that $e\cdot v=\xi\big[S^A(e,v)\big]$. It is non-singular and symmetric because so is S,

$$\xi\big[S^A(v,e)\big]=v\cdot e=e\cdot v.$$

Moreover, an A-linear isomorphism $F\to E$ transforms a non singular metric $\tilde{S}$ into S if and only if it transforms $\tilde{S}^A$ into S^A: *Classification of pairs of metrics, with fixed annihilator $\phi(x)$, is reduced to the classification of non singular metrics on finitely generated A-modules.*

Fix an A-bilinear metric S on a finitely generated A-module E, non singular in the sense that the polarity $\phi\colon E\to \mathrm{Hom}_A(E,A)$ is an isomorphism, and put $e\cdot v=S(e,v)\in A$.

If $\phi(x)=p_1^{n_1}\dots p_s^{n_s}$ is the irreducible factor decomposition, then (p. 177)

$$E=E_1\oplus\ldots\oplus E_s,\ E_i=\ker p_i^{n_i}.$$

First Decomposition Theorem: *This decomposition is orthogonal, $E_i\cdot E_j=0$, $i\neq j$.*

Proof: $E_i\cdot E_j=(p_i^{n_i}A+p_j^{n_j}A)(E_i\cdot E_j)=(p_i^{n_i}E_i)\cdot E_j+E_i\cdot(p_j^{n_j}E_j)=0.$ q.e.d.

Hence, we may assume that $A=k[x]/(p^n)$ is local, and we put $\mathfrak{m}=pA$, $K=A/\mathfrak{m}$.

Any metric $S\colon E\otimes_A E\to A$ defines, by the base change $A\to K$ (p. 132), a K-bilinear metric $\bar{S}\colon \bar{E}\otimes_K\bar{E}\to K$, where $\bar{E}=E/pE$. By definition, $\bar{e}\cdot\bar{v}=\overline{e\cdot v}\in K$.

Remark that, when E is free and S is non singular, then so is $\bar{S}$.

Lemma: *Let $(L,S),(L',S')$ be free A-modules with non singular metrics. Any K-linear isometry $\bar{\sigma}\colon(\bar{L},\bar{S})\to(\bar{L}',\bar{S}')$ may be lifted to an A-linear isometry $\sigma\colon(L,S)\to(L',S')$.*

Proof: By induction on $r = \operatorname{rk} L$ and n. If $L = Ae$, we put $q = p^{n-1}$, and we must lift any isometry $\bar\sigma\colon L/qL \to L/qL'$ to an isometry $\sigma\colon L \to L'$. We fix a representant $e' \in L'$ of $\bar\sigma(\bar e)$.

Since $\bar\sigma$ is an isometry, $e' \cdot e' = (1 + aq)(e \cdot e)$. If we put $b = 1 - \frac{1}{2}aq$, then $(be') \cdot (be') = e \cdot e$ since $q^2 = 0$, and $\sigma(e) = be'$ is the required isometry.

If $r > 1$, we fix $e \in L$ with $\bar e \in \bar L$ non isotropic, and a representant $e' \in L'$ of the non isotropic vector $\bar\sigma(\bar e)$. Then $L = (Ae) \perp F$, $L' = (Ae') \perp F'$.

By induction the isometries $\bar\sigma\colon K\bar e \to K\bar e'$, $\bar\sigma\colon \bar F \to \bar F'$ may be lifted to isometries $Ae \to Ae'$, $F \to F'$, defining the required isometry $\sigma\colon L \to L'$.

Definition: A submodule H is **homogeneous** if $H \simeq A/p^dA \oplus \ldots \oplus A/p^dA$.

In such a case, $p^dH = 0$ and S takes values in $\{a \in A\colon p^da = 0\} = \mathfrak{m}^{n-d} \simeq A/\mathfrak{m}^d$.

Hence we may view S as an $(A/\mathfrak{m}^d)$-bilinear metric on the $(A/\mathfrak{m}^d)$-module H, defining by base change a metric K-vector space $(\bar H, \bar S)$.

We also have a K-bilinear metric $\bar S$ on $(\overline{\ker p^d})$, and the radical contains the image of $\ker p^{d-1}$ and $pE \cap \ker p^d$; hence it projects to the quotient, and we obtain a metric morphism

$$(\bar H, \bar S) \;\longrightarrow\; \Big(\bar E_d = \frac{\overline{\ker p^d}}{[\ker p^{d-1} + (pE \cap \ker p^d)]}, \bar S\Big).$$

Second Decomposition Theorem: *E is an orthogonal sum $E = H_1 \perp \ldots \perp H_n$ of homogeneous modules H_d of annihilator p^d, and uniquely up to isometries.*

Proof: We prove the existence by induction on $\dim E$.

Since $p^{n-1}E \neq 0$, we have $p^{n-1}e \cdot v \neq 0$ for some vectors e, v, so that the metric $\bar S$ on $\bar E = E/pE$ is not zero; hence $e \cdot e$ is invertible in A for some $e \in E$.

Now $E = (Ae) \perp (Ae)^\perp$, and by induction E is an orthogonal sum of monogenous modules.

Grouping terms with equal annihilator, we see that E is an orthogonal sum of homogeneous modules.

Finally, if $E = H_1 \oplus \ldots \oplus H_n$, then the metric morphisms $(\bar H_d, \bar S) \to (\bar E_d, \bar S)$ are isometries (in particular $\bar E_d$ is non singular) because$H_r \cap \ker p^d \subseteq pE \cap \ker p^d$ when $r > d$, and $H_r \subseteq \ker p^{d-1}$ when $r < d$. Now uniqueness follows from the above lemma.

Classification of Metrics on Modules: *Non singular metrics on finitely generated A-modules (E, S) are classified by the invariant factors of E and the non singular metric $(A/\mathfrak{m}_i)$-vector spaces $(\bar H_{i,d}, \bar S)$, where $H_{i,d}$ is a homogeneous component of E of annihilator $\mathfrak{m}_i^d$.*

Classification of Pairs of Metrics: *Pairs of metrics on a k-vector space E (the first non singular) are classified by the invariant factors of the associated endomorphisms T and the non singular metric $k[x]/(p_i)$-vector spaces $(\bar H_{i,d}, \bar S^A)$.*

Corollary: *If k is algebraically closed, pairs of metrics, the first non singular, are classified by the elementary divisors of the associated endomorphism.*

Proof: The dimension classifies non singular metrics on k-vector spaces.

Reduced equations: A pair of metrics with a unique elementary divisor $(x - \alpha)^n$ is

$$S = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad , \quad S' = \begin{pmatrix} 0 & \cdots & 0 & 1 & \alpha \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \iddots & \vdots \\ 1 & \iddots & & & \vdots \\ \alpha & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Definitions: Pencils of quadrics $\lambda S + \mu S'$ in $\mathbb{P}(E)$ are lines in the space of quadrics $\mathbb{P}(S_2E)$. Every point in $\mathbb{P}(E)$ which is not a **base point** (lying on all the quadrics of the pencil) lies on exactly one quadric. The **fundamental points** of a pencil are the points of the vertices of the singular quadrics. *We only may consider pencils with some non singular quadric.*

The endomorphisms associated with two pairs of quadrics in the pencil differ in a homography. Even if the invariant factors are not invariants of the pencil, so are the degrees and multiplicities of the roots (and cross-ratios if there are 4 or more roots).

PENCILS OF CONICS IN $\mathbb{P}_{2,\mathbb{C}}$

1. Annihilator with 3 simple roots. A pencil defined by 2 conics intersecting at 4 points. There are 3 pairs of lines, and 3 non-collinear fundamental points.
$$\lambda(x_0^2 + x_2^2 + x_2^2) + (x_0^2 - x_1^2) = 0.$$
2. Annihilator with a simple and a double root. A pencil defined by 2 conics intersecting at 3 points, with a common tangent at one of them. There are 2 fundamental points, one being a base point, where any conic in the pencil is tangent to the line passing through the fundamental points.
$$\lambda(2x_0x_1 + x_2^2) + (x_0^2 + x_2^2) = 0.$$
3. Annihilator with a triple root. A pencil defined by 2 conics intersecting at 2 points, with a common tangent at one of them. There is a unique fundamental point.
$$\lambda(2x_0x_1 + x_2^2) + 2x_0x_1 = 0.$$
4. Annihilator with 2 simple roots. A pencil defined by 2 conics intersecting at 2 points, with common tangent at both. There is a line of fundamental points, and an exterior fundamental point.
$$\lambda(x_0^2 + x_1^2 + x_2^2) + x_0^2 = 0.$$
5. Annihilator with a double root. A pencil defined by 2 conics intersecting at a unique point, with common tangent. There is a line of fundamental points.
$$\lambda(2x_0x_1 + x_2^2) + x_0^2 = 0.$$

PENCILS OF QUADRICS IN $\mathbb{P}_{3,\mathbb{C}}$

1. Annihilator with 4 simple roots. There are 4 cones, with non coplanar vertices. There are infinite equivalence classes, according to the cross-ratio a of the 4 roots.
$$\lambda(x_0^2 + x_1^2 + x_2^2 + x_3^2) + (ax_0^2 + x_1^2 - x_2^2) = 0\ ; \quad a \neq 0, 1, -1.$$
2. Annihilator with 2 simple roots and a double root. There are 3 cones, with non collinear vertices. The vertex of a cone lies in the other two cones.
$$\lambda(2x_0x_1 + x_2^2 + x_3^2) + (x_0^2 - x_2^2 + x_3^2) = 0.$$
3. Annihilator with 2 double roots. There are 2 cones, and the line joining the vertices is a common generatrix.
$$\lambda(2x_0x_1 + 2x_2x_3) + (x_0^2 + 2x_0x_1 + x_2^2) = 0.$$

4. Annihilator with a simple and a triple root. There are 2 cones, the vertex of a cone lying on the other cone.
$$\lambda(x_1^2 + 2x_0x_2 + x_3^2) + (2x_0x_1 + x_3^2) = 0.$$

5. Annihilator with a root of multiplicity 4. There is a unique cone.
$$\lambda(2x_0x_3 + 2x_1x_2) + (2x_0x_2 + x_1^2) = 0.$$

6. Annihilator with 3 simple roots, one a simple root of ϕ_2. There is a pair of planes, and 2 cones with exterior vertices.
$$\lambda(x_0^2 + x_1^2 + x_2^2 + x_3^2) + (x_2^2 - x_3^2) = 0.$$

7. Annihilator with a double and a simple root, also a simple root of ϕ_2. There is a pair of planes, and a cone with incident vertex.
$$\lambda(2x_0x_1 + x_2^2 + x_3^2) + (2x_0x_1 + x_0^2) = 0.$$

8. Annihilator with a simple and a double root, also a simple root of ϕ_2. There is a pair of planes, and a cone with exterior vertex, tangent to the vertex of the pair of planes.
$$\lambda(2x_0x_1 + x_2^2 + x_3^2) + (x_0^2 + x_3^2) = 0.$$

9. Annihilator with a triple root, also a simple root of ϕ_2. There is a pair of planes, one being tangent to any quadric in the pencil.
$$\lambda(2x_0x_2 + x_1^2 + x_3^2) + 2x_0x_1 = 0.$$

10. Annihilator $\phi_1 = \phi_2$, with 2 simple roots. There are 2 pairs of planes, with skew vertices.
$$\lambda(x_0^2 + x_1^2 + x_2^2 + x_3^2) + (x_0^2 + x_1^2) = 0.$$

11. Annihilator with 2 simple roots, one a simple root of ϕ_2 and ϕ_3. There is a a double plane, and a cone with exterior vertex.
$$\lambda(x_0^2 + x_1^2 + x_2^2 + x_3^2) + x_0^2 = 0.$$

12. Annihilator $\phi_1 = \phi_2$, with a double root. There is a unique pair of planes.
$$\lambda(2x_0x_1 + 2x_2x_3) + (x_0^2 + x_2^2) = 0.$$

13. Annihilator with a double root, also a simple root of $\phi_2 = \phi_3$. There is a double plane.
$$\lambda(2x_0x_1 + x_2^2 + x_3^2) + x_0^2 = 0.$$

6.5 Semisimple Rings

In this section any ring A has a unity, but may be non-commutative, and $A^o = \mathrm{End}_A(A)$ will denote the **opposite** ring: the set A with the same addition, and product $a \cdot b = ba$.

The ring of $n \times n$ matrices with coefficients in A is denoted by $M_n(A)$, and the transposition of matrices defines an isomorphism $M_n(A^{\mathrm{o}}) = M_n(A)^{\mathrm{o}}$.

Except otherwise stated, all modules, submodules and ideals are on the left.

Definitions: A module $S \neq 0$ is **simple** if it has no proper submodules. A finitely generated module M is **semisimple** (resp. **homogeneous** of type S) if it is a sum, hence a finite sum, of simple submodules (resp. submodules isomorphic to S); hence so is any quotient of M.

Example: Let k be a field and $G = \{g_1, \ldots, g_n\}$ a finite group. Then $k[G] = kg_1 \oplus \ldots \oplus kg_n$ is a k-algebra with the product induced by G. If E is a k-vector space, any **linear representation** (that is to say, group morphism) $G \to \mathrm{Aut}_k(E)$ extends by linearity to a morphism of k-algebras $k[G] \to \mathrm{End}_k(E)$, so that E is a $k[G]$-module, and it is simple when the representation is **irreducible** ($E \neq 0$ and any G-invariant vector subspace is trivial).

If the characteristic of k does not divide the order of G, any finite $k[G]$-module is semisimple (**Maschke's theorem**). In fact, it is enough to show that any exact sequence of $k[G]$-modules $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ splits. If a k-linear section s of π is G-invariant, $s = gsg^{-1}$, then it is $k[G]$-linear. Otherwise the arithmetic mean $\tilde{s} = \frac{1}{|G|}\sum_g gsg^{-1}$ is a G-invariant section:

$$\pi\tilde{s} = \tfrac{1}{|G|}\textstyle\sum_{g\in G} g\pi sg^{-1} = \tfrac{1}{|G|}\sum_{g\in G} g(\mathrm{Id}_{M''})g^{-1} = \tfrac{1}{|G|}\sum_{g\in G} \mathrm{Id}_{M''} = \mathrm{Id}_{M''}.$$

Schur's lemma: *Any non-zero morphism $f\colon S \to S'$ between simple A-modules is an isomorphism. In particular, the ring $\mathrm{End}_A(S)$ is a division ring.*

Proof: Since $\mathrm{Ker}\, f \neq S$ and $\mathrm{Im}\, f \neq 0$, we have $\mathrm{Ker}\, f = 0$ and $\mathrm{Im}\, f = S'$.

Lemma: *Let N be a submodule of an A-module M. If $M = N + S_1 + \ldots + S_n$, where the submodules S_i are simple, then $M = N \oplus S_{i_1} \oplus \ldots \oplus S_{i_r}$ for some indices $i_1, \ldots, i_r$. Hence, any semisimple module is a direct sum of simple modules.*

Proof: If $N \cap S_1 = S_1$, we drop S_1. If $N \cap S_1 = 0$, we replace N by $N \oplus S_1$.

We conclude by induction on n.

Theorem: *Any submodule N of a semisimple module M is a direct summand, $M = N \oplus N'$.*

Proof: If $M = \sum_i S_i$, then $M = N + \sum_i S_i$, and we conclude by the lemma.

Corollary: *If a module is semisimple (resp. homogeneous of type S), then so is any submodule. In particular, the type of a homogeneous module is unique.*

Theorem: *Any semisimple module M admits a unique decomposition as a direct sum of homogeneous submodules of different types (the **isotypic** decomposition).*

Proof: If $M = S_1 \oplus \ldots \oplus S_n$, grouping isomorphic summands we obtain the existence.

If $S \subseteq M$ is a simple submodule, the projection $S \to S_i$ is null or an isomorphism by Schur's lemma. Hence S is contained in the component of type S, and the uniqueness follows.

Definitions: A **division ring** is a ring where any non-zero element is invertible.

The **center** of a ring A is the subring $Z(A) = \{a \in A\colon ab = ba,\ \forall b \in A\}$.

When k is a (commutative) field, a k-algebra is a ring A endowed with a ring morphism $k \to Z(A)$, and we say that it is a **central** k-algebra if $k = Z(A)$.

A ring A is **semisimple** if it is a semisimple A-module (so that any finite A-module, being a quotient of A^n, is semisimple), and it is a **simple** ring if it is a homogeneous A-module (so that all finite A-modules are homogeneous of the same type).

The Fundamental Example: Let us see that *the ring* $A = \mathrm{End}_D(E) \simeq M_n(D^{\mathrm{o}})$ *of endomorphisms of a* n*-dimensional vector space* E *over a division ring* D *is simple* and how it determines E and D. A simple A-module is E, and a basis $e_1, \ldots, e_n$ of E gives an isomorphism

$$\mathrm{End}_D(E) = E \oplus \overset{n}{\cdots} \oplus E\ , \quad T \mapsto (Te_1, \ldots, Te_n),$$

hence A is a simple ring, and E is the unique simple A-module (any other being a quotient of A). Moreover $D = \mathrm{End}_A(E)$ and, in particular, $Z(A) = Z(D)$. In fact,

$$M_n(D) = A^{\mathrm{o}} = \mathrm{Hom}_A(A, A) = \mathrm{Hom}_A(\oplus_n E, \oplus_n E) = M_n(\mathrm{End}_A(E)).$$

Skolem-Noether Theorem: *Let* E *be a finite-dimensional vector space over a division ring* D. *Any automorphism of the ring* $A = \mathrm{End}_D(E)$ *is induced by the conjugation with a semi-linear automorphism* $E \to E$, *well-defined up to the product with a homothety* $\lambda\mathrm{Id}$, $\lambda \in D$. *Hence, the group of automorphisms of the* k*-algebra* $\mathrm{End}_k(E)$ *is the group* $PGl(E)$.

Proof: Since E is the unique simple module over the ring $\mathrm{End}_D(E)$, for any ring isomorphism $\phi\colon A = \mathrm{End}_D(E) \to A' = \mathrm{End}_{D'}(E')$ we have a group isomorphism $T\colon E \to E'$ such that $T(ae) = (\phi a)(Te)$, $\forall a \in A, e \in E$. That is to say, $\phi(a) = TaT^{-1}$.

This isomorphism T is well-defined up to A-linear isomorphisms $\lambda\mathrm{Id} \in \mathrm{End}_A(E) = D$.

Moreover, T induces a ring isomorphism $\sigma\colon D = \mathrm{End}_A(E) \to D' = \mathrm{End}_{A'}(E')$ such that, for any $\lambda \in D$, the following square commutes

$$\begin{array}{ccc} E & \xrightarrow{T} & E' \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\sigma(\lambda)} \\ E & \xrightarrow{T} & E' \end{array}$$

Hence T is a semi-linear transformation with associated isomorphism σ.

Finally, condition $T(\lambda\mathrm{Id})T^{-1} = \lambda\mathrm{Id}$ states that T is a linear map.

Theorem: *Let* A *be a semisimple ring. Up to isomorphisms, there is a finite number of simple modules* $S_1, \ldots, S_r$, *and* A *is a direct product of matrix rings over division rings,*

$$A = \mathrm{End}_{D_1}(S_1) \times \ldots \times \mathrm{End}_{D_r}(S_r)\ , \quad D_i = \mathrm{End}_A(S_i).$$

Proof: Let $A = S_1^{n_1} \oplus \ldots \oplus S_r^{n_r}$ be the isotypic decomposition.

Any simple module, being a quotient of A, appears in this decomposition.

Moreover, by Schur's lemma, $\mathrm{Hom}_A(S_i, S_j) = 0$ when $i \neq j$. Hence

$$A^{\mathrm{o}} = \mathrm{Hom}_A(\oplus_i S_i^{n_i}, \oplus_i S_i^{n_i}) = \textstyle\prod_i \mathrm{Hom}_A(S_i^{n_i}, S_i^{n_i}) = \prod_i M_{n_i}(D_i).$$

Corollary: *Let* A *be a simple ring. Up to isomorphisms, there is a unique simple module* S, *and* A *is a matrix ring over a division ring,*

$$A = \mathrm{End}_D(S)\ , \quad D = \mathrm{End}_A(S).$$

Corollary: *Any simple finite algebra A over an algebraically closed field k is $A \simeq M_n(k)$.*

Proof: We have $k \hookrightarrow D = \text{End}_A(S)$ because $k \hookrightarrow Z(A)$. Now, if $\alpha \in D$, then $k[\alpha]$ is an integral finite commutative k-algebra; hence it is a field and $k = k[\alpha]$, so that $k = D$.

Corollary: *Let G be a finite group and k an algebraically closed field whose characteristic does not divide $|G|$. The number of different irreducible representations $G \to \text{Aut}_k(E_i)$ coincides with the number of conjugacy classes in G, and the degrees $d_i = \dim_k E_i$ satisfy that*

$$|G| = d_1^2 + \ldots + d_r^2.$$

Proof: Since $k[G]$ is semisimple and k is algebraically closed, $k[G] = \text{End}_k(E_1) \times \ldots \times \text{End}_k(E_r)$.

Now it is clear that $|G| = \dim_k k[G] = \sum_i d_i^2$, and that r is just the dimension of

$$Z(k[G]) = Z(\text{End}_k(E_1)) \times \ldots \times Z(\text{End}_k(E_r)) = k \times \overset{r}{\ldots} \times k.$$

We have $\sum_i \lambda_i g_i \in Z(k[G])$ if and only if $\sum_i \lambda_i (g g_i g^{-1}) = \sum_i \lambda_i g_i$, $\forall g \in G$. Hence a basis of $Z(k[G])$ is defined by the elements $g_1 + \ldots + g_s$, where $\{g_1, \ldots, g_s\}$ is a conjugacy class.

Theorem: *A ring A is simple if and only if it admits a* ***minimal*** *ideal (a simple submodule) and any two-sided ideal of A is trivial.*

Proof: Let A be a simple ring. If $I \neq A$ is a two-sided ideal, since I annihilates $A/I \simeq S \oplus \ldots \oplus S$, we have $IS = 0$; hence $IA = I(S \oplus \ldots \oplus S) = 0$, and $I = 0$.

Conversely, if A admits a minimal ideal I and any two-sided ideal is trivial, then

$$A = IA = \textstyle\sum_{a \in A} Ia.$$

Since we have an epimorphism $I \to Ia$, we see that Ia is a simple module isomorphic to I whenever non null. Hence A is a homogeneous A-module.

Wedderburn's Theorem: *Let A be a simple ring and S a simple A-module. The category of A-modules is equivalent to the category of right modules over the division ring $D = \text{End}_A(S)$.*

Proof: If M is an A-module, then $F(M) = \text{Hom}_A(S, M)$ is a right D-module.

If E is a right D-module, then $G(E) = E \otimes_D S$ is a left A-module via the action on S.

The natural morphisms $G(F(S)) \to S$, $D \to F(G(D))$ are isomorphisms, and both functors preserve direct sums and inductive limits.

Since any A-module (resp. right D-module) is an inductive limit of direct sums of modules isomorphic to S (resp. D), we conclude that F and G define an equivalence of categories.

6.5.1 The Brauer Group

Definition: An **Azumaya** k-algebra is a finite simple central k-algebra; hence it is isomorphic to a matrix algebra $M_n(D)$ over a division ring D such that $k = Z(D)$.

Theorem: *A finite k-algebra A is an Azumaya algebra if and only if the natural morphism $\phi_A \colon A \otimes_k A^{\text{o}} \to \text{End}_k(A)$, $\phi_A(a \otimes b)(x) = axb$, is an isomorphism.*

Proof: If A is an Azumaya algebra, we consider the image B of ϕ_A,

$$B \hookrightarrow \text{End}_k(A) = A \oplus \ldots \oplus A.$$

Since two-sided ideals of A are just $(A \otimes_k A^o)$-submodules and A is a simple ring, it is a simple B-module. Hence B is a homogeneous B-module: it is a simple ring.

It follows that B is the ring of endomorphisms of the simple module,

$$B = \mathrm{End}_D(A) \ , \quad D = \mathrm{End}_B(A) = \mathrm{End}_{A\otimes A^o}(A) = Z(A) = k,$$

so that $B = \mathrm{End}_k(A)$, and ϕ_A is surjective.

Since $\dim (A \otimes_k A^o) = \dim \mathrm{End}_k(A)$, it is an isomorphism.

Conversely, if ϕ_A is an isomorphism, then A is a simple $(A \otimes_k A^o)$-module, because A is always a simple $\mathrm{End}_k(A)$-module. Hence A is a simple ring, and moreover

$$Z(A) = \mathrm{End}_{A\otimes_k A^o}(A) = \mathrm{End}_{\mathrm{End}_k(A)}(A) = k.$$

Corollary: *If A and B are two Azumaya k-algebras, then so is $A \otimes_k B$.*

Proof: $\phi_{A\otimes_k B} = \phi_A \otimes \phi_B$.

Corollary: *Let $k \to L$ be a commutative extension. A finite k-algebra A is an Azumaya k-algebra if and only if $A \otimes_k L$ is an Azumaya L-algebra.*

Proof: $\phi_{A\otimes_k L} = \phi_A \otimes 1$.

Corollary: *A finite k-algebra A is an Azumaya k-algebra if and only if $A \otimes_k \bar{k} \simeq M_n(\bar{k})$. In particular, the dimension of any Azumaya k-algebra is a perfect square,* $\dim_k A = n^2$.

Definition: The Azumaya k-algebras form an abelian group $\mathrm{Br}(k)$ under the tensor product if we identify $M_n(D)$ and $M_m(D')$ whenever $D \simeq D'$. The neutral element is the class of all **trivial** Azumaya k-algebras $M_n(k)$, and the inverse of a class $[A]$ is the class $[A^o]$. It is the **Brauer group** of k, and its elements correspond to the central extensions $k \to D$. For any commutative extension $k \to L$ we have a group morphism $\mathrm{Br}(k) \to \mathrm{Br}(L)$, $[A] \mapsto [A \otimes_k L]$, the kernel $\mathrm{Br}(L/k)$ being the subgroup defined by the Azumaya k-algebras trivial over L.

Theorem: *An Azumaya k-algebra A of degree n^2 is trivial, $A \simeq M_n(k)$, if and only if contains a trivial commutative subalgebra B of degree n, $B \simeq k\oplus \overset{n}{\dots} \oplus k$.*

Proof: If $A \simeq M_n(k)$, then the diagonal matrices define a subalgebra $B \simeq k\oplus \overset{n}{\dots} \oplus k$.

Conversely, assume that A admits a trivial subalgebra $B \simeq k\oplus \overset{n}{\dots} \oplus k$.

Then we have an isomorphism of A-modules

$$A = A \otimes_B B = A \otimes_B (k\oplus \overset{n}{\dots} \oplus k) = (A \otimes_B k)\oplus \overset{n}{\dots} \oplus(A \otimes_B k) = I_1\oplus \overset{n}{\dots} \oplus I_n,$$

so that $\dim I_i \le n$ for some index i. Now, the ring morphism $A \to \mathrm{End}_k(I_i)$ is injective because A is simple, and it is an isomorphism since $\dim A = n^2 \ge \dim \mathrm{End}_k(I_i)$.

Lemma: *Any Azumaya k-algebra A of degree n^2 contains a separable commutative subalgebra of degree n.*

Proof: If $a \in A$, then the commutative subalgebra $k[a] \simeq k[x]/(q(x))$ is separable of degree n when so is the annihilator polynomial $q(x)$. Hence, it is enough to show that the characteristic polynomial of the endomorphism $a\colon A \to A$ is separable for some $a \in A$.

Now, the discriminant of the characteristic polynomial is a polynomial on A, and it is not the zero polynomial since it does not vanish at some points of $A \otimes_k \bar{k} \simeq M_n(\bar{k})$.

When k is infinite, we conclude that it does not vanish at some point $a \in A$.

When k is a finite field, any extension is separable, and it is enough to show that any division ring D of degree n^2 contains a commutative field of degree n.

Pick a commutative field $L \subset D$ of degree $d < n$.

Since $\oplus \bar{k} \simeq L \otimes_k \bar{k} \subset D \otimes_k \bar{k} \simeq M_n(\bar{k})$, we have $L \otimes_k \bar{k} = \bar{k}[T]$, where the annihilator of the endomorphism T is a separable polynomial of degree d.

Now T is diagonalizable and commutes with all the diagonal matrices.

The commutator of $L \otimes_k \bar{k}$ in $D \otimes_k \bar{k}$ has degree $\geq n$, and the commutator of L in D strictly contains L: there exists $\alpha \in D$, $\alpha \notin L$, such that the field $L[\alpha]$ is commutative.

Theorem: *If A is an Azumaya k-algebra, then there exists a finite Galois extension $k \to L$ such that $A \otimes_k L$ is trivial, $A \otimes_k L \simeq M_n(L)$.*

Proof: If $\dim A = n^2$, pick a separable commutative algebra $B \subset A$ of degree n.

Now (p. 150) B is trivial over a finite Galois extension $k \to L$, and $A \otimes_k L$ contains a trivial commutative L-algebra $B \otimes_k L$ of degree n; hence it is trivial, $A \otimes_k L \simeq M_n(L)$.

Definition: Let $k \to L$ be a Galois extension of Galois group G. An **action** of G on a L-vector space E is a group morphism $\rho\colon G \to \mathrm{Aut}_k(E)$ such that $\rho(g)$ is a semi-linear transformation of automorphism g. Two actions on E and F are **equivalent** if there is a L-linear isomorphism $T\colon E \to F$ preserving the actions, $T(ge) = g(Te)$.

An action of G on the projective space $\mathbb{P}(E)$ is a group morphism ρ of G into the group of all collineations, such that g is the automorphism of the collineation $\rho(g)$. Two actions are equivalent if some projectivity transforms one into the other.

An action of G on a L-algebra A is a group morphism $\rho\colon G \to \mathrm{Aut}_{k\text{-alg}}(A)$ such that $\rho(g)$ is a semi-linear transformation of automorphism g. Two actions on A and B are equivalent if there is an isomorphism of L-algebras $A \to B$ preserving the actions.

Let us consider the ring $L[G] = \oplus_{g \in G} Lg$ with the product $(\lambda_1 g_1)(\lambda_2 g_2) = \lambda_1 g_1(\lambda_2) \cdot (g_1 g_2)$, so that $L[G]$-modules are just L-vector spaces with a G-action, $(\sum_i \lambda_i g_i)e = \sum_i \lambda_i (g_i e)$.

Lemma: *The natural morphism $L[G] \to \mathrm{End}_k(L)$ is an isomorphism. Hence $L[G]$ is a simple ring, and the simple $L[G]$-module is just L with the natural G-action.*

Proof: We have that $\mathrm{Hom}_k(L, L) = \mathrm{Hom}_L(L \otimes_k L, L) = \mathrm{Hom}_L(\oplus L, L)$ and the elements of G correspond to the projections $\oplus L \to L$. Now the result is obvious.

Corollary: *The natural morphism $E^G \otimes_k L \to E$ is an isomorphism for any $L[G]$-module E.*

Proof: It holds when $E = L$, because $k = L^G$ (p. 151), and in general $E \simeq \oplus L$.

Theorem: *Let $k \to L$ be a Galois extension of group G. The algebra of invariants $M_n(L)^G$ of any action of G on the L-algebra $M_n(L)$ is an Azumaya k-algebra of degree n^2 trivial over L, and so we obtain a bijection*

$$\begin{bmatrix} \text{Azumaya } k\text{-algebras of} \\ \text{degree } n^2 \text{ trivial over } L \\ \text{up to isomorphisms} \end{bmatrix} = \begin{bmatrix} G\text{-actions on } M_n(L) \\ \text{up to equivalence} \end{bmatrix} = \begin{bmatrix} G\text{-actions on } \mathbb{P}_{n-1} \\ \text{up to equivalence} \end{bmatrix}$$

Proof: The natural morphism $M_n(L)^G \otimes_k L \to M_n(L)$ is an isomorphism; hence $M_n(L)^G$ is an Azumaya k-algebra of degree n^2 trivial over L, and it determines the action of G on $M_n(L)$ up to equivalence. Moreover, for any Azumaya k-algebra A of degree n^2 trivial over L, we have that $A = A \otimes_k (L^G) = (A \otimes_k L)^G = M_n(L)^G$.

Finally, the last bijection is a direct consequence of the Skolem-Noether theorem.

Definition: Let $k \to L$ be a cyclic Galois extension of degree d and let us fix a generator g of the Galois group G. Now a G-action on a projective space is just a collineation τ of automorphism g and order d. If T is a semi-linear representant, condition $\tau^d = \text{Id}$ states that $T^d = \mu\text{Id}$, $\mu \in L^*$. Since $\mu T = T^d T = T T^d = T\mu$, we have $g(\mu) = \mu$, so that in fact $\mu = \text{inv}(T)$ is in k^*.

But it depends on the semi-linear representant T,

$$\text{inv}(\lambda T) = (\lambda T)^d = \lambda(g\lambda)\dots(g^{d-1}\lambda)T^d = N(\lambda)\text{inv}(T),$$

where $N(\lambda) = \lambda(g\lambda)\dots(g^{d-1}\lambda)$ is the **norm** of $\lambda \in L^*$. Hence the invariant $\text{inv}(\tau) = [\mu]$ is well-defined in the group $k^*/N(L^*)$, but it depends on the fixed generator g of G.

Remark that $\text{inv}(T \oplus \dots \oplus T) = \text{inv}(T)$, so that $\text{inv}(\tau)$ only depends on the class of the Azumaya k-algebra, and we obtain a map $\text{inv}\colon \text{Br}(L/k) \to k^*/N(L^*)$ which is a group morphism because $\text{inv}(T \otimes \bar{T}) = \text{inv}(T) \cdot \text{inv}(\bar{T})$.

Now, the representant $T\colon E \to E$ defines on E a structure of module over the ring

$$L_\mu[G] = L \oplus Lg \oplus \dots \oplus Lg^{d-1} \ ; \quad g^d = \mu,\ g \cdot \lambda = g(\lambda) \cdot g.$$

Lemma: *The ring $L_\mu[G]$ is simple (in fact it is an Azumaya k-algebra).*

Proof: We have an isomorphism $L_\mu[G] \otimes_k \bar{k} \to L[G] \otimes_k \bar{k} = M_d(\bar{k})$, $g \otimes 1 \mapsto g \otimes \sqrt[d]{\mu}$.

Theorem: *If $k \to L$ is a cyclic extension,* $\text{inv}\colon \text{Br}(L/k) \to k^*/N(L^*)$ *is an isomorphism.*

Proof: The Azumaya k-algebra A defined by a simple $L_\mu[G]$-module S has invariant $[\mu]$, and any other Azumaya k-algebra B defined by a semi-linear transformation of invariant μ corresponds to a module $E \simeq S \oplus \overset{n}{\dots} \oplus S$; hence $B \simeq M_n(A)$ and we conclude.

Frobenius Theorem: *There is a unique non-commutative finite extension of* $\mathbb{R}$,

$$\text{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}.$$

Proof: $\text{Br}(\mathbb{R}) = \text{Br}(\mathbb{C}/\mathbb{R}) = \mathbb{R}^*/N(\mathbb{C}^*) = \mathbb{R}^*/\mathbb{R}^*_+ = \{\pm 1\}$.

Wedderburn's Theorem: *Every finite division ring is commutative,* $\text{Br}(\mathbb{F}_q) = 0$.

Proof: Any finite extension $\mathbb{F}_q \to \mathbb{F}_{q^d}$ is cyclic and the Galois group is generated by the Frobënius automorphism $F(x) = x^q$. Hence the norm $N\colon \mathbb{F}^*_{q^d} \to \mathbb{F}^*_q$ is

$$N(x) = xF(x)F^2(x)\dots F^{d-1}(x) = xx^q x^{q^2}\dots x^{q^{d-1}} = x^{\frac{q^d-1}{q-1}}$$

and the kernel of N has order $\leq \frac{q^d-1}{q-1}$, so that N is surjective and $\text{Br}(\mathbb{F}_{q^d}/\mathbb{F}_q) = 0$.

We conclude that $\text{Br}(\mathbb{F}_q) = \bigcup_d \text{Br}(\mathbb{F}_{q^d}/\mathbb{F}_q) = 0$.

Example: Let $G = \{\text{Id}, j\}$ be the Galois group of $\mathbb{C}$ over $\mathbb{R}$, where j stands for the complex conjugation. Since $\mathbb{C}_{-1}[G]$ is a non-trivial Azumaya $\mathbb{R}$-algebra of degree 4, it is the unique non-commutative finite extension of $\mathbb{R}$. We denote it $\mathbb{H}$ and name it the **quaternions**,

$$\mathbb{H} = \mathbb{C}_{-1}[G] = \mathbb{C} \oplus \mathbb{C}j = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij \ ; \quad i^2 = j^2 = -1,\ ji = -ij.$$

Part III

Third Year

Chapter 7

Commutative Algebra

7.1 The Spectrum of a Ring

Let A be a ring and M an A-module.

If U is an open set in $X = \operatorname{Spec} A$, then A_U and $M_U = M \otimes_A A_U$ denote the localization of A and M by the multiplicative system of all functions without zeros in U.

When $U = U_f = \operatorname{Spec} A_f$ is a basic open set, then the functions without zeros in U are invertible in A_f, so that $A_U = A_f$ (hence $M_U = M_f$), and $U = \operatorname{Spec} A_U$.

Now let us consider the map

$$\pi\colon \widetilde{M} = \coprod_{x\in X} M_x \longrightarrow X,\ \pi(m_x) = x.$$

When $x \in U$, we have a morphism $M_U \to M_x$, so that $m \in M_U$ defines a section $\tilde{m}\colon U \to \widetilde{M}$ of π. If two sections $\tilde{m}, \tilde{n}$ coincide at a point, $m_x = n_x$, then $fn = fm$, where $f(x) \neq 0$, and both sections coincide on the neighborhood $U_f \cap U$. The images of these sections $\tilde{m}$ form a base of a topology on $\widetilde{M}$, the sections $\tilde{m}$ are continuous, and π is a local homeomorphism.

Moreover, any continuous section of π locally coincides with some section $\tilde{m}$, and M_x is just the set of germs at x of continuous sections. The continuous sections of π on U form an A_U-module $\operatorname{Hom}_X(U, \widetilde{M})$, and we have a natural morphism of A_U-modules

$$M_U \longrightarrow \operatorname{Hom}_X(U, \widetilde{M}).$$

Theorem: $M_U = \operatorname{Hom}_X(U, \widetilde{M})$, *when* $U = U_f$ *is a basic open set.*

Proof: Since $U_f = \operatorname{Spec} A_{U_f}$, we may assume that $U = X$.

We know that the morphism is injective (p. 142). Let us see that it is surjective.

Let $s\colon X \to \widetilde{M}$ be a continuous section. Since X is compact, there is a finite basic open cover $X = \bigcup U_i$, and elements $s_i \in M_{U_i}$ such that $s|_{U_i} = \tilde{s}_i$.

The sections $\tilde{s}_i$ and $\tilde{s}_j$ coincide on the basic open set $U_i \cap U_j$ and, since the morphism is injective, $s_i = s_j$ in $M_{U_i\cap U_j}$; hence we have to prove the exacteness of the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \xrightarrow{\ i\ } & \bigoplus_i M_{U_i} & \overset{i_1,i_2}{\rightrightarrows} & \bigoplus_{i,j} M_{U_i\cap U_j} \\ & & & \| & & \| \\ & & & M\otimes_A B & & M\otimes_A B\otimes_A B \end{array}$$

where $B = \oplus_i A_{U_i}$, $i(m) = m\otimes 1$, and $i_1(m\otimes b) = m\otimes b\otimes 1$, $i_2(m\otimes b) = m\otimes 1\otimes b$.

It is clear when the natural ring morphism $A \to B$ admits a retract $r\colon B \to A$: if $m\otimes b\otimes 1 = m\otimes 1\otimes b$, applying $1\otimes 1\otimes r$ we obtain $m\otimes b = r(b)m \in M$.

If $A \to B = \oplus_i A_{U_i}$ has no retract, since $X = \bigcup_i U_i$, it is enough to prove the exactness after applying the functor $M \rightsquigarrow M_B := M \otimes_A B$ (p. 142)

$$M_B \longrightarrow M_B \otimes_B B_B \rightrightarrows M_B \otimes_B B_B \otimes_B B_B$$

i.e. the corresponding sequence for the B-module M_B and the morphism $B \to B_B = B \otimes_A B$, $b \mapsto 1 \otimes b$, admitting the obvious retract $\mu\colon B \otimes_A B \to B$, $\mu(b' \otimes b) = b'b$.

Direct proof: Let $U_i = X - (f_i)_0$. Given elements $s_i \in M_{f_i}$, such that $s_i = s_j$ in $M_{f_i f_j}$, we must prove the existence of $m \in M$ such that $s_i = m$ in M_{f_i}.

Put $s_i = \frac{m_i}{f_i^n}$, with a common exponent n.

Since $\frac{m_i}{f_i^n} = \frac{m_j}{f_j^n}$ in $M_{f_i f_j}$, there exists r such that $f_i^r f_j^r f_j^n m_i = f_i^r f_j^r f_i^n m_j$.

Since $\frac{m_i}{f_i^n} = \frac{f_i^r m_i}{f_i^r f_i^n}$, when $n \gg 0$ we may assume that $f_j^n m_i = f_i^n m_j$.

We have $X = \bigcup_j U_j$; hence $A = \sum_j f_j^n A$, and $1 = \sum_j \phi_j f_j^n$, where $\phi_j \in A$.

We put $m = \sum_j \phi_j m_j$, and let us prove that $m = \frac{m_i}{f_i^n}$ in M_{f_i},

$$f_i^n m = \textstyle\sum_j \phi_j f_i^n m_j = \sum_j \phi_j f_j^n m_i = \left(\sum_j \phi_j f_j^n\right) m_i = 1 \cdot m_i. \qquad \text{q.e.d.}$$

1. *If the spectrum disconnects, $X = U \amalg V$, then the ring decomposes, $A = A_U \oplus A_V$.*

   U and V are basic open sets: the section with value 0 on U and 1 on V is continuous.

2. *If M is supported on a finite number of closed points $x_1, \ldots, x_n$* (i.e. $M_x = 0$ when $x \neq x_i$), *then $M = M_{x_1} \oplus \ldots \oplus M_{x_n}$.*

3. *If M is annihilated by some power of a maximal ideal $\mathfrak{m}$, then $M = M_{\mathfrak{m}}$.*

Note: From now on, in these notes the spectrum $\operatorname{Spec} A$ of a ring A will be considered to be the pair $(\operatorname{Spec} A, A)$, and a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is defined to be a ring morphism $A \to B$, so that it induces a continuous map $\operatorname{Spec} B \to \operatorname{Spec} A$, but different morphisms may induce the same continuous map. When A and B are algebras over a field k, then the k-morphisms $\operatorname{Spec} B \to \operatorname{Spec} A$ are defined to be the morphisms of k-algebras $A \to B$, so that $\operatorname{Hom}_k(\operatorname{Spec} B, \operatorname{Spec} A) = \operatorname{Hom}_{k\text{-alg}}(A, B)$.

7.2 Primary Decomposition

Definition: A module is **noetherian** if any submodule is finitely generated (equivalently, any strictly increasing sequence of submodules is finite).

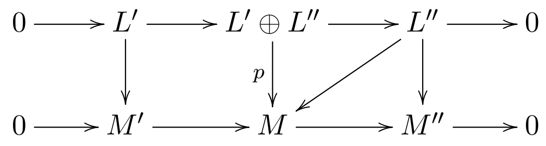
A ring is **noetherian** if any ideal is finitely generated.

Lemma: *If $0 \to M' \to M \to M'' \to 0$ is exact, then M is noetherian $\Leftrightarrow$ so are M' and M''.*

Proof: It is enough to show that M is finitely generated when so are M' and M'', since any submodule of M has an analogous exact sequence.

Let us fix epimorphisms $L' = A^n \to M'$, $L'' = A^m \to M''$.

We extend the last one to a morphism $L'' \to M$, and we conclude since p is surjective,



Theorem: *Any finitely generated module over a noetherian ring is noetherian.*

Proof: If A is noetherian, so is A^n, and any quotient of A^n, by the lemma.

Theorem: *If A is a noetherian ring, so is $A[x]$.*

Proof: Let I be an ideal of $A[x]$.

The leading coefficients of polynomials in I form an ideal $a_1A + \ldots + a_rA$.

Fix polynomials $P_i(x) = a_ix^n + \ldots \in I$, of equal degree n.

If $Q(x) = ax^m + \ldots \in I$, $m \geq n$, then $a = \sum_i b_ia_i$, and $\deg\,(Q - \sum_i b_ix^{m-n}P_i) < m$,

$$I = (P_1, \ldots, P_r) + (I \cap L),$$

where $L = A \oplus Ax \oplus \ldots \oplus Ax^{n-1}$ is a noetherian A-module.

Therefore $I \cap L$ is a finitely generated A-module, and the ideal I is finitely generated.

Corollary: *If A is a noetherian ring, so is any finitely generated A-algebra.*

Proof: The ring $A[x_1, \ldots, x_n]$ is noetherian; hence so is any quotient.

Definitions: An ideal $\mathfrak{q} \neq A$ is **irreducible** if it is not a proper intersection of two ideals.

The **radical** $\operatorname{rad} I = \{a \in A \colon a^n \in I, \text{ for some } n \geq 1\}$ of an ideal I is (p. 141) the intersection of all prime ideals containing I, and it is the greatest ideal such that $(\operatorname{rad} I)_0 = (I)_0$.

An ideal $\mathfrak{q} \neq A$ is **primary** if any homothety $A/\mathfrak{q} \xrightarrow{b\cdot} A/\mathfrak{q}$, $b \in A$, is injective or nilpotent,

$$ab \in \mathfrak{q},\ a \notin \mathfrak{q} \ \Rightarrow\ b^n \in \mathfrak{q}.$$

In such a case $\mathfrak{p} = \operatorname{rad}\mathfrak{q}$ is a prime ideal, and we say that $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal.

Lemma: *If A is noetherian, any irreducible ideal is primary.*

Proof: Let $b\colon A/\mathfrak{q} \to A/\mathfrak{q}$ be a homothety. $\operatorname{Ker} b \subseteq \operatorname{Ker} b^2 \subseteq \ldots \subseteq \operatorname{Ker} b^n \subseteq \ldots$

Since A is noetherian, $\operatorname{Ker} b^n = \operatorname{Ker} b^{n+1}$ for some exponent n,

$$\operatorname{Ker} b \cap \operatorname{Im} b^n = 0.$$

Since $\mathfrak{q}$ is irreducible, we have that $\operatorname{Ker} b = 0$ or $\operatorname{Im} b^n = 0$.

If $\operatorname{Ker} b = 0$, then b is injective. If $\operatorname{Im} b^n = 0$, then b is nilpotent.

Theorem: *Any ideal I of a noetherian ring A is a finite intersection of primary ideals.*

Proof: If I is not irreducible, $I = I_1 \cap I_2$. If some ideal I_i is not irreducible,... The process ends by noetherianness, and I is a finite intersection of irreducible (hence primary) ideals. q.e.d.

Now, since intersections of $\mathfrak{p}$-primary ideals are $\mathfrak{p}$-primary, grouping ideals and eliminating redundancies, I admits a **reduced** primary decomposition $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$, where $\mathfrak{p}_i \neq \mathfrak{p}_j$, and $I \neq \mathfrak{q}_1 \cap \ldots \widehat{\mathfrak{q}}_i \ldots \cap \mathfrak{q}_n$ (we put $\mathfrak{p}_i = \operatorname{rad}\mathfrak{q}_i$).

Corollary: *The number of minimal prime ideals of a noetherian ring is finite.*

Proof: Let $0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a primary decomposition.

We have $\operatorname{Spec} A = (\mathfrak{q}_1)_0 \cup \ldots \cup (\mathfrak{q}_n)_0 = (\mathfrak{p}_1)_0 \cup \ldots \cup (\mathfrak{p}_n)_0$.

Now it is clear that any minimal prime coincides with some ideal $\mathfrak{p}_i$.

Lemma: *Any $\mathfrak{p}$-primary ideal $\mathfrak{q}$ is defined by infinitesimal conditions at the point $\mathfrak{p}$,*

$$\mathfrak{q} = A \cap \mathfrak{q}A_{\mathfrak{p}}.$$

Proof: If $f \in A \cap \mathfrak{q}A_{\mathfrak{p}}$, then $sf \in \mathfrak{q}$, where $s \notin \mathfrak{p} = \operatorname{rad}\mathfrak{q}$; hence $f \in \mathfrak{q}$.

Theorem: *Let $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a reduced primary decomposition of an ideal I of a noetherian ring A. The primary component $\mathfrak{q}_i$ does not depend on the decomposition when $(\mathfrak{p}_i)_0$ is an irreducible component of $(I)_0$.*

Proof: When $j \neq i$, we have $\mathfrak{q}_j A_{\mathfrak{p}_i} = A_{\mathfrak{p}_i}$ since $\mathfrak{p}_j = \operatorname{rad}\mathfrak{q}_j$ intersects $A - \mathfrak{p}_i$.

$$I_{\mathfrak{p}_i} = \mathfrak{q}_1 A_{\mathfrak{p}_i} \cap \ldots \cap \mathfrak{q}_n A_{\mathfrak{p}_i} = \mathfrak{q}_i A_{\mathfrak{p}_i},$$

so that $\mathfrak{q}_i = A \cap \mathfrak{q}_i A_{\mathfrak{p}_i} = A \cap I_{\mathfrak{p}_i}$ only depends on I and $\mathfrak{p}_i$. q.e.d.

The annihilator of $\bar{a} \in A/I$ is denoted by $(I : a) := \{b \in A \colon ba \in I\} \supseteq I$.
If $\mathfrak{q}$ is $\mathfrak{p}$-primary, then $(\mathfrak{q} : a)$ is $\mathfrak{p}$-primary when $a \notin \mathfrak{q}$, and $(\mathfrak{q} : a) = A$ when $a \in \mathfrak{q}$.

Theorem: *Let $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a reduced primary decomposition of an ideal I of a noetherian ring A. The* ***associated*** *primes $\mathfrak{p}_i = \operatorname{rad}\mathfrak{q}_i$ are just the prime ideals coinciding with the annihilator of some element of A/I; hence they do not depend on the decomposition.*

Proof: If $\mathfrak{p} = (I : a) = \bigcap_i (\mathfrak{q}_i : a)$ is prime, taking radicals we see that $\mathfrak{p}$ is an intersection of some ideals $\mathfrak{p}_i$; hence $\mathfrak{p} = \mathfrak{p}_i$ for some index i.

Conversely, if $a \in \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$, $a \notin \mathfrak{q}_1$, then $(I : a) = (\mathfrak{q}_1 : a)$ is a $\mathfrak{p}_1$-primary ideal, and we may consider the first power $\mathfrak{p}_1^r$ contained in $(\mathfrak{q}_1 : a)$.

Now, if $b \in \mathfrak{p}_1^{r-1}$ and $b \notin (\mathfrak{q}_1 : a)$, then $(I : ab) = (\mathfrak{q}_1 : ab) = \mathfrak{p}_1$.

Corollary: *The union of the associated primes of the ideal 0 is the set of all zero divisors.*

Proof: Let $0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$. If a is zero divisor, $ab = 0$ and $b \notin \mathfrak{q}_i$, then $a \in \mathfrak{p}_i$.

Corollary: *Any finitely generated module M over a noetherian ring A admits a chain of submodules $0 = M_0 \subset M_1 \ldots \subset M_n = M$ such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$, with $\mathfrak{p}_i$ a prime ideal.*

Proof: By the above theorem, any monogenous submodule has some element m of prime annihilator $\mathfrak{p}$. We put $M_1 = Am \simeq A/\mathfrak{p}$, and we conclude since M is noetherian.

7.3 Completion

Definition: Let I be a proper ideal of a ring A, and let M be an A-module. We put

$$\|m\| = \begin{cases} e^{-n} & \text{if } m \in I^n M,\ m \notin I^{n+1}M \\ 0 & \text{if } m \in \bigcap I^n M \end{cases}$$

The pseudometric $\mathrm{d}(m, m') = \|m' - m\|$ defines the ***I*-adic topology** of M, and the submodules $I^n M$ form a base of open (and closed) neighborhoods of 0.

Artin-Rees Lemma: *Let A be a noetherian ring. If M' is a submodule of a finitely generated module M, there is an exponent h such that*

$$I^{n-h}(M' \cap I^h M) = M' \cap I^n M,\ n \geq h.$$

Proof: Put $A_D = A \oplus I \oplus \ldots \oplus I^n \oplus \ldots$

$$M_D = M \oplus IM \oplus \ldots \oplus I^n M \oplus \ldots$$

M_D is an A_D-module, with the obvious product $I^n \times I^m M \to I^{n+m}M$.

A_D and M_D are noetherian: If $I = (\xi_1, \ldots, \xi_s)$, then $A_D = A[\xi_1 t, \ldots, \xi_s t] \subset A[t]$, and M_D is a finite A_D-module. The following submodule of M_D,

$$N = M' \oplus M'_1 \oplus \ldots \oplus M'_n \ldots, \quad \text{where } M'_n = M' \cap I^n M,$$

is finite, generated by $e_1, \ldots, e_r$, that we may assume homogeneous of degree $\leq h$.

Now any homogeneous element of N of degree $\geq h$ is

$$\sum_{i=1}^{r} a_i e_i = \sum_{i=1}^{r} \sum_j b_{ij} a_{ij} e_i, \quad \text{where } \deg(a_{ij} e_i) = h.$$

Since $a_{ij} e_i \in M'_h$, we see that $A_D M'_h = \bigoplus_{n \geq h} M'_n$; i.e., $I^{n-h} M'_{n-h} = M'_n$.

Corollary: *The I-adic topology of M induces on M' the I-adic topology of M'.*

Proof: $I^n M' \subseteq M'_n$, and $M'_{n+h} = I^n M'_h \subseteq I^n M'$.

Corollary: *If A is noetherian, any finitely generated A-module M is separated, $\bigcap_n I^n M = 0$.*

Proof: Localizing at a point of $\operatorname{Spec} A$ we may assume that A is local.

Since the topology induced on $N = \bigcap_n I^n M$ is trivial, we have $N = IN$.

Finally, $N = 0$ by Nakayama's lemma.

Definition: $\widehat{M}$ denotes the **completion** of M, formed by classes of Cauchy sequences.

We have a canonical morphism $M \to \widehat{M}$, and any morphism of A-modules $f \colon M \to N$ naturally induces a morphism of $\widehat{A}$-modules $\widehat{f} \colon \widehat{M} \to \widehat{N}$, $\widehat{f}(m_n) = (f(m_n))$.

Proposition: $\widehat{M} = \varprojlim M/I^n M$.

Proof: If (m_n) is a Cauchy sequence, there is a subsequence such that $\|m_i - m_j\| \leq e^{-n}$ when $i, j \geq n$, and $(\bar{m}_n) \in \varprojlim M/I^n M$ since $m_i - m_n \in I^n M$ for all $i \geq n$.

If $(\bar{m}_n) \in \varprojlim M/I^n M$, then $\|m_i - m_j\| \leq e^{-n}$ when $i, j \geq n$, and (m_n) is a Cauchy sequence.

These morphisms are well defined and mutually inverse. q.e.d.

1. Any Cauchy sequence (m_n) is equivalent to a series $\sum_n s_n$, $s_n = m_{n+1} - m_n \in I^n M$.

2. Let $\mathcal{O}$ be the local ring at the origin of $\mathbb{A}_n := \operatorname{Spec} k[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$, $\mathcal{O}/\mathfrak{m}^{d+1} =$ {Polynomials of degree $\leq d$}. The $\mathfrak{m}$-adic completion is $\widehat{\mathcal{O}} = k[[x_1, \ldots, x_n]]$, and the morphism $\mathcal{O} \to \widehat{\mathcal{O}}$ assigns to any germ the Taylor expansion at the origin.

3. Let $(\mathcal{O}, \mathfrak{m})$ be the local ring at the origin of a plane curve $0 = y +$ (terms of degree≥ 2). By Nakayama's lemma $\mathfrak{m} = (x)$; hence $\mathfrak{m}^n = (x^n)$, $k[x]/(x^n) = \mathcal{O}/\mathfrak{m}^n$, and $\widehat{\mathcal{O}} = k[[x]]$.

4. If $\mathcal{O}$ is the ring of germs at the origin of $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$, then $\bigcap \mathfrak{m}^n$ is the ideal of all germs with null Taylor expansion, and $e^{-1/x^2} \in \bigcap \mathfrak{m}^n$. Hence $\mathcal{O}$ is not noetherian.

5. $\widehat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the ring of p-adic numbers.

Theorem: *Let A be a noetherian ring. If $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is an exact sequence of finitely generated A-modules, then so is exact the sequence*

$$0 \longrightarrow \widehat{M}' \xrightarrow{\hat{i}} \widehat{M} \xrightarrow{\hat{p}} \widehat{M}'' \longrightarrow 0$$

Proof: We put $M'_n = M' \cap I^n M$. We have exact sequences

$$0 \longrightarrow M'/M'_n \longrightarrow MI^nM \longrightarrow M''/I^nM'' \longrightarrow 0$$
$$0 \longrightarrow \varprojlim M'/M'_n \xrightarrow{\hat{i}} \widehat{M} \xrightarrow{\hat{p}} \widehat{M}''$$

and $\varprojlim M'/M'_n = \widehat{M}'$ since $\{M'_n\}$ defines, by Artin-Rees, the I-adic topology on M'.

Now $\hat{p}$ is surjective: $I^nM \to I^nM''$ is surjective, and any series $\sum_n s''_n$, with $s''_n \in I^nM''$, is

$$\textstyle\sum_n s''_n = \sum_n p(s_n) = \hat{p}\big(\sum_n s_n\big), \text{ where } s_n \in I^nM.$$

Theorem: $M \otimes_A \widehat{A} = \widehat{M}$*, when A is noetherian and M is finitely generated.*

Proof: If L is free, $L = A^n$, then we have $\widehat{L} = \widehat{A}^n = L \otimes_A \widehat{A}$.

In general, we fix a presentation $A^m = L' \to L \to M \to 0$, and we conclude

$$\begin{array}{ccccccc} L' \otimes_A \widehat{A} & \longrightarrow & L \otimes_A \widehat{A} & \longrightarrow & M \otimes_A \widehat{A} & \longrightarrow & 0 \\ \| & & \| & & \downarrow & & \\ \widehat{L}' & \longrightarrow & \widehat{L} & \longrightarrow & \widehat{M} & \longrightarrow & 0 \end{array}$$

Corollary: *If A is noetherian, $\widehat{A}$ is a flat A-algebra.*

Proof: Let $0 \to M' \to M$ be exact. Now, M is an inductive limit of finitely generated submodules M_i, the submodules $M'_i = M' \cap M_i$ are finitely generated, and $M' = \varinjlim M'_i$. Therefore

$$0 \longrightarrow M'_i \otimes_A \widehat{A} \longrightarrow M_i \otimes_A \widehat{A}$$

and $0 \to M' \otimes_A \widehat{A} \to M \otimes_A \widehat{A}$ is exact because inductive limits preserve exact sequences.

Corollary: $I^n\widehat{A} = \widehat{I}^{\,n}$, $A/I^n = \widehat{A}/\widehat{I}^{\,n}$, $I^n/I^{n+1} = \widehat{I}^{\,n}/\widehat{I}^{\,n+1}$, *when A is noetherian.*

Proof: $0 \longrightarrow I \longrightarrow A$ is exact and, since $\widehat{A}$ is A-flat,

$0 \longrightarrow I \otimes_A \widehat{A} \longrightarrow \widehat{A}$ is exact. Hence $I\widehat{A} = I \otimes_A \widehat{A} = \widehat{I}$, and $I^n\widehat{A} = \big(\widehat{I}\,\big)^n$.

$0 \longrightarrow I^n \longrightarrow A \longrightarrow A/I^n \longrightarrow 0$ is exact; hence so is

$0 \longrightarrow I^n \otimes_A \widehat{A} \longrightarrow \widehat{A} \longrightarrow A/I^n \longrightarrow 0$

since A/I^n is complete (as any other module annihilated by I^n).

This exact sequence shows that $A/I^n = \widehat{A}/I^n\widehat{A} = \widehat{A}/\widehat{I}^n$; hence $I^n/I^{n+1} = \widehat{I}^{\,n}/\widehat{I}^{\,n+1}$.

Corollary: *If A is noetherian, $\widehat{A}$ is complete and separated for the $\widehat{I}$-adic topology.*

If $(\mathcal{O}, \mathfrak{m})$ is a noetherian local ring, $\widehat{\mathcal{O}}$ is a complete local ring of maximal ideal $\widehat{\mathfrak{m}}$.

Proof: $\varprojlim \widehat{A}/\widehat{I}^{\,n} = \varprojlim A/I^n = \widehat{A}$.

To prove that $\widehat{\mathcal{O}}$ is local, of maximal ideal $\widehat{\mathfrak{m}}$, we have to show that $1 + f$ is invertible for any $f \in \widehat{\mathfrak{m}}$. Since $\widehat{\mathcal{O}}$ is complete, the series $(1+f)^{-1} = 1 - f + f^2 - f^3 + \ldots$ converges.

Definitions: An I-**filtration** of M is a chain of submodules $M = M_0 \supseteq M_1 \supseteq \ldots$ such that $IM_n \subseteq M_{n+1}$. The associated **graded module**

$$GM = M/M_1 \oplus M_1/M_2 \oplus \ldots \oplus M_n/M_{n+1} \ldots$$

is a module over the graded ring $G_I A = A/I \oplus I/I^2 \oplus \ldots \oplus I^n/I^{n+1} \ldots$, and the completion $\widehat{M} = \varprojlim M/M_n$ is an $\widehat{A}$-module.

A morphism $f\colon M \to N$ is **compatible** with the filtrations if $f(M_n) \subseteq N_n$, so that it induces a morphism of $G_I A$-modules $f'\colon GM \to GN$, and a morphism of $\widehat{A}$-modules $\hat{f}\colon \widehat{M} \to \widehat{N}$.

Example: Let $\mathfrak{m} = (x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$, and let $A = k[x_1, \ldots, x_n]/I$ be the ring of functions on a subvariety $X = \operatorname{Spec} A$ passing through the origin, $\bar{\mathfrak{m}} = (x_1, \ldots, x_n) \subset A$.

The exact sequence $\mathfrak{m}^r \cap I \to \mathfrak{m}^r \to \bar{\mathfrak{m}}^r \to 0$ induces an exact sequence

$$\mathfrak{m}^r \cap I \longrightarrow \mathfrak{m}^r/\mathfrak{m}^{r+1} \longrightarrow \bar{\mathfrak{m}}^r/\bar{\mathfrak{m}}^{r+1} \longrightarrow 0$$

and $G_{\bar{\mathfrak{m}}} A = k[x_1, \ldots, x_n]/I_{\text{in}}$, where I_{in} is the ideal of the **tangent cone** to X at the origin, $I_{\text{in}} = (f_r)_{f \in I}$, $f = f_r +$ (terms of higher degree).

Now, in Algebra we have a weak Inverse Function Theorem:

Lemma: *If $f'\colon GM \to GN$ is surjective (injective), so is $\hat{f}\colon \widehat{M} \to \widehat{N}$.*

Proof: If the morphisms $M_i/M_{i+1} \to N_i/N_{i+1}$ are surjective, and $n \in \widehat{N}$,

$$\begin{aligned} n &= f(m_0) \ \text{ in } N/N_1, \text{ where } m_0 \in M, \\ n &= f(m_0) + f(m_1) \ \text{ in } N/N_2, \text{ where } m_1 \in M_1, \text{and so on} \end{aligned}$$

so that $m = \sum_i m_i \in \widehat{M}$, and $\hat{f}(m) = \sum_i f(m_i) = n$.

If f' is injective, and $0 \neq (m_i) \in \widehat{M} = \varprojlim M/M_i$, we consider the first index i such that $0 \neq m_i \in M/M_i$, so that $m_i \in M_{i-1}/M_i \hookrightarrow N_{i-1}/N_i$, and we see that $(f(m_i)) \neq 0$.

Lemma: *Let A be a complete and separated ring with the I-adic topology. If $G_I A$ is a noetherian ring, then A is a noetherian ring .*

Proof: Let $\mathfrak{q}$ be an ideal of A, that we filter with $\mathfrak{q} \cap I^n$. The ideal $G\mathfrak{q}$ of $G_I A$ is generated by a finite number of homogeneous elements, and each generator of degree d defines a morphism $A \to \mathfrak{q}$, compatible with the filtrations if we put $A_0 = \ldots = A_d = A$, $A_{d+n} = I^n$.

So we get a morphism $A^r = L \to \mathfrak{q}$ such that $GL \to G\mathfrak{q}$ is surjective and, by the above lemma, $\widehat{L} \to \widehat{\mathfrak{q}}$ is surjective; hence $L \to \mathfrak{q}$ is surjective,

$$\begin{array}{ccc} L & \xrightarrow{\sim} & \widehat{L} \\ \downarrow & & \downarrow \\ \mathfrak{q} & \hookrightarrow & \widehat{\mathfrak{q}} \end{array} \qquad (\mathfrak{q} \text{ is separated since so is } A)$$

Theorem: *If A is a noetherian ring, then $\widehat{A}$ is a noetherian ring.*

Proof: $\widehat{A}$ is complete and separated for the $\widehat{I}$-adic topology, and $G_{\widehat{I}}\widehat{A} = G_I A$ is noetherian.

7.4 Dimension Theory

Definition: Let A be a ring. The (Krull) **dimension** of A is the maximal length of all chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ of prime ideals (or irreducible closed sets in $X = \operatorname{Spec} A$).

If $x, y \in X$, we put $x \leq y$ when $x \in \overline{y}$, i.e. $\mathfrak{p}_y \subseteq \mathfrak{p}_x$, and we say that x is a **specialization** of y. So, $\dim A$ is the maximal length of all specializations $x_0 > x_1 > \ldots > x_n$.

Definitions: Let $A = A_0 \oplus A_1 \oplus \ldots \oplus A_n \oplus \ldots$ be a **graded ring**, $A_m \cdot A_n \subseteq A_{m+n}$.

Let $M = \oplus_n M_n$ be a **graded A-module**, $A_m \cdot M_n \subseteq M_{m+n}$.

If $A = A_0[\xi_1, \ldots, \xi_d]$, where A_0 is a ring of finite length (hence noetherian) and $\deg \xi_i = 1$, and M is a finitely generated A-module, then the A_0-modules M_n are of finite length. The **Hilbert** and **Samuel functions** of M are $H(n) = l(M_n)$ and $S(n) = l(M_0) + \ldots + l(M_{n-1})$; so that $\Delta S(n) = S(n+1) - S(n) = H(n)$.

Theorem: *$H(n)$ is a polynomial function of degree $< d$ when $n \gg 0$; hence $S(n)$ is a polynomial function of degree $\leq d$ when $n \gg 0$.*

Proof: By induction on d. If $A = A_0$, then $M = A_0 m_1 + \ldots + A_0 m_s$, and $M_n = 0$ when n is bigger that the degree of any generator m_i. The Hilbert polynomial is $H = 0$.

If $d \geq 1$, we consider the following exact sequences

$$0 \longrightarrow \mathrm{Ker}_n \longrightarrow M_n \xrightarrow{\xi_d} M_{n+1} \longrightarrow \mathrm{Coker}_{n+1} \longrightarrow 0$$

$$\Delta H(n) = H(n+1) - H(n) = l(\mathrm{Coker}_{n+1}) - l(\mathrm{Ker}_{\,n})$$

Since A is noetherian, $\mathrm{Ker} = \oplus_n \mathrm{Ker}_n$ and $\mathrm{Coker} = \oplus_n \mathrm{Coker}_n$ are finite A-modules, and they are annihilated by ξ_d; hence both are finite $A_0[\xi_1, \ldots, \xi_{d-1}]$-modules.

By induction, $l(\mathrm{Coker}_{n+1})$ and $l(\mathrm{Ker}_n)$, hence the difference $\Delta H(n)$, are polynomial functions of degree $< d-1$ when $n \gg 0$, and $H(n)$ is a polynomial function of degree $< d$.

Example: The Samuel function of $k[x_1, \ldots, x_d]$ is $S(n) = \binom{n+d-1}{d} = \frac{n^d}{d!} + \ldots$.

In fact, $\mathrm{Ker} = 0$, $\mathrm{Coker} = k[x_1, \ldots, x_{d-1}]$, and $\Delta S(n) = l(\mathrm{Coker}_{n+1}) = \binom{n+d-1}{d-1}$ by induction on d. Hence $S(n) = \binom{n+d-1}{d} + c$. Since $S(1) = 1$, the constant c is null.

Theorem: *A noetherian local ring $\mathcal{O}$ has finite length if and only if* $\dim \mathcal{O} = 0$.

Proof: If $l(\mathcal{O}) < \infty$, then $\mathfrak{m}^m = \mathfrak{m}^{n+1}$ for some exponent, and $\mathfrak{m}^n = 0$ by Nakayama's lemma.

The unique prime ideal of $\mathcal{O}$ is $\mathfrak{m}$, and $\dim \mathcal{O} = 0$.

If $\dim \mathcal{O} = 0$, then the unique prime ideal of $\mathcal{O}$ is $\mathfrak{m}$, and some power of $\mathfrak{m} = \operatorname{rad} \mathcal{O}$ is null, $\mathcal{O} = \mathcal{O}/\mathfrak{m}^n$. Hence the $\mathcal{O}/\mathfrak{m}$-vector spaces $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ have finite dimension, and

$$l(\mathcal{O}) = l(\mathcal{O}/\mathfrak{m}^n) = l(\mathcal{O}/\mathfrak{m}) + l(\mathfrak{m}/\mathfrak{m}^2) + \ldots + l(\mathfrak{m}^{n-1}/\mathfrak{m}^n) < \infty.$$

Definition: Let $(\mathcal{O}, \mathfrak{m})$ be a noetherian local ring. If $\mathfrak{q} = (f_1, \ldots, f_d)$ is an ideal of radical $\mathfrak{m}$, then $l(\mathcal{O}/\mathfrak{q}) < \infty$, and $G_{\mathfrak{q}}\mathcal{O} = (\mathcal{O}/\mathfrak{q})[\xi_1, \ldots, \xi_d]$, where $\xi_i = [f_i] \in \mathfrak{q}/\mathfrak{q}^2$ has degree 1.

A $\mathfrak{q}$-filtration $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ is **stable** if there is an index h such that $\mathfrak{q}^n M_h = M_{n+h}$ (recall Artin-Rees). In such a case, if M is finitely generated, then GM is a finitely generated $G_{\mathfrak{q}}\mathcal{O}$-module, with a **Samuel polynomial** $S(n)$ of degree $\leq d$,

$$S(n) = l(M/M_1) + \ldots + l(M_{n-1}/M_n) = l(M/M_n), \; n \gg 0.$$

Lemma: *The degree and first coefficient of $S(n)$ do not depend on the stable $\mathfrak{q}$-filtration.*

Proof: Let us consider the $\mathfrak{q}$-adic filtration, $S'(n) = l(M/\mathfrak{q}^n M)$, $n \gg 0$.

We have $\mathfrak{q}^n M \subseteq M_n$ in any $\mathfrak{q}$-filtration, and $M_{n+h} \subseteq \mathfrak{q}^n M$ if it is stable.

Hence the degree and first coefficient of $S(n)$ and $S'(n)$ coincide,

$$S'(n) \geq S(n),\ S(n+h) \geq S'(n)\ , \quad n \gg 0.$$

Lemma: *The degree of the Samuel polynomial does not depend on the* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$.

Proof: Let $S(n) = l(M/\mathfrak{q}^n M)$, and $S'(n) = l(M/\mathfrak{m}^n M)$ when $n \gg 0$. Since $\mathfrak{m}^k \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, we have $S'(kn) \geq S(n) \geq S'(n)$, $n \gg 0$; so that the degrees of $S(n)$ and $S'(n)$ coincide.

Definition: The **Samuel polynomial** $S_{\mathcal{O}}$ of a local ring $(\mathcal{O}, \mathfrak{m})$ is the Samuel polynomial of the $\mathfrak{m}$-adic filtration, $S_{\mathcal{O}}(n) = l(\mathcal{O}/\mathfrak{m}^n\mathcal{O})$, $n \gg 0$.

Since $S_{\mathcal{O}}(n)$ is integer when $n \gg 0$, we have $S_{\mathcal{O}}(n) = m_d\binom{n}{d} + \ldots + m_1\binom{n}{1} + m_0 = \frac{m_d}{d!}n^d + \ldots$ for some integers $m_0, \ldots, m_d$, and m_d is said to be the **multiplicity** of $\mathcal{O}$.

Example: Let $\mathcal{O}$ be the local ring at the origin of $0 = P_m(x_1, \ldots, x_d) +$ (terms of degree $> m$). We have an exact sequence

$$0 \longrightarrow k[x_1, \ldots, x_d] \xrightarrow{P_m} k[x_1, \ldots, x_d] \longrightarrow k[x_1, \ldots, x_d]/(P_m) = G_{\mathfrak{m}}\mathcal{O} \longrightarrow 0$$

so that $S_{\mathcal{O}}(n) = \binom{n+d-1}{d} - \binom{n+d-m-1}{d} = \frac{m}{(d-1)!}\,n^{d-1} + \ldots$, and the multiplicity is m.

Theorem: $\deg S_{\mathcal{O}/f\mathcal{O}} < \deg S_{\mathcal{O}}$*, when* $f \in \mathfrak{m}$ *is not a zero divisor.*

Proof: $0 \to f\mathcal{O}/(f\mathcal{O} \cap \mathfrak{m}^n) \to \mathcal{O}/\mathfrak{m}^n \to \bar{\mathcal{O}}/\bar{\mathfrak{m}}^n \to 0$ is exact, where $\bar{\mathcal{O}} = \mathcal{O}/f\mathcal{O}$.

$$S_{\bar{\mathcal{O}}}(n) = S_{\mathcal{O}}(n) - l(f\mathcal{O}/(f\mathcal{O} \cap \mathfrak{m}^n))\ , \quad n \gg 0.$$

Now, $\mathfrak{m}^n \cap f\mathcal{O}$ is a stable $\mathfrak{m}$-filtration of $f\mathcal{O}$ by Artin-Res, and $f\mathcal{O} \simeq \mathcal{O}$ since f is not a zero divisor. Hence, $l(f\mathcal{O}/(f\mathcal{O}\cap\mathfrak{m}^n))$ and $S_{\mathcal{O}}(n)$ are polynomials of equal degree and first coefficient, when $n \gg 0$, and the difference has lower degree.

Definition: Some functions $f_1, \ldots, f_d \in \mathcal{O}$ form a **system of parameters** if they generate an ideal of radical $\mathfrak{m}$; i.e., $\dim \mathcal{O}/(f_1, \ldots, f_d) = 0$.

Theorem: *The dimension of a noetherian local ring* $\mathcal{O}$ *coincides with the minimum number of parameters, and with the degree of the Samuel polynomial* $S_{\mathcal{O}}(n)$.

1. $\dim \mathcal{O} \geq$ minimum number of parameters.

   Let $x_1, \ldots, x_s$ be the generic points of the irreducible components of $\operatorname{Spec} \mathcal{O}$. Since functions separate points and closed sets, there exists $f_i \in \mathcal{O}$ vanishing at all of them, except x_i. Now $f = f_1 + \ldots + f_s$ does not vanish at any point x_i, and therefore $\dim \mathcal{O}/f\mathcal{O} < \dim \mathcal{O}$. We conclude by induction on $\dim \mathcal{O}$.

2. Minimum number of parameters $\geq \deg S_{\mathcal{O}}$.

   If $\mathfrak{q} = (f_1, \ldots, f_d)$ is $\mathfrak{m}$-primary, the degree of the Samuel polynomial of any stable $\mathfrak{q}$-filtration is $\leq d$; hence so is the degree of $S_{\mathcal{O}}$ by the above lemma.

3. $\deg S_{\mathcal{O}} \geq \dim \mathcal{O}$.

   By induction on the degree of $S_{\mathcal{O}}$. If it is 0, then $l(\mathcal{O}/\mathfrak{m}^n)$ is constant when $n \gg 0$, and $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Hence $\mathfrak{m}^n = 0$ by Nakayama, and $\dim \mathcal{O} = 0$.

   If $\deg S_{\mathcal{O}} \geq 1$, and $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_d$ is a chain of prime ideals of $\mathcal{O}$, take $f \in \mathfrak{p}_1$ not in $\mathfrak{p}_0$ and put $\bar{\mathcal{O}} = \mathcal{O}/(\mathfrak{p}_0 + f\mathcal{O})$. By the above theorem, $\deg S_{\mathcal{O}} \geq \deg S_{\mathcal{O}/\mathfrak{p}_0} > \deg S_{\bar{\mathcal{O}}}$, and by induction $\deg S_{\bar{\mathcal{O}}} \geq d-1$, since $\bar{\mathfrak{p}}_1 \subset \ldots \subset \bar{\mathfrak{p}}_d$ is a chain of prime ideals of $\bar{\mathcal{O}}$. Hence $\deg S_{\mathcal{O}} \geq d$, and $\deg S_{\mathcal{O}} \geq \dim \mathcal{O}$.

Corollary: 1. *The dimension of a noetherian local ring $\mathcal{O}$ is finite.*
2. $\dim (f)_0 \geq \dim \mathcal{O} - 1$, *when* $f \in \mathfrak{m}$. (**Krull's Theorem**)
3. $\dim \mathcal{O} = \dim \widehat{\mathcal{O}}$.
4. $\dim \mathcal{O} \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2)$, *where* $K = \mathcal{O}/\mathfrak{m}$.

Proof: (1) The minimal number of parameters (or the degree of $S_{\mathcal{O}}$) is finite.

(2) If $\bar{f}_1, \ldots, \bar{f}_r$ is a system of parameters of $\mathcal{O}/f\mathcal{O}$, then $f_1, \ldots, f_r, f$ is a system of parameters of $\mathcal{O}$; hence $\dim \mathcal{O} \leq 1 + \dim \mathcal{O}/f\mathcal{O}$.

(3) $\mathcal{O}$ and $\widehat{\mathcal{O}}$ have equal Samuel functions, since $\mathcal{O}/\mathfrak{m}^n = \widehat{\mathcal{O}}/\widehat{\mathfrak{m}}^n$.

(4) By Nakayama's lemma, a base of $\mathfrak{m}/\mathfrak{m}^2$ defines a generating system of $\mathfrak{m}$.

Definitions: Let $\mathcal{O}$ be a noetherian local ring, of residue field $K = \mathcal{O}/\mathfrak{m}$.
The **tangent cone** is $\operatorname{Spec} G_{\mathfrak{m}}\mathcal{O}$, and the **cotangent space** is $\mathfrak{m}/\mathfrak{m}^2$. The ring $\mathcal{O}$ is **regular** if $\mathfrak{m}$ is generated by a minimal system of parameters, $\dim \mathcal{O} = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$.

Proposition: *A noetherian local ring $\mathcal{O}$ is regular $\Leftrightarrow$ the tangent cone is an affine space,*

$$G_{\mathfrak{m}}\mathcal{O} = K[x_1, \ldots, x_d] \quad \text{(isomorphism of graded rings).}$$

Proof: If $G_{\mathfrak{m}}\mathcal{O} = K[x_1, \ldots, x_d]$, then $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = d = \deg S_{\mathcal{O}} = \dim \mathcal{O}$.
If $\mathcal{O}$ is regular of dimension d, we have an epimorphism

$$K[x_1, \ldots, x_d] \longrightarrow G_{\mathfrak{m}}\mathcal{O} \longrightarrow 0,$$

and the Samuel polynomial of both rings at the origin has degree d.
Since the local ring of $K[x_1, \ldots, x_n]$ is integral, and the degree of the Samuel polynomial of the quotient by a non zero divisor decreases, it is an isomorphism.

Corollary: *A noetherian local ring $\mathcal{O}$ is regular if and only if $\widehat{\mathcal{O}}$ is regular.*

Proof: $G_{\mathfrak{m}}\mathcal{O} = G_{\widehat{\mathfrak{m}}}\widehat{\mathcal{O}}$.

Corollary: *Regular local rings are integral.*

Proof: Let $a, b \in \mathcal{O}$ be non null. Since $0 = \bigcap_n \mathfrak{m}^n$, some $\bar{a} \in \mathfrak{m}^i/\mathfrak{m}^{i+1}$, $\bar{b} \in \mathfrak{m}^j/\mathfrak{m}^{j+1}$ are non null and, $G_{\mathfrak{m}}\mathcal{O}$ being integral, $0 \neq \bar{a} \cdot \bar{b} = [ab] \in \mathfrak{m}^{i+j}/\mathfrak{m}^{i+j+1}$. Hence $ab \neq 0$.

Theorem: *If $\mathcal{O}$ is a regular local k-algebra and $k = \mathcal{O}/\mathfrak{m}$, then $\widehat{\mathcal{O}} = k[[x_1, \ldots, x_d]]$.*

Proof: Let $\mathfrak{m} = (f_1, \ldots, f_d)$, $d = \dim \mathcal{O}$. The morphism $k[x_1, \ldots, x_d] \to \mathcal{O}$, $x_i \mapsto f_i$, induces an isomorphism after completion, since it induces an isomorphism (p. 203)

$$k[x_1, \ldots, x_d] \xrightarrow{\sim} G_{\mathfrak{m}}\mathcal{O} = (\mathcal{O}/\mathfrak{m})[\bar{f}_1, \ldots, \bar{f}_d].$$

Proposition: *Let $\mathcal{O}$ be a noetherian local k-algebra. If $K = \mathcal{O}/\mathfrak{m}$ is a finite separable extension of k, the ring morphism $\widehat{\mathcal{O}} \to K$ admits a section, and $\widehat{\mathcal{O}} = K[[x_1, \ldots, x_d]]$ if $\mathcal{O}$ is regular.*

Proof: The epimorphisms $\pi_0^k(\mathcal{O}/\mathfrak{m}^n) \to \pi_0^k(K)$ are isomorphisms, because $\pi_0^k(\mathcal{O}/\mathfrak{m}^n)$ is a field (p. 158); hence we have a section $K = \pi_0^k(K) = \varprojlim \pi_0^k(\mathcal{O}/\mathfrak{m}^n) \to \varprojlim \mathcal{O}/\mathfrak{m}^n = \widehat{\mathcal{O}}$.

7.5 Finite Morphisms

Definition: A ring morphism $A \to B$ is **finite** if B is a finitely generated A-module,

$$B = Ab_1 + \ldots + Ab_n.$$

Finite morphisms are stable under base changes $A \to C$, since $B \otimes_A C = \sum_i C(b_i \otimes 1)$, and composition: If $B = Ab_1 + \ldots + Ab_n$, and $C = Bc_1 + \ldots + Bc_m$, then

$$C = \left(\sum_i Ab_i\right)c_1 + \ldots + \left(\sum_i Ab_i\right)c_m = \sum_{i,j} Ab_i c_j.$$

Lemma: *If $f\colon M \to M$ is an endomorphism of a finitely generated A-module, then we have a relation $f^n + a_1 f^{n-1} + \ldots + a_n \mathrm{Id} = 0$, where $a_1, \ldots, a_n \in A$.*

Proof: Since M is a quotient of some free module A^n, we may assume that $M = A^n$.

If the lemma holds for f, then it holds for $f \otimes 1$ for any base change $A \to B$; hence we may assume that A is the polynomial ring $\mathbb{Z}[x_{ij}]$ and f is the endomorphism of A^n given by the generic matrix (x_{ij}), because any endomorphism is obtained from f by a base change.

Now, by Hamilton-Cayley's theorem, the lemma holds after the base change $\mathbb{Z}[x_{ij}] \to \mathbb{Q}(x_{ij})$, because $\mathbb{Q}(x_{ij})$ is a field, and we conclude that it also holds in $\mathbb{Z}[x_{ij}]$, because $\mathbb{Z}[x_{ij}] \to \mathbb{Q}(x_{ij})$ is injective.

Theorem: *Let $A \to B$ be a morphism, and $b \in B$. The following conditions are equivalent:*

1. *b satisfies an* ***integral dependence*** *relation $b^n + a_1 b^{n-1} + \ldots + a_n = 0$, $a_i \in A$.*

2. *$A[b]$ is a finite subalgebra of B.*

3. *b is in a finite subalgebra of B.*

Proof: ($1 \Rightarrow 2$) If $b^n + a_1 b^{n-1} + \ldots + a_n$, then $A[b] = A + Ab + \ldots + Ab^{n-1}$.

($3 \Rightarrow 1$) If $C \subseteq B$ is a finite subalgebra, and $b \in C$, then $C \xrightarrow{b\cdot} C$ is an endomorphism of a finitely generated A-module, and by the lemma there exist $a_1, \ldots, a_n \in A$ such that $(b^n + a_1 b^{n-1} + \ldots + a_n)c = 0$, $\forall c \in C$. Now put $c = 1$.

Corollary: *The integral elements of B form a subalgebra (the* ***integral closure*** *of A in B).*

Proof: If $b_1, b_2 \in B$ are integral, the morphisms $A \to A[b_1] \to A[b_1, b_2]$ are finite.

Hence any element of $A[b_1, b_2]$ (in particular $b_1 + b_2$, $b_1 b_2$ and ab_1, $\forall a \in A$) is integral.

Example: The morphism $A = k[x] \to k[x, y]/(P)$ is finite if and only if the curve $P(x, y) = 0$ has no vertical asymptote (if the projective curve $P(x_0, x_1, x_2) = 0$ passes through the point $p = (0, 0, 1)$, the tangent cone at p is the line at infinity, with certain multiplicity).

In fact, y is integral when a multiple of P (hence P) is $y^n + p_1(x)y^{n-1} + \ldots + p_n(x)$. Homogenizing, and then dehomogenizing with x_2 so that $p = (0, 0, 1)$ is the origin, this condition states that, if the curve passes through p, the tangent cone is $x_0^{d-n} = 0$,

$$0 = x_0^{d-n} + (\text{terms of degree} > d - n), \quad \text{where } d = \deg P.$$

Theorem: *Any finite morphism $\pi\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ has finite discrete fibres, and π is surjective when $A \to B$ is injective.*

Proof: In fact $\pi^{-1}(x) = \operatorname{Spec}(B \otimes_A \kappa(x))$, and $B \otimes_A \kappa(x)$ is a finite $\kappa(x)$-algebra (p. 145).
If $A \to B$ is injective, then so is $A_x \to B_x$, and $B_x \neq 0$.
Now $B_x/\mathfrak{p}_x B_x \neq 0$ by Nakayama, and $\pi^{-1}(x) = \operatorname{Spec}(B_x/\mathfrak{p}_x B_x)$ is not void.

Corollary: *If $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a finite morphism, then* $\dim B \leq \dim A$.

Proof: No chain of specializations $y_0 > y_1 > \ldots > y_n$ in $\operatorname{Spec} B$ may have a coincidence $\pi(y_{i-1}) = \pi(y_i)$, since the fibre of $\pi(y_i)$ is discrete.

Going Up Theorem: *Finite morphisms are closed maps.*

Proof: If J is an ideal of B, the morphism $A/I \to B/J$ is finite and injective, where $I = A \cap J$.
Now the following commutative square shows that $\pi((J)_0) = (I)_0$,

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{\pi} & \operatorname{Spec} A \\ \cup & & \cup \\ (J)_0 = \operatorname{Spec} B/J & \longrightarrow & \operatorname{Spec} A/I = (I)_0 \end{array}$$

Corollary: *If $x' < x = \pi(y)$, there exists $y' < y$ such that $\pi(y') = x'$.*

Proof: The closure of $x = \pi(y)$ is contained in $\pi(\overline{y})$ since π is closed.

Corollary: *If $A \to B$ is a finite injective morphism, then* $\dim A = \dim B$.

Proof: Let $x_0 > \ldots > x_n$ be a chain of specializations in $\operatorname{Spec} A$, and $x_0 = \pi(y_0)$.
There are specializations $y_0 > \ldots > y_n$ such that $\pi(y_i) = x_i$; hence $\dim B \geq \dim A$.

Definition: An integral ring A is **normal** if it is integrally closed in the field of fractions Σ_A.
Unique factorization domains are normal (p. 88).

Lemma: *Let A be a normal ring, let L be a normal extension of the field of fractions Σ_A, and let $B \subset L$ be a finite subalgebra. If B is stable under the action of the group $G = \operatorname{Aut}(L/\Sigma_A)$, then G acts transitively on the fibres of $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$.*

Proof: Let $G = \{\sigma_1, \ldots, \sigma_n\}$, and let y, y' be two points of a fibre $\pi^{-1}(x)$.
If $y' \notin Gy$, since the fibres are discrete, there exists $f \in B$ vanishing at y' and without zeros on the orbit Gy (p. 205). The roots of the irreducible polynomial $t^d + c_1 t^{d-1} + \ldots + c_d$ of f over Σ are $\sigma_i(f) \in B$, with certain multiplicities. Since A is normal, $B \cap \Sigma_A = A$, and

$$c_d = \textstyle\prod_i \sigma_i(f)^{m_i} \in B \cap \Sigma_A = A.$$

Now $c_d(x) = c_d(y') = 0$, since $f(y') = 0$, and $c_d(x) = c_d(y) \neq 0$, since $(\sigma_i f)(y) \neq 0$. Absurd.

Going Down Theorem: *Let $A \to B$ be a finite injective morphism between integral rings. If A is normal, then $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is open.*

Proof: Let L be the normal envelop of Σ_B over Σ_A, and put $G = \operatorname{Aut}(L/\Sigma_A)$.

If $B = A[b_1, \dots, b_n]$, we put $B' = A[\sigma_i(b_j)]$, $\sigma_i \in G$. Since $\rho\colon \operatorname{Spec} B' \to \operatorname{Spec} B$ is surjective, we have $U = \rho(\rho^{-1}(U))$, and we may assume that B is G-invariant.

Finally, $\pi(U)$ is open when so is U, because π is a closed map and by the above lemma

$$\operatorname{Spec} A - \pi(U) = \pi\big(\operatorname{Spec} B - \bigcup_{\sigma \in G} \sigma(U)\big).$$

Corollary: *If $x' > x = \pi(y)$, there exists $y' > y$ such that $\pi(y') = x'$.*

Proof: We have to show that y is in the closure F of the fibre of x', since it is finite.

Since x' is not in the open set $\pi(\operatorname{Spec} B - F)$, neither is the specialization $x = \pi(y)$.

Corollary: $\dim B_y = \dim A_x$, *when $x = \pi(y)$.*

Definition: The **affine space** of dimension n over a field k is $\mathbb{A}_n := \operatorname{Spec} k[x_1, \dots, x_n]$. If $A = k[\xi_1 \dots, \xi_n]$ is a finitely generated k-algebra, its spectrum $X = (A, \operatorname{Spec} A)$ is said to be an **affine algebraic variety** over k, and recall that $\operatorname{Hom}_k(\operatorname{Spec} B, \operatorname{Spec} A) := \operatorname{Hom}_{k\text{-alg}}(A, B)$.

Normalization Lemma: *There is a finite injective morphism $k[x_1, \dots, x_d] \to A = k[\xi_1 \dots, \xi_n]$, where $d \le n$, and $d < n$ when $\xi_1 \dots, \xi_n$ are algebraically dependent.*

Proof: If $\xi_1, \dots, \xi_n$ satisfy an algebraic relation, write it in decreasing lexicographical order,

$$a\xi_1^{r_1}\xi_2^{r_2} \cdots \xi_n^{r_n} + \ldots = 0.$$

We put $\xi_i' = \xi_i - \xi_1^{d_i}$, where $d_1 \gg d_2 \gg \ldots \gg d_{n-1}$.

$A = k[\xi_1', \dots, \xi_{n-1}', \xi_n]$, where $\xi_1', \dots, \xi_{n-1}', \xi_n$ satisfy a relation, with leading term in ξ_n,

$$a\xi_n^{r_1 d_1 + \ldots + r_{n-1}d_{n-1} + r_n},$$

so that $k[\xi_1', \dots, \xi_{n-1}'] \hookrightarrow A$ is a finite morphism. By induction, there exists a finite morphism $k[x_1, \dots, x_d] \hookrightarrow k[\xi_1', \dots, \xi_{n-1}']$, $d \le n-1$, and we conclude.

Corollary: $\dim k[x_1, \dots, x_n] = n$.

Proof: By induction on n. If $0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m$ is a chain of prime ideals in $k[x_1, \dots, x_n]$, let $P \in \mathfrak{p}_1$ be non null, so that the dimension of $A = k[x_1, \dots, x_n]/(P)$ is $\ge m-1$.

The proof of the lemma shows the existence of a finite injective morphism $k[y_1, \dots, y_d] \to A$, $d < n$. By induction $\dim A = d$; hence $m \le n$ and $\dim k[x_1, \dots, x_n] \le n$.

We conclude since it is clear that $\dim k[x_1, \dots, x_n] \ge n$.

Corollary: *Any affine algebraic variety X of dimension d admits a finite projection $X \to \mathbb{A}_d$.*

Proof: If $k[x_1, \dots, x_d] \hookrightarrow A$ is finite, then $d = \dim k[x_1, \dots, x_d] = \dim A$.

Theorem of Zeros: *Finitely generated k-algebras of dimension 0 are finite k-algebras. Hence, any residue field $A/\mathfrak{m}$ of a finitely generated k-algebra A is a finite extension of k.*

Proof: If $\dim A = 0$, there exists a finite morphism $k \to A$.

Corollary: *Morphisms $\phi\colon Y \to X$ of affine algebraic varieties preserve closed points.*

Proof: If $\phi(y) = x$, we have $\kappa(x) \hookrightarrow \kappa(y)$. Now, if y is closed, $\kappa(y)$ is a finite extension of k; hence so is $\kappa(x)$, and $A/\mathfrak{p}_x \subseteq \kappa(x)$ is a field.

Corollary: *The radical of A is the intersection of all maximal ideals.*

Proof: If $f \in A$ vanishes at any closed point of X, then the algebraic variety $U_f = X - (f)_0$ has no closed point; hence it is void, and f vanishes on X.

7.6 Valuations and Dedekind Rings

Proposition: *Let $(\mathcal{O}, \mathfrak{m})$ be a noetherian local domain with residue field $K = \mathcal{O}/\mathfrak{m}$. If $\mathfrak{m}$ is a principal ideal, $\mathfrak{m} = t\mathcal{O}$, then any non null ideal of $\mathcal{O}$ is $\mathfrak{m}^n = t^n\mathcal{O}$ for some exponent $n \in \mathbb{N}$.*

Proof: If $\mathfrak{m} = t\mathcal{O}$, and $0 \neq f \in \mathfrak{m}$, then $f = th$, where $f\mathcal{O} \subset h\mathcal{O}$.

Since $\mathcal{O}$ is noetherian, we have $f = t^n u$, where $u \in \mathcal{O}$ is invertible.

Given an ideal $I \neq 0$, consider the first power $t^n \in I$. Clearly $I = t^n\mathcal{O} = \mathfrak{m}^n$.

Definition: A local domain $\mathcal{O}$, with maximal ideal $\mathfrak{m}$ and field of fractions Σ, is a **discrete valuation** ring when it is a principal ideal domain (i.e. $\mathfrak{m} = t\mathcal{O}$), so that any rational function $0 \neq f \in \Sigma$ is $f = ut^n$, where $u \in \mathcal{O}^*$, $n \in \mathbb{Z}$; and we say that n is the **valuation** $v_x(f)$ of f at the point $x \in \operatorname{Spec}\mathcal{O}$ defined by $\mathfrak{m}$.

We agree that $v_x(0) = \infty$, so that $\mathcal{O} = \{f\Sigma\colon v_x(f) \geq 0\}$, and we have

$$v_x(fg) = v_x(f) + v_x(g) \quad , \quad v_x(f+g) \geq \min\big(v_x(f), v_x(g)\big).$$

1. A **discrete valuation** of a field Σ is any map $v\colon \Sigma - \{0\} \to \mathbb{Z}$ such that $v(fg) = v(f) + v(g)$, $v(f+g) \geq \min\big(v(f), v(g)\big)$, so that $\mathcal{O} := \{f \in \Sigma\colon v(f) \geq 0\}$ is a discrete valuation ring with field of fractions Σ, the maximal ideal being $\mathfrak{m} := \{f \in \Sigma\colon v(f) \geq 1\}$, and v is the corresponding valuation (up to a constant factor $d \geq 2$, when the image of v is $d\mathbb{Z}$).

2. A point of an algebraic variety is **simple** or **non singular** when the local ring is regular. The local ring of a curve at a non singular point x is a discrete valuation ring, and $v_x(f)$ is the number of zeros of f at x (or poles, if it is negative).

3. The discrete valuation of $k(t)$ at $t = \infty$ is defined to be $v_\infty\left(\frac{p(t)}{q(t)}\right) = \deg q - \deg p$.

Theorem: *A noetherian local ring $\mathcal{O}$ of dimension 1 is regular if and only if it is normal.*

Proof: If $\mathcal{O}$ is regular, it is a principal ideal domain; hence it is normal.

If $\mathcal{O}$ is normal and $0 \neq f \in \mathfrak{m}$, consider the first power $\mathfrak{m}^n \subseteq f\mathcal{O}$ (it exists since $\mathfrak{m} = \operatorname{rad} f\mathcal{O}$, because $\dim\mathcal{O} = 1$). Take $h \in \mathfrak{m}^{n-1}$, $h \notin f\mathcal{O}$, and put $x = h/f$,

$$x\mathfrak{m} \subseteq \mathcal{O},\ x \notin \mathcal{O}.$$

To conclude that $\mathfrak{m} = (f_1, \ldots, f_n)$ is a principal ideal it is enough to prove that $x\mathfrak{m} = \mathcal{O}$.

Otherwise $x\mathfrak{m} \subseteq \mathfrak{m}$, $xf_j = \sum_i a_{ij} f_j = 0$, so that the matrix $xI - (a_{ij})$, with coefficients in the field of fractions, has null determinant (it defines a linear system with a non zero solution).

Hence x is integral over $\mathcal{O}$, and $\mathcal{O}$ is not a normal ring.

Definitions: An injective morphism $A \to B$ of integral rings is **birrational** if it induces an isomorphism between the field of fractions, $\Sigma_A \xrightarrow{\sim} \Sigma_B$.

If $(\mathcal{O}, \mathfrak{m})$ is a local ring, an injective morphism $\mathcal{O} \to B$ is **dominant** if $\mathfrak{m} = \mathfrak{m}' \cap \mathcal{O}$ for some maximal ideal $\mathfrak{m}'$ of B; i.e., $\mathfrak{m}B \neq B$.

A domain $\mathcal{V}$, of field of fractions Σ, is a **valuation** ring of Σ when $f \in \mathcal{V}$ or $f^{-1} \in \mathcal{V}$, $\forall f \in \Sigma$. The ring $\mathcal{V} = \Sigma$ is the **trivial** valuation.

The ideals of a valuation ring $\mathcal{V}$ are totally ordered (hence $\mathcal{V}$ is local) because if I, J is an incomparable pair, there are $f \in I$, $f \notin J$, $h \notin I$, $h \in J$, so that $\frac{f}{h} \notin \mathcal{V}$, $\frac{h}{f} \notin \mathcal{V}$.

Lemma: *A local integral ring $\mathcal{V}$ is a valuation ring if and only if any dominant birrational morphism $\mathcal{V} \to B$ is an isomorphism.*

Proof: Let $\mathcal{V}$ be a valuation ring and $\mathcal{V} \to B$ a birrational morphism.

If $b \in B$ is not in $\mathcal{V}$, then $b^{-1} \in \mathcal{V}$ is not invertible; hence $b^{-1} \in \mathfrak{m}_{\mathcal{V}}$, and $\mathfrak{m}_{\mathcal{V}}B = B$, so that the morphism is not dominant.

Conversely, let $f \in \Sigma$, $f \notin \mathcal{V}$. Since $\mathcal{V} \to \mathcal{V}[f]$ is not dominant, then $\mathfrak{m}_{\mathcal{V}}\mathcal{V}[f] = V[f]$,

$$1 = a_0 + a_1 f + \ldots + a_n f^n;\ a_i \in \mathfrak{m}_{\mathcal{V}},$$
$$0 = (a_0 - 1)f^{-n} + a_1 f^{n-1} + \ldots + a_n,$$

Since $a_0 - 1 \in \mathcal{V}^*$, the morphism $\mathcal{V} \to \mathcal{V}[f^{-1}]$ is finite; hence dominant, and $f^{-1} \in \mathcal{V}$.

Theorem: *Let Σ be the field of fractions of a domain A. The integral closure B of A in a finite extension L of Σ is the intersection of all the valuation rings $\mathcal{V}_i$ of L containing A.*

Proof: Valuation rings are normal since finite birrational morphisms are dominant (p. 208); hence $B \subseteq \bigcap \mathcal{V}_i$. On the other hand, $f^{-1} \in L$ is invertible in $A[f^{-1}]$,

$$1 = f^{-1}(a_0 + a_1 f^{-1} + \ldots + a_n f^{-n}),$$
$$f^n = a_0 f^{n-1} + a_1 f^{n-2} + \ldots + a_n,$$

just when f is integral over A. Hence, if $f \notin B$, then f^{-1} is not invertible in $A[f^{-1}]$.

Localizing $A[f^{-1}]$ at a prime ideal containing f^{-1}, and considering a maximal ring for domination (Zorn's lemma), we obtain a valuation ring $\mathcal{V}_i$ not containing f.

Lemma: *Let $A \to B$ be a morphism between integral rings, and let C be the integral closure of A in B. If S is a multiplicative system of A, then C_S is the integral closure of A_S in B_S.*

Hence, an integral ring A is normal if and only if A_x is normal at any point $x \in \operatorname{Spec} A$.

Proof: Clearly $C_S = C \otimes_A A_S$ is integral over A_S. Moreover, if $\frac{b}{s} \in B_S$ is integral over A_S, then we have $\left(\frac{b}{s}\right)^n + \frac{a_1}{s_1}\left(\frac{b}{s}\right)^{n-1} + \ldots + \frac{a_n}{s_n} = 0$, and we may assume that $s = s_1 = \cdots = s_n$.

Multiplying by s^n we obtain an integral dependence relation of $b/1$ over A; hence tb is integral over A for some $t \in S$, and $\frac{b}{s} = \frac{tb}{ts} \in C_S$.

Finally, A is normal if and only if $\bar{A}/A = 0$, and we conclude (p. 142).

Finiteness Theorem: *Let A be a noetherian normal ring and L a finite separable extension of the field of fractions Σ. The integral closure B of A in L is a finitely generated A-module.*

Proof: If $b \in B$, we have $\operatorname{tr}(b) = \sigma_1(b) + \ldots + \sigma_d(b)$, where $\sigma_1, \ldots, \sigma_d \colon L \to L'$ are the points of L with values in a trivializing extension L'. Hence $\operatorname{tr}(b) \in \Sigma$ is integral over A, so that $\operatorname{tr}(B) \subseteq A$.

By the lemma, some $b_1, \dots, b_n \in B$ form a base of L over Σ. Let us consider the dual base $\omega_1, \dots, \omega_n \in L^*$.

The trace metric being non singular (p. 149), it induces a Σ-linear (hence A-linear) isomorphism $T\colon L \to L^*$. Since A is noetherian, it is enough to show that $T(B) \subseteq A\omega_1 + \ldots + A\omega_n$. Now, if $b \in B$, we have

$$b = \operatorname{tr}(b_1 b)\omega_1 + \ldots + \operatorname{tr}(b_n b)\omega_n \in A\omega_1 + \ldots + A\omega_n.$$

Theorem: *Let Σ be the field of fractions of an integral finitely generated k-algebra A. The integral closure of A in any finite extension L of Σ is a finitely generated A-module.*

Proof: We fix an injective finite morphism $k[x_1, \dots, x_n] \to A$. The integral closure of A in L is the integral closure of $k[x_1, \dots, x_n]$ in L, so that we may assume that $A = k[x_1, \dots, x_n]$.

If $\operatorname{char} k = 0$, it follows from the above theorem.

If $\operatorname{char} k = p$, we may assume that L is a normal extension of Σ (p. 158).

If $G = \operatorname{Aut}(L/\Sigma)$, then L is a separable extension of L^G by Artin's theorem, and L^G is a purely inseparable extension of Σ (p. 158). If we show that the integral closure B of A in L^G is a finitely generated A-module, then B is a noetherian normal ring, and the above theorem let us conclude that the integral closure of B in L is a finitely generated B-module, hence a finitely generated A-module.

We may assume that $A = k[x_1, \dots, x_n]$ and L is a purely inseparable extension of Σ; hence (p. 158) there is a power $q = p^r$ such that $L = \Sigma(\alpha_1, \dots, \alpha_r)$, $\alpha_i^q \in \Sigma$.

If $\alpha_i^q = P_i/Q_i$, then $(Q_i\alpha_i)^q = P_i Q_i^{q-1}$, and we may assume that $\alpha_i^q \in A$:

$$\alpha_i^q = \sum_{j_1 \dots j_n} \lambda_{i,j_1 \dots j_n} x_1^{j_1} \dots x_n^{j_n},$$
$$\alpha_i = \sum_{j_1 \dots j_n} \sqrt[q]{\lambda_{i,j_1 \dots j_n}}\, y_1^{j_1} \dots y_n^{j_n}\ , \quad y_j = \sqrt[q]{x_j}.$$

Now $L = k(x_1, \dots, x_n, \alpha_1, \dots, \alpha_r) \to K(y_1, \dots, y_n)$, where $K = k(\sqrt[q]{\lambda_{i,j_1 \dots j_n}}\,)$.

Since $K[y_1, \dots, y_n]$ is normal, and it is finite over $k[x_1, \dots, x_n] = k[y_1^q, \dots, y_n^q]$, it is the integral closure of $A = k[x_1, \dots, x_n]$ in $K(y_1, \dots, y_n)$, and it is a finitely generated A-module.

Dedekind Domains

Definition: An integral noetherian ring A is a **Dedekind domain** if all the local rings A_x, $x \in \operatorname{Spec} A$, are principal ideal domains.

In such a case, any valuation ring $\mathcal{V}$ of the field of fractions Σ_A containing A is a discrete valuation ring A_x: If $\mathfrak{m}_{\mathcal{V}} \cap A = \mathfrak{p}_x$, then $\mathcal{V}$ dominates A_x, and $\mathcal{V} = A_x$. Hence,

$$A = \{f \in \Sigma_A \colon v_x(f) \geq 0,\ \forall x \in \operatorname{Spec} A\}.$$

An integral curve $\operatorname{Spec} A$ has no singular point when A is a Dedekind domain.

The integral closure A of $\mathbb{Z}$ in a finite extension L of $\mathbb{Q}$ is a noetherian ring by the finiteness theorem, hence it is a Dedekind domain.

Theorem: *Any non null ideal I of a Dedekind domain A uniquely decomposes as a product of powers of prime ideals, $I = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_r^{n_r}$.*

Proof: At any point of $(I)_0 = \{x_1, \dots, x_r\}$ we have that I_{x_i} coincides with some power $\mathfrak{p}_i^{n_i}$, since A_{x_i} has no other non null ideals.

Now $I = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_r^{n_r}$ since both ideals coincide at any point of $\operatorname{Spec} A$.
Uniqueness is obvious. q.e.d.

Let A be a Dedekind ring. The support of a finitely generated torsion A-module T is formed by a finite number of closed points; hence T decomposes as a direct sum of the localization at such points and, the local rings A_x being principal ideal domains, T decomposes, uniquely up to isomorphisms, as a direct sum of primary monogenous modules (p. 178),

$$T = \bigoplus_{ij} A/\mathfrak{p}_i^{n_{ij}}.$$

If T is the torsion submodule of a finitely generated A-module M, then $L = M/T$ is locally free since it is torsion free (p. 176); hence it is projective (we shall prove it in p. 218), and $M = T \oplus L$ since the following exact sequence splits

$$0 \longrightarrow T \longrightarrow M \longrightarrow L \longrightarrow 0.$$

If $\operatorname{rk} L = 1$, localizing at the generic point we see that L is a submodule of the field of fractions Σ; hence $fL \subseteq A$, and L is isomorphic to a non null ideal.

Definitions: An isomorphism $I \simeq I'$ between non null ideals induces at the generic point an isomorphism $\Sigma \simeq \Sigma$; hence an homothety, and we see that $I \simeq I' \Leftrightarrow I' = fI$ for some $f \in \Sigma$.

Non null ideals $I = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_r^{n_r}$ correspond to **effective** divisors $D = n_1 x_1 + \ldots + n_r x_r$, $n_i \geq 0$, and if we introduce the divisor of a rational function $0 \neq f \in \Sigma$,

$$D(f) = \sum_{x \in \operatorname{Spec} A} v_x(f)\, x,$$

then $I' \simeq I$ if and only if the corresponding divisors are **linearly equivalent**,

$$D' = D(f) + D \text{ for some } f \in \Sigma.$$

The **Picard group** of isomorphism classes of non null ideals is $\operatorname{Pic}(A) = \operatorname{Div}(A)/\{D(f)\}$ since any divisor is equivalent to an effective divisor. In fact, by the Chinese theorem

$$A/\mathfrak{p}_1^{n_1} \dots \mathfrak{p}_r^{n_r} = (A/\mathfrak{p}_1^{n_1}) \oplus \ldots \oplus (A/\mathfrak{p}_r^{n_r})$$

and we see that, given closed points $x_1, \dots, x_r$ and natural numbers $m_1, \dots, m_r$, there exists a function $f \in A$ such that $v_{x_i}(f) = m_i$.

In general, when $\operatorname{rk} L = r > 1$, we take $m \in L$ non null, and we put

$$L' = \{m' \in L \colon am' \in Am \text{ for some } a \in A\},$$

so that L/L' is a torsion free module of rank $r - 1$.

By induction on the rank, we see that L is a direct sum of ideals (not uniquely),

$$L = I_1 \oplus \ldots \oplus I_r.$$

Lemma: *Given closed points $x_1, \dots, x_r \in \operatorname{Spec} A$, and natural numbers $m_1, \dots, m_r$, there exists $f \in \Sigma$ such that $v_{x_i}(f) = -m_i$, and $v_x(f) \geq 0$ at any other point $x \in \operatorname{Spec} A$.*

Proof: Let $a \in A$ be such that $v_{x_i}(a) = m_i$. If a^{-1} has other poles, we take $b \in A$ vanishing at them with equal order, and $v_{x_i}(b) = 0$. Now take $f = b/a$. q.e.d.

Given ideals I, J without common zeros, $I + J = A$, the exact sequence

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow A \longrightarrow 0$$

shows that $I \oplus J \simeq A \oplus (I \cap J)$. When J has common zeros with I, by the above lemma there exists $f \in \Sigma$ such that fJ and I have no common zeros; hence $I \oplus J \simeq A \oplus I'$ for some ideal I', and we see that $I_1 \oplus \ldots \oplus I_r \simeq I \oplus A^{r-1}$ for some ideal I.

Theorem: *Any finitely generated A-module M decomposes, uniquely up to isomorphisms,*

$$M = I \oplus A^{r-1} \oplus \left(\oplus_{ij} A/\mathfrak{p}_i^{n_{ij}}\right).$$

Proof: We have just proved the existence.

The uniqueness follows from this fact: If $L = I \oplus A^{r-1}$, then $I = \Lambda^r L$.

Corollary: *The K-group of finitely generated locally free A-modules is*

$$K(A) = \mathbb{Z} \oplus \operatorname{Pic}(A).$$

Proof: We have a natural morphism $K(A) \to \mathbb{Z} \oplus \operatorname{Pic}(A)$, $[L] \mapsto (\operatorname{rk} L, \Lambda^{\operatorname{rk} L} L)$, the inverse being the morphism $\mathbb{Z} \oplus \operatorname{Pic}(A) \to K(A)$, $(r, [I]) \mapsto I \oplus A^{r-1}$.

7.7 Birrational Finite Morphisms

Let Σ be the field of rational functions on an integral **curve** $C = \operatorname{Spec} A$ (i.e. the field of fractions of an integral finitely generated k-algebra A of dimension 1).

The integral closure $\bar{A}$ of A in Σ is a finitely generated A-module (p. 212); hence it is a finitely generated k-algebra, and $\bar{C} = \operatorname{Spec} \bar{A}$ is a non singular curve (named desingularization of C) endowed with a finite birrational morphism $\bar{C} \to C$.

Definition: The finitely generated A-module $\mathfrak{C} = \bar{A}/A$ is the **conductor**, and it vanishes at a point x just when $A_x = \bar{A}_x$; i.e., when x is a simple point.

Theorem: *The number of singular points of any integral curve is finite.*

Proof: Localizing at the generic point p_g the exact sequence

$$0 \longrightarrow A \longrightarrow \bar{A} \longrightarrow \mathfrak{C} \longrightarrow 0,$$

we obtain $\mathfrak{C}_{p_g} = \Sigma/\Sigma = 0$, so that $\mathfrak{C}$ is a finite torsion module; hence of finite support. q.e.d.

$\bar{C} \to C$ is an isomorphism at any simple point of C. Let us study it at a singular point x.

Let $\mathcal{O} = A_x$ be the local ring of C at x, and let $\mathfrak{m}$ be the maximal ideal.

Theorem: *$\widehat{\mathcal{O}}$ is reduced, and the minimal primes of $\widehat{\mathcal{O}}$ correspond to the maximal ideals of $\bar{\mathcal{O}}$.*

Proof: $\mathfrak{C} = \bar{\mathcal{O}}/\mathcal{O}$ is complete, since it is annihilated by some power of $\mathfrak{m}$.

$$0 \longrightarrow \mathcal{O} \longrightarrow \bar{\mathcal{O}} \longrightarrow \mathfrak{C} \longrightarrow 0$$

$$0 \longrightarrow \widehat{\mathcal{O}} \longrightarrow \widehat{\bar{\mathcal{O}}} \longrightarrow \mathfrak{C} \longrightarrow 0$$

$\bar{\mathcal{O}}$ is a Dedekind domain; hence $\mathfrak{m}\bar{\mathcal{O}} = \mathfrak{m}_1^{n_1} \ldots \mathfrak{m}_r^{n_r}$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals of $\bar{\mathcal{O}}$. By the Chinese remainder theorem,

$$\widehat{\bar{\mathcal{O}}} = \varprojlim \bar{\mathcal{O}}/\mathfrak{m}^n\bar{\mathcal{O}} = \varprojlim(\bar{\mathcal{O}}/\mathfrak{m}_1^{n_1 n} \oplus \ldots \oplus \bar{\mathcal{O}}/\mathfrak{m}_r^{n_r n}) = \widehat{\mathcal{O}}_{x_1} \oplus \ldots \oplus \widehat{\mathcal{O}}_{x_r},$$

where $\mathcal{O}_{x_i}$, the localization of $\bar{\mathcal{O}}$ at $\mathfrak{m}_i$, is regular; hence $\widehat{\mathcal{O}}_{x_i}$ is regular, so that it is integral.

So we see that $\widehat{\bar{\mathcal{O}}}$ is reduced (hence so is the subring $\widehat{\mathcal{O}}$) and the minimal primes correspond to the maximal ideals (corresponding to the maximal ideals of $\bar{\mathcal{O}}$).

Now, we have $\mathfrak{C}_\mathfrak{p} = 0$ at any minimal prime $\mathfrak{p}$ of $\widehat{\mathcal{O}}$, since $\mathfrak{C}$ is annihilated by a power of $\mathfrak{m}\widehat{\mathcal{O}}$; hence $\widehat{\mathcal{O}}_\mathfrak{p} = \widehat{\bar{\mathcal{O}}}_\mathfrak{p}$, and $\operatorname{Spec} \widehat{\bar{\mathcal{O}}}$ has a unique point over $\mathfrak{p}$.

Hence, the minimal primes of $\widehat{\mathcal{O}}$ correspond to the minimal primes of $\widehat{\bar{\mathcal{O}}}$.

Definitions: The minimal primes of $\widehat{\mathcal{O}}$ are the **analytical branches** of the curve C at x, and they correspond to the points x_i of the fibre of $\bar{C}$ over x.

The **intersection multiplicity** at x of C with a hypersurface H of equation $f = 0$ is defined to be $(C \cap H)_x = l(\mathcal{O}/f\mathcal{O})$.

Lemma: $l_\mathcal{O}(\mathcal{O}/f\mathcal{O}) = l_\mathcal{O}(\bar{\mathcal{O}}/f\bar{\mathcal{O}})$.

Proof: We have $l(\bar{\mathcal{O}}/\mathcal{O}) = l(f\bar{\mathcal{O}}/f\mathcal{O})$ since $\bar{\mathcal{O}}/\mathcal{O} \xrightarrow{f\cdot} f\bar{\mathcal{O}}/f\mathcal{O}$ is an isomorphism.

The additive character of length, and the following commutative square let us conclude,

$$\begin{array}{ccc} f\mathcal{O} & \longrightarrow & f\bar{\mathcal{O}} \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \bar{\mathcal{O}} \end{array}$$

Theorem: $(C \cap H)_x = \sum_{x_i \to x} v_{x_i}(f) \cdot [\kappa(x_i) \colon \kappa(x)]$.

Proof: If f is invertible in $\mathcal{O}$, it is obvious.

If $f \in \mathfrak{m}$, then $\bar{\mathcal{O}}/f\bar{\mathcal{O}}$ has dimension 0, and it decomposes as a direct sum of localizations,

$$\bar{\mathcal{O}}/f\bar{\mathcal{O}} = \mathcal{O}_{x_1}/f\mathcal{O}_{x_1} \oplus \ldots \oplus \mathcal{O}_{x_n}/f\mathcal{O}_{x_n},$$
$$(C \cap H)_x = l_\mathcal{O}(\mathcal{O}_{x_1}/f\mathcal{O}_{x_1}) + \ldots + l_\mathcal{O}(\mathcal{O}_{x_n}/f\mathcal{O}_{x_n}).$$

Now, the length of the $\mathcal{O}$-module $\mathcal{O}_{x_i}/f\mathcal{O}_{x_i}$ is the product of the length $v_{x_i}(f)$ as an $\mathcal{O}_{x_i}$-module, by the length $[\kappa(x_i) \colon \kappa(x)]$ of the unique simple $\mathcal{O}_{x_i}$-module $\kappa(x_i)$.

Quadratic Transformations

Now we assume that k is infinite, so that no vector space is a finite union of proper vector subspaces (p. 148). If m_i is the minimal value at $\mathfrak{m}$ of the valuation v_{x_i}, $x_i \in \operatorname{Spec} \bar{\mathcal{O}}$, then $\{a \in \mathfrak{m} \colon v_{x_i}(a) > m_i\}$ is a proper vector subspace of $\mathfrak{m}$; hence there exists $f \in \mathfrak{m}$ of minimal value at any valuation v_{x_i}. If $\mathfrak{m} = (f_1, \ldots, f_d)$, then $v_{x_i}(f_j/f) \geq 0$ and $f_j/f \in \bar{\mathcal{O}}$,

$$\mathcal{O} \longrightarrow \mathcal{O}_1 = \mathcal{O}\left[\tfrac{f_1}{f}, \ldots, \tfrac{f_d}{f}\right] \longrightarrow \bar{\mathcal{O}}.$$

Definitions: The finite birrational morphism $\mathcal{O} \to \mathcal{O}_1$ is the **quadratic transformation** or **blowup** of $\mathcal{O}$ at x. The ring $\mathcal{O}_1$ is **semilocal** (with a finite number of maximal ideals), and the ideal $\mathfrak{m}\mathcal{O}_1 = f\mathcal{O}_1$ is principal, since $f_j = f\frac{f_j}{f} \in f\mathcal{O}_1$.

This construction holds even if $\mathcal{O}$ is a semilocal subring of $\bar{\mathcal{O}}$, so that we may blow-up again $\mathcal{O}_1$ at a point, and so on.

Lemma: $\mathcal{O}_1$ *does not depend on the function* $f \in \mathfrak{m}$ *of minimal valuation.*

Proof: If $f' \in \mathfrak{m}$, and $v_{x_i}(f') = v_i(f)$, then $f'/f \in \mathcal{O}_1$ does not vanish at any point of $\operatorname{Spec}\bar{\mathcal{O}}$, nor at any point of $\operatorname{Spec}\mathcal{O}_1$, since the morphism $\mathcal{O}_1 \hookrightarrow \bar{\mathcal{O}}$ is finite.

Hence f'/f is invertible in $\mathcal{O}_1$, and $f_j/f' = (f_j/f)(f/f') \in \mathcal{O}_1$,

$$\mathcal{O}\left[\tfrac{f_1}{f'},\dots,\tfrac{f_d}{f'}\right] \subseteq \mathcal{O}_1 = \mathcal{O}\left[\tfrac{f_1}{f},\dots,\tfrac{f_d}{f}\right].$$

By symmetry, we also have the reverse inclusion.

Theorem: *Any curve C desingularizes with a finite number of quadratic transformations.*

Proof: If $\mathcal{O} \to \mathcal{O}_1$ is an isomorphism, then x is simple point: $\mathfrak{m} = \mathfrak{m}\mathcal{O}_1 = f\mathcal{O}_1 = f\mathcal{O}$.

Since the length of $\bar{\mathcal{O}}/\mathcal{O}$ is finite, after of a finite number of blow-ups

$$\mathcal{O} \longrightarrow \mathcal{O}_1 \longrightarrow \dots \longrightarrow \mathcal{O}_r$$

we obtain a ring $\mathcal{O}_r \subseteq \bar{\mathcal{O}}$ without singular points; hence it is normal, and $\mathcal{O}_r = \bar{\mathcal{O}}$.

Stability Lemma: $\mathfrak{m}^n = \mathfrak{m}^n\mathcal{O}_1$, *when* $n \gg 0$.

Proof: The $\mathcal{O}$-module $\mathcal{O}_1$ is generated by the elements $\frac{f_1^{n_1}\dots f_d^{n_d}}{f^{n_1+\dots+n_d}}$.

Since $\mathcal{O}_1$ is a finite $\mathcal{O}$-module, when $n \gg 0$, we may fix a generating system of the form $\frac{a}{f^n}$, with $a \in \mathfrak{m}^n$. Now $\mathfrak{m}^n\mathcal{O}_1 = f^n\mathcal{O}_1 \subseteq \mathfrak{m}^n$, and we conclude.

Theorem: $S_{\mathcal{O}}(n) = mn - c$*; where* $m = l(\mathcal{O}/f\mathcal{O})$, $c = l(\mathcal{O}_1/\mathcal{O})$.

Proof: When $n \gg 0$, we have $\mathfrak{m}^n = \mathfrak{m}^n\mathcal{O}_1 = f^n\mathcal{O}_1$. Hence we have an exact sequence

$$0 \longrightarrow \mathcal{O}/\mathfrak{m}^n \longrightarrow \mathcal{O}_1/f^n\mathcal{O}_1 \longrightarrow \mathcal{O}_1/\mathcal{O} \longrightarrow 0$$

$$S_{\mathcal{O}}(n) = l(\mathcal{O}/\mathfrak{m}^n) = l(\mathcal{O}_1/f^n\mathcal{O}_1) - l(\mathcal{O}_1/\mathcal{O}) = l(\mathcal{O}/f^n\mathcal{O}) - l(\mathcal{O}_1/\mathcal{O}) = nl(\mathcal{O}/f\mathcal{O}) - l(\mathcal{O}_1/\mathcal{O}).$$

Corollary: *The multiplicity m is the intersection number of the blow-up of the curve with the exceptional fibre $f = 0$, counting each point y with the degree $[\kappa(y) : \kappa(x)]$,*

$$m = (C_1 \cap E).$$

Proof: $m = l_{\mathcal{O}}(\mathcal{O}/f\mathcal{O}) = l_{\mathcal{O}}(\mathcal{O}_1/f\mathcal{O}_1)$.

Corollary: *The multiplicity is 1 if and only if x is a simple point.*

Proof: If $1 = m = l(\mathcal{O}/f\mathcal{O})$, then $\mathfrak{m} = f\mathcal{O}$ is a principal ideal. q.e.d.

In this process, over a singular point x appear some points y, that we draw as a tree (the end points being the branches at x) with certain multiplicity m_y at each node y.

Corollary: *If k is algebraically closed, and C is a plane curve,*

$$\dim_k(\bar{\mathcal{O}}/\mathcal{O}) = \sum_y \binom{m_y}{2}.$$

Proof: We have $l_{\mathcal{O}}(M) = \dim_k M$ when $k = \kappa(x)$, and at a point of multiplicity m of a plane curve (p. 205), the Samuel polynomial is $\binom{n+1}{2} - \binom{n+1-m}{2} = mn - \binom{m}{2}$.

Example: Let k be algebraically closed, so that, after an axes change, we may assume that our point is the origin. If C is a plane curve, and $x=0$ is not tangent to C at the origin,

$$0 = P(x,y) = a_0 y^m + a_1 x y^{m-1} + \ldots + a_m x^m + \sum_{i+j>m} a_{ij} x^i y^j, \quad a_0 \neq 0,$$

dividing by x^m we obtain an integral dependence relation of $\frac{y}{x}$ over the local ring $\mathcal{O}$ of C at the origin. Hence $v_i(x) \leq v_i(y)$ for any discrete valuation v_i centered at the origin, and $\mathcal{O}_1 = \mathcal{O}\left[\frac{y}{x}\right]$. We put $z = \frac{y}{x}$, so that

$$0 = P(x,xz) = x^m P_1(x,z) = x^m \Big(a_0 z^m + a_1 z^{m-1} + \ldots + a_m + \sum_{i+j>m} a_{ij} x^{i+j-m} z^j \Big),$$

and we see that $\mathcal{O}_1$ is the semilocal ring of the curve $P_1(x,z)=0$ at intersection points with the exceptional fibre $x=0$, which are points $x=0$, $z=\lambda$, where $y=\lambda x$ is a line of the tangent cone $a_0 y^m + a_1 x y^{m-1} + \ldots + a_m x^m = 0$.

For example, let C be the complex plane curve (integral by Eisenstein criterion)

$$P(x,y) = y^7 - x^9 y - x^{10} y + x^{11} + x^{12} = 0.$$

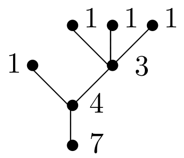
The tangent cone is $y^7 = 0$ and, blowing-up with $z = \frac{y}{x}$, appears the point $x=z=0$,

$$\begin{aligned} 0 = P(x,xz) =& x^7(z^7 - x^3 z - x^4 z + x^4 + x^5), \\ 0 = P_1(x,z) =& x^4 - x^3 z + x^5 - x^4 z + z^7. \end{aligned}$$

The tangent cone is $x^3(x-z) = 0$, and blowing-up with $s = \frac{x}{z}$,

$$0 = P_2(z,s) = z^3 - s^3 + s^4 - zs^4 + zs^5,$$

appear the simple point $z=0$, $s=1$ ($\frac{\partial P_2}{\partial s}$ does not vanishes), and the point $z=s=0$, of multiplicity 3. Blowing-up it appear 3 points, all simple since the blow-up of the curve intersects the exceptional fibre with multiplicity 1. The blow-up tree is



$$l(\bar{\mathcal{O}}/\mathcal{O}) = \binom{7}{2} + \binom{4}{2} + \binom{3}{2} = 30$$

Theorem: *The intersection multiplicity of C with a hypersurface H of multiplicity r is the product of the multiplicities plus the intersection multiplicities of the blow-ups at the common points of the exceptional fibre,*

$$(C \cap H)_x = mr + (C_1 \cap H_1).$$

Proof: If the equation of H is $P(x_1,\ldots,x_n) = 0$, and $f = x_1$ is of minimal valuation, then the birrational transformation is $z_j = \frac{x_j}{x_1}$, and we have

$$P(x_1,\ldots,x_n) = x_1^r P_1(x_1, z_2, \ldots, z_n),$$

where $P_1 = 0$ is the equation of the hypersurface H_1. Now

$$(C \cap H)_x = l(\mathcal{O}/P) = l(\mathcal{O}_1/P) = l(\mathcal{O}_1/f^r P_1) = rl(\mathcal{O}_1/f) + l(\mathcal{O}_1/P_1) = mr + (C_1 \cap H_1),$$

where $(C_1 \cap H_1) = l(\mathcal{O}_1/P_1)$ is just the sum of the intersection multiplicities of C_1 and H_1 at the common points over x.

7.8 Faithfully Flat Morphisms

Theorem: *On finitely generated modules over a local ring, the conditions of being free, projective and flat are equivalent.*

Proof: Always free $\Rightarrow$ projective $\Rightarrow$ flat, so we must prove flat $\Rightarrow$ free.
Let $\{m_1, \dots, m_n\}$ be a minimal generating system of a finite flat $\mathcal{O}$-module M.
If $f_1 m_1 + \dots + f_n m_n = 0$, we put $I = (f_1, \dots, f_n)$.
By flatness $I \otimes_{\mathcal{O}} M \to M$ is injective, and $f_1 \otimes m_1 + \dots + f_r \otimes m_r = 0$.
The base change $\mathcal{O} \to k = \mathcal{O}/\mathfrak{m}$ shows that $\bar{f}_1 \otimes \bar{m}_1 + \dots + \bar{f}_r \otimes \bar{m}_r = 0$ in the vector space

$$(I/\mathfrak{m}I) \otimes_k (M/\mathfrak{m}M) = (I/\mathfrak{m}I) \otimes_k (k\bar{m}_1 \oplus \dots \oplus k\bar{m}_n).$$

Hence $\bar{f}_i = 0$, and $I/\mathfrak{m}I = 0$. By Nakayama $I = 0$, and $m_1, \dots, m_n$ is a base of M.

Lemma: $\mathrm{Hom}_A(M,N)_x = \mathrm{Hom}_{A_x}(M_x, N_x)$*, for all* $x \in \mathrm{Spec}\, A$*, when* M *is an* A*-module of* ***finite presentation*** (it admits an exact sequence $A^m \to A^n \to M \to 0$).

Proof: Just consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_A(A^m,N)_x & \longleftarrow & \mathrm{Hom}_A(A^n,N)_x & \longleftarrow & \mathrm{Hom}_A(M,N)_x & \longleftarrow & 0 \\
\| & & \| & & \downarrow & & \\
\mathrm{Hom}_{A_x}(A_x^m,N_x) & \longleftarrow & \mathrm{Hom}_{A_x}(A_x^n,N_x) & \longleftarrow & \mathrm{Hom}_{A_x}(M_x,N_x) & \longleftarrow & 0
\end{array}$$

Theorem: *On finitely presented* A*-modules, the conditions of being locally free, projective and flat are equivalent.*

Proof: Since flat modules are locally free by the above theorem, we only have to show that any locally free module P is projective.
Now, if $\mathbf{E}$ is an exact sequence of A-modules, so is $\mathrm{Hom}_A(P,\mathbf{E})_x = \mathrm{Hom}_{A_x}(P_x,\mathbf{E}_x)$, since P_x is a free A_x-module; hence $\mathrm{Hom}_A(P,\mathbf{E})$ is exact (p. 142).

Theorem: *If* P *is a projective finitely generated* A*-module, any point* $x \in \mathrm{Spec}\, A$ *has a basic neighborhood* U *such that* P_U *is a free* A_U*-module.*

Proof: Since P_x is free, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & L & \longrightarrow & P & \longrightarrow & \mathrm{Coker} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Ker})_x & \longrightarrow & L_x & \xrightarrow{\sim} & P_x & \longrightarrow & (\mathrm{Coker})_x & \longrightarrow & 0
\end{array}$$

where L is free. Since Coker is finitely generated and $\mathrm{Coker}_x = 0$, in a neighborhood U we have $\mathrm{Coker}_U = 0$ (the support of any element is closed, p. 142).
Since P_U projective, $L_U = \mathrm{Ker}_U \oplus P_U$, and Ker_U is finitely generated.
Since $\mathrm{Ker}_x = 0$, we may fix U so that $\mathrm{Ker}_U = 0$, and $P_U = L_U$ is free.

Theorem: *If* P *is a flat* A*-module, the following conditions are equivalent,*

1. P *is* ***faithfully flat****: any sequence of* A*-modules* $\mathbf{E}$ *is exact* $\Leftrightarrow$ $\mathbf{E} \otimes_A P$ *is exact.*
2. $P/\mathfrak{m}P \neq 0$ *for any maximal ideal* $\mathfrak{m}$ *of* A.
3. $M \otimes_A P = 0 \ \Leftrightarrow \ M = 0$.

Proof: (1 ⇒ 2) $0 \to A/\mathfrak{m} \to 0$ is not exact, hence $0 \to P/\mathfrak{m}P \to 0$ is not exact.

(2 ⇒ 3) If $m \in M$ is non null, and we take an epimorphism $Am \to A/\mathfrak{m} \to 0$, the epimorphism $(Am) \otimes_A P \to (A/\mathfrak{m}) \otimes_A P \to 0$ shows that $(Am) \otimes_A P \neq 0$.

Since P is flat, $(Am) \otimes_A P$ is a submodule of $M \otimes_A P$; hence $M \otimes_A P \neq 0$.

(3 ⇒ 1) Let us consider $M' \xrightarrow{i} M \xrightarrow{p} M''$ and $M' \otimes_A P \xrightarrow{i\otimes 1} M \otimes_A P \xrightarrow{p\otimes 1} M'' \otimes_A P$.

Since P is flat, $(\operatorname{Im} i) \otimes_A P = \operatorname{Im}(i \otimes 1)$, $(\operatorname{Ker} p) \otimes_A P = \operatorname{Ker}(p \otimes 1)$, and

$$\left(\frac{\operatorname{Im} i + \operatorname{Ker} p}{\operatorname{Im} i}\right) \otimes_A P = \frac{\operatorname{Im}(i \otimes 1) + \operatorname{Ker}(p \otimes 1)}{\operatorname{Im}(i \otimes 1)}$$
$$\left(\frac{\operatorname{Im} i + \operatorname{Ker} p}{\operatorname{Ker} p}\right) \otimes_A P = \frac{\operatorname{Im}(i \otimes 1) + \operatorname{Ker}(p \otimes 1)}{\operatorname{Ker}(p \otimes 1)}$$

hence $\operatorname{Im} i = \operatorname{Ker} p$ if and only if $\operatorname{Im}(i \otimes 1) = \operatorname{Ker}(p \otimes 1)$.

Corollary: *Let $A \to B$ be a faithfully flat morphism. An A-module M is finitely generated (resp. flat) if and only if the B-module M_B is finitely generated (resp. flat).*

Proof: If M_B is finite, it is generated by some elements of M, defining a morphism $A^n \to M$ such that $B^n \to M_B$ is surjective; hence so is $A^n \to M$, and M is finitely generated.

If $\mathbf{E}$ is an exact sequence of A-modules, then $\mathbf{E}_B$ is exact; and if M_B is a flat B-module, then $\mathbf{E}_B \otimes_B M_B = (\mathbf{E} \otimes_A M)_B$ is exact. Hence $\mathbf{E} \otimes_A M$ is exact, and M is flat.

Corollary: *A flat morphism $A \to B$ is faithfully flat if and only if $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof: If $A \to B$ is faithfully flat, then the fibres of ϕ are non void by the fibre formula, since $B \otimes_A \kappa(x) \neq 0$. Conversely, if the fibres are non void and $x = \phi(y)$, then the flat morphism $A_x \to B_y$ is faithfully flat by the theorem; hence so is $A \to B$.

Theorem: *If $A \to B$ is faithfully flat, then the sequence $A \to B \rightrightarrows B \otimes_A B$ is exact.*

Proof: The argument of p. 197 holds.

7.9 Galois Theory of Rings

Let A be a noetherian ring such that $S = \operatorname{Spec} A$ is connected.

Definitions: A **covering** of S is an **unramified** finite flat morphism $X = \operatorname{Spec} B \to S$ (i.e. B is a finite flat A-module and $\Omega_{B/A} = 0$). Then the degree of the $\kappa(x)$-algebra $B \otimes_A \kappa(x)$ is locally constant; hence constant if S is connected, and it is the **degree** of the covering.

The **trivial** covering of degree n is $S \oplus \overset{n}{\ldots} \oplus S := \operatorname{Spec} A^n \to S$.

Any covering of degree 1 is an isomorphism: it is the trivial covering of degree 1.

Theorem: *The concept of covering is stable under base changes (if $A \to B$ is a covering, so is $C \to B \otimes_A C$), and it is local with respect to faithfully flat morphisms (if $A \to C$ is faithfully flat, then $A \to B$ is a covering if and only if so is $C \to B \otimes_A C$).*

Proof: Finite flat morphisms, and the module of differentials, are stable under base changes.

Lemma: *Any connected component of a covering also is a covering.*

Proof: If $X = X' \amalg X''$, then $B = B' \oplus B''$ (p. 198). Since B is finite and flat, so is B', and it is unramified since $0 = \Omega_{B/A} = \Omega_{B'/A} \oplus \Omega_{B''/A}$.

Lemma: *Any section of a connected covering $X \to S$ is an isomorphism.*

Proof: Any section $0 \longrightarrow I \longrightarrow B \xrightarrow{\sigma} A \longrightarrow 0$ defines a derivation

$$D\colon B \longrightarrow I/I^2,\ D(b) = b - \sigma(b) \ \ (\text{modulo } I^2).$$

Since $\Omega_{B/A} = 0$, we have $I = I^2$; and $I_x = 0$ or $I_x = B_x$ at any point $x \in X$.
Hence I vanishes on a closed open set. Since X is connected, $I = 0$. q.e.d.

When X is *connected*, for any covering $Y \to S$ we obtain the **Points Formula**:

$$\mathrm{Hom}_S(X,Y) = \mathrm{Hom}_X(X, X \times_S Y) = \begin{bmatrix} \text{Connected components of } X \times_S Y \\ \text{isomorphic to } X \text{ via the first projection} \end{bmatrix}$$

Hence the degree of Y bounds the number of S-morphisms $X \to Y$, and both are equal if and only if Y is trivial over X; i.e., $X \times_S Y = \amalg X$.

Any morphism $f\colon X \to Y$ also is a covering, since it is $X \xrightarrow{1\times f} X \times_S Y \xrightarrow{\pi_2} Y$, where $1 \times f$ is an isomorphism onto a connected component, and π_2 is a covering.

In particular, any morphism $X \to X$ is an automorphism, since the degree is 1. Hence,

$$\mathrm{Aut}(B/A) := \mathrm{Aut}_{A\text{-alg}}(B) = \begin{bmatrix} \text{Connected components of } X \times_S X \\ \text{isomorphic to } X \text{ via the first projection} \end{bmatrix}$$

Definition: A connected covering $X = \mathrm{Spec}\, B \to S$ is a **Galois** covering when $X \times_S X \to X$ is a trivial covering, $B \otimes_A B = \oplus_G B$,

$$X \times_S X = X \amalg \overset{G}{\ldots} \amalg X =: X \times G,$$

where $G = \mathrm{Aut}(B/A) = \mathrm{Hom}_{A\text{-alg}}(B,B)$ is the **Galois group**. Since $A \to B$ is faithfully flat, the exact sequence $A \to B \rightrightarrows B \otimes_A B = \oplus_G B$ shows that $A = B^G$,

$$X/G := \mathrm{Spec}\, B^G = S.$$

Moreover, when a finite group G acts on an A-algebra B, the argument of page 150 shows that $B^G \otimes_A C = (B \otimes_A C)^G$ when the base change $Y = \mathrm{Spec}\, C \to S$ is flat; i.e.

$$(X/G) \times_S Y = (X \times_S Y)/G.$$

Therefore, when a covering $Y = \mathrm{Spec}\, C \to S$ is trivial over a Galois covering $X \to S$ of group G, it is fully determined by the finite G-set

$$F(Y) = \mathrm{Hom}_{A\text{-alg}}(C,B) = \mathrm{Hom}_S(X,Y) = [\text{Connected components of } X \times_S Y]$$

because $X \times F(Y) := X \amalg \overset{F(Y)}{\ldots} \amalg X = X \times_S Y$ and we have

$$Y = (X/G) \times_S Y = (X \times_S Y)/G = (X \times F(Y))/G.$$

Hence, given a finite G-set Δ, since G also acts on X, we may define the **associated covering** to be

$$R(\Delta) = (X \times \Delta)/G := (\amalg_\Delta X)/G := \mathrm{Spec}\ (\oplus_\Delta B)^G\,,$$

which is a covering of S of degree $|\Delta|$ trivial over X. In fact we have

$$X \times_S R(\Delta) = X \times_S (X \times \Delta)/G = ((X \times_S X) \times \Delta)/G = (X \times G \times \Delta)/G,$$

where G acts via X-morphisms, so that $(X \times G \times \Delta)/G = X \times (G \times \Delta)/G = X \times \Delta$.

Now the points formula shows that $\Delta = \mathrm{Hom}_S(X, R(\Delta)) = FR(\Delta)$ and we obtain the following result:

Galois Theorem: *If* $X = \mathrm{Spec}\, B \to S = \mathrm{Spec}\, A$ *is a Galois covering of group* $G = \mathrm{Aut}(B/A)$, *then the above functors* F *and* R *define an equivalence of categories*

$$\begin{bmatrix}\text{Coverings of } S \\ \text{trivial over } X\end{bmatrix} \leftrightsquigarrow \begin{bmatrix}\text{Finite} \\ G\text{-sets}\end{bmatrix}, \quad \begin{matrix} R \circ F = \mathrm{Id} \\ F \circ R = \mathrm{Id}\end{matrix}$$

Artin's Theorem: *If a connected covering* $\mathrm{Spec}\, B \to S$ *admits a group of automorphisms* G *such that* $A = B^G$, *then it is a Galois covering of group* $G = \mathrm{Aut}(B/A)$.

Proof: $X \times_S X$ has a connected component $\simeq X$ for each element of G, and

$$X = X \times_S (X/G) = (X \times_S X)/G = (X \amalg \overset{G}{\ldots} \amalg X)/G \oplus (X_1 \amalg \ldots \amalg X_r)/G$$

is connected. The components X_i do not exist, and $X \times_S X = X \amalg \overset{G}{\ldots} \amalg X$.

Theorem: *Any covering* $Y \to S$ *is trivial over some Galois covering* $X \to S$.

Proof: If $X \to S$ is a connected covering and $X \times_S Y = X \amalg \ldots \amalg X \amalg X' \amalg \ldots$ is not trivial, then

$$X' \times_S Y = X' \times_X (X \times_S Y) = X' \amalg \ldots \amalg X' \amalg (X' \times_X X') \amalg \ldots$$

and $X' \times_X X' = X' \amalg \ldots$ by the points formula; hence $X' \times_S Y$ has more trivial components than $X \times_S Y$. Starting with $X = S$, finally we obtain a connected component X of a product $Y \times_S \ldots \times_S Y$ such that $X \times_S Y = \amalg X$ is trivial.

Now, since $X \times_S X$ is a closed open subspace of $(Y \times_S \ldots \times_S Y) \times_S X = \oplus X$, we conclude that X is a Galois covering of S.

7.9.1 The Fundamental Group

Definition: Let $s\colon \mathrm{Spec}\, \bar{k} \to S$ be a **geometric point**, where $\bar{k}$ is an algebraically closed field (for example an algebraic closure of a residue field). The **geometric fibre** of a covering $Y = \mathrm{Spec}\, C \to S$ over s is

$$\mathrm{Hom}_S(\mathrm{Spec}\, \bar{k}, Y) = \mathrm{Hom}_{\mathrm{Spec}\, \bar{k}}(\mathrm{Spec}\, \bar{k}, \mathrm{Spec}\, \bar{k} \times_S Y) = \mathrm{Spec}\, (C \otimes_A \bar{k}),$$

because $C \otimes_A \bar{k} = \oplus \bar{k}$ (it is a separable $\bar{k}$-algebra).

The geometric fibre has as many points as the degree of Y over S.

When $X \to S$ is a connected covering, if two S-morphisms $X \rightrightarrows Y$ coincide at point of the geometric fibre, they coincide, since any two connected components of $X \times_S Y$ are disjoint.

Pairs X_x, where $X = \mathrm{Spec}\, B \to S$ is a Galois covering and $x\colon \mathrm{Spec}\, \bar{k} \to X$ is a geometric point over s, form a projective system with the morphisms

$$\phi_i^j\colon (X_j = \mathrm{Spec}\, B_j)_{x_j} \longrightarrow (X_i = \mathrm{Spec}\, B_i)_{x_i},\ \phi_i^j(x_j) = x_i,$$

because, given two pairs X_x and $\bar{X}_{\bar{x}}$, we take the connected component X' of $X \times_S \bar{X}$ passing through the geometric point $x' = (x, \bar{x})$, so that we have morphisms $X'_{x'} \to X_x$, $X'_{x'} \to \bar{X}_{\bar{x}}$, and X' is a Galois covering. In fact, X and $\bar{X}$ are trivial over X'; hence so is $X \times_S \bar{X}$ and its connected component X'; i.e., $X' \times_S X' = \amalg X'$.

Definition: Put $G_i = \mathrm{Aut}(B_i/A)$. Any morphism $\phi_i^j\colon (X_j)_{x_j} \to (X_i)_{x_i}$ induces an epimorphism $G_j \to G_i$, and the **fundamental group** of S at the geometric point s is

$$\pi_1(S,s) = \varprojlim G_i.$$

Any covering $Y \to S$ is trivial over some Galois covering $(X_i)_{x_i}$, and x_i defines a canonical bijection of the fibre $F(Y) = \mathrm{Hom}_S(\mathrm{Spec}\,\bar{k}, Y)$ of Y over s with the finite G_i-set, hence finite $\pi_1(S,s)$-set, $\mathrm{Hom}_S(X_i, Y)$ classifying Y. But so we only obtain finite $\pi_1(S,s)$-sets where the action factors through some quotient $\pi_1(S,s) \to G_i$.

To characterize these actions, we consider $\pi_1(S,s)$ as a projective limit of discrete groups. Any open subgroup $U \subset \pi_1(S,s)$ contains the kernel K_i of a projection $\pi_1(S,s) \to G_i$, so that any continuous action $\pi_1(S,s) \times \Delta \to \Delta$ on a finite discrete space is induced by some action $G_i \times \Delta \to \Delta$. In fact, the isotropy subgroups I_x, $x \in \Delta$, being open, so is $\bigcap_x I_x \supseteq K_i$, and the action factors through G_i. Now, according to the Galois theorem,

Galois Theorem: *The fibre functor $F(Y) = \mathrm{Hom}_S(\mathrm{Spec}\,\bar{k}, Y)$ defines an equivalence of the category of coverings of S with the category of finite $\pi_1(S,s)$-sets (with continuous action).*

Definition: Let G be a finite group. A G-**principal** covering is a covering $X \to S$ (not necessarily connected) with an action of G defining an isomorphism $X \times G := \amalg_G X \xrightarrow{\sim} X \times_S X$, so that it is a Galois covering of group G when it is connected.

Isomorphisms of G-principal coverings are S-isomorphisms commuting with the action of G.

If we fix a geometric point of the fibre of s, we say that it as a pointed principal covering.

Now, for any finite (discrete) group G the argument of p. 255 gives

Corollary: $\mathrm{Hom}_{\mathbf{TopGr}}(\pi_1(S,s), G) = \begin{bmatrix}\text{Pointed } G\text{-principal}\\ \text{coverings of } S\end{bmatrix}.$

Finally, we show that the fundamental group is a functor:

Let $\phi\colon S' \to S$ be a morphism ($S' = \mathrm{Spec}\,A'$ connected), $s'\colon \mathrm{Spec}\,\bar{k} \to S'$ a geometric point, and $s = \phi(s')\colon \mathrm{Spec}\,\bar{k} \to S$. Given a pointed Galois covering $(X_i)_{x_i}$ of S, then $X_i \times_S S'$ is a G_i-principal covering of S', pointed with the geometric point $x' = (x_i, s')$. Hence it defines a continuous group morphism $\pi_1(S', s') \to G_i$, and we obtain a continuous group morphism

$$\phi_*\colon \pi_1(S',s') \to \varprojlim G_i = \pi_1(S,s).$$

Example: In the case of a field k, coverings $\mathrm{Spec}\,A \to \mathrm{Spec}\,k$ are separable finite k-algebras A, Galois coverings are Galois extensions, and to fix a geometric point $s\colon \mathrm{Spec}\,\bar{k} \to \mathrm{Spec}\,k$ is just to fix an algebraically closed extension $k \to \bar{k}$. Any Galois extension $k \to L$ admits an embedding $L \to \bar{k}$, so that the **separable closure** $\bar{k}^{\mathrm{sep}} = \{\alpha \in \bar{k}\colon \alpha \text{ is algebraic and separable over } k\}$ is just $\bar{k}^{\mathrm{sep}} = \bigcup_i L_i$, where L_i runs over all Galois extensions of k. Since L_i is invariant under any k-automorphism of $\bar{k}^{\mathrm{sep}}$, the fundamental group of $\mathrm{Spec}\,k$ is just the **absolute** Galois group

$$\pi_1(\mathrm{Spec}\,k, s) = \varprojlim \mathrm{Aut}(L_i/k) = \mathrm{Aut}(\bar{k}^{\mathrm{sep}}/k).$$

When $k = \mathbb{F}_q$ is the finite field with q elements, then (p. 153) the fundamental group is

$$\pi_1(\mathrm{Spec}\,\mathbb{F}_q) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \textstyle\prod_p \widehat{\mathbb{Z}}_p,$$

and the Frobënius automorphism $F(\alpha) = \alpha^q$ generates a dense subgroup.

Now, any point $p\colon \operatorname{Spec}\mathbb{F}_q \to S$ defines a morphism $\pi_1(\operatorname{Spec}\mathbb{F}_q, s) \to \pi_1(U, s)$, where U is any open neighborhood of p in S, and the image of F is the **Frobënius automorphism** F_p at the point p, so generalizing the definition of p. 154.

Chapter 8

Topology

Unless otherwise stated, from now on all topological spaces are assumed to be T_0: any point is determined by the open subsets containing it.

8.1 Lattice Semirings

Definitions: A set A endowed with two operations $(+, \cdot)$ and two elements $0, 1$ is a **commutative semiring** when $(A, +, 0)$ and $(A, \cdot, 1)$ are abelian **semigroups** (associative and commutative operations with neutral element) and the maps $A \xrightarrow{a\cdot} A$ are endomorphisms of the additive semigroup,

$$a(b + c) = ab + ac \quad , \quad a \cdot 0 = 0.$$

A subset $B \subseteq A$ is a **subsemiring** when $a + b, ab \in B$; $\forall a, b \in B$, and $0, 1 \in B$.

A map $f\colon A \to A'$ between two semirings is a **morphism** of semirings when

$$\begin{aligned} f(a + b) &= f(a) + f(b) \quad , \quad f(0) = 0, \\ f(a \cdot b) &= f(a) \cdot f(b) \quad , \quad f(1) = 1, \end{aligned}$$

so that the image is a subsemiring of A'.

A semiring A is a **lattice semiring** if moreover $a^2 = a$, $1 + a = 1$; $\forall a \in A$.

In this course all semirings are assumed to be lattice semirings.

Theorem: *If $(A, +, \cdot, 0, 1)$ is a lattice semiring, then $A^* = (A, \cdot, +, 1, 0)$ also is a lattice semiring, the **dual** semiring of A.*

Proof: $a + a = a(1 + 1) = a \cdot 1 = a$.

$(a + b)(a + c) = a + ba + ac + bc = a(1 + b + c) + bc = a + bc$.

Main Example: The semiring $\mathbb{K} = \{0, 1\}$, where $1 + 1 = 1$, is a lattice semiring; and both operations are continuous when we consider on $\mathbb{K}$ the topology with a closed point 0 and a dense point 1 (representing the generic non null scalar). Given a topological space X, the set $A_X = A(X)$ of all continuous functions $f\colon X \to \mathbb{K}$ inherits a structure of lattice semiring, and any continuous map $\phi\colon Y \to X$ induces a morphism of semirings $\phi^*\colon A(X) \to A(Y)$, $\phi^*(f) = f \circ \phi$. More examples:

1. The lattice of all subsets of a set X, when the addition is the union and the product is the intersection, is a lattice semiring where $0 = \emptyset$ and $1 = X$. In general, any lattice of subsets of X; i.e. a family of subsets, closed under finite unions and intersections (in particular $\emptyset$ and X are in the family), is a lattice semiring.

2. When X is a topological space, any continuous function $X \to \mathbb{K}$ is the indicator function $\mathbf{1}_U$ of a unique open set $U \subseteq X$ (i.e. $\mathbf{1}_U$ vanishes on $X - U$), and

$$\mathbf{1}_{U\cup V} = \mathbf{1}_U + \mathbf{1}_V \quad , \quad \mathbf{1}_{U\cap V} = \mathbf{1}_U \cdot \mathbf{1}_V \ ,$$

so that $A(X)$ is canonically isomorphic to the lattice of all open sets in X; hence to the dual semiring of the lattice of closed sets. In particular any series $\sum_{i\in I} f_i$ of continuous $\mathbb{K}$-valued functions (even with a non countable index set) is a well-defined continuous function (because arbitrary unions of open sets are open sets), the condition of X being compact stating that if a series $\sum_i f_i$ is the constant function 1, then so is some finite sum $f_{i_1} + \ldots + f_{i_n}$.

3. The **zero-set** $z(f) := \{x \in X : f(x) = 0\}$ of a continuous function $f : X \to \mathbb{K}$ is a closed set, and the correspondence $f \leftrightarrow z(f)$ defines an isomorphism of $A(X)$ with the semiring of all closed sets in X (dual to the lattice of closed sets: $+ = \cap$, $\cdot = \cup$, $0 = X$ and $1 = \emptyset$).

4. When $A(X)$ is viewed as a semiring of open or closed sets, the morphism $\phi^* : A(X) \to A(Y)$ induced by a continuous map $\phi : Y \to X$ is just the inverse image, $\phi^* = \phi^{-1}$.

5. A subsemiring $B \subseteq A(X)$ is said to be a **base** of the topology of X when the corresponding open sets form a base in the usual sense (any continuous function $h \in A(X)$ is a series $h = \sum_i f_i$ with $f_i \in B$); i.e. any closed subset Y is an intersection of zero-sets $z(f)$ with $f \in B$: for any point $x \notin Y$, there is a function $f \in B$ such that $f(Y) = 0$ and $f(x) = 1$.

Definition: If A is a semiring, $I \subseteq A$ is an **ideal** when $0 \in I$, $I + I \subseteq I$ and $A \cdot I \subseteq I$; so that the kernel of any semiring morphism $A \to B$ is an ideal of A.

Any ideal I defines an equivalence relation on A,

$$a \equiv b \ (\text{mod. } I) \ \Leftrightarrow \ a + x = b + y, \ \text{for some } x, y \in I$$

(or $a + z = b + z$ for some $z \in I$; just take $z = x + y$).

The quotient set $\pi : A \to A/I$ is a semiring with the natural operations

$$[a] + [c] = [a + c] \quad , \quad [a] \cdot [c] = [ac] \quad , \quad 0 = [0] \quad , \quad 1 = [1],$$

$\operatorname{Ker} \pi = I$ (if $a + x = 0 + y$, then $a = a(1 + x) = a(a + x) = ay \in I$) and the usual universal property (p. 48) holds; but the isomorphism theorem fails.

For example, any closed subset $Y \subseteq X$ defines an ideal $I_Y = \{f \in A(X) : f(Y) = 0\}$ of $A(X)$, the natural morphism $A(X)/I_Y \to A(Y)$, $[h] \mapsto h|_Y$ is an isomorphism, and I_Y is a principal ideal, $I_Y = (\mathbf{1}_U)$ where $U = X - Y$. However, if $i : Z \to X$ is a dense subspace, then $i^* : A(X) \to A(Z)$ is a surjective morphism with null kernel, but usually it is not injective.

Definition: A multiplicative system S of A (i.e., $1 \in S$ and $S \cdot S \subseteq S$) defines an equivalence relation on $A \times S$:

$$(a, s) \equiv (b, t) \ \Leftrightarrow \ atu = bsu \text{ for some } u \in S,$$

and the quotient set A_S, with the natural operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$, is the **localization**[1] of A at S. It is a semiring with the usual universal property (p. 88).

Example: If $\mathfrak{p}$ is the prime ideal of all continuous $\mathbb{K}$-valued functions vanishing at a given point $x \in X$, then $A(X)_{\mathfrak{p}} := A(X)_S$ (where $S = \{f \in A(X) : f(x) \neq 0\}$) is the semiring of germs at

[1]In fact $a/s = a/1$, because $as = as^2$; and $a/1 = b/1$ when $as = bs$, $\exists s \in S$. Hence A_S is the quotient of A by the equivalence relation $a \equiv b \ \Leftrightarrow \ as = bs$ for some $s \in S$, dual of the relation defined by an ideal.

x of continuous $\mathbb{K}$-valued functions; i.e. the lattice of germs of open sets, dual to the lattice of germs of closed sets. If U is an open set and $f = \mathbf{1}_U$, then $A(X)_f = A(U)$, $\frac{h}{1} \mapsto h|_U$.

Definitions: Integral semirings, semifields, prime and maximal ideals, and dimension are defined as in the case of rings. The **spectrum** of a semiring A is the set $\operatorname{Spec} A$ of all prime ideals, with the **Zariski topology** (closed sets are zeros $(I)_0 := \{\mathfrak{p} \in \operatorname{Spec} A \colon I \subseteq \mathfrak{p}\}$ of ideals $I \subseteq A$), and any semiring morphism $f \colon A \to B$ induces a natural continuous map

$$f^* \colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A\ , \quad f^*(\mathfrak{p}) = \mathfrak{p} \cap A := \{a \in A \colon f(a) \in \mathfrak{p}\}.$$

Basic closed sets are $(a)_0$, and basic open sets are $U_a := \operatorname{Spec} A - (a)_0$; where $a \in A$.

Many properties of rings, and their proofs, hold in the case of (lattice) semirings:

1. *Ideals of A/I correspond to ideals of A containing I* (p. 71); hence $\operatorname{Spec}(A/I) = (I)_0$.
2. *Maximal ideals are prime ideals* (p. 71). The subspace of all maximal ideals is the **maximal spectrum** $\operatorname{Spec}_{\mathrm{m}} A \subseteq \operatorname{Spec} A$.
3. *Any ideal $I \neq A$ is contained in a maximal ideal* (p. 124).
4. *The closure of a point $x \in \operatorname{Spec} A$ is $(\mathfrak{p}_x)_0$. Hence $\operatorname{Spec} A$ is a T_0 space and $\operatorname{Spec}_{\mathrm{m}} A$ is the subspace of closed points of $\operatorname{Spec} A$* (p. 139).
5. *The spectrum $\operatorname{Spec} A$ is a T_0 compact space* (p. 139).
6. *Prime ideals of A_S correspond to prime ideals of A not intersecting S, hence prime ideals of $A_{\mathfrak{p}}$ correspond to prime ideals contained in $\mathfrak{p}$* (p. 140).
7. $\operatorname{Spec}(A \times B) = (\operatorname{Spec} A) \amalg (\operatorname{Spec} B)$, (p. 140).
8. *Irreducible closed sets in $\operatorname{Spec} A$ correspond to prime ideals of A, and the dimension of A is the supremum of the lengths of all chains of irreducible closed sets (or specializations $x_0 > x_1 \ldots > x_n$) in $\operatorname{Spec} A$* (p. 204).

but (lattice) semirings have some additional nice properties:

1. *If $a \in bA$, then $a = ab$.*

   If $a = cb$, then $ab = cb \cdot b = cb^2 = cb = a$.
2. *The unity is the unique invertible element. By duality, $a + b = 0 \Rightarrow a = b = 0$.*

   If $1 \in aA$, then $1 = a \cdot 1 = a$.
3. $\mathbb{K} = \{0, 1\}$ *is the unique (lattice) semifield.*
4. *Let I be an ideal. If $a + b \in I$, then $a, b \in I$.*
5. *Any finite intersection of principal ideals is a principal ideal*: $(a) \cap (b) = (ab)$.

   If $c \in aA \cap bA$, then $c = ac$ and $c = bc$, so that $c = ac = abc$.
6. *Any finitely generated ideal is a principal ideal*: $(a, b) = (a + b)$.

   In fact $a = a(1 + b) = a^2 + ab = a(a + b)$ and $b = b(a + b)$.
7. *Any semiring is a union of finite semirings.*

   Any finite family generates a finite subring because $a^2 = a$ and $a + a = a$.

8. *Any nilpotent element is null.*

   If $a^n = 0$, then $a = 0$ because $a^n = a$.

9. *The canonical localization morphism $A \to A_S$ always is surjective.*

   We have $\frac{a}{s} = \frac{a}{1}$ because $as = as^2$.

with many surprising and very agreeable consequences:

1. *Principal ideals have a unique generator: $aA = bA \ \Rightarrow \ a = b$.*

   If $a \in bA$ and $b \in aA$, then $a = ab$ and $b = ba$; hence $a = ab = b$.

2. *Any element $a \neq 1$ belongs to some maximal ideal.*

3. *Basic closed sets (hence basic open sets) are stable under finite unions and intersections,*

   $$(a+b)_0 = (a)_0 \cap (b)_0 \ , \ (ab)_0 = (a)_0 \cup (b)_0 \quad ; \quad U_{a+b} = U_a \cup U_b \ , \ U_{ab} = U_a \cap U_b \ .$$

4. *An ideal I is maximal if and only if $A/I = \mathbb{K} = \{0,1\}$.*

5. *A prime ideal $\mathfrak{p}$ is minimal if and only if $A_{\mathfrak{p}} = \mathbb{K} = \{0,1\}$.*

6. *Any ideal is an intersection of prime ideals.*

7. *If I is an ideal of A and $(I)_0 = \operatorname{Spec} A$, then $I = 0$.*

8. *Any closed open set in* Spec A *is a basic closed set*, (p. 198).

   If $\operatorname{Spec} A = (I)_0 \cup (J)_0 = (I \cap J)_0$ and $\emptyset = (I)_0 \cap (J)_0 = (I+J)_0$, then $I + J = A$ and $I \cap J = 0$, so that $a + b = 1$, $ab = 0$ for some $a \in I$, $b \in J$; and $(I)_0 = (a)_0$.

9. *Prime ideals $\mathfrak{p}$ of A correspond to prime ideals $A - \mathfrak{p}$ of A^*.* The spectrum of the dual semiring A^* is just Spec A, but the basic open sets are $(f)_0$ and the basic closed sets are U_f.

Spectral Representation Theorem: *Any (lattice) semiring A is canonically isomorphic to the lattice of basic open sets in the spectrum* Spec A.

Proof: If $(a)_0 = (b)_0$, then $aA = bA$, since any ideal is an intersection of prime ideals, and $a = b$ since the generator of a principal ideal is unique.

Note: Now it is clear that lattice semirings, with the order $a \leq b \Leftrightarrow a + b = b$ (i.e. $a = ab$), are lattices –hence the name– where $\max(a,b) = a + b$ and $\min(a,b) = ab$.

Theorem: *The functor* Spec *is representable,* $\operatorname{Spec} A = \operatorname{Hom}_{\text{sring}}(A, \mathbb{K})$.

Proof: The kernel of any morphism $A \to \mathbb{K}$ is a prime ideal, and each prime ideal $\mathfrak{p}$ is just the kernel of the morphism $f \colon A \to \mathbb{K}$, $f(\mathfrak{p}) = 0$, $f(A - \mathfrak{p}) = 1$.

Corollary: $\operatorname{Spec}(\varinjlim A_i) = \varprojlim(\operatorname{Spec} A_i)$, *and* $\dim(\varinjlim A_i) \leq \sup\{\dim A_i\}$.

Proof: Any representable functor transforms inductive limits[2] into projective limits.

If $\mathfrak{p}$ is a prime ideal of $A := \varinjlim A_i$, we put $\mathfrak{p}_i = A_i \cap \mathfrak{p}$. If $\mathfrak{p} \neq \mathfrak{q}$, there exists an index i such that $\mathfrak{p}_i \neq \mathfrak{q}_i$; hence, if $\mathfrak{p}_0 \subset \mathfrak{p}_1 \ldots \subset \mathfrak{p}_n$ is a chain of primes in A, there exists an index i such that $(\mathfrak{p}_0)_i \subset (\mathfrak{p}_1)_i \ldots \subset (\mathfrak{p}_n)_i$, and $n \leq \dim A_i$.

[2]The inductive limit of semirings is just the inductive limit as sets, with the obvious operations. In general $\varinjlim A(X_i)$ is not a base of $X = \varprojlim X_i$, since X may be void even if the spaces X_i are non void.

Corollary: *Any* ***spectral*** *topological space (i.e. the spectrum of a semiring) is a projective limit of finite topological spaces.* (The converse also holds, see p. 236).

Proof: Any semiring A is a union of finite semirings, $A = \varinjlim A_i$; hence $\operatorname{Spec} A = \varprojlim(\operatorname{Spec} A_i)$.

Tensor Product: Let $\{A_i\}$ be a family of semirings, and put $X_i = \operatorname{Spec} A_i$, $X = \prod_i X_i$.

By the spectral representation theorem, A_i is a family of continuous $\mathbb{K}$-valued functions on X_i, hence on X composing with the projection $X \to X_i$.

The semiring of continuous $\mathbb{K}$-valued functions on X generated by these families is denoted by $\bigotimes_i A_i$, since we have canonical morphisms $A_i \to \bigotimes_i A_i$ with the universal property

$$\operatorname{Homsring}\left(\bigotimes_i A_i, B\right) = \prod_i \operatorname{Homsring}(A_i, B).$$

In fact, given morphisms $f_i \colon A_i \to B$, the continuous maps $f_i^* \colon Y = \operatorname{Spec} B \to X_i$ define a continuous map $\phi \colon Y \to X$; hence a morphism $f \colon \bigotimes_i A_i \subseteq A(X) \to A(Y)$.

By the spectral representation theorem $B \subseteq A(Y)$, and by construction $f(A_i) = f_i(A_i) \subseteq B$; hence $f \colon \bigotimes_i A_i \to B$ induces on each semiring A_i the given morphism f_i.

Corollary: $\operatorname{Spec}\left(\bigotimes_i A_i\right) = \prod_i(\operatorname{Spec} A_i)$, $\operatorname{Spec}_{\mathrm{m}}\left(\bigotimes_i A_i\right) = \prod_i(\operatorname{Spec}_{\mathrm{m}} A_i)$.

Proof: The first formula holds because Spec is a representable functor, and the second one because a point $(x_i) \in \prod_i X_i$ is closed if and only if so is any point x_i.

Corollary: $\dim A \otimes B = \dim A + \dim B$.

Proof: Put $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. In $X \times Y$ we have a specialization $(x, y) \geq (x', y')$ if and only if $x \geq x'$ and $y \geq y'$. Now it is clear that the supremum of the lengths of all chains of specializations in $X \times Y$ is just $\dim A + \dim B$.

Theorem: *A semiring A has dimension 0 (any prime ideal is a maximal ideal) if and only if A is a* ***Boolean algebra*** (for any $a \in A$ there exists $a' \in A$ such that $a' + a = 1$, $a'a = 0$).

Proof: If $\dim A = 0$ and $a \neq 1$ is in a prime $\mathfrak{p}$, since $A_{\mathfrak{p}} = \mathbb{K} = \{0, 1\}$ because any prime is minimal, we see that a is annihilated by an element of $A - \mathfrak{p}$.

That is to say, no prime ideal contains $\operatorname{Ann}(a) + aA$; hence $\operatorname{Ann}(a) + aA = A$.

There exists $a' \in \operatorname{Ann}(a)$ such that $1 = a' + ab$ for some $b \in A$, so that

$$a' + a = a' + a(1 + b) = a + a' + ab = a + 1 = 1.$$

Now let us prove the converse. If $\mathfrak{p}$ is a prime ideal and $a \in \mathfrak{p}$, then $a' \notin \mathfrak{p}$ since $a' + a = 1$, and $\frac{a}{s} = 0$ in $A_{\mathfrak{p}}$ since a is annihilated by $a' \in A - \mathfrak{p}$.

Then $A_{\mathfrak{p}} = \{0, 1\}$, and $\mathfrak{p}$ is a minimal prime. Since any prime is minimal, $\dim A = 0$.

Corollary: *In any Boolean algebra, the* ***complement*** *a' is unique, and it defines an isomorphism onto the dual semiring; $(a + b)' = a'b'$, $(ab)' = a' + b'$, $0' = 1$, $1' = 0$.*

Proof: The equalities $a' + a = 1$ and $a'a = 0$, show that in the spectral representation a' is just the complement of a, and now all is obvious.

Corollary: *Any Boolean algebra is a union of finite Boolean algebras, and any finite Boolean algebra is isomorphic to the semiring of all subsets of a finite set.*

Proof: In a Boolean algebra A, any finite family $a_1, \dots, a_n$ is contained in the semiring generated by $a_1, \dots, a_n, a'_1, \dots, a'_n$, which is a finite Boolean algebra.

If moreover A is finite, then $A = A(\operatorname{Spec} A)$, and closed sets in $\operatorname{Spec} A$ are open.

Since $X = \operatorname{Spec} A$ is T_0 and any point is closed, the topology is discrete.

Hence A coincides with the semiring of all subsets of the finite set X.

Stone's Representation Theorem: *The functor* Spec *defines an antiequivalence of the category of Boolean algebras with the category of* ***profinite spaces*** (projective limits of finite discrete spaces) *the inverse functor being* $X \rightsquigarrow$ Lattice of closed open sets in X.

Proof: Any Boolean algebra A is isomorphic to the semiring of closed open sets in $\operatorname{Spec} A$, because any closed open set is a basic open set.

Moreover, any profinite space is $X = \varprojlim X_i = \varprojlim(\operatorname{Spec} A_i) = \operatorname{Spec}(\varinjlim A_i) = \operatorname{Spec} A$, where $A = \varinjlim A_i$ is a Boolean algebra, so that A is the semiring of closed open sets in $\operatorname{Spec} A = X$.

8.2 Compact Spaces

Let $B \subseteq A(X)$ be a base of a topological space X (and recall that X is assumed to be T_0).

Any point $x \in X$ defines a prime ideal $\mathfrak{p}_x = \{f \in B \colon f(x) = 0\}$ of B (the ideal of all closed sets in B containing x when B is viewed as a semiring of closed sets) and this map

$$X \longrightarrow \operatorname{Spec} B, \; x \mapsto \mathfrak{p}_x,$$

is injective because X is T_0, continuous since $z(f) = (f)_0 \cap X$, and a homeomorphism onto the image (B is a base), which is dense: $X \subseteq (f)_0 \Rightarrow f = 0 \Rightarrow (f)_0 = \operatorname{Spec} B$.

Theorem: *A topological space X is compact if and only if* $\operatorname{Spec}_{\mathrm{m}} B \subseteq X$.

Proof: If $\mathfrak{m}$ is a maximal ideal of B and $f_1, \dots, f_n \in \mathfrak{m}$, then $f_1 + \ldots + f_n \neq 1$, so that

$$z(f_1) \cap \ldots \cap z(f_n) = z(f_1 + \ldots + f_n) \neq \emptyset.$$

When X is compact, the intersection of all zero sets $z(f)$, $f \in \mathfrak{m}$, is non void; so that $\mathfrak{m} \subseteq \mathfrak{p}_x$ for some $x \in X$, and $\mathfrak{m} = \mathfrak{p}_x$ since $\mathfrak{m}$ is maximal.

Conversely, if a closed set $(I)_0$ of $\operatorname{Spec} B$ does not intersect $\operatorname{Spec}_{\mathrm{m}} B$, it is void since any ideal $I \neq B$ is contained in some maximal ideal.

Since $\operatorname{Spec} B$ is compact, any subspace containing $\operatorname{Spec}_{\mathrm{m}} B$ also is compact.

Corollary: *A topological space X is compact and T_1 if and only if* $\operatorname{Spec}_{\mathrm{m}} B = X$.

Proof: If X is compact and T_1, let us show that the prime ideals $\mathfrak{p}_x$ of B are maximal ideals; i.e. $fB + \mathfrak{p}_x = B$ for any continuous function $f \in B$ such that $f(x) \neq 0$.

Since x is a closed point, there are functions $f_i \in \mathfrak{p}_x$ such that $\sum_i f_i$ only vanishes at the point x; hence $f + \sum_i f_i = 1$ and, X being compact, some finite sum is $f + f_{i_1} + \ldots + f_{i_n} = 1$. We conclude that $fB + \mathfrak{p}_x = B$.

Tychonoff's Theorem: *Any direct product $\prod_i X_i$ of compact spaces is compact.*

Proof: Put $A_i = A(X_i)$. By definition $A = \bigotimes_i A_i$ is a base of $\prod_i X_i$.

Now, $\operatorname{Spec}_{\mathrm{m}} A_i \subseteq X_i$ when X_i is compact, and we conclude since (p. 229)

$$\operatorname{Spec}_{\mathrm{m}} A = \textstyle\prod_i \operatorname{Spec}_{\mathrm{m}} A_i \subseteq \prod_i X_i \subseteq \operatorname{Spec} A.$$

Corollary: *Any projective limit of non void compact separated spaces is a non void compact separated space.*

Proof: The limit of a projective system $\{X_i, \phi_{ji}\}$ is the subspace $\bigcap_{i,j} Y_{ji}$ of $\prod_i X_i$, where

$$Y_{ji} = \{(x_i) \in \prod_i X_i \colon x_i = \phi_{ji}(x_j)\},$$

and all finite intersections are non void since the set of indices is a filtered order.

Now, Y_{ji} is closed when X_i is separated; hence $\bigcap_{i,j} Y_{ji}$ is a non void closed set.

Corollary: *Any profinite space is a compact separated space.*

Lemma: *Let $\phi\colon X \to Y$ be a closed continuous map. The morphism $\phi^{-1}\colon A_Y/(A_Y \cap I) \to A_X/I$ is injective for any ideal I of $A(X)$.*

Proof: We view A_X and A_Y as semirings of closed sets (dual to the lattices of closed sets).

Let $a, b \in A_Y$.

If $\phi^{-1}(a) + c = \phi^{-1}(b) + c$, $c \in I$, then $\phi(c) \in A_Y$ (ϕ is a closed map) and

$$a + \phi(c) = a \cap \phi(c) = \phi(\phi^{-1}(a) \cap c) = \phi(\phi^{-1}(b) \cap c) = b + \phi(c),$$

where $\phi(c) \in A_Y \cap I$, because $\phi^{-1}(\phi(c)) \in I$ since it contains $c \in I$.

Proposition: *Let $\{X_i, \phi_{ji}\}$ be a projective system. If the maps ϕ_{ji} are closed, then*

$$\operatorname{Spec}_{\mathrm{m}} (\varinjlim A_i) = \varprojlim (\operatorname{Spec}_{\mathrm{m}} A_i) \ , \quad A_i := A(X_i).$$

Proof: We have $\operatorname{Spec}(\varinjlim A_i) = \varprojlim \operatorname{Spec} A_i$, $\mathfrak{p} = (\mathfrak{p}_i)_{i \in I}$, where $\mathfrak{p}_i := A_i \cap \mathfrak{p}$.

By the lemma, the morphisms $\phi_{ji}^*\colon A_i/\mathfrak{p}_i \to A_j/\mathfrak{p}_j$ are injective, so that

$$A/\mathfrak{p} = \varinjlim (A_i/\mathfrak{p}_i) = \bigcup_i A_i/\mathfrak{p}_i$$

and now it is clear that $A/\mathfrak{p} = \mathbb{K}$ if and only if $A/\mathfrak{p}_i = \mathbb{K}$ for any index i.

Theorem: *Let $\{X_i, \phi_{ji}\}$ be a projective system of non void compact T_0 spaces. If the maps ϕ_{ji} are closed, then the projective limit is a non void compact T_0 space.*

Proof: Put $A_i = A(X_i)$. We have $A := \varinjlim A_i \neq 0$ since the spaces X_i are non void and

$$\operatorname{Spec}_{\mathrm{m}} A = \varprojlim (\operatorname{Spec}_{\mathrm{m}} A_i) \subseteq \varprojlim X_i \subseteq \operatorname{Spec} A.$$

Definitions: If X is a topological space, and Y a metric space, then $\operatorname{Hom}_{\mathrm{Sets}}(X, Y)_c$ is the set of maps $f\colon X \to Y$ with the **compact convergence** topology, defined by the "pseudometrics[3]"

$$\mathrm{d}_K(f, h) = \sup_{x \in K} \mathrm{d}\big(f(x), h(x)\big) \in [0, \infty]$$

where K runs over the compact sets in X, and $\mathcal{C}(X, Y)_c$ is the set of continuous maps, with the induced topology. A base of neighborhoods of a map f are the balls $B_{\mathrm{d}_K}(f, \varepsilon)$.

When we consider X with the discrete topology, the compact sets are the finite subsets, and we obtain the **weak** or **pointwise convergence** topology on $\operatorname{Hom}_{\mathrm{Sets}}(X, Y)_w = \prod_X Y$: it is just the product topology.

[3]In fact d_K is not a true pseudometric, because it may take the value ∞, but however it clearly defines a topology in $\operatorname{Hom}_{\mathrm{Sets}}(X, Y)$ in the usual way.

When we consider X with the trivial topology, all the subsets are compact, and we obtain $\mathrm{Hom}_{\mathrm{Sets}}(X,Y)_u$, the set of all maps with the topology of **uniform convergence** on X.

A family of maps $\mathcal{F} \subseteq \mathrm{Hom}_{\mathrm{Sets}}(X,Y)$ is **equicontinuous** when, given $\varepsilon > 0$, any point $p \in X$ has a neighborhood U_p in X such that $\mathrm{d}\big(f(x), f(p)\big) < \varepsilon$, $\forall x \in U_p$, $f \in \mathcal{F}$ (in particular, any map $f \in \mathcal{F}$ is continuous).

Then the closure $\bar{\mathcal{F}}$ in $\mathrm{Hom}_{\mathrm{Sets}}(X,Y)_w$ also is equicontinuous: given $\varepsilon > 0$, for any function $\bar{f} \in \bar{\mathcal{F}}$ and any point $x \in U_p$, there exists $f \in \mathcal{F}$ such that $\mathrm{d}\big(\bar{f}(x), f(x)\big)$ and $\mathrm{d}\big(\bar{f}(p), f(p)\big)$ are $< \varepsilon$; since $\mathrm{d}\big(f(x), f(p)\big) < \varepsilon$, we conclude that $\mathrm{d}\big(\bar{f}(x), \bar{f}(p)\big) < 3\varepsilon$.

Lemma: *The compact and pointwise convergence induce the same topology on any equicontinuous family $\mathcal{F}$.*

Proof: Given $\varepsilon > 0$ and a compact set $K \subseteq X$, there is a finite open cover $K \subseteq U_1 \cup \ldots \cup U_n$ and points $x_i \in U_i$ such that $\mathrm{d}\big(f(x), f(x_i)\big) < \varepsilon$, $\forall x \in U_i,\ f \in \mathcal{F}$.

Now, if $x \in K$, then $x \in U_i$ for some index i, so that

$$\mathrm{d}\big(f(x), h(x)\big) \leq \mathrm{d}\big(f(x), f(x_i)\big) + \mathrm{d}\big(f(x_i), h(x_i)\big) + \mathrm{d}\big(h(x_i), h(x)\big) < 2\varepsilon + \mathrm{d}\big(f(x_i), h(x_i)\big)$$

for every function $h \in \mathcal{F}$. Hence $B_{\mathrm{d}_{x_1,\ldots,x_n}}(f, \varepsilon) \cap \mathcal{F} \subseteq B_{\mathrm{d}_K}(f, 3\varepsilon) \cap \mathcal{F}$ and we conclude.

Ascoli's Theorem: *Let $\mathcal{F}$ be an equicontinuous family of maps from a topological space X to a metric space Y. If $\mathcal{F}(x) = \{f(x);\ f \in \mathcal{F}\}$ has compact closure in Y for any point $x \in X$, then $\mathcal{F}$ has compact closure in $\mathcal{C}(X,Y)_c$.*

Proof: The closure $\bar{\mathcal{F}}$ in $\mathrm{Hom}_{\mathrm{Sets}}(X,Y)_w = \prod_X Y$ is compact because it is a closed subspace of the compact space $\prod_x \overline{\mathcal{F}(x)}$.

Now, $\bar{\mathcal{F}}$ is an equicontinuous family; hence $\mathcal{F} \subseteq \bar{\mathcal{F}} \subseteq \mathcal{C}(X,Y)$ and the topology of the compact space $\bar{\mathcal{F}}$ is induced by the topology of $\mathcal{C}(X,Y)_c$.

8.2.1 Proper Morphisms

Theorem: *A topological space K is compact if and only if $\pi_2 \colon K \times T \to T$ is a closed map for every topological space T.*

Proof: If $C \subseteq K \times T$ is closed, and $t \in T - \pi_2(C)$, then any point $(x,t) \in K \times t$ has an open neighborhood $U_x \times V_x$ not intersecting C. When K is compact, the fibre $K \times t$ admits a finite open cover $K \times t \subseteq (U_1 \times V_1) \cup \ldots \cup (U_n \times V_n)$, so that $\bigcap_i V_i$ is a neighborhood of t not intersecting $\pi_2(C)$, and we see that $\pi_2(C)$ is closed.

Conversely, put $T = \mathrm{Spec}\, A(K)$, with the topology generated by the sets $(f)_0$ (open sets are arbitrary unions of sets $(f)_0$, it is just the spectrum of the dual semiring).

The map $\phi \colon K \to T = \mathrm{Spec}\, A(K)$, $\phi(x) = \mathfrak{p}_x$, is not continuous, but it has dense image.

Let us consider the graphic $\Gamma_\phi \subseteq K \times T$. Since $\pi_2 \colon K \times T \to T$ is closed, $\pi_2(\bar{\Gamma}_\phi)$ is a closed set containing $\pi_2(\Gamma_\phi) = \mathrm{Im}\,\phi$; hence $\pi_2(\bar{\Gamma}_\phi) = T$. For any prime ideal $\mathfrak{p}$ of $A(K)$ there is a point $x \in K$ such that $(x, \mathfrak{p}) \in \bar{\Gamma}_\phi$, so that for any neighborhood U of x, and any function $f \in \mathfrak{p}$ we have $(U \times (f)_0) \cap \Gamma_\phi \neq \emptyset$; i.e. $U \cap z(f) \neq \emptyset$. Since $z(f)$ is closed, $f(x) = 0$ and $\mathfrak{p} \subseteq \mathfrak{p}_x$.

When $\mathfrak{p}$ is maximal, we conclude that $\mathfrak{p} = \mathfrak{p}_x$, so that K is compact.

Definition: A continuous map $f \colon X \to S$ is **proper** if it is universally closed; i.e. for any continuous map $T \to S$ the base change $\pi_2 \colon X \times_S T \to T$ is a closed map.

1. *If $f \colon X \to S$ is a proper morphism, so is $\pi_2 \colon X \times_S T \to T$ for any base change $T \to S$.*

   In fact $(X \times_S T) \times_T Z = X \times_S Z$.

2. *If $Y \to X$ and $X \to S$ are proper morphisms, so is the composition $Y \to S$.*

   The composition of closed maps also is closed.

3. *Let $S = \bigcup_i U_i$ be an open cover. A morphism $f\colon X \to S$ is proper if and only if so is $f^{-1}(U_i) \to U_i$ for any index i.*

   A subset $Y \subseteq S$ is closed if and only every $U_i \cap Y$ is closed in U_i.

4. The above theorem states that *a topological space K is compact if and only if the projection $X \to *$ onto the one point space $*$ is a proper morphism.*

5. *If $f\colon X \to S$ is a proper morphism, and $K \subseteq S$ is compact, then $f^{-1}(K)$ is compact.*

   In fact, $f^{-1}(K) \to K$ and $K \to \mathrm{pt}$ are proper; hence so is $f^{-1}(K) \to \mathrm{pt}$.

6. *If K is compact, and S is separated, then any morphism $K \to S$ is proper.*

   For any base change $T \to S$, the subspace $K \times_S T \subseteq K \times T$ is closed, because it is the inverse image of the diagonal by the natural map $K \times T \to S \times S$; hence the composition $K \times_S T \hookrightarrow K \times T \to T$ is a closed map.

7. *Let S be a separated locally compact space, and $f\colon X \to S$ a continuous map. If any point $p \in S$ has a compact neighborhood K_p such that $f^{-1}(K_p)$ is compact, then f is proper.*

   The morphism $f^{-1}(K_p) \to K_p$ is proper ($f^{-1}(K_p)$ is compact and K_p is separated); hence so is $f^{-1}(U_p) \to U_p$, where U_p is the interior of K_p, and $S = \bigcup_p U_p$.

Lemma: *Let $f\colon X \to S$ be a continuous map. If $f \times \mathrm{Id}\colon X \times T \to S \times T$ is closed for any topological space T, then f is a proper morphism.*

Proof: Any morphism $h\colon T \to S$ is the composition of $\Gamma_h\colon T \to S \times T$, $\Gamma_h(t) = \big(h(t), t\big)$, with the projection $S \times T \to S$.

The map $X \times T = X \times_S (S \times T) \to S \times T$ is closed by hypothesis; hence so is after the base change $\Gamma_h\colon T \to S \times T$, because Γ_h is an isomorphism onto a subspace of $S \times T$.

Theorem: *Proper morphisms are just closed continuous maps with compact fibres.*

Proof: The necessity is clear. Conversely, if $f\colon X \to S$ is a closed morphism with compact fibres, let us see that $f \times \mathrm{Id}\colon X \times T \to S \times T$ is closed for any topological space T.

Let $C \subseteq X \times T$ be a closed set and put $C' = (f \times \mathrm{Id})(C) \subseteq S \times T$. If $(s,t) \notin C'$, then the compact set $f^{-1}(s) \times t$ does not intersect the closed set C; hence there is an open set $U \times V_t$ in $X \times T$ such that $f^{-1}(s) \times t \subseteq U \times V_t$ and $C \cap (U \times V_t) = \emptyset$. Then $V = S - f(X - U)$ is an open set, and $V \times V_t$ is a neighborhood of (s,t) not intersecting C'. We see that C' is closed.

8.3 Separation Properties

Definition: Two prime ideals $\mathfrak{p}_1 \neq \mathfrak{p}_2$ of a semiring A **hybridize** if $\mathfrak{p}_1 \cap \mathfrak{p}_2$ contains a prime ideal of A.

In terms of the primes $\mathfrak{p}_i^* = A - \mathfrak{p}_i$ of the dual semiring A^*, we have that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ do not hybridize if no prime of A^* contains $\mathfrak{p}_1^*$ and $\mathfrak{p}_2^*$; i.e., $\mathfrak{p}_1^* + \mathfrak{p}_2^* = A^*$.

In terms of A, there exist $f_1, f_2 \in A$ such that $f_1 f_2 = 0$ and $f_i \notin \mathfrak{p}_i$; i.e., both primes define points of $\operatorname{Spec} A$ with disjoint basic neighborhoods U_{f_1} and U_{f_2} in $\operatorname{Spec} A$.

Proposition: *Let B be a base of a topological space X. The space X is Hausdorff if and only if any two points of $X \subseteq \operatorname{Spec} B$ do not hybridize.*

Proof: Let $x_1, x_2 \in X$. There are functions $f_1, f_2 \in B$ such that $f_1 f_2 = 0$ and $f_i(x_i) \neq 0$ just when there are open sets U_1, U_2 in the given base such that $U_1 \cap U_2 = \emptyset$ and $x_i \in U_i$.

Proposition: *A T_1 space X is **regular** (open sets separate points and closed sets) if and only if no point of X hybridize with a point of* $\operatorname{Spec_m} A(X)$.

Proof: Since any point $x \in X$ is closed, its ideal $\mathfrak{p}_x$ is a maximal ideal of $A(X)$.

Let $\mathfrak{m}$ be another maximal ideal, and consider a function $f \in \mathfrak{m}$ such that $f(x) \neq 0$.

If X is regular, there are functions $g, h \in A(X)$ separating $z(f)$ and x:

$$gh = 0 \quad , \quad f + g = 1 \quad , \quad h(x) \neq 0 \; ;$$

hence $g \notin \mathfrak{m}$, since $f \in \mathfrak{m}$, and we see that $\mathfrak{p}_x, \mathfrak{m}$ do not hybridize.

Conversely, if $f \in A(X)$ and $f(x) \neq 0$, by hypothesis for any maximal ideal $\mathfrak{m}_i$ containing f there are functions $g_i, h_i \in A(X)$ such that

$$g_i h_i = 0 \quad , \quad g_i \notin \mathfrak{m}_i \quad , \quad h_i(x) \neq 0 \; .$$

Now $(f)_0 \cap (\bigcap_i (g_i)_0)$ does not intersect $\operatorname{Spec_m} A(X)$, so that it is void.

Since $\operatorname{Spec} A(X)$ is compact, there exists a finite family $g_1, \ldots, g_n$ such that

$$\emptyset = (f)_0 \cap (g_1)_0 \cap \ldots \cap (g_n)_0 = (f + g_1 + \ldots + g_n)_0;$$

hence $1 = f + g_1 + \ldots + g_n$.

Put $g = g_1 + \ldots + g_n$, $h = h_1 \ldots h_n$. Then $gh = 0$, $f + g = 1$, and $h(x) \neq 0$.

Now $U_h \cap X$ and $U_g \cap X$ are disjoint neighborhoods of x and the closed set $z(f)$.

Definition: A semiring A is **normal** if for each pair $a_1, a_2 \in A$ with sum $a_1 + a_2 = 1$ there is some pair $b_1, b_2 \in A$ such that

$$a_1 + b_1 = 1 \quad , \quad a_2 + b_2 = 1 \quad , \quad b_1 b_2 = 0 \; ,$$

and a space X is **normal** if so is $A(X)$ (disjoint closed sets have disjoint neighborhoods).

Examples: Any Boolean algebra is normal (take $b_i = a_i'$).

$\mathbb{K} = \{0, 1\}$, with a dense point 1, is a normal space; but it is not T_1.

Theorem: *Let A be a semiring. The following conditions are equivalent,*

1. *A is a normal semiring.*
2. $\operatorname{Spec} A$ *is a normal space.*
3. *Any prime ideal of A is contained in a unique maximal ideal.*

Proof: ($1 \Rightarrow 2$) If $\emptyset = (I_1)_0 \cap (I_2)_0 = (I_1 + I_2)_0$, then $I_1 + I_2 = A$.

Hence $a_1 + a_2 = 1$ for some $a_i \in I_i$.

By hypothesis there are $b_1, b_2 \in A$ such that $b_1 b_2 = 0$, $a_1 + b_1 = 1$, $a_2 + b_2 = 1$.

Now U_{b_1} and U_{b_2} are disjoint neighborhoods of $(I_1)_0$ and $(I_2)_0$.

($2 \Rightarrow 3$) Closed points of $\operatorname{Spec} A$ have disjoint neighborhoods; hence they do not hybridize.

($3 \Rightarrow 1$) By hypothesis the closure of each point has a unique closed point, and any two closed points (defined by two maximal ideals $\mathfrak{m}, \mathfrak{m}'$) have disjoint basic neighborhoods (because $(A_{\mathfrak{m}})_{\mathfrak{m}'} = 0$); hence any two points with disjoint closures have disjoint basic neighborhoods.

If $a_1 + a_2 = 1$, then $C_1 = (a_1)_0$ and $C_2 = (a_2)_0$ are disjoint.

If $x \in C_1$, any point of C_2 and x have disjoint basic neighborhoods.

Since C_2 is compact (and basic open sets are stable under finite unions and intersections) C_2 and x have disjoint basic neighborhoods.

Since C_1 is compact, $C_1 = (a_1)_0$ and $C_2 = (a_2)_0$ have disjoint basic neighborhoods U_{b_1}, U_{b_2}:

$$(a_i)_0 \subseteq U_{b_i} \quad , \quad \emptyset = (a_i)_0 \cap (b_i)_0 = (a_i + b_i)_0 \quad , \quad a_i + b_i = 1$$
$$\emptyset = U_{b_1} \cap U_{b_2} \quad , \quad \operatorname{Spec} A = (b_1)_0 \cup (b_2)_0 = (b_1 b_2)_0 \quad , \quad b_1 b_2 = 1$$

Corollary: *Any base B of a compact separated space X is normal.*

Proof: We have $X = \operatorname{Spec}_{\mathrm{m}} B$, and any two points of X do not hybridize.

Lemma: *If A is normal, disjoint closed sets in* $\operatorname{Spec}_{\mathrm{m}} A$ *have disjoint neighborhoods in* $\operatorname{Spec} A$.

Proof: If two closed sets $(I_1)_0, (I_2)_0$ in $\operatorname{Spec} A$ intersect $\operatorname{Spec}_{\mathrm{m}} A$ at disjoint closed sets, then $(I_1)_0 \cap (I_2)_0 = (I_1 + I_2)_0 = \emptyset$ because any ideal $\neq A$ is contained in some maximal ideal, and $(I_1)_0, (I_2)_0$ have disjoint neighborhoods since $\operatorname{Spec} A$ is normal.

Corollary: *If A is normal, then* $\operatorname{Spec}_{\mathrm{m}} A$ *is a Hausdorff space.*

Corollary: *If A is a Boolean algebra, then* $\operatorname{Spec} A$ *is a compact Hausdorff space.*

Proof: Any Boolean algebra is normal; hence $\operatorname{Spec} A = \operatorname{Spec}_{\mathrm{m}} A$ is T_2.

Continuous Retract Theorem: *If A is a normal semiring, the map $r\colon \operatorname{Spec} A \to \operatorname{Spec}_{\mathrm{m}} A$, where $r(\mathfrak{p})$ is the unique maximal ideal containing $\mathfrak{p}$, is a continuous retract.*

Proof: Remark that each point $x \in \operatorname{Spec}_{\mathrm{m}} A$ is in the closure of any point of the fibre $r^{-1}(x)$; hence any neighborhood of x in $\operatorname{Spec} A$ contains $r^{-1}(x)$.

Let U be an open set in $\operatorname{Spec}_{\mathrm{m}} A$ and let C be the complement.

By the lemma, any point $x \in U$ and C have disjoint neighborhoods V_x, W_x in $\operatorname{Spec} A$.

Hence $r^{-1}(x) \subseteq V_x$, $r^{-1}(C) \subseteq W_x$, so that $V_x \subseteq r^{-1}(U)$.

We conclude that $r^{-1}(U) = \bigcup_{x \in U} V_x$ is open, and r is continuous.

8.4 Noetherian and Finite Spaces

Definition: A semiring A is **noetherian** if any ideal is finitely generated (hence principal).

A T_0 topological space X is **noetherian** if any descending chain of closed sets is finite (any ascending chain of open sets is finite).

Any noetherian space X is compact, and

1. *Any subspace Y of X is noetherian (and compact).*

   An infinite chain of closed sets $C_0 \supset C_1 \supset \ldots$ in Y would define an infinite chain of closed sets $\overline{C}_0 \supset \overline{C}_1 \supset \ldots$ in X, because $C_i = Y \cap \overline{C}_i$.

2. *X has a finite number of irreducible components.*

   If X is not a finite union of irreducible closed sets, then $X = Y_1 \cup Y_2$, where some of the two closed sets $Y_i \neq X$ also has such property, so obtaining an infinite decreasing chain of closed sets in X.

3. *$A(X)$ is a noetherian semiring, and it is the unique base of X.*

   An element a of an ideal I generates I when no element of I is strictly greater than a; hence a non principal ideal of $A(X)$ would define an infinite increasing chain of open sets. Any base of X is $A(X)$ since the union of any family of open sets coincides with the union of a finite subfamily.

4. *Irreducible closed sets in X correspond to prime ideals in $A(X)$.*

   A closed set Y of a topological space X is irreducible if and only if the principal ideal (χ_Y) is a prime ideal of $A(X)$, where $\chi_Y\colon X \to \mathbb{K}$ is the function just vanishing on Y.

   If X is noetherian, any prime ideal of $A(X)$ is principal, and we conclude.

5. *The spectrum of a noetherian ring is a noetherian topological space.*

   Any infinite chain of closed sets $Y_0 \supset Y_1 \supset \dots$ in $\operatorname{Spec} A$ defines an infinite chain of ideals $I_0 \subset I_1 \subset \dots$ in A, where $I_i := I_{Y_i}$ (the inclusions are strict because $Y_i = (I_i)_0$).

   Hence, the finiteness of the number of irreducible components gives another proof of the existence of a finite number of minimal primes in a noetherian ring, p. 140.

If A is a noetherian semiring, then $\operatorname{Spec} A$ is a noetherian space since ideals of A correspond to closed sets in the spectrum, $(I)_0 = (J)_0 \Leftrightarrow I = J$; hence (by 3)

$$A = A(\operatorname{Spec} A).$$

However, when X is noetherian, the inclusion $X \subseteq \operatorname{Spec} A_X$ is strict if some irreducible closed set in X has no dense point.

Theorem: *The functors $A \rightsquigarrow \operatorname{Spec} A$, $X \rightsquigarrow A_X$ define an antiequivalence of the category of noetherian semirings with the category of noetherian spaces where irreducible closed sets have a generic (i.e. dense) point.*

$$\begin{bmatrix}\text{Noetherian}\\ \text{semirings}\end{bmatrix}^{\text{op}} \leftrightsquigarrow \begin{bmatrix}\text{Noetherian } T_0 \text{ spaces}\\ \text{with generic points}\end{bmatrix}, \quad \begin{matrix} A = A(\operatorname{Spec} A)\\ X = \operatorname{Spec}(A_X)\end{matrix}$$

Finite Spaces: Any finite ordered set X has a topology: a subset C is closed if it is stable under specialization, $x \leq y \in C \Rightarrow x \in C$. This topology determines the order, and a map $f\colon X \to Y$ is order-preserving, $x \leq y \Rightarrow f(x) \leq f(y)$, just when it is continuous.

Conversely, in a finite T_0 topological space X closed sets are (finite) unions of closures of points, so that the topology of X is fully determined by the order relation $x \leq y \Leftrightarrow x \in \overline{\{y\}}$.

Theorem: *The functors $A \rightsquigarrow \operatorname{Spec} A$, $X \rightsquigarrow A(X)$ define an antiequivalence of categories*

$$\begin{bmatrix}\text{Finite}\\ \text{semirings}\end{bmatrix}^{\text{op}} \leftrightsquigarrow \begin{bmatrix}\text{Finite } T_0\\ \text{spaces}\end{bmatrix} = \begin{bmatrix}\text{Finite}\\ \text{orders}\end{bmatrix}$$

Corollary: *Projective limits of non void finite T_0 spaces are non void spectral spaces.*

Proof: If $X_i = \operatorname{Spec} A_i$, then $\varprojlim X_i = \varprojlim(\operatorname{Spec} A_i) = \operatorname{Spec}(\varinjlim A_i)$.

Definitions: The **barycentric subdivision** βX of a finite order X is the space of all chains $x_0 < x_1 < \dots < x_n$ (with $n \geq 0$), ordered by inclusion.

We have a natural continuous map $\beta X \to X$ transforming any chain into the last element, and the inverse image of any closed set Y is βY. Inductively we define the iterated subdivisions, $\beta^n X = \beta(\beta^{n-1}X)$, and we obtain a projective system

$$\ldots \longrightarrow \beta^{n+1}X \longrightarrow \beta^n X \longrightarrow \ldots \longrightarrow \beta X \longrightarrow X$$

The **geometric realization** $|X|$ of a finite order X is the subspace of all closed points of $\varprojlim \beta^n X$, and it is clear that $|X| = |\beta X| = |\beta^n X|$.

If A_n denotes the topology of $\beta^n X$, the continuous maps $\beta^{n+1}X \to \beta^n X$ induce morphisms $A_n \to A_{n+1}$, and we have $\varprojlim \beta^n X = \varprojlim(\operatorname{Spec} A_n) = \operatorname{Spec}(\varinjlim A_n)$. Hence

$$|X| = \operatorname{Spec}_{\mathrm{m}}(\varinjlim A_n).$$

A continuous map $f\colon X \to X'$ induces a closed map

$$\beta f\colon \beta X \longrightarrow \beta X'$$

transforming any chain $x_0 < x_1 < \ldots < x_n$ of X into the greatest chain of βX contained in $f(x_0) \le f(x_1) \le \ldots \le f(x_n)$.

We have closed maps $\beta^n f\colon \beta^n X \to \beta^n X'$ and, by the lemma of p. 231, the continuous map

$$\operatorname{Spec}(\varinjlim A_n) = \varprojlim \beta^n X \longrightarrow \varprojlim \beta^n X' = \operatorname{Spec}(\varinjlim A'_n)$$

transforms closed points into closed points, hence it defines a continuous map

$$|f|\colon |X| \longrightarrow |X'|.$$

When $Y \to X$ is a closed embedding, $|Y| \to Y \times_X |X|$ is a homeomorphism.

Theorem: *The geometric realization of the finite order $\Delta_n = \{0, 1, \ldots, n\}$ is the n-tetrahedron,* i.e. the subspace $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}\colon t_0 + \ldots + t_n = 1, t_i \ge 0\}$.

Proof: Let A_1 be the semiring of all finite unions of closed simplices (vertices, edges, faces,...) of the n-tetrahedron. Simplices are just generators of prime ideals in A_1, so that $\operatorname{Spec} A_1$ has a point for each simplex, and the order is the incidence relation.

Once we number the vertices of the tetrahedron, we have that $\operatorname{Spec} A_1 = \beta\Delta_n$, where each simplex is identified with the sequence formed by the vertices.

Let A_{i+1} be the lattice of all finite unions of closed simplices of the i-th barycentric subdivision of the tetrahedron (the new vertices being the barycentres of the former simplices).

Now $\operatorname{Spec} A_i = \beta^i \Delta_n$, and we have commutative squares

$$\begin{array}{ccc} \operatorname{Spec} A_{i+1} & \longrightarrow & \operatorname{Spec} A_i \\ || & & || \\ \beta^{i+1}\Delta_n & \longrightarrow & \beta^i \Delta_n \end{array}$$

$$\varprojlim \beta^i \Delta_n = \varprojlim \operatorname{Spec}(A_i) = \operatorname{Spec}(\varinjlim A_i)$$
$$|\Delta_n| = \operatorname{Spec}_{\mathrm{m}}(\varinjlim A_i) = \operatorname{Spec}_{\mathrm{m}} B$$

The tetrahedron is a compact T_1 space, hence it coincides (p. 230) with the maximal spectrum of any base, and B is a base. q.e.d.

Remark that B is a base of a compact Hausdorff space; hence it is normal (p. 235) and we have a continuous retract $r\colon \varprojlim \beta^i \Delta_n = \operatorname{Spec} B \longrightarrow \operatorname{Spec}_{\mathrm{m}} B = |\Delta_n|$.

Theorem: *The geometric realization of any finite T_0 space X is a finite* ***polyhedron*** (i.e., we have a homeomorphism $|X| \simeq P$ with a finite union P of closed simplices of a tetrahedron, and such homeomorphism is named a **triangulation** of $|X|$).

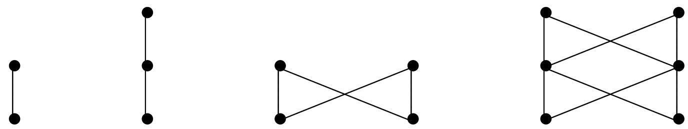
Proof: If we number the points of X, then βX is a closed set in $\beta\Delta_n$.

Hence $|\beta X| = |X|$ is a closed set in $|\beta\Delta_n| = |\Delta_n|$ formed by some simplices.

Examples: The above proof shows that the polyhedron $|X|$ has a vertex for each point of X, an edge for any chain $x_0 < x_1$ (joining the vertices x_0, x_1), a face for any chain $x_0 < x_1 < x_2$ (joining the edges $x_0 < x_1, x_1 < x_2, x_0 < x_2$), and so on.

We represent a finite space with a diagram of points at different levels, with an edge joining a point y with an inferior point x when $x < y$; i.e., the closure of a point is formed by the points under it (and closed points are at the bottom).

For example, the geometric realization of the following finite spaces



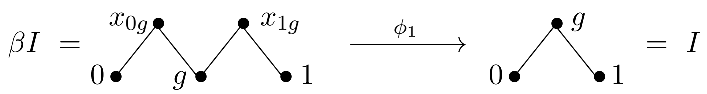
are a closed interval, a triangle, a circle and a sphere respectively.

Any semiring is an inductive limit of finite semirings, so that, in a certain sense, any space X may be approximated by finite polyhedrons, since it is a dense subspace of a projective limit of finite spaces, $X \subseteq \operatorname{Spec} A(X) = \operatorname{Spec}(\varinjlim A_i) = \varprojlim(\operatorname{Spec} A_i)$.

Uryshon's Lemma: *If C_0 and C_1 are disjoint closed sets in a normal space X, there exists a continuous function $f\colon X \to [0,1]$ such that $f(C_0) = 0$, $f(C_1) = 1$.*

Proof: $I = \beta\Delta_1 = \{0, 1, g\}$ is a finite space with two closed points 0,1 and a generic point g.

A continuous map $\phi\colon X \to I$ is just a pair of disjoint closed sets $C_0 = \phi^{-1}(0), C_1 = \phi^{-1}(1)$, and the existence of disjoint open neighborhoods U_1, U_2 in X amounts to lift ϕ to a continuous map $\phi_1\colon X \to \beta I$,



where $U_1 = \phi_1^{-1}\{0, x_{0g}\}, U_2 = \phi_1^{-1}\{1, x_{1g}\}$, and therefore $\phi_1^{-1}(g) = X - (U_1 \cup U_2)$.

Since βI is formed by two copies of I, reiterating the argument we may lift ϕ_1 to a continuous map $\phi_2\colon X \to \beta^2 I$, and so on.

Now the continuous function f is given by the continuous retract theorem,

$$X \longrightarrow \varprojlim \beta^n\Delta_1 \xrightarrow{\ r\ } |\Delta_1| = [0,1].$$

Spectral Spaces: A topological space X is a **spectral** (resp. **Stone**) space when it is homeomorphic to the spectrum of a semiring (resp. Boolean algebra), $X = \operatorname{Spec} A$.

1. Any spectral space is a compact T_0 space. If it is T_1, then it is a Stone space (p. 229); hence it is a compact T_2 space (p. 235) with a base of closed open sets. Conversely, if the Boolean algebra B of closed open sets in a compact separated space K is a base of the topology, then the map $K \to \operatorname{Spec} B = \operatorname{Spec}_{\mathrm{m}} B$ is a homeomorphism, and K is a Stone space.

2. Since any semiring (resp. Boolean algebra) is an inductive limit of finite semirings (resp. finite Boolean algebras) we see that spectral (resp. Stone) spaces are just the projective limits of finite (resp. finite discrete) T_0 spaces.

   Hence any spectral space $\varprojlim X_i$ is a continuous image $\varprojlim (X_i)_{\text{dis}} \to \varprojlim X_i$ of a Stone space and, by the Continuous Retract Theorem, any compact separated space is a continuous image of a Stone space.

3. Compact open sets $U \subseteq \operatorname{Spec} A$ are basic because $U_{a_1} \cup \ldots \cup U_{a_n} = U_{a_1+\ldots+a_n}$; hence any spectral space X is homeomorphic to the spectrum of a unique (up to isomorphisms) semiring: the lattice of all compact open sets.

   Since compact sets in a Stone space are just closed sets, any Stone space is the spectrum of a unique Boolean algebra: the algebra of all closed open sets. Hence any continuous map $\operatorname{Spec} B \to \operatorname{Spec} A$ between Stone spaces is induced by a semiring morphism $A \to B$, and we see that *the category of Boolen algebras is dual to the category of Stone spaces.*

4. The spectrum of any commutative ring is a spectral space:

Theorem: *A topological space X is a spectral space if and only if it admits a base of compact open sets (closed under finite unions and intersections, in particular X is compact) and any irreducible closed set has a generic (i.e. dense) point.*

Proof: Let B be a base of compact open sets. Any ideal I of B is just the ideal of all open sets $b \in B$ contained in $c = \bigcup_{a \in I} a$ because, b being compact, b is contained in a finite union of elements of I, so that $b \in I$. Hence $I = I_Y := \{f \in B \colon f(Y) = 0\}$ for some closed set Y.

We see that the lattice of ideals of B is dual to the lattice of closed sets in X, and any prime ideal $\mathfrak{p} \subset B$ corresponds to an irreducible closed set $Y = \bar{x}$, so that $\mathfrak{p} = \mathfrak{p}_x$.

The natural map $X \to \operatorname{Spec} B$ is bijective, and an homeomorphism because B is a base.

8.5 Compactifications

Definition: A continuous map $X \to K$ is a **compactification** of X if it is a homeomorphism onto a dense subspace of a compact space K.

Any T_0 space X admits the compactification $X \to \operatorname{Spec} A(X)$ and, when it is T_1, the T_1 compactification $X \to \operatorname{Spec}_{\text{m}} A(X)$. Now we study Hausdorff compactifications.

Zeros $z(f) := \{x \in X \colon f(x) = 0\}$ of continuous functions $f \colon X \to \mathbb{R}$ form a subring $Z(X)$ of A_X (but it may be not a base of the topology of X) because

$$\begin{array}{ll} z(f) + z(h) = z(f^2 + h^2) & , \quad z(0) = 0 \\ z(f)z(h) = z(fh) & , \quad z(1) = 1 \end{array}$$

Lemma: *The semiring $Z(X)$ is normal.*

Proof: If $z(f)$ and $z(h)$ are disjoint, the continuous function $g = |f| - |h|$ is positive on $z(h)$ and negative on $z(f)$. If we put $f' = \frac{1}{2}(g - |g|)$, $h' = \frac{1}{2}(g + |g|)$, then

$$\begin{aligned} z(f) + z(f') &= 1, \\ z(h) + z(h') &= 1, \\ z(f') \cdot z(h') &= 0. \end{aligned}$$

q.e.d.

$\beta X = \operatorname{Spec}_{\mathrm{m}} Z(X)$ is a compact Hausdorff space (p. 235) and we have a continuous retract

$$r\colon \operatorname{Spec} Z(X) \longrightarrow \operatorname{Spec}_{\mathrm{m}} Z(X).$$

Composing it with the continuous map of dense image $X \to \operatorname{Spec} Z(X)$ (it may be not injective if $Z(X)$ is not a base of the topology of X) we obtain a canonical map of dense image $j\colon X \to \beta X$, which is universal[4]:

Theorem: $\operatorname{Hom}(\beta X, K) = \operatorname{Hom}(X, K)$, $f \mapsto fj$, *for any compact Hausdorff space* K*.*

Proof: A continuous map $\phi\colon X \to K$ defines a morphism $\phi^{-1}\colon Z(K) \to Z(X)$, and by Uryshon's lemma $Z(K)$ is a base of K; hence $K = \operatorname{Spec}_{\mathrm{m}} Z(K)$.

The required extension to βX (unique since the image of $X \to \beta X$ is dense and K is T_2) is

$$\beta X = \operatorname{Spec}_{\mathrm{m}} Z(X) \longrightarrow \operatorname{Spec} Z(X) \longrightarrow \operatorname{Spec} Z(K) \xrightarrow{\ r\ } \operatorname{Spec}_{\mathrm{m}} Z(K) = K. \qquad \text{q.e.d.}$$

Now let $\mathcal{C}(X)$ be the ring of real continuous functions on X, and $\mathcal{C}^*(X)$ the ring of bounded real continuous functions.

Any point $x \in X$ defines a maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{C}(X)\colon f(x) = 0\}$, and this map

$$X \longrightarrow \operatorname{Spec}_{\mathrm{m}} \mathcal{C}(X)$$

is continuous and has dense image since $(f)_0 \cap X = z(f)$.

We also have a continuous map $X \longrightarrow \operatorname{Spec}_{\mathrm{m}} \mathcal{C}^*(X)$ of dense image.

Theorem: $\beta X = \operatorname{Spec}_{\mathrm{m}} \mathcal{C}^*(X)$.

Proof: $\mathcal{C}^*(X) = \mathcal{C}(\beta X)$ by the universal property of βX, and $\operatorname{Spec}_{\mathrm{m}} \mathcal{C}(\beta X) = \beta X$, since βX is a compact Hausdorff space (p. 260).

Theorem: $\beta X = \operatorname{Spec}_{\mathrm{m}} \mathcal{C}(X)$.

Proof: Any ideal I of the semiring $Z(X)$ defines an ideal $I' = \{f \in \mathcal{C}(X)\colon z(f) \in I\}$ of $\mathcal{C}(X)$, and any ideal J of $\mathcal{C}(X)$ defines an ideal $z(J) = \{z(f)\colon f \in J\}$ of $Z(X)$.

Now, if $\mathfrak{m}$ is a maximal ideal of $Z(X)$, so is $\mathfrak{m}'$ since for any ideal $\mathfrak{m}' \subset J \subset \mathcal{C}(X)$ we have that $\mathfrak{m} = z(\mathfrak{m}') \subset z(J) \subset Z(X)$ (if $z(f) = \emptyset$, then f is invertible).

Analogously, if $\mathfrak{m}$ is a maximal ideal of $\mathcal{C}(X)$, so is $z(\mathfrak{m})$ since for any ideal $z(\mathfrak{m}) \subset I \subset Z(X)$ we obviously have $\mathfrak{m} \subset I' \subset \mathcal{C}(X)$.

So we obtain a natural bijection $\operatorname{Spec}_{\mathrm{m}} Z(X) = \operatorname{Spec}_{\mathrm{m}} \mathcal{C}(X)$, and it is a homeomorphism since the zeros of $z(f)$ correspond to the zeros of f.

Proposition: *A* T_1 *topological space* X *is normal if and only if* $\beta X = \operatorname{Spec}_{\mathrm{m}} A(X)$*.*

Proof: If X is normal, then $\operatorname{Spec}_{\mathrm{m}} A_X$ is a compact Hausdorff space.

The extension of a continuous map $X \to K$ to $\operatorname{Spec}_{\mathrm{m}} A_X$ (unique since $X \to \operatorname{Spec}_{\mathrm{m}} A_X$ has dense image and K is T_2) is just

$$X \hookrightarrow \operatorname{Spec}_{\mathrm{m}} A_X \longrightarrow \operatorname{Spec} A_X \longrightarrow \operatorname{Spec} A_K \xrightarrow{\ r\ } \operatorname{Spec}_{\mathrm{m}} A_K = K.$$

[4]The existence of a universal continuous map into compact Hausdorff spaces also follows from the representability theorem.

Conversely, if $\beta X = \operatorname{Spec_m} A_X$, then $\operatorname{Spec_m} A_X$ is Hausdorff, so that maximal ideals of A_X do not hybridize; hence A_X is normal.

Definition: A T_0 space X is **completely regular** when $Z(X)$ is a base of the topology of X (continuous functions separate points and closed sets).

Any subspace of a completely regular space so is completely regular, and by Uryshon's lemma any normal T_1 space is completely regular.

Any completely regular space is regular, hence it is Hausdorff.

Theorem: *If X is a topological space, the following conditions are equivalent,*

1. *X is completely regular.*

2. *βX is a compactification of X* (the **Stone-Cěch compactification**).

3. *X admits a Hausdorff compactification.*

Proof: ($1 \Rightarrow 2$) If X is completely regular, $j\colon X \to \beta X$ is a compactification since $Z(X)$ is a base of the topology of X.

($2 \Rightarrow 3$) is obvious, and finally, if X admits a Hausdorff compactification $X \to K$, since K is completely regular, so is the subspace X.

Definition: We say that a subalgebra $B \subseteq \mathcal{C}(X)$ separates points of X if for each pair of points $x \neq y$ of X there is a continuous function $f \in B$ such that $f(x) \neq f(y)$. In such a case, for any pair of real numbers $a \neq b$ there exists $h \in B$ such that $h(x) = a$, $h(y) = b$.

Stone-Weierstrass Theorem: *Let K be a compact separated space. A subalgebra B of $\mathcal{C}(K)$ is dense if and only if it separates points of K.*

Proof: If B is dense, and we take $f \in \mathcal{C}(K)$ such that $f(x) = 0$, $f(y) = 1$ (it exists by Urysohn's lemma), any function $h \in B$ such that $\|f - h\| < \frac{1}{2}$ separates the points x, y.

Conversely, if B separates points, considering the closure we may assume that B is closed, and we have to show that for all $f \in \mathcal{C}(K)$, $\varepsilon > 0$, there is $h \in B$ such that $\|f - h\| < \varepsilon$.

Given $x \in K$, for any $y \in K$ there is $h_y \in B$ such that $h_y(x) = f(x)$, $h_y(y) = f(y)$, and on an open neighborhood U_y we have $h < f + \varepsilon$.

There are points $y_1, \dots, y_n$ such that $K = \bigcup_i U_{y_i}$, and we put $h_x = \min(h_{y_1}, \dots, h_{y_n})$.

It is clear that $h_x(x) = f(x)$ and $h_x < f + \varepsilon$, and by the following lemma $h_x \in B$.

On an open neighborhood V_x of x we have $f - \varepsilon < h_x$.

There are points $x_1, \dots, x_m$ such that $K = \bigcup_i V_{x_i}$.

Now $h = \max(h_{x_1}, \dots, h_{x_m}) \in B$, and $f - \varepsilon < h < f + \varepsilon$.

Lemma: *Let B be a closed subalgebra of $\mathcal{C}(K)$. If $f, h \in B$, then* $\max(f, h), \min(f, h) \in B$.

Proof: Since $\max(f, h) = \frac{1}{2}(f + h + |f - h|)$ and $\min(f, h) = f + h - \max(f, h)$, it is enough to show that $|f|$ is in the closure of B whenever $f \in B$. Now, the series

$$\sqrt{1 - t} = 1 - \tfrac{1}{2}t + \dots$$

uniformly converges on $|t| \leq 1$. If $P_n(t)$ is the Taylor polynomial of degree n, for any $\varepsilon > 0$ there exists n such that $|P_n(t) - \sqrt{1 - t}\,| < \varepsilon$ on $|t| \leq 1$.

If $f \in B$, dividing by a constant we have that $|f| \leq 1$, and if we put $t = 1 - f(x)^2$,

$$|P_n(1-f^2) - |f|| = |P_n(1-f^2) - \sqrt{1-(1-f^2)}\,| < \varepsilon \ ,$$

hence $|f|$ is adherent to B, since $P_n(1-f^2) \in B$.

Corollary: *Let X be a σ-compact space. A subalgebra B of $\mathcal{C}(X)$ is dense (with the topology of uniform convergence on compact sets) if and only if it separates points of X.*

Proof: If B separates points of X, for any compact set $K \subset X$ we have that the image of the restriction morphism $B \to \mathcal{C}(K)$ separates points of K.

Hence it is dense in $\mathcal{C}(K)$, and B is dense in $\mathcal{C}(X)$.

Tietze's Extension Theorem: *Let X be a T_1 normal space. If Y is a closed set in X, then the restriction morphism $\mathcal{C}^*(X) \to \mathcal{C}^*(Y)$ is surjective.*

Proof: Since X is normal and T_1, the Stone-Cĕch compactification is $\beta X = \operatorname{Spec}_{\rm m} A_X$.

Since Y is closed, it is normal and T_1, so that $\beta Y = \operatorname{Spec}_{\rm m} A_Y \subset \operatorname{Spec}_{\rm m} A_X$, because $A_Y = A_X/I_Y$.

Replacing X, Y by $\beta X, \beta Y$, we may assume that X is compact.

Let us consider the image of the restriction morphism $\mathcal{C}(X) \to \mathcal{C}(Y)$. It is a subalgebra which separates points of Y, so that it is dense.

If $h \in \mathcal{C}(Y)$, there are functions $f_n \in \mathcal{C}(X)$ such that $|h - f_n| < 2^{-n}$ on Y.

Let $\phi_n \colon X \to [0,1]$ be a continuous function with value 1 on Y and vanishing on the closed set $|f_{n+1} - f_n| \geq 2^{1-n}$ (both sets do not intersect since the functions f_n, f_{n+1} differ from h less that 2^{-n} on Y). Now the series

$$f = f_1 + \sum_{n=1}^{\infty} (f_{n+1} - f_n)\phi_n$$

uniformly converges, since $|(f_{n+1} - f_n)\phi_n| \leq 2^{1-n}$, and it defines a continuous function on X coinciding with h on Y, since on Y the n-th partial sum is f_{n+1}, and $f_n \to h$.

8.6 Dimension Theory

Definition: The **dimension** of a topological space is the minimal (Krull) dimension of a base of open sets. The dimension may be infinite, and $\dim X = -1$ if and only if $X = \emptyset$.

Theorem: $\dim X = 0$ *if and only if $X \neq \emptyset$ and it admits a base of open closed sets.*

Proof: The open closed sets in X form a base if and only if some base of X is a Boolean algebra; i.e., a semiring of dimension 0 (p. 229).

Examples: Rational numbers and irrational numbers define two spaces of dimension 0; but the union $\mathbb{R}$ is connected, hence of dimension > 0.

Any space X of dimension 0 admits a base B which is a Boolean algebra; hence X is a subspace of the compact Hausdorff space $\operatorname{Spec} B$ (p. 235), and X is completely regular.

Products and projective limits of spaces of dimension 0 have dimension 0. In particular any profinite space is a **Stone space** (a compact separated space of dimension 0). Conversely, any Stone space is $K = \operatorname{Spec}_{\rm m} B = \operatorname{Spec} B$ for some Boolean algebra; hence it is a profinite space.

Let K be a compact Hausdorff space. Then $A_K = \bigcup_i A_i$, where the semirings A_i are finite. Let $\overline{X}_i$ be the finite set $\operatorname{Spec} A_i$ with the discrete topology. By the continuous retract theorem, K is a continuous image of a compact 0-dimensional space:

$$\varprojlim \overline{X}_i \longrightarrow \varprojlim(\operatorname{Spec} A_i) = \operatorname{Spec}(\varinjlim A_i) = \operatorname{Spec} A_K \xrightarrow{\ r\ } \operatorname{Spec_m} A_K = K$$

Lemma: *If we have a surjective morphism $A \to A'$, then* $\dim A' \leq \dim A$.

Proof: Any chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ in A' defines a chain of prime ideals $\mathfrak{p}_0 \cap A \subset \mathfrak{p}_1 \cap A \subset \ldots \subset \mathfrak{p}_n \cap A$ in A.

Theorem: *If Y is a subspace of X, then* $\dim Y \leq \dim X$.

Proof: If B is a base of X, the image of the restriction morphism $B \to A_Y$ is a base of Y.
Hence $\dim Y \leq \dim B$.

Theorem: $\dim(\operatorname{Spec} A) = \dim A$, *for any semiring A.*

Proof: $\dim(\operatorname{Spec} A) \leq \dim A$ since A defines a base of $\operatorname{Spec} A$.
On the other hand, if B is a base of $\operatorname{Spec} A$, then $\operatorname{Spec} A$ is a subspace of $\operatorname{Spec} B$.
Since the dimension of a semiring is the supremum of the lengths of all chains of specializations in the spectrum, we conclude that $\dim A \leq \dim B$.

Corollary: *The dimension of a noetherian space X is the supremum of the lengths of all chains of irreducible closed sets*[5] *in X.*

Proof: If X is noetherian, it has a unique base $A(X)$, and prime ideals in $A(X)$ correspond to irreducible closed sets in X (p. 236).

Corollary: *The dimension of a finite T_0 space X is the maximal length of a chain in X (considered as a finite order), hence* $\dim X = \dim(\beta X)$.

Theorem: $\dim(X \times Y) \leq \dim X + \dim Y$.

Proof: If A and B are bases of X and Y, the image of the morphism $A \otimes B \to A(X \times Y)$ is a base of $X \times Y$; hence $\dim(X \times Y) \leq \dim A \otimes B = \dim A + \dim B$ (p. 229).

Theorem: $\dim(\varprojlim X_i) \leq \sup\{\dim X_i\}$, *when the spaces X_i are finite and T_0.*

Proof: If $X_i = \operatorname{Spec} A_i$, then $\varprojlim X_i = \operatorname{Spec}(\varinjlim A_i)$, and (p. 228)

$$\dim \operatorname{Spec}(\varinjlim A_i) = \dim(\varinjlim A_i) \leq \sup\{\dim A_i\}.$$

Corollary: $\dim |X| \leq \dim X$, *for any finite T_0 space X.*

Proof: $\dim |X| \leq \dim(\varprojlim \beta^n X) \leq \sup\{\dim \beta^n X\} = \dim X$.

Fundamental Theorem of Dimension Theory: $\dim \mathbb{R}^n = n$.

Proof: $\dim \mathbb{R}^n \leq \dim |\Delta_n| \leq n$; but the proof of the equality is postponed to p. 345.

[5]The supremum of the lengths of all chains of irreducible closed sets in a space X is the **combinatorial** or **Grothendieck dimension** of X.

Corollary: *Any non empty open set $U \subseteq \mathbb{R}^n$ has dimension n. Hence, it is not homeomorphic to an open subset of $\mathbb{R}^m$ when $n \neq m$.*

Proof: Take an open ball $B \subseteq U$.
Since B is homeomorphic to $\mathbb{R}^n$, we have $n = \dim B \leq \dim U \leq \dim \mathbb{R}^n = n$.

Corollary: $\dim |X| = \dim X$, *for any finite T_0 space X.*

Proof: If $\dim X = n$, then $|X|$ contains a n-tetrahedron; hence $\mathbb{R}^n$, so that $n \leq \dim |X|$.

Lemma: *Let K be a compact separated space. If A is a dense subalgebra of $\mathcal{C}(K)$,*

$$\dim K \leq \dim A.$$

Proof: Let B be the semiring generated by the closed sets $f^{-1}(I)$, where $f \in A$ and I is a closed interval, eventually unbounded (recall that $+ = \cap$, $\cdot = \cup$).
Since A is dense, B is a base of X. If $\mathfrak{p}$ is a prime ideal of B, then $\mathfrak{p}' = \{f \in A \colon z(f) \in \mathfrak{p}\}$ is a prime ideal of A, and we have to show that $\mathfrak{p}' \subset \mathfrak{q}'$ when $\mathfrak{p} \subset \mathfrak{q}$.
The closed sets $f^{-1}[0, \infty)$ generate B, so that $\mathfrak{q}$ contains a closed set $f^{-1}[0, \infty) \notin \mathfrak{p}$.
Since $\mathfrak{p}$ is prime, and $0 = f^{-1}(-\infty, 0] \cdot f^{-1}[0, \infty)$, we have that $f^{-1}(-\infty, 0] \in \mathfrak{p} \subset \mathfrak{q}$, and $z(f) = f^{-1}(-\infty, 0] + f^{-1}[0, \infty) \in \mathfrak{q}$.
Since $z(f) \notin \mathfrak{p}$, because $f^{-1}[0, \infty) \supseteq z(f)$ is not in $\mathfrak{p}$, we see that $f \in \mathfrak{q}'$, and $f \notin \mathfrak{p}'$.

Theorem: *The dimension of any finite polyhedron P is the minimal dimension of a dense subalgebra of $\mathcal{C}(P)$.*

Proof: P is a finite union of some simplices σ of a tetrahedron in $\mathbb{R}^n$.
The algebra A of polynomial functions on P separates points of P, and $A \simeq \mathbb{R}[x_1, \ldots, x_n]/I$, where I is the ideal of all polynomials vanishing on P.
Since $I = \bigcap_\sigma \mathfrak{p}_\sigma$, where $\mathfrak{p}_\sigma$ is the ideal of all polynomials vanishing on a given simplex σ of P, the dimension of A is the maximal dimension of $\mathbb{R}[x_1, \ldots, x_n]/\mathfrak{p}_\sigma$.
Now, $\mathbb{R}[x_1, \ldots, x_n]/\mathfrak{p}_\sigma \simeq \mathbb{R}[y_1, \ldots, y_d]$, $d = \dim \sigma$, because $\mathfrak{p}_\sigma$ is the ideal of the linear subvariety spanned by σ, so that $\dim A = \dim P$.

8.7 Uniform Spaces

Let X be a set. Given subsets $\mathcal{E}, \delta \subseteq X \times X$ we put:

$$\mathcal{E}^* = \{(x, y) \in X \times X \colon (y, x) \in \mathcal{E}\},$$
$$\mathcal{E} \circ \delta = \{(x, z) \in X \times X \colon (x, y) \in \mathcal{E},\ (y, z) \in \delta \text{ for some } y \in X\},$$
$$n\mathcal{E} = \mathcal{E} \circ \overset{n}{\ldots} \circ \mathcal{E},$$
$$\mathcal{E}[x] = \{y \in X \colon (x, y) \in \mathcal{E}\}\ , \quad x \in X.$$

A **filter** in a set X is a proper ideal $\mathcal{F}$ of the dual algebra of subsets of X: it is closed under finite intersections, $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, and if $A \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ for any subset $B \subseteq X$.

Definition: A **uniform space** is a set X endowed with a filter of subsets of $X \times X$, named **entourages** satisfying the following conditions:

1. Any entourage contains the **diagonal** $\Delta = \{(x, x);\ x \in X\}$.
2. If $\mathcal{E}$ is an entourage, then so is $\mathcal{E}^*$.

3. If $\mathcal{E}$ is an entourage, there is an entourage δ such that $2\delta \subseteq \mathcal{E}$.

A family of entourages is a **base** of the uniformity when any entourage contains a member of such family. The **symmetric** (i.e. $\mathcal{E}^* = \mathcal{E}$) entourages form a base, because $\mathcal{E}^* \cap \mathcal{E}$ is symmetric. Moreover, if $\{\mathcal{E}_i\}$ is a base of entourages, then so is $\{n\mathcal{E}_i\}$ for any $n \in \mathbb{N}$.

A map $f\colon X \to Y$ between uniform spaces is **uniformly continuous** when, for any entourage $\mathcal{E}$ in Y, we have that $(f \times f)^{-1}(\mathcal{E})$ is an entourage in X; i.e. there is an entourage δ in X such that $(f \times f)(\delta) \subseteq \mathcal{E}$.

Induced Topology: A uniform structure defines a topology on X, the filter of neighborhoods of a point $x \in X$ being $\{\mathcal{E}(x)\}$, where $\mathcal{E}$ runs on the entourages of X.

In fact, these are the filters of neighborhoods of a topology because, if $2\delta \subseteq \mathcal{E}$, then for any $y \in \delta[x] \subseteq \mathcal{E}[x]$ we have $\delta[y] \subseteq (2\delta)[x] \subseteq \mathcal{E}[x]$, so that $\delta[x]$ is in the interior of $\mathcal{E}[x]$.

Let us consider on $X \times X$ the product topology. If $\mathcal{E}$ is symmetric, then $3\mathcal{E}$ is a neighborhood of $\mathcal{E}$, because $\mathcal{E}[x] \times \mathcal{E}[y] \subseteq 3\mathcal{E}$, $\forall (x,y) \in \mathcal{E}$. Hence, *the open entourages form a base of the uniformity*, and any entourage is a neighborhood of the diagonal. Also $\overline{\mathcal{E}} \subseteq 3\mathcal{E}$, so that *the closed entourages form a base of the uniformity*, and any point admits a base of closed neighborhoods.

Uniformly continuous maps are continuous: $f^{-1}\big(\mathcal{E}[y]\big) = \big((f \times f)^{-1}(\mathcal{E})\big)[x]$, $y := f(x)$.

The T_0 Quotient: The topology of a uniform space X is T_0 when, for any pair of points $x \neq y$, there is an entourage $\mathcal{E}$ such that (x,y) or (y,x) is not in $\mathcal{E}$. In such case, if δ is a symmetric uniform neighborhood such that $2\delta \subseteq \mathcal{E}$, then $\delta[x] \cap \delta[y] = \emptyset$, and we see that X is separated, and the intersection of all entourages is just the diagonal.

In general, the intersection of all entourages is the set of pairs $(x,y) \in X \times X$ such that $\bar{x} = \bar{y}$. Hence, if we identify such points, the canonical projection $\pi\colon X \to X_s$ induces an isomorphism $A(X_s) = A_X$ between the lattices of closed sets, and we see that any point $x \in X$ defines a maximal ideal of A_X, that we denote $\mathfrak{m}_x = \{A \in A_X \colon x \in A\}$.

Now $\mathcal{E}_s \subseteq X_s \times X_s$ is defined to be an entourage when $(\pi \times \pi)^{-1}(\mathcal{E}_s)$ is an entourage in X; i.e. when $\mathcal{E}_s = (\pi \times \pi)(\mathcal{E})$ for some entourage $\mathcal{E}$ in X. So we obtain a uniformity on X_s because $(\pi \times \pi)^{-1}(\mathcal{E}_s) \subseteq 3\mathcal{E}$, so that $2\delta_s \subseteq \mathcal{E}_s$ when $6\delta \subseteq \mathcal{E}$.

The uniform space X_s is separated, $\pi\colon X \to X_s$ is uniformly continuous, and any uniformly continuous map $X \to Y$ to a separated uniform space Y uniquely factors through π.

Examples: (1) Any pseudometric d on a set X defines a uniform structure on X, such that a base of entourages is defined by the sets $U_\varepsilon = U_{\mathrm{d},\varepsilon} := \{(x,y) \in X \times X \colon \mathrm{d}(x,y) < \varepsilon\}$, $\varepsilon \in \mathbb{R}_+$, and it is a uniform structure because $U_\varepsilon^* = U_\varepsilon$ and $U_\varepsilon \circ U_\varepsilon \subseteq U_{2\varepsilon}$. The corresponding topology is just the topology defined by d. In general, any family $\{\mathrm{d}_i\}$ of pseudometrics on X defines a uniform structure, a base of entourages being the finite intersections $U_{\mathrm{d}_{i_1},\varepsilon_1} \cap \ldots \cap U_{\mathrm{d}_{i_n},\varepsilon_n}$.

(2) A topological vector space E (or a topological abelian group) has a natural uniform structure, a base of entourages being the sets $\tilde{U} = \{(x,y) \in E \times E \colon y - x \in U\}$, where U runs over the neighborhoods of 0 in E. In fact $(\tilde{U})^* = (-U)^\sim$, and $\tilde{V} \circ \tilde{V} \subseteq \tilde{U}$ when V is a neighborhood of 0 such that $V + V \subseteq U$.

(3) The **initial uniformity** of a family of maps $f_i\colon X \to Y_i$ to uniform spaces Y_i is defined by the filter generated by the sets $(f_i \times f_i)^{-1}(\mathcal{E}_i) \subseteq X \times X$, where $\mathcal{E}_i$ runs over the entourages of the spaces Y_i, so that a map $f\colon T \to X$ is uniformly continuous if and only if so are the maps $f_i \circ f$. The sets $(f_{i_1} \times f_{i_1})^{-1}(\mathcal{E}_{i_1}) \cap \ldots \cap (f_{i_n} \times f_{i_n})^{-1}(\mathcal{E}_{i_n})$ form a base of entourages, and the induced topology is just the initial topology of the continuous maps f_i.

In particular, any subset Y of a uniform space X inherits a uniform structure.

Theorem: *A compact separated space K admits a unique uniformity inducing its topology, the entourages being all the neighborhoods of the diagonal.*

Proof: Any continuous function $f\colon K \to \mathbb{R}$ defines a pseudometric $\mathrm{d}_f(x,y) = |f(y) - f(x)|$, and the uniformity defined by these pseudometrics induces the topology of K because K is a completely regular space (p. 241).

Finally, given a uniformity inducing the topology of K, if U is a neighborhood of the diagonal, then $U^c \cap \left(\bigcap \mathcal{E}\right) = U^c \cap \Delta = \emptyset$, where $\mathcal{E}$ runs over the closed entourages and $U^c := K - U$.

Since $K \times K$ is compact, there is a finite empty intersection $U^c \cap \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_n = \emptyset$.

Hence $U \supseteq \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_n$ is an entourage of the considered uniformity.

Corollary: *Any continuous map f from a compact separated space K to a uniform space Y is uniformly continuous.*

Proof: If $\mathcal{E}$ is an entourage in Y, then $\mathcal{E}$ is a neighborhood of the diagonal in $Y \times Y$; hence $(f \times f)^{-1}(\mathcal{E})$ is a neighborhood of the diagonal in $K \times K$, so that it is an entourage in K.

Definitions: A filter $\mathcal{F}$ in a topological space X **converges** to a point $x \in X$ (unique when X is separated) if it contains the filter $\mathcal{N}_x$ of neighborhoods of x. In particular, a limit point x is adherent to any set $A \in \mathcal{F}$.

A filter $\mathcal{F}$ in a uniform space X is a **Cauchy filter** if for any entourage $\mathcal{E}$ there is a **$\mathcal{E}$-small** set $A \in \mathcal{F}$ (i.e. $A \times A \subseteq \mathcal{E}$); hence a closed $\mathcal{E}$-small set: if $\delta \subseteq \mathcal{E}$ is a closed entourage and A is a δ-small set, then $\overline{A} \times \overline{A} \subseteq \overline{\delta} = \delta \subseteq \mathcal{E}$. So, if a point $x \in X$ is adherent to any set in $\mathcal{F}$ then $\mathcal{F}$ converges to x, because $x \in \overline{A} \subseteq \mathcal{E}(x)$, so that $\mathcal{E}(x) \in \mathcal{F}$: *A Cauchy filter $\mathcal{F}$ converges to x when the maximal ideal $\mathfrak{m}_x$ of the semiring A_X contains the ideal $A_X \cap \mathcal{F}$.*

1. Given a map $f\colon X \to Y$ and a filter $\mathcal{F}$ in X, then $f(\mathcal{F}) := \{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\}$ is a filter in Y. When f is continuous and $\mathcal{F}$ converges to $x \in X$, then $f(\mathcal{F})$ converges to $f(x)$. When f is uniformly continuous and $\mathcal{F}$ is a Cauchy filter on X, then $f(\mathcal{F})$ is a Cauchy filter on Y.

2. In a uniform space X, the filter $\mathcal{N}_x$ of neighborhoods of $x \in X$, and the filter $\mathcal{F}_x$ of all subsets containing x, are always Cauchy filters.

3. Given a sequence (x_n) in a topological space X, the filter $\mathcal{F}_{(x_n)}$ of subsets containing all the terms x_n up to a finite number converges to a point $x \in X$ if and only if $x_n \to x$.

   A sequence (x_n) in a uniform space X is defined to be a **Cauchy sequence** when $\mathcal{F}_{(x_n)}$ is a Cauchy filter: for any entourage $\mathcal{E}$ there is an index k such that $(x_n, x_m) \in \mathcal{E}$, $\forall n, m \geq k$.

Definition: A uniform space X is **complete** when it is T_0 (hence separated) and any Cauchy filter $\mathcal{F}$ on X converges to a point $x \in X$.

In the case of metric spaces, this definition coincides with the older one (p. 28):

Lemma: *A separated uniform space X with a countable base of entourages $\{\mathcal{E}_n\}$ is complete if and only if any Cauchy sequence in X converges.*

Proof: If (x_n) is a Cauchy sequence in X, then $\mathcal{F}_{(x_n)}$ is a Cauchy filter. If X is complete, then $\mathcal{F}_{(x_n)}$ converges to a point $x \in X$, so that $x_n \to x$.

Conversely, replacing $\mathcal{E}_n$ by $\mathcal{E}_1 \cap \ldots \cap \mathcal{E}_n$, we may assume that $\mathcal{E}_{n+1} \subseteq \mathcal{E}_n$.

Let $\mathcal{F}$ be a Cauchy filter and let A_n be a closed set in $\mathcal{F}$ such that $A_n \times A_n \subseteq \mathcal{E}_n$.

Replacing A_n by $A_1 \cap \ldots \cap A_n$, we may assume that $A_{n+1} \subseteq A_n$. Pick $x_n \in A_n$.

Then $(x_n, x_m) \in A_k \times A_k \subseteq \mathcal{E}_k$, $\forall n, m \geq k$, so that (x_n) is a Cauchy sequence and it converges to a point $x \in \bar{A}_n = A_n$. Since $A_n \subseteq \mathcal{E}_n(x)$, we conclude that $\mathcal{F}$ converges to x.

Proposition: *Any compact separated uniform space K is complete.*

Proof: If $\mathcal{F}$ is a Cauchy filter in K, then $A_K \cap \mathcal{F}$ is a proper ideal of A_K; hence it is contained in some maximal ideal $\mathfrak{m}_x$ of A_K (p. 230) so that $\mathcal{F}$ converges to $x \in K$.

Proposition: *Any closed subspace $i\colon Y \hookrightarrow X$ of a complete uniform space is complete.*

Proof: If $\mathcal{G}$ is a Cauchy filter on Y, then $i(\mathcal{G})$ is a Cauchy filter on X; hence $i(\mathcal{G})$ converges to some point $x \in X$, so that $x \in \bar{Y} = Y$ and $\mathcal{G}$ converges to x.

Proposition: *Any complete subspace Y of a separated uniform space X is a closed subspace.*

Proof: If $x \in \bar{Y}$, then $\emptyset \notin \mathcal{N}_x \cap Y$, so that $\mathcal{N}_x \cap Y$ is a Cauchy filter on Y.

Hence $\mathcal{N}_x \cap Y$ converges to some point $y \in Y$, and $y = x$ because X is separated.

Lemma: *Let $\iota\colon Y \to X$ be a dense subspace of a uniform space X. If any Cauchy filter $\mathcal{G}$ on Y converges in X, in the sense that $\iota(\mathcal{G}) = \{A \subseteq X\colon A \cap Y \in \mathcal{G}\}$ converges, then any Cauchy filter $\mathcal{F}$ on X converges.*

Proof: The filter $\mathcal{F}^{\rm o} = \{A \subseteq X\colon \overset{\rm o}{A} \in \mathcal{F}\}$ is a Cauchy filter: if ε is a symmetric entourage and $A \in \mathcal{F}$ is a ε-small, then the open set $\bigcup_{a\in A} \varepsilon(a)$ is 3ε-small.

Since $\mathcal{F}^{\rm o} \subseteq \mathcal{F}$, it is enough to see that $\mathcal{F}^{\rm o}$ converges.

Now $\mathcal{F}^{\rm o} \cap Y$ is a filter because Y is dense, and it is a Cauchy filter on Y.

Hence $\iota(\mathcal{F}^{\rm o} \cap Y)$ converges to some point $x \in X$, adherent to any subset in $\mathcal{F}^{\rm o} \subseteq \iota(\mathcal{F}^{\rm o} \cap Y)$, and we see that $\mathcal{F}^{\rm o}$ converges to x.

Completion of a Uniform Space: Let X be a uniform space, and $\tilde{X}$ the set of Cauchy filters on X. If ε is a symmetric entourage in X, then $\tilde{\varepsilon} \subseteq \tilde{X} \times \tilde{X}$ denotes the set of pairs $(\mathcal{F}, \mathcal{F}')$ of Cauchy filters with a common ε-small set. The sets $\tilde{\varepsilon}$ form a base of entourages for a uniformity on $\tilde{X}$: Clearly $\tilde{\varepsilon}$ contains the diagonal, it is symmetric, $(\varepsilon \cap \delta)^{\sim} \subseteq \tilde{\varepsilon} \cap \tilde{\delta}$, and $\tilde{\varepsilon} \circ \tilde{\delta} \subseteq (\varepsilon \circ \delta)^{\sim}$.

In fact, if $(\mathcal{F}, \mathcal{F}')$ have a common ε-small set A and $(\mathcal{F}', \mathcal{F}'')$ have a common δ-small set B, then $\emptyset \neq A \cap B \in \mathcal{F}'$, so that $A \cup B$ is a $(\varepsilon \circ \delta)$-small set, and $A \cup B$ is in $\mathcal{F}$ and $\mathcal{F}''$.

Now, X has the initial uniformity of the map $\iota\colon X \to \tilde{X}$, $\iota(x) = \mathcal{F}_x = \{A \subseteq X\colon x \in A\}$, because $\delta \subseteq (X \times X) \cap \tilde{\varepsilon} \subseteq \varepsilon$ whenever δ is a closed symmetric entourage and $2\delta \subseteq \varepsilon$.

In fact, if $(x, y) \in \delta$, then $x, y \in \delta(x)$ and $\delta(x)$ is a closed ε-small set in $\mathcal{F}_x$ and $\mathcal{F}_y$.

Let us see that $\iota(X)$ is dense and that any Cauchy filter $\mathcal{F}$ on X converges in $\tilde{X}$:

For every symmetric entourage ε in X, there is a ε-small set $A \in \mathcal{F}$. If $x \in A$, then $\big(\iota(x), \mathcal{F}\big) \in \tilde{\varepsilon}$, so that $\iota(A) \subseteq \tilde{\varepsilon}(\mathcal{F})$; hence $\mathcal{F} \in \tilde{X}$ is in the closure of $\iota(X)$, and the filter $\iota(\mathcal{F})$ contains any neighborhood $\tilde{\varepsilon}(\mathcal{F})$ of $\mathcal{F}$ in $\tilde{X}$.

By the lemma, the separated quotient $\widehat{X} = (\tilde{X})_s$ is a complete space, named **completion** of the uniform space X, endowed with a canonical uniformly continuous map $i = \pi\iota\colon X \to \widehat{X}$ with dense image, such that X has the initial uniformity.

When X is separated, $i\colon X \to \widehat{X}$ is injective; hence it is an isomorphism onto a dense subspace of $\widehat{X}$. If moreover X is complete, then any Cauchy filter $\mathcal{F}$ on X converges to a point $x \in X$, so that $(\mathcal{F}, \mathcal{F}_x) \in \tilde{\varepsilon}$ for any symmetric entourage ε in X; hence $\pi(\mathcal{F}) = \pi(\mathcal{F}_x) = i(x)$ and we see that $i\colon X \to \widehat{X}$ is an isomorphism.

If $f\colon X \to Y$ is uniformly continuous and $\mathcal{F}$ is a Cauchy filter in X, then $f(\mathcal{F})$ is a Cauchy filter in Y, so that f induces a natural uniformly continuous map $\tilde{f}\colon \tilde{X} \to \tilde{Y}$; hence a uniformly continuous map $\widehat{f}\colon \widehat{X} = (\tilde{X})_s \to (\tilde{Y})_s = \widehat{Y}$ and a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \widehat{X} & \xrightarrow{\ \widehat{f}\ } & \widehat{Y} \end{array}$$

Universal Property: *If $f\colon X\to Y$ is a uniformly continuous map and Y is complete, then there exists a unique uniformly continuous map $\widehat{f}\colon \widehat{X}\to Y$ such that $f=\widehat{f}\circ i$.*

$$\mathrm{Hom}_{\mathrm{unif}}(\widehat{X},Y)=\mathrm{Hom}_{\mathrm{unif}}(X,Y)\ ,\quad \phi\mapsto\phi\circ i.$$

Proof: The existence of $\widehat{f}$ follows form the above commutative square, because $Y=\widehat{Y}$, and this continuous map $\widehat{f}$ is unique since it is fully determined on the dense set $i(X)$, and Y is separated.

Theorem: *If X is a subspace of a complete uniform space, then the completion of X is just the closure, $\widehat{X}=\bar{X}$.*

Proof: Let us see that the inclusion map $i\colon X\to\bar{X}$ has the above universal property.

Given a point $x\in\bar{X}$, the Cauchy filter $f(\mathcal{N}_x\cap X)$ converges to a unique point $\phi(x)\in Y$, so that $\phi(x)$ is in the closure of $f(U_x\cap X)$ for any neighborhood U_x of x in $\bar{X}$, and we only have to show that such extension $\phi\colon\bar{X}\to Y$ (obviously unique since X is dense in $\bar{X}$ and Y is separated) is uniformly continuous.

Given a closed entourage $\mathcal{E}$ in Y, there exists a symmetric entourage δ in $\bar{X}$ such that $(f\times f)\big((3\delta)\cap X\times X\big)\subseteq\mathcal{E}$. If $(x,x')\in\delta$, then $\big(\delta(x)\cap X\big)\times\big(\delta(x')\cap X\big)\subseteq(3\delta)\cap X\times X$ and, $\mathcal{E}$ being closed, we conclude that $(\phi\times\phi)(\delta)\subseteq\mathcal{E}$:

$$\big(\phi(x),\phi(x')\big)\in\overline{f\big(\delta(x)\cap X\big)}\times\overline{f\big(\delta(x')\cap X\big)}\subseteq\overline{(f\times f)\big((3\delta)\cap X\times X\big)}\subseteq\overline{\mathcal{E}}=\mathcal{E}.$$

Theorem: *A uniform space X is* ***precompact*** (with compact completion) *if and only if it admits, for every entourage $\mathcal{E}$, a finite cover by $\mathcal{E}$-small sets.*

Proof: Consider an entourage $\mathcal{E}'$ in $\widehat{X}$ such that $(X\times X)\cap\mathcal{E}'\subseteq\mathcal{E}$. If $\widehat{X}$ is compact, it admits a finite cover $\widehat{X}=\bigcup_n U'_n$ by $\mathcal{E}'$-small sets; hence $X=\bigcup_n(U'_n\cap X)$, and $U'_n\cap X$ is $\mathcal{E}$-small.

Conversely, to prove that $\widehat{X}$ is compact, since it is complete, it is enough to show (p. 230) that any maximal ideal $\mathfrak{m}$ of the semiring $A_{\widehat{X}}$ generates a Cauchy filter. For any closed entourage $\mathcal{E}'$ in $\widehat{X}$, put $\mathcal{E}:=(X\times X)\cap\mathcal{E}'$. There is a finite cover $X=\bigcup_n U_n$ by $\mathcal{E}$-small sets, and each closure $\overline{i(U_n)}$ in $\widehat{X}$ is $\mathcal{E}'$-small because $\mathcal{E}'$ is closed. Now $\widehat{X}=\overline{i(X)}=\bigcup_n\overline{i(U_n)}$, and $\overline{i(U_n)}\in\mathfrak{m}$ for some index n because $\mathfrak{m}$ is a prime ideal.

Corollary: *Any direct product of precompact spaces also is precompact.*

Corollary: *A complete uniform space is compact if and only if it admits a finite cover by $\mathcal{E}$-small sets for every entourage $\mathcal{E}$. Hence, in a complete uniform space, compact sets are just closed sets admitting a finite cover by $\mathcal{E}$-small sets for every entourage $\mathcal{E}$.*

8.8 Galois Theory of Coverings

Now we consider the category of spaces over a fixed topological space S.

Definition: A continuous map $X\to S$ is a **trivial covering** if there is a discrete space F and an isomorphism $X\simeq S\times F=\amalg_F S$, and it is a **covering** if we may cover S with open sets U where $X\times_S U\to U$ is a trivial covering.

Examples: The maps $e^{2\pi it}\colon\mathbb{R}\to S_1$, $e^z\colon\mathbb{C}\to\mathbb{C}^*=\mathbb{C}-0$, $z^n\colon\mathbb{C}^*\to\mathbb{C}^*$, $\pi\colon S_n\to\mathbb{P}_n(\mathbb{R})$, and $\pi\colon\mathbb{C}\to\mathbb{C}/\Gamma$, where $\Gamma=\mathbb{Z}e_1+\mathbb{Z}e_2$ is discrete, are coverings.

If $X\to S$ is a covering, X inherits the local structure of S. When S is a smooth manifold, a Riemann surface, a riemannian manifold, etc., so is X.

1. *Coverings are stable under base changes. If $X \to S$ is a covering, so is $T \times_S X \to T$ for any continuous map $T \to S$.*

2. *The concept of covering is local. If $R \to S$ is an open cover (or even a covering), then $X \to S$ is a covering if and only if $R \times_S X \to R$ is a covering.*

3. *If some maps $X_i \to S$ are coverings, so is $\amalg_i X_i \to S$.*

4. *The cardinal of the fibres of a covering $X \to S$ is locally constant; hence it is constant (the **degree** of the covering) when S is connected.* The degree may be infinite, contrary to what has been assumed in the case of fields (p. 151) and noetherian rings (p. 221).

5. *If $X \to S$ is a surjective covering, any continuous map $X \to T$ constant on the fibres induces a continuous map $S \to T$. In particular, the sequence $X \times_S X \rightrightarrows X \to S$ is exact.*

   The induced map $S \to T$ is continuous because $X \to S$ is a local homeomorphism.

6. *The functor "connected components" defines an equivalence of the category of trivial coverings of a given connected space S with the category of sets.*

7. *When S is a locally connected space, if a group G acts on a covering $X \to S$, then the quotient space X/G is stable under base changes, $T \times_S (X/G) = (T \times_S X)/G$.*

   To show that the natural bijection $(T \times_S X)/G \to T \times_S (X/G)$ is a homeomorphism we may assume that S is connected and $X = S \times F$ is a trivial covering, so that G acts through an action on the discrete space F and $X/G = S \times (F/G)$. In such a case, $(T \times_S X)/G = (T \times F)/G = T \times (F/G) = T \times_S (X/G)$.

From now on we assume that S is *connected* and *locally connected*.

Then coverings of S are locally connected, morphisms between coverings of S are coverings, and any connected component of a covering of S also is a covering.

Moreover, any section of a connected covering $X \to S$ is an isomorphism; hence morphisms of a connected covering X into another covering Y are given by the **Points Formula**:

$$\mathrm{Hom}_S(X,Y) = \mathrm{Hom}_X(X, X\times_S Y) = \begin{bmatrix} \text{Connected components of } X\times_S Y \\ \text{isomorphic to } X \text{ via the first projection}\end{bmatrix}$$

where each morphism $f\colon X \to Y$ correspond to its graphic $\Gamma_f := \{(x, f(x))\}_{x\in X}$.

Hence, if two morphisms $X \rightrightarrows Y$ coincide at a point, then they coincide (if two connected components have a common point, they coincide).

Definition: A connected covering $X \to S$ is a **Galois** covering when $\pi_1\colon X\times_S X \to X$ is a trivial covering,

$$X \times_S X = \amalg_G X = X \times G\ , \quad G := \mathrm{Hom}_S(X,X).$$

Since $\pi_2\colon X\times_S X \to X$ also is a trivial covering, we see that $\pi_2\colon \Gamma_f \to X$ is an isomorphism for any morphism $f\colon X\to X$. Hence f is an automorphism and $G = \mathrm{Aut}(X/S)$ is a group.

Now, the exact sequence $X\times G = X\times_S X \rightrightarrows X \to S$ shows that

$$S = X/G.$$

Therefore, when a covering $Y \to S$ is trivial over a Galois covering $X \to S$, it is fully determined by the natural right action of $G = \mathrm{Aut}(X/S)$ (hence a left action of the opposite group $\pi := \mathrm{Aut}(X/S)^{\mathrm{op}}$) on the set

$$F(Y) = \mathrm{Hom}_S(X,Y) = [\text{Connected components of } X\times_S Y].$$
$$Y = (X/G)\times_S Y = (X\times_S Y)/G = (X\times F(Y))/G,$$

where the action of G on $X \times F(Y)$ is just $\tau(x,f) = (\tau x, f\tau^{-1})$.

Hence, given a right G-set Δ, we define the **associated covering** to be

$$R(\Delta) = (X \times \Delta)/G$$

where Δ is considered with the discrete topology, and G acts on both factors.

The natural map $R(\Delta) \to S$ is a covering trivial over X since

$$X \times_S R(\Delta) = X \times_S (X \times \Delta)/G = ((X \times_S X) \times \Delta)/G = (X \times G \times \Delta)/G = X \times (G \times \Delta)/G = X \times \Delta,$$

because G acts via X-morphisms.

Now the points formula shows that $\Delta = \mathrm{Hom}_S(X, R(\Delta)) = FR(\Delta)$ and we obtain the following result:

Galois Theorem: *Let S be a connected and locally connected space, $X \to S$ a Galois covering of group G, and $\pi = \mathrm{Aut}(X/S)^{\mathrm{op}}$. The above functors F and R define an equivalence of categories*

$$\begin{bmatrix}\text{Coverings of } S \\ \text{trivial over } X\end{bmatrix} \rightsquigarrow \begin{bmatrix} G\text{-sets}\end{bmatrix}, \quad \begin{matrix} R \circ F = \mathrm{Id} \\ F \circ R = \mathrm{Id}\end{matrix}$$

Definition: Let S be a connected and locally connected space. A connected covering $\bar{S} \to S$ is a **universal covering** when any covering of $\bar{S}$ is trivial.

In particular any covering of S is trivial over $\bar{S}$, so that $\bar{S} \times_S \bar{S} \to \bar{S}$ is a trivial covering and $\bar{S} \to S$ is a Galois covering and the group $\pi = \mathrm{Aut}(X/S)^{\mathrm{op}}$ classifies coverings of S:

Theorem: *If S admits a universal covering $\bar{S} \to S$, then the category of coverings of S is equivalent to the category of π-sets, where $\pi = \mathrm{Aut}(\bar{S}/S)^{\mathrm{op}}$.*

If it exists, the universal covering is unique up to (non canonical) isomorphisms: If $\bar{S}_1 \to S$ is another universal covering, then $\bar{S} \times_S \bar{S}_1$ is a trivial covering of both factors; hence any connected component is isomorphic to $\bar{S}$ and $\bar{S}_1$.

The isomorphism is unique once we fix points of $\bar{S}$ and $\bar{S}_1$ over a given point of S.

8.8.1 The Fundamental Group

Definitions: A continuous map $\gamma \colon I = [0,1] \to X$ is a **path** of end points $p = \gamma(0), q = \gamma(1)$, and if γ' is a path of end points $q = \gamma'(0), r = \gamma'(1)$, the composition $\gamma'\gamma$ is the path

$$(\gamma' \cdot \gamma)(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma'(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Two paths γ_0, γ_1 from p to q are **homotopic**, $\gamma_0 \equiv \gamma_1$, if there is a continuous map $H \colon I \times I \to X$, named **homotopy** and we put $H_s(t) = H(t,s)$, such that

$$H_0 = \gamma_0, \; H_1 = \gamma_1, \; H_s(0) = p, \; H_s(1) = q,$$

and this is an equivalence relation, compatible with the composition of paths. It is clearly reflexive and symmetric, and if $H \colon \gamma_0 \equiv \gamma_1$, $H' \colon \gamma_1 \equiv \gamma_2$, then $H'' \colon \gamma_0 \equiv \gamma_2$, where

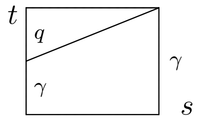
$$H''_s = \begin{cases} H_{2s} & 0 \leq s \leq \frac{1}{2} \\ H'_{2s-1} & \frac{1}{2} \leq s \leq 1 \end{cases}$$

and if $H\colon \gamma \equiv \gamma'$, $\bar{H}\colon \bar{\gamma} \equiv \bar{\gamma}'$, then $H'\colon \bar{\gamma}\gamma \equiv \bar{\gamma}'\gamma'$, where

$$H'_s(t) = \begin{cases} H_s(2t) & 0 \le t \le \frac{1}{2} \\ \bar{H}_s(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

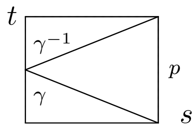
If q is the constant path $q(t) = q$, then $H\colon q\gamma \equiv \gamma$ (analogously $\gamma p \equiv \gamma$), where

$$H(t,s) = \begin{cases} \gamma\left(\frac{2t}{1+s}\right) & 2t \le 1+s \\ q & 2t \ge 1+s \end{cases}$$



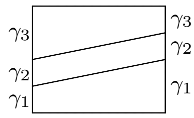
If γ^{-1} is the inverse path $\gamma^{-1}(t) = \gamma(1-t)$, then $H\colon \gamma^{-1}\gamma \equiv p$, where

$$H(t,s) = \begin{cases} \gamma(2t) & 2t \le 1-s \\ \gamma(1-s) & 1-s \le 2t \le 1+s \\ \gamma^{-1}(2t) & 2t \ge 1+s \end{cases}$$



and composition of paths is associative, $H\colon (\gamma_3\gamma_2)\gamma_1 \equiv \gamma_3(\gamma_2\gamma_1)$, where

$$H(t,s) = \begin{cases} \gamma_1\left(\frac{4t}{1+s}\right) & t \le \frac{1+s}{4} \\ \gamma_2\,(4t-s-1) & \frac{1+s}{4} \le t \le \frac{2+s}{4} \\ \gamma_3\left(\frac{4t-s-2}{2-s}\right) & \frac{2+s}{4} \le t \end{cases}$$



When $p = q$, the path is a **loop** at p. Homotopy classes of loops at p, with the composition of maps, form a group $\pi_1(X,p)$, the **fundamental group** of X at p.

Any continuous map $f\colon X \to Y$ naturally induces a morphism of groups

$$f_*\colon \pi_1(X,p) \longrightarrow \pi_1\big(Y, f(p)\big),\ \ f_*[\sigma] = [f \circ \sigma],$$

well defined: if $H\colon \sigma_0 \equiv \bar{\sigma}$, then $H'(t,s) = f(H(t,s))$ is a homotopy $f \circ \sigma \equiv f \circ \bar{\sigma}$.

Any path γ from p to q induces a group isomorphism (non canonical, it depends on the path, except when the fundamental group is abelian)

$$\gamma\colon \pi_1(X,p) \xrightarrow{\ \sim\ } \pi_1(X,q),\ \ [\sigma] \mapsto [\gamma\sigma\gamma^{-1}].$$

Two continuous maps $f', f\colon X \to Y$ are said to be **homotopic** if there is a continuous map $H\colon X \times I \to Y$ such that

$$f(x) = H(x,0) \quad , \quad f'(x) = H(x,1).$$

This is an equivalence relation compatible with the composition of maps, so that topological spaces, with homotopy classes of continuous maps, form a category.

Isomorphisms in this category are named **homotopical equivalences**, and a space is **contractible** if it is homotopically equivalent to a point.

Lemma: *If γ is the path $\gamma(s) = H(p,s)$, the following square is commutative*

$$\begin{array}{ccc} \pi_1(X,p) & \xrightarrow{f_*} & \pi\big(Y, f(p)\big) \\ \| & & \wr|\gamma \\ \pi_1(X,p) & \xrightarrow{f'_*} & \pi\big(Y, f'(p)\big) \end{array}$$

Proof: If we consider $F\colon I\times I \xrightarrow{\sigma\times 1} X\times I \xrightarrow{H} Y$, then

$$F(t,0)=f(\sigma(t))\ ,\ F(t,1)=f'(\sigma(t)),$$
$$F(0,s)=F(1,s)=\gamma(s).$$

Since the upper side of the square $I\times I$ is homotopic to the path formed by the three remaining sides, composing with F we obtain that $f'\circ\sigma\equiv\gamma(f\circ\sigma)\gamma^{-1}$.

Theorem: *If $f\colon X\to Y$ is a homotopical equivalence, then $f_*\colon \pi_1(X,p)\to\pi_1(Y,f(p))$ is an isomorphism. In particular $\pi_1(X,p)=0$ when X is contractible.*

Definition: X is **simply connected** if it is **path-connected** (paths join any pair of points) and $\pi_1(X,p)=0$, and it is **locally simply connected** if any point has a base of simply connected open neighborhoods (for example, any topological manifold).

Lemma: *Let $X\to S$ be a trivial covering, and let $f\colon T\to S$ be a continuous map, T connected. Any continuous lifting $\tilde f\colon C\to X$ of f, defined on a connected subspace $C\subset T$, admits a unique extension to T.* (Recall that S is connected and locally connected.)

Proof: Extensions of f correspond to continuous sections of $X\times_S T\to T$; but the image of a continuous section defined on a connected subspace is contained in a connected component of $X\times_S T=\amalg T$, and now the statement is obvious.

Lemma: *Let $\pi\colon X\to S$ be a covering, and let $\gamma\colon I\to S$ be a path with origin at p. If $\pi(x)=p$, there exists a unique continuous lifting $\tilde\gamma\colon I\to X$ of γ with origin at x.*

Moreover, if $\gamma\equiv\gamma'$, then $\tilde\gamma\equiv\tilde\gamma'$.

Proof: The existence and uniqueness of the lifting $\tilde\gamma$ follows from the above lemma, considering a partition of I with images contained in open sets of S where X is trivial.

Moreover, the existence of a lifting of any homotopy $H\colon I\times I\to S$ between γ and γ' also follows from the above lemma, considering a partition of $I\times I$ into squares with images contained in open sets of S where X is trivial, since the intersection of any square with the union of the former squares is connected.

Lemma: *Any covering $X\to S$ of a simply connected locally path-connected space is trivial.*

Proof: It is enough to show that any connected covering X has degree 1; and X is path-connected because it is connected and locally path-connected.

If x,y are two points of the fibre of p, projecting onto S a path joining x to y we have a loop at p which is not homotopic to a point, since the lifting is not a loop.

Theorem: *Let S be a connected and locally path-connected space. If $\tilde S\to S$ is a simply connected covering, then it is a universal covering of S, and any point $\tilde p$ of the fibre of $p\in S$ defines an isomorphism*

$$\pi_1(S,p)=\mathrm{Aut}(\tilde S/S)^{\mathrm{op}},$$

and the functor "fibre over p" defines an equivalence of the category of coverings of S with the category of $\pi_1(S,p)$-sets.

Proof: Any loop σ has a lifting $\tilde\sigma$ with origin at $\tilde p$ and end point at $\tau(\tilde p)$, $\tau\in\mathrm{Aut}(\tilde S/S)$.

The map $\pi_1(S,p)\to\mathrm{Aut}(\tilde S/S)^{\mathrm{op}}$, $[\sigma]\mapsto\tau$, is well defined (by the lemma), surjective ($\tilde S$ is path-connected, and any two points of the fibre of p may be joined by a path), injective (any loop in $\tilde{S}$ is homotopic to a point) and a group morphism: If $\tilde{\sigma}_1$ joins $\tilde{p}$ with $\tau_1(\tilde{p})$, and $\tilde{\sigma}_2$ joins $\tilde{p}$ with $\tau_2(\tilde{p})$, then the lifting of $\sigma_2\sigma_1$ is $\tau_1(\tilde{\sigma}_2)\cdot\tilde{\sigma}_1$, with end point $\tau_1\tau_2(\tilde{p})$.

Finally, the point $\tilde{p}$ over p defines a bijection of the $\pi_1(S,p)$-set $F(X)=\mathrm{Hom}_S(\tilde{S},X)$ with the fibre X_p of any covering X over p: for any $x\in X_p$ there is a unique morphism $f\colon \tilde{S}\to X$ such that $f(\tilde{p})=x$.

Corollary: *$\pi_1(S_1,p)=\mathbb{Z}$, generated by the loop $e^{2\pi ti}\colon[0,1]\to S_1$.*

Proof: The map $e^{2\pi ti}\colon \mathbb{R}\to S_1$ is a simply connected covering. q.e.d.

1. *There is no continuous retract $r\colon D_2\to S_1$ of a disc onto the boundary.*

   $\mathbb{Z}=\pi_1(S_1)\xrightarrow{i_*}\pi_1(D_2)=0\xrightarrow{r_*}\pi_1(S_1)=\mathbb{Z}$ would be the identity.

2. **Brouwer's Theorem:** *Any continuous map $f\colon D_2\to D_2$ has a fixed point.*

   Otherwise we have a continuous retract $r\colon D_2\to S_1$, where $r(x)$ is the intersection point of S_1 with the half-line with origin at x passing through $f(x)$.

3. *Any non constant polynomial with complex coefficients has a complex root.*

   If $P(z)=z^n+a_1z^{n-1}+\ldots+a_n$ has no complex root, it defines a homotopy

   $$H(z,s)=s^nP\left(\tfrac{1-s}{s}z\right)=(1-s)^nz^n+a_1s(1-s)^{n-1}z^{n-1}+\ldots+a_ns^n$$

   between the continuous maps $h_0(z)=z^n$ and $h_1(z)=a_n\colon S_1\to\mathbb{C}^*$. But $h_{1,*}=0$, while $h_{0,*}\colon\mathbb{Z}=\pi_1(S_1)\to\pi_1(\mathbb{C}^*)=\pi_1(S_1)=\mathbb{Z}$ is the multiplication by n.

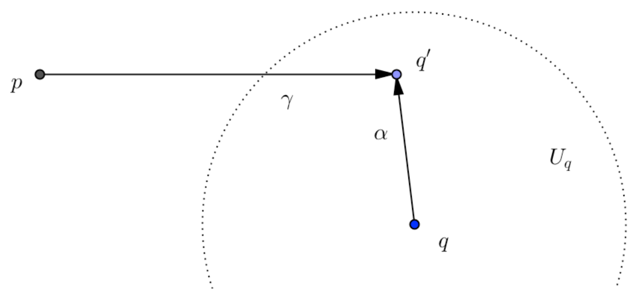
4. *Two toruses $\mathbb{C}/\Gamma$, $\mathbb{C}/\Gamma'$ are analytically isomorphic if and only if $\Gamma'=a\Gamma$, $a\in\mathbb{C}$.*

   An isomorphism $\mathbb{C}/\Gamma'\simeq\mathbb{C}/\Gamma$ induces an isomorphism $\tau\colon\mathbb{C}\simeq\mathbb{C}$ of the universal coverings, and we may assume that $\tau(0)=0$. Now, $\tau(z)=az$ since any analytic automorphism of $\mathbb{C}$ is an affinity (p. 279).

The Universal Covering: Assume that S is a connected and locally simply connected space, consider the set $\tilde{S}$ of homotopy classes of paths with origin at $p\in S$, and $\pi\colon\tilde{S}\to S$, $\pi(\gamma)=\gamma(1)$. Over any simply connected neighborhood U_q of $q\in S$ we have a bijection

$$\pi^{-1}(U_q)=U_q\times F_q,\quad\text{where } F_q=\begin{bmatrix}\text{homotopy classes}\\ \text{of paths from } p \text{ to } q\end{bmatrix}$$

transforming $[\gamma]$ into the pair $\big(q'=\gamma(1),[\alpha^{-1}]\gamma\big)$, where α is a path in U_q joining q with q' (it does not depend on α since all choices are homotopic).



These bijections define topologies on the open sets $\pi^{-1}(U_q)$, considering F_q as a discrete space, so that π is a covering (trivial over U_q). In fact, when $U_q \subseteq U_{\bar{q}}$, the topology defined on $\pi^{-1}(U_q)$ coincides with the topology induced by $\pi^{-1}(U_{\bar{q}})$; i.e., the composition

$$U_q \times F_q = \pi^{-1}(U_q) = U_q \times F_{\bar{q}}$$

is a homeomorphism, since it transforms the component $U_q \times [\gamma]$ into $U_q \times [\beta\gamma]$, where β is a path in $U_{\bar{q}}$ joining q with $\bar{q}$.

Theorem: *Let S be a connected and locally simply connected space. The space $\pi\colon \tilde{S} \to S$ of homotopy classes of paths with origin at p is a simply connected covering; hence a universal covering of S.*

Proof: If γ is a path in S with origin at p, the lifting $\tilde{\gamma}$ with origin at $\tilde{p}$ is $\tilde{\gamma}(s) = [\gamma_s]$,

$$\gamma_s(t) = \begin{cases} \gamma(t) & t \le s \\ \gamma(s) & t \ge s \end{cases}$$

and the end point is $\tilde{\gamma}(1) = [\gamma_1] = [\gamma]$; hence $\tilde{S}$ is path-connected.

Moreover, the homotopy class of the constant loop at p defines a point $\tilde{p} \in \tilde{S}$ is in the fibre of p, and any loop $\tilde{\sigma}$ at $\tilde{p}$ is the lifting of $\sigma = \pi\tilde{\sigma}$, so that $[\sigma] = \tilde{\sigma}(1) = \tilde{p}$. Hence the lifting $\tilde{\sigma}$ is homotopic to the constant loop: $\tilde{S}$ is simply connected.

Corollary: *If a connected locally simply connected space $S = U_1 \cup U_2$ is the union of two simply connected open sets and $U_1 \cap U_2$ is connected, then S is simply connected.*

Proof: Let us show that any covering of S admits a continuous section.

We have some section on U_1 since it is simply connected, and the induced section on the connected set $U_1 \cap U_2$ may be extended to U_2 by the lemma of p. 252.

Corollary: $\pi_1(S_n, p) = 0$*, and* $\pi_1(\mathbb{P}_{n,\mathbb{R}}, p) = \mathbb{Z}/2\mathbb{Z}$*; when* $n \ge 2$*.*

Proof: We have a covering $S_n \to \mathbb{P}_{n,\mathbb{R}}$ of degree 2, and S_n is simply connected because it is the union of two disks with connected intersection when $n \ge 2$. q.e.d.

1. $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *when* $n \neq 2$*.*

   $\pi_1(\mathbb{R}^2 - p) = \pi_1(S_1) = \mathbb{Z}$, and $\pi_1(\mathbb{R}^n - p) = \pi_1(S_{n-1}) = 0$ when $n \ge 3$.

2. *There is no continuous map $f\colon S_n \to S_1$, $n \ge 2$, preserving antipodes.*

   If $f(-x) = -f(x)$, it induces a continuous map $h\colon \mathbb{P}_n \to \mathbb{P}_1$ and a commutative square

$$\begin{array}{ccc} S_n & \xrightarrow{f} & S_1 \\ \downarrow & & \downarrow \\ \mathbb{P}_n & \xrightarrow{h} & \mathbb{P}_1 \end{array}$$

   If we take a path $\bar{\gamma}$ joining two antipodal points $x, -x \in S_n$, the image γ is a loop in $\mathbb{P}_n$; hence $h(\gamma)$ is homotopic to a point, since $h_*\colon \mathbb{Z}/2\mathbb{Z} = \pi_1(\mathbb{P}_n) \to \pi_1(\mathbb{P}_1) = \mathbb{Z}$ is null, and the lifting $f(\bar{\gamma})$ is a loop; absurd since $f(-x) = -f(x) \neq f(x)$.

3. *Any continuous map $S_n \to S_2$, $n \ge 2$, preserving antipodes is surjective.*

   If the north pole is out of the image, so is the south pole, and projecting onto the equator along the meridians we obtain a continuous map $S_n \to S_1$ preserving antipodes.

4. **Borsuk-Ulam Theorem:** *Any continuous map $f\colon S_n \to \mathbb{R}^2$, $n \geq 2$, coincides on two antipodal points.*

   Otherwise the continuous map $\frac{f(x)-f(-x)}{|f(x)-f(-x)|}\colon S_n \to S_1$ preserves antipodes.

5. *Any continuous map $f\colon S_n \to \mathbb{R}^2$, $f(-x) = -f(x)$, vanishes at some point.*

6. *No compact set in $\mathbb{R}^2$ is homeomorphic to S_2.*

Definition: A **principal** covering of group G is a covering $P \to S$ (not necessarily connected) with a free G-action transitive on any fibre; i.e., the action defines an isomorphism

$$G \times P \xrightarrow{\sim} P \times_S P,\ (g, x) \mapsto (x, gx).$$

Isomorphisms of principal coverings are isomorphisms of coverings which are also isomorphisms of G-sets.

If we fix a point of the fibre of $p \in S$, we say that it as a pointed principal covering.

When S is connected, isomorphisms of pointed principal coverings are unique.

Proposition: *If S is a connected and locally simply connected space,*

$$\mathrm{Hom}_{\mathrm{gr}}(\pi_1(S,p), G) = \begin{bmatrix}\text{Pointed principal}\\ G\text{-coverings of } S\end{bmatrix}$$

Proof: Principal G-coverings of S correspond to (say right) $\pi_1(S,p)$-sets F with a free transitive G-action, $g(x\gamma) = (gx)\gamma$. Once we fix a point $x \in F$, the group G may be identified with F, and such actions correspond to morphism of groups $f\colon \pi_1(S,p) \to G$, $x\gamma = f(\gamma)x$,

$$f(\gamma_1\gamma_2)x = x\gamma_1\gamma_2 = f(\gamma_1)x\gamma_2 = f(\gamma_1)f(\gamma_2)x.$$

Corollary: $\mathrm{Hom}_{\mathrm{gr}}(\pi_1(S,p), A) = \begin{bmatrix}\text{Principal coverings}\\ \text{of } S \text{ of group } A\end{bmatrix}$, *when A is an abelian group.*

Proof: Principal G-coverings of S correspond to morphisms $\pi_1(S,p) \to G$, modulo inner automorphisms of G. In fact, if we fix another point $y = gx$ in the fibre of p, the corresponding morphism is gfg^{-1},

$$y\gamma = gx\gamma = gf(\gamma)x = gf(\gamma)g^{-1}y.$$

Van-Kampen Theorem: *If U_1, U_2 are two connected and locally simply connected open sets, and $U_1 \cap U_2$ is connected, then the fundamental group of the union is the coproduct (in the category of groups) of the fundamental groups,*

$$\pi_1(U_1 \cup U_2, p) = \pi_1(U_1, p) *_{\pi_1(U_1 \cap U_2, p)} \pi_1(U_2, p).$$

Proof: Since isomorphisms of pointed principal coverings are unique, to give a pointed principal G-covering over $U_1 \cup U_2$ is just to give one over each open set U_i, coinciding over the intersection.

That is to say, we have a fibred product

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{gr}}(\pi_1(U_1 \cup U_2, p), G) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}}(\pi_1(U_1, p), G) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{gr}}(\pi_1(U_1, p), G) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}}(\pi_1(U_1 \cap U_2, p), G)
\end{array}$$

8.8.2 Triangulated Compact Surfaces

A triangulated compact surface X is a connected 2-dimensional polyhedron (a connected finite union of vertices, closed edges and faces of a tetrahedron $|\Delta_n|$) such that each edge only belongs to two faces and the faces with any given vertex admit a cyclic ordering such that each face has a common edge with the following one (hence with the former one). Given two triangulated compact surfaces X, Y we put $X \equiv Y$ when there is some homeomorphism $X \to Y$.

Since X is connected, we may order the faces $T_0, \ldots, T_n$ so that each triangle T_i, $i \geq 1$, has a common edge with a former one. Inductively, one sees that $T_0 \cup \ldots \cup T_i$ is homeomorphic (with a homeomorphism preserving the triangulations) to a triangulated plane polygon with an even number of sides, where the sides are identified by pairs. If we fix an orientation of the boundary of such polygon, put a letter $a^{\pm 1}$ on each pair of identified sides (with the same exponent when the identification preserves the orientation, and different exponent otherwise) and read the letters starting from a vertex of the boundary, we see that X is fully determined by a **symbol** $a_1^{\pm 1} \ldots a_n^{\pm 1}$, where each letter appears exactly two times.

A pair of first kind is a pair $\ldots x \ldots x^{-1} \ldots \equiv \ldots x^{-1} \ldots x \ldots$, and a pair of second kind is a pair $\ldots x \ldots x \ldots \equiv \ldots x^{-1} \ldots x^{-1} \ldots$

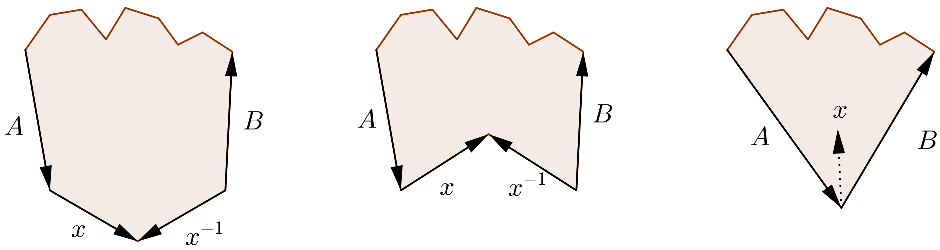
For example, the symbol aa^{-1} represents a sphere, the symbol aa a projective plane, the symbol $aba^{-1}b^{-1}$ a torus, and the symbol $aba^{-1}b$ a Klein bottle.

But different symbols may define homeomorphic compact surfaces. Apart of the rule that we may replace a letter by any new symbol, we have the following transformation rules:

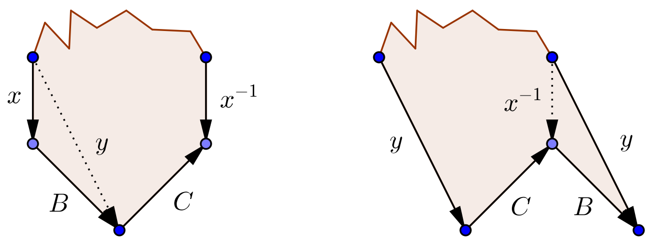
1. *Cyclic permutation*: $a_1 a_2 \ldots a_n \equiv a_2 \ldots a_n a_1$.

2. *Reverse orientation of a side*: $AxBxC \equiv Ax^{-1}Bx^{-1}C$, $AxBx^{-1}C \equiv Ax^{-1}BxC$.

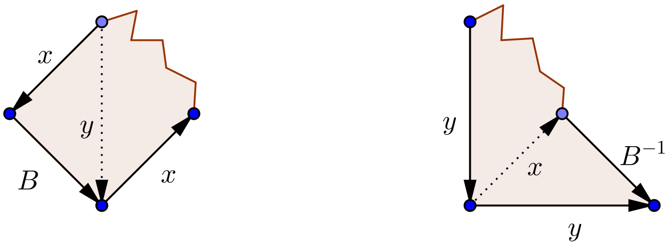
3. *The pair xx^{-1} may be simplified*: $Axx^{-1}B \equiv AB$.



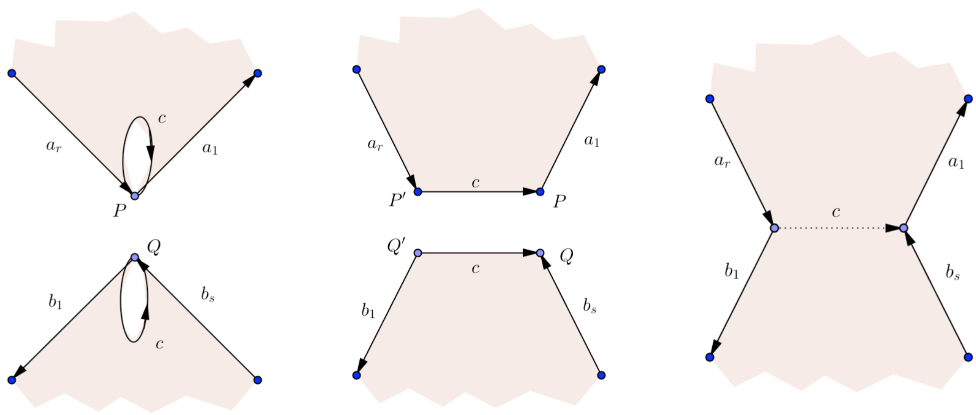
4. *Cyclic permutation inside a pair of first kind*: $AxBCx^{-1}D \equiv AxCBx^{-1}D$. Hence, also cyclic permutations outside a pair of first kind:
$$AxBCx^{-1}D \stackrel{1}{\equiv} x^{-1}DAxBC \stackrel{4}{\equiv} x^{-1}ADxBC \stackrel{1}{\equiv} DxBCx^{-1}A.$$



5. *Grouping a pair of second kind*: $AxBxC \equiv AxxB^{-1}C$, $AxBxC \equiv AB^{-1}xxC$, where we put $(b_1 \ldots b_s)^{-1} = b_s^{-1} \ldots b_1^{-1}$.



Given two symbols A and B, we say that the surface AB is the **connected sum** of A and B, and it is obtained removing a small open disk in each surface A, B and then identifying the boundaries of the disks (and rule 3 states that the connected sum of a surface A with a sphere is just the surface A):



Lemma: *The connected sum of a projective plane and a torus is the connected sum of three projective planes,*

$$Axxaba^{-1}b^{-1}B \equiv AxxbbaaB.$$

Proof: $Axxaba^{-1}b^{-1}B \overset{5}{\equiv} Axa^{-1}xba^{-1}b^{-1}B \overset{5}{\equiv} Axa^{-1}a^{-1}b^{-1}x^{-1}b^{-1}B$
$\overset{5}{\equiv} Axa^{-1}a^{-1}b^{-1}b^{-1}xB \overset{5}{\equiv} AxxbbaaB.$

Theorem: *Any triangulated compact surface is homeomorphic to one (and only one) of the following surfaces*:

1. *A sphere,* aa^{-1}.

2. *A connected sum* τ_g *of* g *toruses,* $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$.

3. *A connected sum* π_g *of* g *projective planes,* $a_1a_1\dots a_ga_g$.

Proof: By the lemma, we only have to show that any symbol is equivalent to a sphere xx^{-1} or to a connected sum of projective planes and toruses:

$$xx\dots zzaba^{-1}b^{-1}\dots cdc^{-1}d^{-1}.$$

1. *All pairs of second kind may be grouped at the beginning, so that the symbol is* $xx\dots zzA$, *where* A *is a symbol with no pairs of second kind.*

$$AxBxC \overset{5}{\equiv} AxxB^{-1}C \overset{5}{\equiv} xA^{-1}xB^{-1}C \overset{5}{\equiv} xxAB^{-1}C.$$

Now, if there is another second kind pair, we iterate:

$$xxAyByC \overset{5}{\equiv} xxAyyB^{-1}C \overset{5}{\equiv} xxyA^{-1}yB^{-1}C \overset{5}{\equiv} xxyyAB^{-1}C.$$

2. *If a symbol A has no pairs of second kind, then $XA \equiv Xaba^{-1}b^{-1}\dots cdc^{-1}d^{-1}$.*

   If the length l of A is 2, then $A = aa^{-1}$, and $XA \equiv X$. If $l \geq 4$, consider a pair $a\dots a^{-1}$ at minimal distance. If they are adjacent, cancel them with rule 3. Otherwise, there is some letter b between them, $a\dots b\dots a^{-1}$, and the other letter $b^{\pm 1}$ in A is not between the pair $a\dots a^{-1}$, since they are at minimal distance. There are two possible cases

$$X\dots a\dots b\dots a^{-1}\dots b^{-1}\dots \qquad \text{or} \qquad X\dots b^{-1}\dots a\dots b\dots a^{-1}\dots$$

   and both are equivalent by rule 2 (change b by b^{-1}). Now we repeatedly apply rule 4, underlining the cyclic permutation:

$$\underline{X\dots}a\dots b\dots a^{-1}\underline{\dots b^{-1}\dots} \overset{4}{\equiv} \dots Xa\dots b\underline{\dots a^{-1}\dots}b^{-1}\dots \overset{4}{\equiv} \dots Xa\underline{\dots b\dots}a^{-1}b^{-1}\dots$$
$$\overset{4}{\equiv} \underline{\dots Xa\dots}ba^{-1}\dots b^{-1}\underline{\dots} \overset{4}{\equiv} \dots Xaba^{-1}b^{-1}\dots \overset{1}{\equiv} Xaba^{-1}b^{-1}\dots$$

Finally, let us see that all these surfaces S_2, τ_g and π_g are not homeomorphic, because they have different fundamental group. Consider a small open disk D around the center p of the polygon, and let U be the complement of p. Then U is homotopic to the image of the boundary of the polygon, and the image of the natural map $\mathbb{Z} = \pi_1(U \cap D) \to \pi_1(U)$ is generated by the loop defined by the symbol.

Since $\pi(D) = 0$, the Van-Kampen theorem shows that the fundamental group of a surface of symbol A is isomorphic to the quotient of the fundamental group of the boundary (with the sides identified according to A) by the normal subgroup generated by the symbol.

In the case of τ_g (resp. π_g), the boundary are $2g$ (resp. g) circles with a common point.

In particular, the universal abelian quotient (the quotient by the normal subgroup generated by the commutators $ghg^{-1}h^{-1}$) is

$$\begin{aligned}
&\pi_1(S_2)_{\mathrm{ab}} = 0,\\
&\pi_1(\tau_g)_{\mathrm{ab}} = \mathbb{Z}^{2g}/(\textstyle\sum_i a_i + b_i - a_i - b_i)\mathbb{Z} = \mathbb{Z}^{2g},\\
&\pi_1(\pi_g)_{\mathrm{ab}} = \mathbb{Z}^{g}/(2a_1 + \dots + 2a_g)\mathbb{Z} \simeq \mathbb{Z}^{g-1} \oplus (\mathbb{Z}/2\mathbb{Z}).
\end{aligned}$$

Example: The Klein bottle K has symbol $aba^{-1}b$; hence $\pi_1(K)_{\mathrm{ab}} = \mathbb{Z}^2/2b\mathbb{Z} \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$, so that K is homeomorphic to the connected sum π_2 of two projective planes.

Chapter 9

Analysis III

9.1 Rings of Smooth Functions

Definitions: Let X be a topological space. Two continuous real functions, defined on some neighborhoods of $x \in X$, have equal **germ** at x if they coincide on a neighborhood of x.

The ring $\mathcal{O}_x$ of germs at x of continuous functions is defined to be

$$\mathcal{O}_x = \varinjlim_{x \in U} \mathcal{C}(U)$$

where U runs over the open neighborhoods of the point x, and the germ at x of a continuous function $f \in \mathcal{C}(U)$ will be denoted by f_x. The **support** of a function $f \colon X \to \mathbb{R}$ is

$$\operatorname{supp} f = \overline{\{x \in X \colon f(x) \neq 0\}} = \{x \in X \colon f_x \neq 0\}.$$

A **partition of unity** subordinate to an open cover $X = \bigcup_i U_i$ is a family of continuous real functions $\phi_i \in \mathcal{C}(X)$ such that

1. For any index i, we have $\operatorname{supp} \phi_i \subseteq U_i$, and $\phi_i \geq 0$.
2. The family $\{\operatorname{supp} \phi_i\}$ is locally finite (finite in a neighborhood of each point).
3. $1 = \sum_i \phi_i$, (the sum is senseful by condition 2).

Lemma: *Let K be a compact Hausdorff space. If V is a neighborhood of $p \in K$, there is a non negative continuous function $f \colon K \to \mathbb{R}$ such that $f(p) > 0$ and $\operatorname{supp} f \subseteq V$.*

Proof: By Urysohn's lemma, if $Q \subseteq V$ is a compact neighborhood of p, there exists $f \in \mathcal{C}(K)$ such that $f(Q^c) = 0$, $f(p) > 0$.

Theorem: *Any open cover of a σ-compact space admits a subordinate partition of unity.*

Proof: Put $X = \bigcup_n K_n$, with K_n compact and $K_n \subseteq \overset{\circ}{K}_{n+1}$ (p. 100), and $Q_n := K_n - \overset{\circ}{K}_{n-1}$

If $p \in Q_n$, there is a continuous function $f \geq 0$ with support in $U_i \cap (\overset{\circ}{K}_{n+1} - K_{n-2})$ not vanishing at p, and we choose a finite number of these functions not vanishing simultaneously at any point of Q_n.

Varying n we obtain functions f_j such that the family $\{\operatorname{supp} f_j\}$ is locally finite and $h = \sum_j f_j$ has no zero in $\bigcup_n Q_n = X$. Replacing f_j by f_j/h, we may assume that $1 = \sum_j f_j$.

Now, for any index j, fix $\sigma(j)$ such that $\operatorname{supp} f_j \subseteq U_{\sigma(j)}$, and put $\phi_i = \sum\limits_{\sigma(j)=i} f_j$.

Lemma: *If V is a neighborhood of the origin in $\mathbb{R}^n$, there is a smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, non negative and with compact support contained in V, such that $f(0) > 0$.*

Proof: The neighborhood of radius ε is the support of $f(x_1, \dots, x_n) = e(x_1^2 + \dots + x_n^2 - \varepsilon^2)$, where $e(t)$ is the $\mathcal{C}^\infty$ function

$$e(t) = \begin{cases} e^{1/t} & t < 0 \\ 0 & t \geq 0 \end{cases}$$

Now, the above proof (smooth manifolds are always assumed to be σ-compact) gives:

Theorem: *Any open cover of a smooth manifold admits a $\mathcal{C}^\infty$ subordinate partition of unity.*

Corollary: *Let Y_1, Y_2 be disjoint closed sets of a smooth manifold X. There exists a smooth global function $0 \leq f \leq 1$ such that $f(Y_1) = 0$, $f(Y_2) = 1$.*

Proof: Let ϕ_1, ϕ_2 be a partition of unity subordinate to the open cover $X = U_1 \cup U_2$, where $U_i = X - Y_i$. Now $f = \phi_1$.

Corollary: *Let U be a neighborhood of $x \in X$. There is a function $\phi \in \mathcal{C}^\infty(X)$, such that $\phi = 1$ on a neighborhood of x, $\operatorname{supp} \phi \subseteq U$, and $0 \leq \phi \leq 1$.*

Proof: There exists $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$, such that $\phi = 1$ on the closed neighborhood of radius ε, and vanishing on the complement of the open neighborhood of radius 2ε.

Corollary: *If $f \in \mathcal{C}^\infty(U)$ and $x \in U$, there exists $F \in \mathcal{C}^\infty(X)$ with equal germ, $f_x = F_x$.*

Proof: If ϕ is a plateau function, and we extend ϕf by 0 on $X - U$, we obtain a global smooth function F with equal germ at x.

Corollary: *There is a smooth function $f \colon X \to \mathbb{R}$ such that $f^{-1}([a, b])$ is compact, $\forall a, b \in \mathbb{R}$.*

Proof: Put $X = \bigcup_n K_n$, $K_n \subset \overset{\circ}{K}_{n+1}$, and take $f = \sum_n h_n$, where h_n is a non negative smooth function such that $h_n(K_{n-1}) = 0$ and $h_n(X - \overset{\circ}{K}_{n+1}) = 1$.

Definitions: The **maximal spectrum** of a ring A is the subspace $\operatorname{Spec_m} A \subseteq \operatorname{Spec} A$ of all maximal ideals. The **real spectrum** of a $\mathbb{R}$-algebra A is the subspace $\operatorname{Spec_{\mathbb{R}}} A \subseteq \operatorname{Spec_m} A$ of all **real** maximal ideals $\mathfrak{m}$ (the natural morphism $\mathbb{R} \to A/\mathfrak{m}$ is an isomorphism).

When A is a subalgebra of $\mathcal{C}(X)$, we have a surjective morphism $A \to \mathbb{R}$, $f \mapsto f(x)$, for any point $x \in X$; hence the kernel $\mathfrak{m}_x = \{f \in A \colon f(x) = 0\}$ is a maximal ideal of residue field $\mathbb{R}$.

This map $X \to \operatorname{Spec_{\mathbb{R}}} A$ is continuous, since zeros of continuous functions

$$z(f) = (f)_0 \cap X = \{x \in X \colon f(x) = 0\}$$

are closed sets. It is injective when functions in A separate points (if $x \neq y$, then $f(x) \neq f(y)$ for some $f \in A$), and it is a homeomorphism onto the image when moreover any closed set in X is an intersection of zeros; i.e., when functions in A separate points from closed sets: if $x \notin \overline{Y}$, then there is $f \in A$ such that $f(Y) = 0$, $f(x) \neq 0$.

Theorem: $K = \operatorname{Spec_m} \mathcal{C}(K)$, *when K is a compact Hausdorff space.*

Proof: Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{C}(K)$. Finite intersections of zeros of functions in $\mathfrak{m}$ are non void since $z(f_1) \cap \ldots \cap z(f_n) = z(f_1^2 + \ldots + f_n^2)$, and $f_1^2 + \ldots + f_n^2 \in \mathfrak{m}$ is not invertible.

Hence there is a point $x \in K$ where all the functions in $\mathfrak{m}$ vanish, $\mathfrak{m} \subseteq \mathfrak{m}_x$.

Finally, $\mathfrak{m} = \mathfrak{m}_x$ since $\mathfrak{m}$ is a maximal ideal.

Theorem[1]**:** $X = \operatorname{Spec}_{\mathbb{R}} \mathcal{C}^\infty(X)$, *when X is a smooth manifold.*

Proof: Let us consider $f \in \mathcal{C}^\infty(X)$ with compact fibres. If $\mathcal{C}^\infty(X)/\mathfrak{m} = \mathbb{R}$, then $f - a \in \mathfrak{m}$ for some $a \in \mathbb{R}$. Since finite intersections of zeros of functions in $\mathfrak{m}$ are non void, and $z(f-a)$ is compact, all the functions in $\mathfrak{m}$ vanish at some $x \in z(f-a)$; hence $\mathfrak{m} = \mathfrak{m}_x$.

Theorem: $\operatorname{Hom}(X,Y) = \operatorname{Hom}_{\mathbb{R}\text{-alg}}\big(\mathcal{C}^\infty(Y), \mathcal{C}^\infty(X)\big)$, *for any two smooth manifolds X, Y.*

Proof: A smooth map $\phi\colon X \to Y$ induces a morphism of $\mathbb{R}$-algebras $\phi^*\colon \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$, $\phi^*(f) = f\phi$. On the other hand, any morphism of $\mathbb{R}$-algebras $\mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$ induces a continuous map $\phi\colon X = \operatorname{Spec}_{\mathbb{R}} \mathcal{C}^\infty(X) \to \operatorname{Spec}_{\mathbb{R}} \mathcal{C}^\infty(Y) = Y$, and it is smooth.

In fact, ϕ transforms smooth functions on Y into smooth functions on X, and if f is a smooth function on an open set $V \subset Y$, and $x \in \phi^{-1}(V)$, then f coincides on a neighborhood of $\phi(x)$ with a smooth function $F \in \mathcal{C}^\infty(Y)$; hence $\phi^*(f)$ coincides with the smooth function $\phi^*(F)$ on a neighborhood of x, and $\phi^*(f) \in \mathcal{C}^\infty(\phi^{-1}V)$.

Definition: Let X be a smooth manifold. If K is a compact set contained in a coordinate open set $(U; x_1, \ldots, x_d)$, we may consider on $\mathcal{C}^m(X)$, $1 \leq m \leq \infty$, the seminorms

$$\|f\|_{K,r} = \max_{|\alpha| \leq r} \left(\frac{1}{\alpha!} \, \|\partial_\alpha f\|_K \right) \quad ; \quad r \leq m, \ r < \infty,$$

(see p. 104 for notations). The seminorms p_n of a countable family of compact sets contained in open neighborhoods, whose interior sets cover X, define a linear topology on $\mathcal{C}^m(X)$, independent of the compact cover, and $\mathcal{C}^m(X)$ *is metrizable and complete,* $1 \leq m \leq \infty$ (p. 120).

Replacing p_n by $\max\{p_1, \ldots, p_n\}$, we may always assume that $p_{n-1} \leq p_n$.

Theorem: *Let U be an open set in a σ-compact space X. Any function $f \in \mathcal{C}(U)$ is a quotient, $f = g/h$, of two continuous global functions, $z(h) = X - U$. Hence $\mathcal{C}(U) = \mathcal{C}(X)_S$, where S is the multiplicative system of all functions without zeros in U.*

Proof: Put $U = \bigcup_n K_n$, $K_n \subseteq \overset{\circ}{K}_{n+1}$, and consider the seminorms $p_n = \|\ \|_{K_n}$ on $\mathcal{C}(X)$.

Let $0 \leq \phi_n \in \mathcal{C}(X)$ such that $\phi_n(X - \overset{\circ}{K}_{n+1}) = 0$, $\phi_n(K_n) = 1$.

Since $\operatorname{supp} \phi_n f \subseteq K_{n+1}$, the function $\phi_n f$ is continuous when extended by 0 outside of U, and we may consider the convergent series

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + p_n(\phi_n))} \cdot \frac{\phi_n f}{1 + p_n(\phi_n f)}$$

$$h = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + p_n(\phi_n f))} \cdot \frac{\phi_n}{1 + p_n(\phi_n)}$$

[1]This theorem shows that the $\mathbb{R}$-algebra $\mathcal{C}^\infty(X)$ reconstructs the topological space X; but it also determines the sheaf of smooth functions, since it clearly determines global smooth functions, and smooth functions on an open set $U \subset X$ are just continuous functions locally coinciding with global smooth functions. Hence, the functors $\mathcal{C}^\infty$ and $\operatorname{Spec}_{\mathbb{R}}$ define an antiequivalence of the category of smooth (σ-compact) manifolds with a certain category of $\mathbb{R}$-algebras (namely, $\mathbb{R}$-algebras isomorphic to some $\mathcal{C}^\infty(X)$). And the former theorem shows that the category of compact Hausdorff spaces is antiequivalent to a certain category of $\mathbb{R}$-algebras.

where h vanishes exactly at $X - U$. Now, this epimorphism $\mathcal{C}(X)_S \to \mathcal{C}(U)$ is injective:

If a fraction $\frac{a}{s}$ vanishes on U, then $ha = 0$, so that $\frac{a}{s} = 0$ in $\mathcal{C}(X)_S$. q.e.d.

This argument also proves the analogous result for smooth manifolds:

Theorem: *Let U be an open set in a smooth manifold X. Any smooth function f on U is a quotient, $f = g/h$, of two smooth global functions, $z(h) = X - U$. Hence $\mathcal{C}^\infty(U) = \mathcal{C}^\infty(X)_S$, where S is the multiplicative system of all smooth functions without zeros in U.*

9.2 Ordinary Differential Equations

Definition: If D is a continuous vector field on a smooth manifold X, a derivable curve $\sigma\colon I \to X$ is an **integral curve** of D if the tangent vector $T_t \in T_{\sigma(t)}X$ coincides with $D_{\sigma(t)}$ at any $t \in I$.

In a coordinate neighborhood we have $D = \sum_i f_i(x_1, \ldots, x_n)\partial_i$, where the functions f_i are continuous, and $\sigma(t) = (x_1(t), \ldots, x_n(t))$ is an integral curve if the functions $x_i(t)$ define a solution of the system of differential equations $\sigma'(t) = f(\sigma(t))$, where $f = (f_1, \ldots, f_n)$,

$$\begin{cases} x_1' = f_1(x_1, \ldots, x_n) \\ \quad \cdots\cdots\cdots\cdots \\ x_n' = f_n(x_1, \ldots, x_n) \end{cases}$$

In particular σ' is continuous, and if f is $\mathcal{C}^m$, so is σ', and the solutions are $\mathcal{C}^{m+1}$.

If we impose an initial condition $\sigma(t_0) = x$, the equality $\sigma' = f(\sigma)$ states that

$$\sigma(t) = x + \int_{t_0}^{t} f(\sigma(t))\mathrm{d}t.$$

Lemma: *Any contractive map T of a complete metric space E (i.e., there is a constant $k < 1$ such that $d(T\sigma, T\varphi) \leq kd(\sigma, \varphi)$ for all $\sigma, \varphi \in E$) has a unique fixed point.*

Proof: If $\sigma_0 \in E$, then $\sigma_n = T^n(\sigma_0)$ is a Cauchy sequence,

$$d(\sigma_n, \sigma_{n+1}) \leq kd(\sigma_{n-1}, \sigma_n) \leq \ldots \leq k^n d(\sigma_0, \sigma_1) = k^n c,$$
$$d(\sigma_n, \sigma_{n+m}) \leq c(k^n + \ldots + k^{n+m-1}) \leq c\tfrac{k^n}{1-k} < \varepsilon \quad \text{when } n \gg 0,$$

and $\sigma = \lim \sigma_n$ is a fixed point, $T(\sigma) = \lim T(\sigma_n) = \lim \sigma_{n+1} = \sigma$.

There is no other fixed point since $d(\sigma, T\varphi) = d(T\sigma, T\varphi) < d(\sigma, \varphi)$.

Theorem: *If a vector field is $\mathcal{C}^1$, then through any point x passes an integral curve σ such that $\sigma(t_0) = x$, and it is unique (any two coincide on the common definition interval).*

Proof: If $X = \mathbb{R}^n$ and $\operatorname{supp} D$ is compact, the continuous functions $\partial f_i/\partial x_j$ have compact support and by the mean value theorem there is a constant k such that $|f(x) - f(y)| \leq k|x - y|$.

We put $I = [t_0 - \varepsilon, t_0 + \varepsilon]$ and we consider on $E = \mathcal{C}(I, \mathbb{R}^n)$ the maximum module norm.

Solutions are just fixed points of the following operator $T\colon E \to E$,

$$T(\sigma) = x + \int_{t_0}^{t} f\big(\sigma(t)\big)\mathrm{d}t,$$
$$\|T(\varphi) - T(\sigma)\| = \sup_{t\in I} \left| \int_{t_0}^{t} [f(\varphi) - f(\sigma)]\, \mathrm{d}t \right| \leq \varepsilon k \sup_{t\in I} |\varphi(t) - \sigma(t)| \leq k\varepsilon \|\varphi - \sigma\|,$$

and it is contractive when $k\varepsilon < 1$. This proves the existence of integral curves, and that any two coinciding at some instant, coincide on a neighborhood.

Since this statement is local, it also holds in any smooth manifold X, and since the coincidence locus is closed, any two solutions coincide on the common definition interval.

Definition: Through any point $x \in X$ passes a maximal integral curve $\sigma_x \colon I_x \to X$, $\sigma_x(0) = x$. The **flow** τ of the vector field D is well-defined on the subspace $W = \{(t,x) \colon t \in I_x\} \subseteq \mathbb{R} \times X$,

$$\tau \colon W \longrightarrow X,\ \tau(t,x) = \tau_t(x) = \sigma_x(t).$$

We have $\tau_0(x) = x$, and $\tau_{t+s}(x)$ is an integral curve passing by $y = \tau_s(x)$ at $t = 0$, so that $I_x \subseteq I_y + s$. Since $x = \tau_{-s}(y)$, then $I_y \subseteq I_x - s$; hence $I_y + s = I_x$, and

$$\tau_t(\tau_s x) = \tau_{t+s}(x).$$

Lemma: *The flow of a $\mathcal{C}^1$ vector field is continuous on a neighborhood of* $0 \times X$.

Proof: Since it is a local question, we may assume that $\operatorname{supp} D$ is compact and $X = \mathbb{R}^n$.

We fix a compact $\Lambda \subset \mathbb{R}^n$ (a cube, ball,...) and we put $I = [-\varepsilon, \varepsilon]$.

We consider on $E = \mathcal{C}(I \times \Lambda, \mathbb{R}^n)$ the maximum module norm, and we repeat the argument of the previous theorem with the initial condition $\sigma(0,x) = x$,

$$T(\sigma) = x + \int_0^t f(\sigma(t,x))\mathrm{d}t,$$

$$\|T\varphi - T\sigma\| = \sup_{(t,x)\in I\times\Lambda} \left| \int_0^t [f(\varphi) - f(\sigma)]\,\mathrm{d}t \right| \leq k\varepsilon \|\varphi - \sigma\|.$$

When $k\varepsilon < 1$, the map T is contractive, and the fixed point provides a continuous family of solutions $\tau \colon (-\varepsilon, \varepsilon) \times \Lambda \to \mathbb{R}^n$, such that $\tau(0,x) = x$.

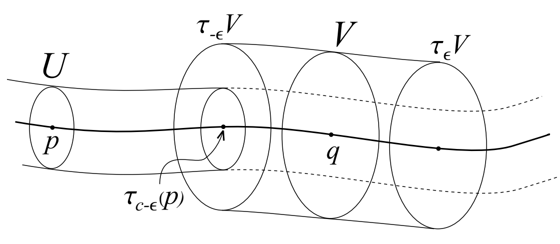
Theorem: *If a vector field is $\mathcal{C}^1$, then $W \subseteq \mathbb{R} \times X$ is open, and τ is continuous.*

Proof: Let us show that τ is defined and it is continuous on a neighborhood of any point of W. Otherwise, we fix a point $p \in X$ and the first positive instant $c \in I_p$ where it is false (the negative case being analogous), and we put $q = \tau_c(p)$.

By the above lemma, τ is continuous on a neighborhood $(-2\varepsilon, 2\varepsilon) \times V$ of $(0,q)$.

On a neighborhood U of p, the map $\tau_{c-\varepsilon} \colon U \to X$ is continuous, and we may assume that $\tau_{c-\varepsilon}(U) \subseteq \tau_{-\varepsilon}(V)$; hence $(c-\varepsilon, c+\varepsilon) \times U \subset W$, and τ is continuous on this open set (against the choice of c) since $\tau_c = \tau_\varepsilon \tau_{c-\varepsilon} \colon U \to V$ is continuous, and

$$\tau(t,x) = \tau(t-c, \tau_c(x)).$$



Lemma: *The flow of a $\mathcal{C}^{m+1}$ vector field, $m < \infty$, is $\mathcal{C}^m$ on a neighborhood of* $0 \times X$.

Proof: Since it is a local question, we may assume that $\operatorname{supp} D$ is compact and $X = \mathbb{R}^n$.

Let $\Lambda \subset \mathbb{R}^n$ be an open ball of radius $r \gg 0$, $I = [-\varepsilon, \varepsilon]$, and let E be the complete metric space of all $\mathcal{C}^m$ maps $\sigma\colon I \times \Lambda \to \mathbb{R}^n$ such that

$$\|\sigma\| = \sup_{\substack{(t,x)\in I\times\Lambda \\ 0\le|\alpha|\le m}} |D_\alpha \sigma| \le 2r.$$

Let us see the case $m = 1$. We put $x_0 = t$, and for any $i = 0, \ldots, n$ we have

$$D_i(f(\sigma_1, \ldots, \sigma_n)) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\sigma) \cdot \frac{\partial \sigma_j}{\partial x_i},$$

and, since the maps $\partial f/\partial x_j$ are $\mathcal{C}^1$, there is a constant k such that

1. $\|f(\sigma)\| \le 2kr$, when $\|\sigma\| \le 2r$.
2. $\|f(\varphi) - f(\sigma)\| \le k\, \|\varphi - \sigma\|$, when $\|\varphi\|, \|\sigma\| \le 2r$.

Now we repeat the argument of the contractive map, and we put $\varepsilon = \frac{1}{2k}$,

$$\|T(\sigma)\| \le r + \left\| \int_0^t f(\sigma)\, \mathrm{d}t \right\| \le r + \int_0^t \|f(\sigma)\|\, \mathrm{d}t \le r + 2rk\varepsilon = 2r,$$

$$\|T\varphi - T\sigma\| = \left\| \int_0^t [f(\varphi) - f(\sigma)]\, \mathrm{d}t \right\| \le \int_0^t \|f(\varphi) - f(\sigma)\|\, \mathrm{d}t \le k\varepsilon \|\varphi - \sigma\| = \tfrac{1}{2}\|\varphi - \sigma\|.$$

In general, when $m > 1$, we have that $D_\alpha(f(\sigma))$ is a polynomial on the derivatives of σ of order $\le m$ with coefficients derivatives of f of order $\le m$ (which are $\mathcal{C}^1$-functions with compact support), and there is also a constant k with properties 1 and 2.

Theorem: *If a vector field is $\mathcal{C}^{m+1}$, $1 \le m \le \infty$, then $W \subseteq \mathbb{R} \times X$ is open and the flow is $\mathcal{C}^m$.*

Proof: We may repeat the argument of the continuous case.

Corollary: *Tangent vector fields with compact support are* ***complete****, $W = \mathbb{R} \times X$.*

Proof: If $D_x = 0$, the constant map $x\colon \mathbb{R} \to X$ is an integral curve, and $I_x = \mathbb{R}$.

If $D = 0$ outside a compact set, W being open, then $(-2\varepsilon, 2\varepsilon) \times X \subset W$ for some $\varepsilon > 0$, and any integral curve may be extended for a time ε; hence $I_x = \mathbb{R}$ at any point x.

9.2.1 Uniparametric Groups and Lie Derivative

Definitions: A smooth map $\tau\colon \mathbb{R} \times X \to X$, $(t, x) \mapsto \tau_t(x)$, is a **uniparametric group** if

1. $\tau_0(x) = x$.
2. $\tau_t(\tau_s x) = \tau_{t+s}(x)$.

(It is a smooth action of the group $\mathbb{R}$ on X). If it is only defined on an open neighborhood of $0 \times X$, intersecting any line $\mathbb{R} \times x$ in an interval, we say that it is a **local** uniparametric group (condition 2 is assumed only whenever both terms are defined).

The **infinitesimal generator** of τ is the vector field D such that D_p is the tangent vector to the curve $\tau_t(p)$ at the instant $t = 0$, so that it is defined by the derivation

$$Df = \left.\frac{\partial(f \circ \tau)}{\partial t}\right|_{t=0} \quad , \quad D_p f = (Df)(p) = \lim_{t\to 0} \frac{f(\tau_t p) - f(p)}{t}.$$

Hence, the flow of any vector field D is a local uniparametric group with infinitesimal generator D. As well, any local uniparametric group is (an open subset of) the flow of the infinitesimal generator D, since the curve $\sigma(t) = \tau_t(p)$ is an integral curve of the vector field D passing through p at the instant $t = 0$. In fact, since $\sigma(t+\varepsilon) = \tau_\varepsilon(\sigma(t))$, the tangent vector to σ at the instant t is $D_{\sigma(t)}$. *Vector fields correspond to (maximal) local uniparametric groups and, on compact manifolds, to uniparametric groups.*

Local Classification of Vector Fields: *If $D_p \neq 0$, then we have $D = \frac{\partial}{\partial u_1}$ in some local coordinate system $(u_1, \ldots, u_n)$ at p.*

Proof: Since it is a local question, we may assume that p is the origin of $\mathbb{R}^n$ and that $\operatorname{supp} D$ is compact. We may also assume that D_p is not tangent to the hyperplane H of equation $x_1 = 0$. Now, the smooth map

$$\mathbb{R} \times H \longrightarrow \mathbb{R}^n, \ (t, x) \mapsto \tau_t(x),$$

transforms integral curves of $\frac{\partial}{\partial t}$ into integral curves of D; hence it transforms $\frac{\partial}{\partial t}$ into D.

We conclude since this map is a local diffeomorphism at p: the tangent map at p transforms $(\frac{\partial}{\partial t})_p$ into D_p, and it is the identity on T_pH.

Definition: The **Lie bracket** of two vector fields D, D' on a manifold X is the vector field defined by the derivation (check that so it is)

$$[D, D'] = D \circ D' - D' \circ D$$
$$\left[\textstyle\sum_i f_i \partial_i, \sum_i h_i \partial_i\right] = \textstyle\sum_{ij} (f_j \partial_j h_i - h_j \partial_j f_i) \partial_i$$

1. $[D, D'] = -[D', D]$, and therefore $[D, D] = 0$.

2. $[D, fD'] = (Df)D' + f[D, D']$.

3. $[[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2] = 0$ (**Jacobi's Identity**)

Definition: The **Lie derivative** of a tensor field T with a vector field D of flow $\{\tau_t\}$ is the following tensor field (a calculation in coordinates shows that it is $\mathcal{C}^\infty$, since so is the flow)

$$(D^L T)_x = \lim_{t \to 0} \frac{(\tau_t^* T)_x - T_x}{t} \cdot$$

Theorem: *If $D^L T = 0$, then the tensor field T is invariant, $\tau_t^* T = T$.*

If $D^L T = fT$, then $\tau_t^* T$ and T are proportional, $\langle \tau_t^* T \rangle = \langle T \rangle$.

Proof: Let us consider in $T_x X$ the curve $\sigma(t) = (\tau_t^* T)_x$. The tangent vector at $t = 0$ is $(D^L T)_x$, and the tangent vector at any instant t is $\tau_t^* \left[(D^L T)_{\tau_t x} \right]$ since

$$\sigma(t+\varepsilon) = (\tau_{t+\varepsilon}^* T)_x = \tau_t^* \left[(\tau_\varepsilon^* T)_{\tau_t x} \right].$$

If $D^L T = 0$, then $\sigma'(t) = 0$, and the curve is constant, $\tau_t^* T = T$.

If $D^L T = fT$, then $\sigma'(t) = h(t)\sigma(t)$, where $h(t) = f(\tau_t x)$. Hence $\tau_t^* T$ is proportional to T,

$$\sigma(t) = e^{H(t)} \sigma(0), \quad \text{where } H(t) = \textstyle\int_0^t h(t) \mathrm{d}t.$$

q.e.d.

1. $D^L f = Df$.

2. $D^L (T + T') = D^L T + D^L T'$.

3. $D^L(T\otimes T') = (D^LT)\otimes T' + T\otimes(D^LT')$.

   $D^L(T\otimes T') = \lim_{t\to 0}\frac{\tau_t^*T\otimes\tau_t^*T' - T\otimes T'}{t} = \lim_{t\to 0}\frac{\tau_t^*T\otimes\tau_t^*T' - T\otimes\tau_t^*T' + T\otimes\tau_t^*T' - T\otimes T'}{t}$

   $= \lim_{t\to 0}\frac{\tau_t^*T - T}{t}\otimes\tau_t^*T' + \lim_{t\to 0} T\otimes\frac{\tau_t^*T' - T'}{t} = (D^LT)\otimes T' + T\otimes(D^LT')$.

4. $D^L(\omega_p\wedge\omega_q) = (D^L\omega_p)\wedge\omega_q + \omega_p\wedge(D^L\omega_q)$.

5. $D^L(C_i^jT) = C_i^j(D^LT)$.

6. $(D^LT)(D_1,\ldots,\omega_q) = D(T(D_1,\ldots,\omega_q)) - T(D^LD_1,\ldots,\omega_q) - \ldots - T(D_1,\ldots,D^L\omega_q)$.

   Just derive $T(D_1,\ldots,\omega_q) = C_1^1\ldots C_1^1(D_1\otimes\ldots\otimes D_p\otimes T\otimes\omega_1\otimes\ldots\otimes\omega_q)$.

7. $D^L\omega = D\circ\omega - \omega\circ D^L$; that is to say, $(D^L\omega)(D') = D(\omega(D')) - \omega(D^LD')$.

8. $D^L(\mathrm{d}f) = \mathrm{d}(Df)$.

   $D^L(\mathrm{d}f) = \lim_{t\to 0}\frac{\tau_t^*\mathrm{d}f - \mathrm{d}f}{t} = \lim_{t\to 0}\frac{\mathrm{d}\tau_t^*f - \mathrm{d}f}{t} = \lim_{t\to 0}\mathrm{d}\Big(\frac{\tau_t^*f - f}{t}\Big) = \mathrm{d}\Big(\lim_{t\to 0}\frac{\tau_t^*f - f}{t}\Big) = \mathrm{d}(Df)$.

9. $D^LD' = [D, D']$.

   $(D^LD')f = (\mathrm{d}f)(D^LD') = D((\mathrm{d}f)(D')) - (D^L(\mathrm{d}f))(D')$

   $= D(D'f) - (\mathrm{d}(Df))(D') = D(D'f) - D'(Df) = [D, D']f$.

10. **Jacobi's Identity**: $D^L[D_1, D_2] = [D^LD_1, D_2] + [D_1, D^LD_2]$.

11. $(D_1 + D_2)^LT = D_1^LT + D_2^LT$.

    $(D_1 + D_2)^LD = [D_1 + D_2, D] = [D_1, D] + [D_2, D]$.

    $(D_1+D_2)^L\omega = (D_1+D_2)\circ\omega - \omega\circ(D_1+D_2)^L = (D_1+D_2)\circ\omega - \omega\circ(D_1^L + D_2^L) = D_1^L\omega + D_2^L\omega$.

    We conclude since both $(D_1 + D_2)^L$ and $D_1^L + D_2^L$ derive tensor products.

12. $[D_1, D_2]^L = [D_1^L, D_2^L]$ on tensor fields: $[D_1, D_2]^LT = D_1^L(D_2^LT) - D_2^L(D_1^LT)$.

    When T is a vector field, it is just Jacobi's identity. Now, when T is a 1-form, it follows from property 7, and we conclude since both $[D_1, D_2]^L$ and $[D_1^L, D_2^L]$ derive tensor products.

Corollary: $[D, \bar{D}] = 0$ *if and only if* $\tau_t\bar{\tau}_s = \bar{\tau}_s\tau_t$; ($D, \bar{D}$ complete vector fields).

Proof: If $D^L\bar{D} = [D, \bar{D}] = 0$, then $\tau_t\bar{D} = \bar{D}$, and τ_t transforms integral curves of $\bar{D}$ into integral curves of $\bar{D}$. The converse is obvious.

9.2.2 Pfaff Systems

Definitions: Let $\mathcal{O}$ be the local ring of germs at a point x of smooth functions on a smooth manifold, and let $\mathcal{T}$ be the module of germs at x of vector fields. A **distribution** of rank r is a free submodule $\mathcal{D} = \langle D_1,\ldots,D_r\rangle \subseteq \mathcal{T}$ generated by r vector fields, linearly independent at x.

Once we fix representants of the germs D_i, they are linearly independent on a neighborhood U of x, and they define at any point $y\in U$ a vector subspace of dimension r,

$$\Delta_y = \{\lambda_1D_{1,y} + \ldots + \lambda_rD_{r,y}\} \subseteq T_yX,$$

so that the distribution may be viewed as the germ at x of a smooth family of r-planes.

We say that $\mathcal{D}$ is **integrable** if $\mathcal{D} = \langle\partial_1,\ldots,\partial_r\rangle$ in some local coordinate system.

Let Ω be the $\mathcal{O}$-module of germs of 1-forms. A **Pfaff system** of rank r is a free submodule $\mathcal{P} = \langle \omega_1, \ldots, \omega_r \rangle \subseteq \Omega$ generated by r forms, linearly independent at x, so that it may be viewed as the germ at x of a smooth family of r-planes,

$$P_y = \{\lambda_1 \omega_{1,y} + \ldots + \lambda_r \omega_{r,y}\} \subseteq T^*_y X,$$

and its incident $\mathcal{P}^{\mathrm{o}}$ is a distribution of rank $n-r$. Moreover, the incident $\mathcal{D}^{\mathrm{o}}$ of any distribution $\mathcal{D}$ is a Pfaff system, and we have $(\mathcal{D}^{\mathrm{o}})^{\mathrm{o}} = \mathcal{D}$ and $(\mathcal{P}^{\mathrm{o}})^{\mathrm{o}} = \mathcal{P}$.

A Pfaff system $\mathcal{P}$ is **projectable** to a subring $B \subset \mathcal{O}$ if it is generated by 1-forms $\sum_i f_i \mathrm{d}h_i$; with $f_i, h_i \in B$, and **integrable** if $\mathcal{P} = \langle \mathrm{d}x_1, \ldots, \mathrm{d}x_r \rangle$ in a local coordinate system $x_1, \ldots, x_n$.

Lemma: *If $D^L\mathcal{P} \subseteq \mathcal{P}$, then we have $\tau_t(P_y) = P_{\tau_t(y)}$ in a neighborhood of x.*

Proof: Let $\mathcal{P} = \langle \omega_1, \ldots, \omega_r \rangle$. If $D^L\mathcal{P} \subseteq \mathcal{P}$, then $D^L(\omega_1 \wedge \ldots \wedge \omega_r) = f\omega_1 \wedge \ldots \wedge \omega_r$ and $\Lambda^r\mathcal{P}$ is invariant under the flow $\{\tau_t\}$ of D by the former theorem; hence so is $\mathcal{P}$.

Definition: The **characteristic system** of $\mathcal{P}$ is formed by the incident vector fields D such that $D^L\mathcal{P} \subseteq \mathcal{P}$ (i.e., $i_D\omega = 0$, $i_D\mathrm{d}\omega \in \mathcal{P}$, for all $\omega \in \mathcal{P}$).

Projection Theorem: *If $D_x \neq 0$, a Pfaff system $\mathcal{P}$ is projectable to the ring of first integrals of the field D if and only if D is in the characteristic system of $\mathcal{P}$.*

Proof: If $D_x \neq 0$, by the local classification of vector fields, on a small neighborhood U of x we have a projection $\pi\colon U \to V$ admitting a smooth section $\sigma\colon V \to U$ passing through x, and first integrals of D are just smooth functions on V. If $\mathcal{P}$ is incident to D and $D^L\mathcal{P} \subseteq \mathcal{P}$, let us see that $\mathcal{P} = \langle \omega_1, \ldots, \omega_r \rangle$ coincides with $\langle \pi^*\sigma^*\omega_1, \ldots, \pi^*\sigma^*\omega_r \rangle$.

Both systems coincide at the points of the section, since both coincide on the hyperplane tangent to the section and both vanish on the supplement defined by D.

Since both are invariant under the flow of D, they coincide on a neighborhood of x.

Conversely, if $\mathcal{P}$ is projectable, then the vector field D is in the characteristic system since, whenever $Df_i = Dh_i = 0$, we have

$$0 = \Big(\textstyle\sum_i f_i \mathrm{d}h_i\Big)(D),$$
$$0 = D^L\Big(\textstyle\sum_i f_i \mathrm{d}h_i\Big).$$

Definition: A distribution $\mathcal{D}$ is **involutive** when $D_1, D_2 \in \mathcal{D} \Rightarrow [D_1, D_2] \in \mathcal{D}$.

If $\mathcal{P} = \mathcal{D}^{\mathrm{o}}$ is the incident Pfaff system, this condition states that the characteristic system of $\mathcal{P}$ is just $\mathcal{P}^{\mathrm{o}}$, since $(D_1^L\omega)(D_2) = D_1(\omega(D_2)) - \omega([D_1, D_2])$.

Theorem: *Let $\mathcal{P}$ be a Pfaff system. The following conditions are equivalent,*

1. *$\mathcal{P}$ is integrable.*
2. *The ideal generated by $\mathcal{P}$ in the algebra of differential forms is stable under the exterior differential, $\mathrm{d}\mathcal{P} \subseteq \mathcal{P} \wedge \Omega^\bullet$.*
3. *The distribution $\mathcal{P}^{\mathrm{o}}$ is involutive (the characteristic system of $\mathcal{P}$ is $\mathcal{P}^{\mathrm{o}}$).*

Proof: $(1 \Rightarrow 2)$ If $\mathcal{P} = \langle \mathrm{d}x_1, \ldots, \mathrm{d}x_r \rangle$, then $\mathrm{d}(\sum_i f_i \mathrm{d}x_i) = \sum_i \mathrm{d}f_i \wedge \mathrm{d}x_i \in \mathcal{P} \wedge \Omega^\bullet$.

$(2 \Rightarrow 3)$ Let $\mathcal{P} = \langle \omega_1, \ldots, \omega_r \rangle$. If $D \in \mathcal{P}^{\mathrm{o}}$, and $\omega \in \mathcal{P}$, then $D^L\omega \in \mathcal{P}$,

$$D^L\omega = i_D\mathrm{d}\omega + \mathrm{d}i_D\omega = i_D(\textstyle\sum_i \theta_i \wedge \omega_i) = \sum_i (i_D\theta_i)\omega_i - (i_D\omega_i)\theta_i = \sum_i (i_D\theta_i)\omega_i.$$

(3 ⇒ 1) By induction on $n = \dim X$. If the incident of $\mathcal{P}$ is involutive, it coincides with the characteristic system of $\mathcal{P}$, and by the projection theorem there is a projection $\pi\colon U \to V$ and a Pfaff system $\mathcal{P}' = \langle \omega'_1, \ldots, \omega'_r \rangle$ on V such that $\mathcal{P} = \langle \pi^*\omega'_1, \ldots, \pi^*\omega'_r \rangle$.

Moreover, any vector field D' on V is the projection of some vector field D on U.

If $D'_1, D'_2 \in (\mathcal{P}')^{\mathrm{o}}$ are the projections of $D_1, D_2 \in \mathcal{P}^{\mathrm{o}}$, then $[D'_1, D'_2]$ is the projection of $[D_1, D_2] \in \mathcal{P}^{\mathrm{o}}$, and $[D'_1, D'_2] \in (\mathcal{P}')^{\mathrm{o}}$. By induction $\mathcal{P}'$ is integrable; hence so is $\mathcal{P}$.

Frobënius Theorem: *Involutive distributions are integrable.*

Proof: If $\mathcal{D} = (\mathcal{D}^{\mathrm{o}})^{\mathrm{o}}$ is involutive, then $\mathcal{D}^{\mathrm{o}} = \langle \mathrm{d}x_{r+1}, \ldots, \mathrm{d}x_n \rangle$, and $\mathcal{D} = \langle \partial_{x_1}, \ldots, \partial_{x_r} \rangle$.

Corollary: *A 1-form germ ω is $\omega = f\mathrm{d}h$ if and only if $\omega \wedge \mathrm{d}\omega = 0$.* (Assuming $\omega_x \neq 0$).

Proof: If $\omega_x \neq 0$ and $\omega \wedge \mathrm{d}\omega = 0$, then $\mathrm{d}\omega = \omega \wedge \omega'$ for some germ ω'.

Note: Given a matrix $\big(f_{ij}\big) \in M_{m\times n}(\mathcal{O})$, if the vector spaces $\{(a_1, \ldots, a_n) \in \mathbb{R}^n \colon \sum_j f_{ij}(x)a_j = 0\}$ have constant dimension r in a neighborhood of the given point $x \in \mathbb{R}^n$, then Cramer's rule shows that the solutions of the linear system $\sum_j f_{ij}X_j = 0$ define a free submodule of $\mathcal{O}^n$ of rank r, generated by a basis linearly independent at x.

Definition: A germ $\omega \in \Omega$ is **regular** of class m if it does not vanish at x and the vector subspaces $\Delta_y = \{D \in T_yX \colon i_D\omega = 0,\ i_D\mathrm{d}\omega = 0\}$ have constant codimension m; hence define a distribution $\mathcal{D}_\omega$, necessarily integrable: If $D, \bar{D} \in \mathcal{D}_\omega$, then

$$\omega([D, \bar{D}]) = D(\omega(\bar{D})) - (D^L\omega)(\bar{D}) = 0,$$
$$[D, \bar{D}]^L\omega = D^L\bar{D}^L\omega - \bar{D}^L D^L\omega = 0.$$

According to Frobënius theorem, in a local coordinate system $\mathcal{D}_\omega = \langle \partial_{m+1}, \ldots, \partial_n \rangle$, and ω is projectable to a germ ω' in dimension m such that $\mathcal{D}_{\omega'} = 0$.

Darboux's Theorem: *Let $\omega \in \Omega$ be a regular germ of class m.*

If $m = 2k+1$ is odd, there are local coordinates $(z, x_1, \ldots, x_k, p_1, \ldots, p_k, \ldots)$ where

$$\omega = \mathrm{d}z - p_1\mathrm{d}x_1 - \ldots - p_k\mathrm{d}x_k.$$

If $m = 2k$ is even, there are local coordinates $(x_1, \ldots, x_k, p_1, \ldots, p_k, \ldots)$ where

$$\omega = p_1\mathrm{d}x_1 + \ldots + p_k\mathrm{d}x_k.$$

Proof: We may assume that $\dim X = m$ and that $\mathcal{D}_\omega = 0$.

We proceed by induction on m, and the case $m = 1$ is trivial.

If m is odd, then $\mathrm{d}\omega$ has non null radical, defining a distribution $\langle Z \rangle$ of rank 1 transversal to $\omega = 0$. Dividing Z by $\omega(Z)$, we may assume that $\omega(Z) = 1$.

Let us fix a local coordinate system where $Z = \partial_z$.

Adding to z a first integral of Z we may assume that $\theta = \mathrm{d}z - \omega$ does not vanish at x.

Now $i_Z\theta = 0$, $i_Z\mathrm{d}\theta = i_Z\mathrm{d}\omega = 0$, and θ projects onto a 1-form θ' in dimension $m-1$, and $\mathcal{D}_{\theta'} = 0$, since the radical of $\mathrm{d}\theta = \mathrm{d}\omega$ is generated by Z.

By induction $\theta' = p_1\mathrm{d}x_1 + \ldots + p_k\mathrm{d}x_k$ and we conclude.

If m is even, the radical of $\mathrm{d}\omega$ is null (otherwise the dimension is ≥ 2, and $\mathcal{D}_\omega \neq 0$), and vectors D such that $i_D\mathrm{d}\omega = \lambda\omega$ (hence $i_D\omega = 0$) define a distribution of rank 1, which is the characteristic system of $\langle \omega \rangle$.

Hence $\langle\omega\rangle$ is projectable, $\omega = p_1(\pi^*\omega')$, to dimension $m-1$.

Clearly p_1 does not vanish at x and ω' does not vanish at $\pi(x)$.

Moreover, if $i_{D'}\omega' = 0$, $i_{D'}\mathrm{d}\omega' = 0$, and we take a vector field D projecting onto D', we have that $i_D\omega = 0$, $i_D\mathrm{d}\omega \in \langle\omega\rangle$; hence D is in the characteristic system of $\langle\omega\rangle$, and $D' = 0$.

By induction $\omega' = \mathrm{d}x_1 + \bar{p}_2\mathrm{d}x_2 + \ldots + \bar{p}_k\mathrm{d}x_k$, and if we put $p_i = p_1\bar{p}_i$,

$$\omega = p_1\mathrm{d}x_1 + \ldots + p_k\mathrm{d}x_k.$$

Since the radical of $\mathrm{d}\omega = \mathrm{d}p_1\wedge\mathrm{d}x_1 + \ldots + \mathrm{d}p_k\wedge\mathrm{d}x_k$ is null, $\mathrm{d}p_1, \mathrm{d}x_1, \ldots, \mathrm{d}p_k, \mathrm{d}x_k$ are linearly independent at x.

9.3 Integration of Differential Forms

Definitions: The **volume forms** on a smooth manifold X of dimension n are the n-forms not vanishing at any point, and X is **orientable** if it admits a volume form.

Two volume forms, ω, ω' define the same **orientation** of X if $\omega' = f\omega$ for some smooth function $f > 0$, and orientations of X are just equivalence classes.

Proposition: *Let $\{U_i, [\omega_i]\}$ be an oriented open cover of X. If the orientations coincide on intersections, $[\omega_i|_{U_i\cap U_j}] = [\omega_j|_{U_i\cap U_j}]$, then there is a unique orientation $[\omega]$ of X inducing on each open set U_i the given orientation, $[\omega|_{U_i}] = [\omega_i]$.*

Proof: Just take $\omega = \sum_i \phi_i\omega_i$, where $\{\phi_i\}$ is a partition of unity subordinate to $\{U_i\}$.

Definition: A closed set $\Omega \subseteq X$ is a **manifold with boundary** when

1. The boundary $\partial\Omega$ is void or a smooth submanifold of dimension $n-1$.

2. $\partial \overset{\circ}{\Omega} = \partial\Omega$.

Lemma: *If $p \in \partial\Omega$, in a coordinate neighborhood $(U; x_1, \ldots, x_n)$ we have that*

$$U \cap \Omega = \{p \in U : x_1(p) \le 0\}.$$

Proof: Let U be a coordinate neighborhood where the equation of $U \cap \partial\Omega$ is $x_1 = 0$, and $U - (U \cap \partial\Omega)$ has two connected components,

$$U - (U \cap \partial\Omega) = U_+ \cup U_-,$$

of equations $x_1 > 0$, and $x_1 < 0$. Since $X - \partial\Omega = \overset{\circ}{\Omega} \cup \Omega^c$, intersecting with U,

$$U_+ \cup U_- = U - (U \cap \partial\Omega) = (U \cap \overset{\circ}{\Omega}) \cup (U \cap \Omega^c).$$

Hence $U_- = U \cap \overset{\circ}{\Omega}$, and in this case $U \cap \Omega$ is $x_1 \le 0$, or $U_+ = U \cap \overset{\circ}{\Omega}$, and in this case $U \cap \Omega$ is $x_1 \ge 0$, and we change the sign of x_1.

Definitions: A vector $D_p \in T_pX$, $p \in \partial\Omega$, points **outside** of the manifold with boundary Ω if, in the coordinate system of the above lemma,

$$D_p = \lambda_1\partial_1 + \ldots + \lambda_n\partial_n,\ \lambda_1 > 0\,;$$

i.e., for any curve $\sigma: I \to X$, tangent to D_p at $t = 0$, there is $\varepsilon > 0$ such that $\sigma(-\varepsilon, 0) \subseteq \Omega$, $\sigma(0, \varepsilon) \subseteq X - \Omega$.

Now each orientation $[\omega]$ of X induces an orientation on the boundary $\partial\Omega$, considering on $T_p(\partial\Omega)$ the orientation defined by $i_{D_p}\omega_p = \omega_p(D_p, \ldots)$, where D_p points outside of Ω.

In the local coordinate system of the lemma, if $[\omega] = [\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n]$, then the orientation induced on the boundary is $[\omega] = [\mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n]$, so that it does not depend on the vector D_p and, on a neighborhood of any point, orientations are defined by a differential form; hence we obtain an intrinsic orientation on $\partial\Omega$.

Let ω be a smooth differential n-form with compact support on an oriented smooth manifold X of dimension n. We shall always consider local coordinate systems $(u_1, \ldots, u_n)$ where the fixed orientation is $[\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n]$.

If the support of ω is contained in a coordinate neighborhood $(U; u_1, \ldots, u_n)$, then

$$\omega = f(u_1, \ldots, u_n)\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n,$$

where f is a smooth function with compact support on an open subset of $\mathbb{R}^n$, and we put

$$\int_X \omega := \int_{\mathbb{R}^n} f\mathrm{d}u_1 \ldots \mathrm{d}u_n.$$

This definition is intrinsic (does not depend on coordinates, nor on U) because, if $(x_1, \ldots, x_n)$ is another coordinate system and $u_i = h_i(x_1, \ldots, x_n) = h_i(x)$, we have

$$\omega = f(u_1, \ldots, u_n)\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n = f(h_1(x), \ldots, h_n(x))J\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$$

(where the jacobian $J = \det\big(\frac{\partial h_i}{\partial x_j}\big)$ is positive since $\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n$ and $\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ define the same orientation) and the change of variables formula states that

$$\int_{\mathbb{R}^n} f\mathrm{d}u_1 \ldots \mathrm{d}u_n = \int_{\mathbb{R}^n} f(h_1(x), \ldots, h_n(x))J\mathrm{d}x_1 \ldots \mathrm{d}x_n \ .$$

In general we consider a partition of unity $\{\phi_i\}$ subordinate to a cover $X = \bigcup_i U_i$ by coordinate open sets, and we put

$$\int_X \omega = \sum\nolimits_i \int_X \phi_i\omega,$$

where the sum is finite since the support of ω is compact and the family $\{\operatorname{supp}\phi_i\}$ is locally finite. This definition does not depend on the cover nor the partition of unity:

In fact, if $\{\varphi_j\}$ is a partition of unity subordinate to another cover $X = \bigcup_j V_j$, we have

$$\sum_i \int_X \phi_i\omega = \sum_i \int_X \sum_j \phi_i\varphi_j\omega = \sum_{i,j} \int_X \phi_i\varphi_j\omega = \sum_j \int_X \sum_i \varphi_j\phi_i\omega = \sum_j \int_X \varphi_j\omega.$$

Even if we have assumed that ω is smooth, these definitions are full sense when the local expression of ω in any local coordinate system is $f(u_1, \ldots, u_n)\mathrm{d}u_1 \wedge \ldots \wedge \mathrm{d}u_n$, where f is integrable; for example when ω is a continuous n-form. Now, *any submanifold $Y \subset \mathbb{R}^n$ of codimension $r \geq 1$ has null measure*, because locally it is $U \cap Y = \{x \in U \colon u_1(x) = \ldots = u_r(x) = 0\}$. Hence, we may define the integral of a smooth n-form ω on any manifold with boundary Ω by (the indicator function I_Ω vanishes outside of Ω, where the value is 1)

$$\int_\Omega \omega = \int_X I_\Omega\omega.$$

Stokes Theorem: $\displaystyle\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega$, *for any $(n-1)$-form ω of class $\mathcal{C}^1$ with compact support.*

Proof: Using partitions of unity, it is enough to show that any point $p \in X$ has a neighborhood U where the theorem holds for any form with compact support contained in U.

We distinguish 3 cases.

1. If $p \in \Omega^c$, we put $U = X - \Omega$, and $\int_\Omega \mathrm{d}\omega = 0 = \int_{\partial\Omega} \omega$ when $\operatorname{supp}\omega \subseteq U$.

2. If $p \in \overset{\circ}{\Omega}$, we may assume that $\mathbb{R}^n = X = \Omega = U$. In this case $\int_{\partial\Omega} \omega = \int_\emptyset \omega = 0$, and we may assume that $\omega = f\mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$, where the support of f is compact,

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}\mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_1}\mathrm{d}x_1 \right) \mathrm{d}x_2 \ldots \mathrm{d}x_n$$
$$= \int_{\mathbb{R}^{n-1}} \left(f(b, x_2, \ldots, x_n) - f(-a, x_2, \ldots, x_n) \right) \mathrm{d}x_2 \ldots \mathrm{d}x_n = 0.$$

3. If $p \in \partial\Omega$, we may assume that $\mathbb{R}^n = X = U$, and that Ω is the closed set $x_1 \leq 0$, and $\partial\Omega$ is the hyperplane $x_1 = 0$.

   If $\omega = f\mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_i} \ldots \wedge \mathrm{d}x_n$, the argument of case 2 shows that $\int_\Omega \mathrm{d}\omega = 0$, and $\int_{\partial\Omega} \omega = 0$, since $\mathrm{d}x_1$ vanishes on $\partial\Omega$. When $\omega = f\mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$,

$$\int_\Omega \mathrm{d}\omega = \int_\Omega \frac{\partial f}{\partial x_1}\mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial f}{\partial x_1}\mathrm{d}x_1 \right) \mathrm{d}x_2 \ldots \mathrm{d}x_n$$
$$= \int_{\mathbb{R}^{n-1}} f(0, x_2, \ldots, x_n)\, \mathrm{d}x_2 \ldots \mathrm{d}x_n = \int_{\partial\Omega} \omega.$$

Definitions: The **volume form** of an oriented riemannian manifold X is the smooth n-form $\mathrm{d}X$ such that $\mathrm{d}X(D_1, \ldots D_n) = 1$ for any direct orthonormal base (p. 69).

The integral of a function f on a compact manifold with boundary $\Omega \subseteq X$ is $\int_\Omega f\mathrm{d}X$, and the **volume** of Ω is $\int_\Omega \mathrm{d}X$.

The **divergence** of a vector field D is defined to be $D^L(\mathrm{d}_X) = (\operatorname{div} D)\mathrm{d}X$, and the **gradient** of a function f is the vector field $\operatorname{grad} f$ such that $Df = (\operatorname{grad} f) \cdot D$ for any vector field D.

Divergence Theorem: *The integral of the divergence of a vector field D of class $\mathcal{C}^1$ on a compact manifold with boundary Ω is the flux through the boundary $S = \partial\Omega$; that is to say, if N_p is the unique unitary orthogonal vector to T_pS pointing outside of Ω,*

$$\int_\Omega (\operatorname{div} D)\mathrm{d}X = \int_S (D \cdot N)\mathrm{d}S.$$

Proof: $\displaystyle\int_\Omega \operatorname{div} D = \int_\Omega D^L\mathrm{d}X = \int_\Omega (\mathrm{d}i_D + i_D\mathrm{d})\mathrm{d}X = \int_\Omega \mathrm{d}i_D(\mathrm{d}X) = \int_S i_D\mathrm{d}X$

and we have to show that the restriction of $i_D\mathrm{d}X$ to S coincides with $(D \cdot N)\mathrm{d}S$.

If $(D_2, \ldots, D_n)$ is a direct orthonormal base in T_pS, then $(N, D_2, \ldots, D_n)$ is a direct orthonormal base in T_pX, and

$$(i_D\mathrm{d}X)(D_2, \ldots, D_n) = \mathrm{d}X(D, D_2, \ldots, D_n) = \mathrm{d}X((D \cdot N)N + \ldots, D_2, \ldots, D_n)$$
$$= (D \cdot N)\mathrm{d}X(N, D_2, \ldots, D_n) = D \cdot N = (D \cdot N)\mathrm{d}S(D_2, \ldots, D_n).$$

Corollary: $\displaystyle\int_\Omega (\operatorname{div} \operatorname{grad} f)\mathrm{d}X = \int_S (Nf)\mathrm{d}S.$

9.3.1 Harmonic Functions

Definition: Let $U \subseteq \mathbb{R}^n$ be an open set, $n \geq 2$. A function $u \in \mathcal{C}^2(U)$ is **harmonic** if it has null laplacian:

$$0 = \Delta u = (\partial_1^2 + \ldots + \partial_n^2)u = \operatorname{div}(\operatorname{grad} u).$$

An harmonic function on $\mathbb{R}^n - \{p\}$ is $\|x-p\|^{2-n}$ when $n \geq 3$, and $\ln \|x-p\|$ when $n=2$.

Lemma: *If f is a continuous function on a closed ball $B = B(x,r)$, then*

$$\int_B f\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_0^r \left(\int_{S_\rho} f\,\mathrm{d}S_\rho \right) \mathrm{d}\rho\ , \quad S_\rho := S(x,\rho).$$

Proof: Let S be the unit sphere. We have a diffeomorphism $(0,r) \times S \simeq \overset{\circ}{B}(x,r) - \{x\}$, and by Fubini's theorem

$$\int_B f\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_0^r \left(\int_{S_\rho} i_N(f\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n) \right) \mathrm{d}\rho,$$

where N is the unitary orthogonal vector to the spheres S_ρ. We conclude because the restriction of $i_N(\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n)$ to S_ρ is just $\mathrm{d}S_\rho$.

Theorem: *Let u be an harmonic function on an open set $U \subseteq \mathbb{R}^n$. The value $u(p)$ at any point $p \in U$ is the mean value on any closed ball $B_r \subset U$ of radius r with center at p, and the mean value on the sphere $S_r = \partial B_r$:*

$$u(p) = \frac{1}{\mathrm{Vol}\, S_r} \int_{S_r} u\mathrm{d}S_r = \frac{1}{\mathrm{Vol}\, B_r} \int_{B_r} u\mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Proof: Let S be the unit sphere and consider the function $M(r) = \int_S u(p+rx)\mathrm{d}S$.

By the differentiation rule under the integral sign and the divergence theorem,

$$M'(r) = \int_S \partial_r\big(u(p+rx)\big)\mathrm{d}S = \int_{S_r} (Nu)\frac{\mathrm{d}S_r}{r^{n-1}} = \frac{1}{r^{n-1}} \int_{B_r} (\Delta u)\mathrm{d}x_1 \dots \mathrm{d}x_n = 0.$$

Hence, $M(r)$ is constant. Now, $M(r) \to u(p)(\mathrm{Vol}\, S)$ when $r \to 0$, and we conclude that

$$\frac{1}{\mathrm{Vol}\, S_r} \int_{S_r} u\mathrm{d}S_r = \frac{1}{\mathrm{Vol}\, S} \int_S u(p+rx)\mathrm{d}S = u(p).$$

Now, u and the constant function $u(p)$ have the same integral on any sphere S_ρ, $0 < \rho \leq r$; hence, by the lemma

$$u(p)(\mathrm{Vol}\, B_r) = \int_{B_r} u(p)\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_{B_r} u\mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Corollary: *If an harmonic function u on a connected open set $U \subseteq \mathbb{R}^n$ attains a maximum (or a minimum), then it is constant.*

Proof: The points where u attains the maximum M form an open set because if u is not the constant M on a ball $B \subset U$, then $\int_B u\mathrm{d}m < M$.

Maximum Principle: *Let $U \subset \mathbb{R}^n$ be an open set. If $\bar{U}$ is compact and $u \in \mathcal{C}(\bar{U})$ is harmonic on U, then the maximum and minimum of u are attained at some points of ∂U.*

Corollary: *Let $U \subset \mathbb{R}^n$ be a bounded open set. If $u \in \mathcal{C}(\bar{U})$ is harmonic on U and it vanishes on ∂U, then $u = 0$.*

Corollary: *If an harmonic function u on $\mathbb{R}^n$ vanishes at infinity (for any $\varepsilon > 0$ there is a compact set K such that $|u(x)| < \varepsilon$, $\forall x \in \mathbb{R}^n - K$), then $u = 0$.*

Proof: Let S_r be the sphere of radius r with center at a given point $p \in \mathbb{R}^n$. When $r \gg 0$, we have $|u(x)| < \varepsilon$, $\forall x \in S_r$; hence $|u(p)| < \varepsilon$ by the theorem.

Theorem: *A continuous function u on an open set $U \subseteq \mathbb{R}^n$ is an harmonic function of class $\mathcal{C}^\infty$ when for any closed ball $B(p,r) \subset U$ we have*

$$u(p) = \frac{1}{\operatorname{Vol} S} \int_S u \mathrm{d}S \ , \qquad S = S(p,r).$$

Proof: Let $h(x) = \phi(\|x\|^2)$ be a smooth function with $\operatorname{supp} h \subseteq B(0,\varepsilon)$ and $\int_{\mathbb{R}^n} h \mathrm{d}x_1 \ldots \mathrm{d}x_n = 1$.

Let V be the open set of all points $x \in U$ such that $B(x,\varepsilon) \subseteq U$. On the spheres with centre at $x \in V$, the functions $u(y)h(y-x)$ and $u(x)h(y-x)$ have the same integral, so that

$$\int_{\mathbb{R}^n} u(y)h(y-x)\mathrm{d}y_1 \ldots \mathrm{d}y_n = \int_{\mathbb{R}^n} u(x)h(y-x)\mathrm{d}y_1 \ldots \mathrm{d}y_n = u(x) \int_{\mathbb{R}^n} h(y-x)\mathrm{d}y_1 \ldots \mathrm{d}y_n = u(x).$$

By the differentiation rule under the integral sign, the left hand side is a $\mathcal{C}^\infty$ function of x, because so is h. Hence $u \in \mathcal{C}^\infty(V)$.

Now, given a point $x \in U$, it is in V when ε is small; hence $u \in \mathcal{C}^\infty(U)$.

Finally, given a point $x \in U$ and a closed ball $B(x,r) \subset U$, in the proof of the above theorem we have seen that

$$\int_{B(x,r)} (\Delta u)\mathrm{d}x_1 \ldots \mathrm{d}x_n = r^{n-1}M'(r) \quad , \quad \text{where } M(r) = \int_S u(p+rx)\omega_S,$$

and by hypothesis $M(r)$ is constant.

Since Δu has null integral on any ball $B(x,r) \subset U$, we conclude that $(\Delta u)(x) = 0$.

Corollary: *Any harmonic function is of class $\mathcal{C}^\infty$.*

Corollary: *The family $\mathcal{H}(U)$ of all harmonic functions is closed in $\mathcal{C}(U)$.*

Proof: If $f \in \mathcal{C}(U)$ is the limit of a sequence $u_n \in \mathcal{H}(U)$, then for any closed ball $B(p,r) \subset U$ we have

$$f(p) = \lim u_n(p) = \lim \frac{1}{\operatorname{Vol} S} \int_S u_n \mathrm{d}S = \frac{1}{\operatorname{Vol} S} \int_S (\lim u_n)\mathrm{d}S = \frac{1}{\operatorname{Vol} S} \int_S f \mathrm{d}S.$$

9.3.2 De Rham Cohomology

Definitions: A differential p-form ω is **closed** if $\mathrm{d}\omega = 0$, and it is **exact** if $\omega = \mathrm{d}\omega'$ for some $(p-1)$-form ω'. Any exact form is closed since $\mathrm{dd}\omega' = 0$, and the p-th **De Rham** cohomology group of a smooth manifold X is the real vector space

$$H^p_{DR}(X) = \frac{\{\text{closed } p\text{-forms}\}}{\{\text{exact } p\text{-forms}\}}.$$

The cohomology ring of X is the anticommutative graded $\mathbb{R}$-algebra

$$H^\bullet_{DR}(X) = \textstyle\bigoplus_p H^p_{DR}(X), \ \ [\omega_p] \cdot [\omega_q] = [\omega_p \wedge \omega_q],$$

and any smooth map $f\colon X \to Y$ naturally induces a morphism of $\mathbb{R}$-algebras

$$f^*\colon H^\bullet_{DR}(Y) \longrightarrow H^\bullet_{DR}(X),\ \ f^*[\omega] = [f^*\omega].$$

Example: Let X be a compact oriented manifold X of dimension n. If $[\omega]$ is the orientation, then $\int_X \omega > 0$. Hence $H^n_{DR}(X) \neq 0$, because exact n-forms have null integral,

$$\int_X \mathrm{d}\omega' = \int_{\partial X} \omega' = \int_{\emptyset} \omega' = 0.$$

Definitions: Two smooth maps $f_0, f_1\colon X \to Y$ are **homotopic**, and we put $f_0 \equiv f_1$, if there is an interval $I = (-\varepsilon, 1+\varepsilon)$ and a smooth map $H\colon X \times I \to Y$ such that

$$f_0(x) = H(x,0) \quad , \quad f_1(x) = H(x,1).$$

A smooth map $f\colon X \to Y$ is said to be a **homotopical equivalence** if there is a smooth map $g\colon Y \to X$ such that $fg \equiv \mathrm{Id}_Y$, $gf \equiv \mathrm{Id}_X$. A smooth manifold X is **contractible** to a point $p \in X$ if the inclusion $p \to X$ is a homotopical equivalence.

If ω is a p-form on $X \times I$, in local coordinates $\omega = \sum_\alpha f_\alpha(x,t)\,\mathrm{d}x_\alpha + (\text{terms with } \mathrm{d}t)$, where $\mathrm{d}x_\alpha = \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p}$, and we consider on X the p-form $I(\omega) = \sum_\alpha \left(\int_0^1 f_\alpha(x,t)\mathrm{d}t\right)\mathrm{d}x_\alpha$.

By a direct calculation, $I(\mathrm{d}\omega) = \mathrm{d}(I(\omega))$.

Lemma: *$I(\partial_t^L \omega) = j_1^*\omega - j_0^*\omega$; where $j_t\colon X \to X \times I$, $j_t(x) = (x,t)$.*

Proof: Barrow's rule and the equality $\partial_t^L\omega = \sum_\alpha (\partial_t f_\alpha)(x,t)\,\mathrm{d}x_\alpha + (\text{terms with } \mathrm{d}t)$.

Theorem: *If $f_0 \equiv f_1$, then $f_0^* = f_1^*\colon H^\bullet_{DR}(Y) \longrightarrow H^\bullet_{DR}(X)$.*

Proof: Let $H : f_0 \equiv f_1$. If ω is a closed p-form closed on Y, then $\bar\omega = H^*\omega$ is closed, and

$$\begin{aligned} f_1^*\omega - f_0^*\omega &= j_1^*H^*\omega - j_0^*H^*\omega = j_1^*\bar\omega - j_0^*\bar\omega = I(\partial_t^L\bar\omega) \\ &= I(\mathrm{d}i_{\partial_t}\bar\omega - i_{\partial_t}\mathrm{d}\bar\omega) = I(\mathrm{d}i_{\partial_t}\bar\omega) = \mathrm{d}I(i_{\partial_t}\bar\omega). \end{aligned}$$

Corollary: *If $f\colon X \to Y$ is a homotopical equivalence, $f^*\colon H^\bullet_{DR}(Y) \xrightarrow{\sim} H^\bullet_{DR}(X)$.*

Poincaré's Lemma: *$H^p_{DR}(\mathbb{R}^n) = 0$, $p \geq 1$.*

Proof: The homotopy $H(x,t) = tx$ shows that $\mathbb{R}^n$ is contractible. q.e.d.

1. *Any closed form is locally exact.*

2. *Let $(D_1, \ldots, D_n)$ be a base of vector fields on a manifold X. If $[D_i, D_j] = 0$, any point has a coordinate neighborhood where $D_1 = \partial_1, \ldots, D_n = \partial_n$.*

   Proof: If $(\omega_1, \ldots, \omega_n)$ is the dual base, by Cartan's formula

   $$\mathrm{d}\omega_k(D_i, D_j) = D_i(\omega_k(D_j)) - D_j(\omega_k(D_i)) - \omega_k([D_i, D_j]) = 0 - 0 - 0,$$

   and by Poincaré's lemma, on a neighborhood of any point x the 1-forms ω_i are exact, $\omega_i = \mathrm{d}u_i$, and $u_1, \ldots, u_n$ form a local coordinate system at x since the differentials define a base of T_x^*X. In these coordinates $D_i = \partial_{u_i}$.

3. *If n is even, any vector field D on the sphere S_n vanishes at a point.*

   If D has no zero, then we may assume that $|D| = 1$, so that it defines a smooth map $\phi\colon S_n \to S_n$ such that $\phi(x)$ is orthogonal to x. Now $H(x,t) = \cos(\pi t)\, x + \sin(\pi t)\, \phi(x)$ defines a homotopy between the automorphism $\tau\colon S_n \to S_n$, $\tau(x) = -x$, and the identity of S_n; so that $\tau^*\colon H^n_{DR}(S_n) \to H^n_{DR}(S_n)$ is the identity. Absurd, τ inverts orientations when n is even. When n is odd, $\phi(x_0, x_1, \dots, x_n) = (-x_1, x_0, \dots, -x_n, x_{n-1})$ defines a tangent vector field on S_n not vanishing at any point.

4. *Put $B_{n,1+\varepsilon} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon \sum_i x_i^2 < 1 + \varepsilon\}$. The inclusion $i\colon S_{n-1} \to B_{n,1+\varepsilon}$ has no smooth retract, not even up to homotopies.*

   If $r\colon B_{n,1+\varepsilon} \to S_{n-1}$ is smooth and $ri \equiv \mathrm{Id}$, then the composition
   $$H^{n-1}_{DR}(S_{n-1}) \xrightarrow{r^*} H^{n-1}_{DR}(B_{n,1+\varepsilon}) \xrightarrow{i^*} H^{n-1}_{DR}(S_{n-1})$$
   is the identity. Absurd since $H^{n-1}_{DR}(B_{n,1+\varepsilon}) = 0$, and $H^{n-1}_{DR}(S_{n-1}) \neq 0$.

5. *If a vector field D points outside at any point of the sphere S_{n-1}, then it vanishes at a point of the ball B_n.*

   Otherwise $r\colon B_{n,1+\varepsilon} \to S_{n-1}$, $r(x) = D_x/|D_x|$, is smooth and $r(x) \neq -x$ for all $x \in S_n$. Hence, up to a homotopy, r is a retract of the inclusion $i\colon S_{n-1} \to B_{n,1+\varepsilon}$, a homotopy between ri and the identity being
   $$H(x,t) = \frac{t \cdot r(x) + (1-t)x}{|t \cdot r(x) + (1-t)x|}\,.$$

Proposition: $H^1_{DR}(X) = 0$, *when X is a simply connected smooth manifold.*

Proof: Let ω be a closed 1-form on X, and let P_x be the non empty set (Poincaré's lemma) of germs at x of primitives of ω (functions such that $\mathrm{d}f = \omega$), so that such germs always differ in a constant. We put (as in p. 197)

$$\pi\colon \widetilde{P} = \textstyle\coprod_x P_x \longrightarrow X,\ \pi(f_x) = x.$$

Any primitive $f \in \mathcal{C}^\infty(U)$ defines a local section $U \to \widetilde{P}$ of π, and if two sections coincide at a point, $f_x = h_x$, they coincide on a neighborhood.

The images of these sections form a base of a topology on $\widetilde{P}$, so that π is a covering (with uncountable fibres), and continuous sections of π are just primitives of ω.

If X is simply connected, π admits a continuous global section, and ω is exact.

9.4 Functions of a Complex Variable

Let $U \subseteq \mathbb{C}$ be an open set. A function $f = u + iv\colon U \to \mathbb{C}$ is $\mathcal{C}^m$ if so are u and v,

$$\mathcal{C}^m(U,\mathbb{C}) = \mathcal{C}^m(U) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{C}^m(U) \oplus i\,\mathcal{C}^m(U).$$

Complex p-forms at a point $z_0 \in U$ are alternate $\mathbb{R}$-multilinear maps of $T_{z_0}U$ into $\mathbb{C}$; i.e., $\omega_p = \omega'_p + i\omega''_p$, where ω'_p and ω''_p are ordinary p-forms. We put

$$\mathrm{d}\omega_p = \mathrm{d}\omega'_p + i\mathrm{d}\omega''_p$$
$$\int_\Omega \omega_p = \int_\Omega \omega'_p + i\int_\Omega \omega''_p$$

so that Stokes theorem holds. Moreover, we have

$$(T^*_{z_0}U)_{\mathbb{C}} := (T^*_{z_0}U) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\mathrm{d}x + \mathbb{C}\mathrm{d}y = \mathbb{C}\mathrm{d}z + \mathbb{C}\mathrm{d}\bar{z},$$

where $z = x + iy$, $\bar{z} = x - iy$, and we define $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$ by the equality $\mathrm{d}f = \frac{\partial f}{\partial z}\,\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z}$.

So we have $\mathbb{C}$-derivations $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)\colon \mathcal{C}^\infty(U, \mathbb{C}) \longrightarrow \mathbb{C}$, and

$$\mathbb{R}\partial_x + \mathbb{R}\partial_y = T_{z_0}U \hookrightarrow \mathrm{Der}_{\mathbb{C}}\big(\mathcal{C}^\infty(U,\mathbb{C}),\mathbb{C}\big) = (T_{z_0}U)_{\mathbb{C}} = \mathbb{C}\partial_x + \mathbb{C}\partial_y = \mathbb{C}\partial_z + \mathbb{C}\partial_{\bar{z}}.$$

Theorem: *If $f = u + iv$ is $\mathcal{C}^1$ on U, the following conditions are equivalent,*

1. *At any point $z_0 \in U$ there exists the limit $f'(z_0) = \lim\limits_{\varepsilon\to 0} \frac{f(z_0+\varepsilon)-f(z_0)}{\varepsilon}$, $\varepsilon \in \mathbb{C}$.*

2. *$f_*\colon \mathbb{C} = T_{z_0}U \longrightarrow \mathbb{C}$ is $\mathbb{C}$-linear at any point $z_0 \in U$;* i.e., $\partial_{\bar{z}} f = 0$,

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \quad (\textbf{Cauchy-Riemann Equations})$$

3. *The complex 1-form $f(z)\mathrm{d}z$ is closed.*

4. *If a circle γ surrounds a point $z_0 \in U$, and it is in a disk contained in U,*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z. \qquad (\textbf{Cauchy's Formula})$$

5. *Locally f is a power series, $f(z) = \sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$,* ($f$ is **analytic**).

Proof: ($1 \Rightarrow 2$) As ε goes to 0 along the real axis or the imaginary axis,

$$f'(z_0) = \partial_x f = u_x + iv_x,$$
$$f'(z_0) = \tfrac{1}{i}\,\partial_y f = -iu_y + v_y.$$

($2 \Rightarrow 3$) $\mathrm{d}(f\mathrm{d}z) = \mathrm{d}f \wedge \mathrm{d}z = (\frac{\partial f}{\partial z}\,\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z}) \wedge \mathrm{d}z = \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z = 0.$

($3 \Rightarrow 4$) When γ is centered at z_0, in polar coordinates $z = z_0 + re^{i\theta}$ we see that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\,\mathrm{d}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,\mathrm{d}\theta$$

is the mean value of $f(z)$ on γ; hence it goes to $f(z_0)$ as $r \to 0$.

But the integrals coincide[2] when two circles surrounding z_0 are in a disk $D \subset U$ and (with reverse orientations) they form the boundary of an annulus: just apply Stokes theorem to the 1-form $\frac{f(z)}{2\pi i(z-z_0)}\,\mathrm{d}z$, which is closed because so is $f(z)\mathrm{d}z$ and $\mathrm{d}\frac{1}{z-z_0} = -(z-z_0)^{-2}\mathrm{d}z$.

($4 \Rightarrow 5$) Let $D \subset U$ be a closed disc centered at z_0. If $z \in \overset{\circ}{D}$, $\eta \in \partial D$,

$$\frac{1}{\eta - z} = \frac{1}{(\eta - z_0) - (z - z_0)} = \frac{1}{\eta - z_0}.\frac{1}{1 - \frac{z-z_0}{\eta-z_0}} = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\eta-z_0)^{n+1}}$$

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta = \frac{1}{2\pi i}\int_{\partial D} \sum_{n=0}^{\infty} \frac{f(\eta)(z-z_0)^n}{(\eta-z_0)^{n+1}}\,\mathrm{d}\eta$$

[2]If a non constant polynomial $P(z)$ has no complex root, then $f(z) = \frac{1}{P(z)}$ is derivable at any point and the mean value of $f(z)$ on the circle $|z| = r$ goes to 0 as $r \to \infty$. Hence $f(0) = 0$, a contradiction proving again **D'Alembert's theorem**.

and we may integrate term by term because the series uniformly converges on ∂D, since it is bounded by a convergent geometric series (K is the maximum of $|f(z)|$ on D)

$$\left|\frac{f(\eta)(z-z_0)^n}{(\eta-z_0)^{n+1}}\right| \le \frac{K}{R}\,\frac{|z-z_0|^n}{R^n}; \quad \frac{|z-z_0|}{R} < 1, \ (R \text{ the radius of } D).$$

Therefore, on the interior of the disc D we have a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \ , \quad \text{where} \ \ a_n = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\eta)}{(\eta-z_0)^{n+1}}\,\mathrm{d}\eta.$$

Cauchy's Inequalities: $|a_n| \le \frac{M_R}{R^n}$, *where* M_R *is the maximum of* $|f(\eta)|$ *on* ∂D.

In fact, in polar coordinates $\eta = z_0 + Re^{i\theta}$,

$$|a_n| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})Rie^{i\theta}}{R^{n+1}e^{i\theta(n+1)}}\,\mathrm{d}\theta\right| \le \frac{1}{2\pi R^n}\int_0^{2\pi} |f(z_0+Re^{i\theta})|\,\mathrm{d}\theta.$$

($5 \Rightarrow 1$) Any series $\sum_n a_n(z-z_0)^n$ is derivable in the interior of the disc of convergence, and the derivative coincides with $\sum_n a_n n(z-z_0)^{n-1}$, with equal radius of convergence.

Since the derivative is again a power series, *analytic functions are infinitely derivable* (and, by the Cauchy-Riemann equations, the real and imaginary part u, v are harmonic functions).

Liouville's Theorem: *Any bounded analytic function on* $\mathbb{C}$ *is constant.*

Proof: If $|f(z)| \le M$ on $\mathbb{C}$, then $|a_n| \le \frac{M}{R^n}$ for any $R > 0$. Hence $a_n = 0$ when $n \ge 1$.

Cauchy-Goursat Formula: *If* U *simply connected, and* $f\colon U \to \mathbb{C}$ *is analytic, then for any closed curve* γ *in* U *we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof: Since U is simply connected and $f(z)\,\mathrm{d}z$ is closed, it is exact (p. 275).

Definition: A topological space X, with a sheaf $\mathcal{O}_X$ of complex valued continuous functions, is a **Riemann surface** if it is locally isomorphic to an open set of $\mathbb{C}$ with the sheaf of analytic functions. We say that $\mathcal{O}_X(U)$ is the ring of **analytic functions** on U, and morphisms $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ between Riemann surfaces (see p. 287) are said to be **analytic** maps. An analytic functions on U is just an analytic map $U \to \mathbb{C}$.

An analytic function $f \in \mathcal{O}_X(U)$ is a **coordinate** if it defines an isomorphism of U onto an open set of $\mathbb{C}$, and it is a **local coordinate** at $p \in U$ if it is a coordinate on some neighborhood of p (i.e., $f'(p) \neq 0$ when U is an open set of $\mathbb{C}$).

Examples: Open sets in $\mathbb{C}$, $\mathbb{P}_{1,\mathbb{C}}$, toruses $\mathbb{C}/(\mathbb{Z}e_1 + \mathbb{Z}e_2)$, coverings of a Riemann surface,...

We shall always assume that all Riemann surfaces are connected and σ-compact[3].

Theorem: *Let* $f\colon X \to Y$ *be a non constant analytic map. If* $p \in X$, *there are coordinate neighborhoods* (U, u) *and* (V, v) *of* $p = 0$ *and* $f(p) = 0$ *such that* $f(u) = u^n$, *for some natural number* $n \ge 1$, the **ramification index** $\mathrm{ind}_p f$ of f at p.

Proof: Let us fix local coordinates u, v so that $p = f(p) = 0$. Then the map is $v = z^n h(z)$, where $h(0) \neq 0$.

In a neighborhood of $p = 0$ the n-th root of $h(z)$ is analytic, and changing the coordinate z by $u = z\sqrt[n]{h(z)}$ we have that $v = z^n h(z) = u^n$. q.e.d.

This local classification of morphisms has some important and obvious consequences:

[3]In fact, according to a Radó's theorem, any separated Riemann surface is σ-compact.

1. *The fibres of any non constant analytic map are discrete. In particular the zeroes of a non constant analytic function are discrete.*

2. *If two analytic functions coincide on a non void open set, then they are equal.*

3. *Any non constant analytic map is open.*

4. **Maximum Principle:** *If $f \in \mathcal{O}(X)$ is not constant, then $|f|$ has no local maximum.*

5. *Any analytic function on a compact Riemann surface is constant.*

6. *Any injective morphism $X \to Y$ is an isomorphism onto an open subset of Y. Hence, any bijective analytic map is an isomorphism.*

9.4.1 Meromorphic Functions

Let $f: U \to \mathbb{C}$ be analytic, except at a point $z_0 \in U$; i.e., analytic on $U - z_0$. In calculations we assume that $z_0 = 0$; but we state the results in general.

Let us consider an annulus $\Omega = \{r \leq |z| \leq R\}$. The boundary is formed by the circles γ and Γ of radius r and R. The argument of Cauchy's formula now gives

$$z \in \Omega; \quad f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta - \frac{1}{2\pi i}\int_\gamma \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta,$$

$$\eta \in \Gamma; \quad \frac{1}{\eta - z} = \sum_{n=0}^{\infty} \frac{z^n}{\eta^{n+1}}, \; \frac{1}{2\pi i}\int_\Gamma \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\Gamma \frac{f(\eta)}{\eta^{n+1}}\,\mathrm{d}\eta\right) z^n,$$

$$\eta \in \gamma; \quad \frac{1}{z - \eta} = \sum_{n=0}^{\infty} \frac{\eta^n}{z^{n+1}}, \; \frac{1}{2\pi i}\int_\Gamma \frac{f(\eta)}{z - \eta}\,\mathrm{d}\eta = \sum_{n=-1}^{-\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\eta)}{\eta^{n+1}}\,\mathrm{d}\eta\right) z^n,$$

and the function $f(z)$ may be expanded as a **Laurent series** on the annulus Ω,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \; a_n = \frac{1}{2\pi i}\int_\sigma \frac{f(\eta)}{(\eta - z_0)^{n+1}}\,\mathrm{d}\eta,$$

where σ is γ, Γ or an intermediate circle, since the 1-form $f(\eta)(\eta - z_0)^{-n-1}\mathrm{d}\eta$ is closed on $U - z_0$. Hence the expansion holds on a neighborhood (except z_0), and Cauchy's inequalities hold, $|a_n| \leq r^{-n} M_r$, where $M_r = \max\limits_{|z - z_0| = r} |f(z)|$.

Definitions: If the coefficients a_n are null when $n < 0$, we say that z_0 is a **removable singularity** of f, if they are null up to a finite number, z_0 is a **pole** of f, (and the **order** is the greatest m such that $a_{-m} \neq 0$), and if there are infinite non null coefficients, z_0 is an **essential singularity**. A function is **meromorphic** if it is analytic up to a discrete set of poles.

Examples: If f has a pole of order m, on a neighborhood we have $f = h/z^m$, where h is analytic, so that meromorphic functions locally coincide with quotients of analytic functions; hence the meromorphic functions on X form a field.

If f is a meromorphic function on $\mathbb{P}_1$, subtracting the singular part of the Laurent expansion at each pole (a rational function with a unique pole) we obtain an analytic function on $\mathbb{P}_1$; hence it is constant. *Meromorphic functions on $\mathbb{P}_1$ are rational functions.*

Removable Singularity Theorem: *If $|f|$ is bounded on a neighborhood of z_0, it is a removable singularity. Hence, if f may be continuously extended to z_0, the extension is analytic.*

Proof: If $M_\varepsilon \leq M$ for any small ε, then $|a_{-n}| \leq M\varepsilon^n$, and $a_{-n} = 0$.

Weierstrass Theorem: *A singularity is essential if and only if the image of any neighborhood is dense in* $\mathbb{C}$.

Proof: If the origin is a pole, $f(z) = h(z)z^{-m}$, where $h(0) \neq 0$.

Hence $f(U - \{0\})$ is not dense in $\mathbb{C}$ when U is small, since $\lim_{z\to 0} f(z) = \infty$.

Conversely, if there is a neighborhood U such that $f(U - \{0\})$ is not dense in $\mathbb{C}$, we may assume that the disc of radius R centered at the origin does not intersect $f(U - \{0\})$.

The module of $f(z)^{-1}$ is bounded by R^{-1}; hence $h(z) = f(z)^{-1}$ is analytic on U, and $f(z) = h(z)^{-1}$ is meromorphic. The singularity is a pole.

Theorem: *The meromorphic functions on a Riemann surface X are the analytic maps $X \to \mathbb{P}_1$ (except the constant map ∞).*

Proof: If $f\colon X \to \mathbb{P}_1$ is a morphism, and we consider the usual coordinate neighborhoods $(U_0; z)$, $(U_\infty; \frac{1}{z})$ of the origin and the point at infinity, the function f is analytic on $V_0 = f^{-1}(U_0)$, and $1/f$ is analytic on $V_\infty = f^{-1}(U_\infty)$. Hence f is meromorphic on $V_0 \cup V_\infty = X$.

Conversely, if we extend a meromorphic function f it with the value ∞ at any pole, the map $f\colon X \to \mathbb{P}_1$ is analytic because $\frac{1}{f}$ is an analytic function on a neighborhood of each pole.

Corollary: *The analytic automorphisms of $\mathbb{P}_1$ are just*[4] *the homographies $\tau(z) = \frac{az+b}{cz+d}$.*

Corollary: *The analytic automorphisms of $\mathbb{C}$ are just the affinities $\tau(z) = az + b$.*

Proof: If we extend an automorphism $\tau\colon \mathbb{C} \to \mathbb{C}$ to $\mathbb{P}_1$, $\tau(\infty) = \infty$, we obtain a meromorphic function on $\mathbb{P}_1$, since the singularity at ∞ is not essential.

Hence it is a homography fixing the point at infinity.

Definitions: A complex 1-form ω on a Riemann surface is **analytic** if locally $\omega = f(z)\mathrm{d}z$, where $f(z)$ is an analytic function (hence $\mathrm{d}\omega = 0$). If ω is analytic on $U - z_0$, the integral along a small closed curve γ around z_0 (oriented as boundary of the region containing z_0) does not depend on the curve and, with a factor $\frac{1}{2\pi i}$, it is the **residue** of ω at z_0,

$$\mathrm{Res}(\omega, z_0) = \frac{1}{2\pi i}\int_\gamma \omega.$$

If $\omega = f(z)\mathrm{d}z$, with $f(z)$ meromorphic and Laurent expansion $f(z) = \sum_n a_n(z - z_0)^n$, we may integrate term by term since the series uniformly converges on γ,

$$\int_\gamma (z - z_0)^n \mathrm{d}z = \frac{1}{n+1}\int_\gamma \mathrm{d}(z - z_0)^{n+1} = 0, \text{ when } n \neq -1.$$

$$\mathrm{Res}(\omega, z_0) = \frac{a_{-1}}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - z_0} = a_{-1}.$$

Residue Theorem: *Let X be a Riemann surface, and let ω be an analytic 1-form up to a discrete set. If $\Omega \subseteq X$ is a compact manifold with boundary and ω has no singular point on $\partial\Omega$,*

$$\frac{1}{2\pi i}\int_{\partial\Omega} \omega = \sum_{z_i\in\Omega} \mathrm{Res}(\omega, z_i).$$

[4] so that the concept of Riemann surface captures the elusive structure of the complex projective line.

Proof: If we consider a small disc D_i containing each singularity $z_i \in \Omega$, then ω is closed in a neighborhood of $\Omega - \bigcup_i D_i$, and by Stokes theorem

$$0 = \int_{\partial\Omega} \omega - \sum\nolimits_i \int_{\partial D_i} \omega = \int_{\partial\Omega} \omega - 2\pi i \sum\nolimits_i \mathrm{Res}(\omega, z_i).$$

Corollary: $\sum_{z_i \in X} \mathrm{Res}(\omega, z_i) = 0$, *when X is a compact Riemann surface.*

Corollary: *Let f be a meromorphic function on a Riemann surface X. If $\Omega \subseteq X$ is a compact manifold with boundary and f has no zero or pole on $\partial\Omega$, then the number of zeroes of f in Ω, minus the number of poles, all counted with multiplicity, is just*

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{\mathrm{d}f}{f}.$$

Proof: If $f(z) = a_m z^m + a_{m+1} z^{m+1} + \ldots$, $a_m \neq 0$, then

$$\omega = \frac{\mathrm{d}f}{f} = \frac{(m a_m z^{m-1} + \ldots)\mathrm{d}z}{a_m z^m + a_{m+1} z^{m+1} + \ldots} = \left(\frac{m}{z} + \ldots\right) \mathrm{d}z,$$

and the residue of ω at $z = 0$ is just m. The sum of the residues of ω is the number of zeros of f minus the number of poles, all counted with multiplicity.

Corollary: *The number of poles of a non constant meromorphic function f on a compact Riemann surface coincides with the number of zeros*[5]*.*

9.4.2 Convergence in $\mathcal{O}(X)$

If X is a Riemann surface and $p \in X$, the differential $\mathrm{d}_p z \colon T_p X \simeq \mathbb{C}$ of a local coordinate $z = x + iy$ defines a complex structure on $T_p X$, and it does not depend on the local coordinate: if u is another local coordinate, then $\mathrm{d}_p u = a \mathrm{d}_p z$, $a \neq 0$. As any complex structure on a real vector space, it is defined by a $\mathbb{R}$-linear isomorphism $J \colon T_p X \to T_p X$ such that $J^2 = -1$; and we have $J(\partial_x) = \partial_y$, $J(\partial_y) = -\partial_x$. The Riemann-Cauchy equations state that a smooth map $\varphi \colon X \to Y$ between Riemann surfaces is an analytic morphism if and only if $\varphi_{*,p} \colon T_p X \to T_{\varphi p} Y$ is $\mathbb{C}$-linear (i.e. commutes with J).

On $T_p X$ we have an orientation, requiring that $D, J(D)$ are oriented pairs, and a scalar product, well-defined up to a positive factor, such that $D \cdot J(D) = 0$, $\forall D \in T_p X$.

In particular X inherits a natural **conformal structure** (a riemannian metric g, well-defined up to the product by a positive function) and we say that g is a compatible metric, or that X admits the metric g). When two Riemann surfaces X, Y are endowed with compatible metrics, any oriented isometry $X \simeq Y$ is an analytic isomorphism.

Now, if $\varphi \colon Y \to X$ is an analytic map, then $\varphi^* g$ is a compatible metric on Y.

In a local coordinate $z = x + iy$, any compatible metric (viewed as a quadratic form) is just $g = f^2(\mathrm{d}x^2 + \mathrm{d}y^2) = f^2 \mathrm{d}z\mathrm{d}\bar{z}$ for some smooth function f without zeroes.

The curvature K of such riemannian metric is just

$$K = -\frac{\Delta(\ln f)}{f^2} = -\frac{(\partial_x^2 + \partial_y^2)(\ln f)}{f^2}.$$

Definition: A **hyperbolic metric** on a Riemann surface is a compatible riemannian metric with constant curvature $K = -1$ (i.e., if and only if $\Delta(\ln f) = f^2$).

[5]Since a polynomial of degree n defines a meromorphic function on $\mathbb{P}_1$ with a pole of order n at $z = \infty$, this result turns obvious **D'Alembert's theorem**.

Definition: The **Poincaré metric** on the open disk $\mathbb{D}_r \subset \mathbb{C}$ of radius $r \in \mathbb{R}_+$ with center 0 is the hyperbolic metric

$$g_r = g_{\mathbb{D}_r} := \left(\frac{2r}{r^2 - |z|^2}\right)^2 \mathrm{d}z\mathrm{d}\bar{z}.$$

The disks $\mathbb{D}_r$ are isometric, by means of a homothety. The **unit disk** is $\mathbb{D} = \mathbb{D}_1$, $g_{\mathbb{D}} = g_1$.

Lemma: *Let $\varphi\colon (X, g_X) \to (Y, g_Y)$ be an analytic map between Riemann surfaces endowed with hyperbolic metrics, so that $\varphi^* g_Y = F g_X$ for some $F \in \mathcal{C}^\infty(X)$. If F attains a maximum M at some point $p \in X$, then $M \leq 1$.*

Proof: If $F(p) = M \neq 0$, then $\varphi^* g_Y \neq 0$ at p, so that φ is a local diffeomorphism at p, and $\varphi^* g_Y$ is hyperbolic in a neighborhood of p. In a local complex coordinate z we have

$$\varphi^* g_Y = h^2 \mathrm{d}z\mathrm{d}\bar{z} \ , \quad g_X = f^2 \mathrm{d}z\mathrm{d}\bar{z} \ , \quad F = h^2/f^2.$$

Since $\ln F$ attains a maximum at p, we have $\partial_x^2(\ln F) \leq 0$ and $\partial_y^2(\ln F) \leq 0$ at p; hence

$$0 \geq (\Delta \ln F)(p) = 2(\Delta \ln h)(p) - 2(\Delta \ln f)(p) = 2h^2(p) - 2f^2(p),$$

and we conclude that $M = h^2(p)/f^2(p) \leq 1$.

Ahlfors lemma: *If g_Y is a hyperbolic metric on a Riemann surface Y, then any analytic map $\varphi\colon \mathbb{D} \to Y$ is contractive: $\varphi^* g_Y \leq g_{\mathbb{D}}$.*

Proof: Since $g_{\mathbb{D}} = \lim_{r\to 1} g_r$, we only have to show that the restriction $\varphi\colon \mathbb{D}_r \to Y$ is contractive, for all $r < 1$. Now we have

$$\varphi^* g_Y = h^2 \mathrm{d}z\mathrm{d}\bar{z} = F g_r \ , \quad F = h^2 \left(\frac{2r}{r^2 - |z|^2}\right)^{-2},$$

where $F \in \mathcal{C}^\infty(\mathbb{D}_r)$ clearly goes to 0 when $z \to \partial\mathbb{D}_r$. Hence F attains a maximum M at a point of $\mathbb{D}_r$. By the lemma $M \leq 1$, and we conclude.

Corollary: *Any analytic map $\mathbb{D} \to \mathbb{D}$ is contractive with respect to the Poincaré metric.*

Corollary: *The Poincaré metric $g_{\mathbb{D}}$ is the biggest hyperbolic metric on $\mathbb{D}$.*

Corollary: *The analytic isomorphisms of $\mathbb{D}$ are the oriented isometries of $g_{\mathbb{D}}$:*

$$\tau(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} \ ; \quad \theta \in \mathbb{R}, \ |a| < 1.$$

Proof: We have $\tau(\mathbb{D}) = \mathbb{D}$ because $\tau(\partial\mathbb{D}) = \partial\mathbb{D}$ and $\tau(a) = 0$.

In fact $|\tau(1)| = |\tau(-1)| = |\tau(i)| = 1$ and any homography transforms a circle into a circle (or a line, a circle through ∞) because so do any similarity $az + b$ and the inversion $\frac{1}{z}$, and these homographies generate the group $PGL(1, \mathbb{C})$.

Moreover, with a suitable value of θ, we obtain any oriented isometry $\tau_{*,a}\colon T_a\mathbb{D} \to T_0\mathbb{D}$, and any isometry (p. 303) $\tau\colon \mathbb{D} \to \mathbb{D}$ is fully determined by $\tau(0)$ and the oriented isometry $\tau_{*,0}$.

Schwarz's Lemma: *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an analytic function. If $f(0) = 0$, then*

$$|f(z)| \leq |z| \quad , \quad |f'(0)| \leq 1.$$

Proof: Since f is contractive, we have $\mathrm{d}\big(0, f(z)\big) \leq \mathrm{d}(0, z)$, with the Poincaré metric.

Since $\mathrm{d}(0, z)$ is an increasing function of the Euclidean distance $|z|$, we have $|f(z)| \leq |z|$.

Finally, by Ahlfors lemma $f_*\colon T_0\mathbb{D} \to T_0\mathbb{D}$ is contractive.

Since f_* is the multiplication by $f'(0)$, we conclude that $|f'(0)| \leq 1$.

Corollary: *If $f(0) = 0$ and $|f'(0)| = 1$, then $f(z) = f'(0)z$ is a rotation.*

Proof: We have $f(z) = zh(z)$ and, by the Schwarz's lemma, $|h(z)|$ attains a maximum at $z = 0$. Hence $h(z)$ is constant and $f(z) = az = f'(0)z$ is a rotation.

Corollary: *If a morphism $f\colon \mathbb{D} \to \mathbb{D}$ is an isometry at a point, then it is a global isometry.*

Proof: Composing with an isometry of $\mathbb{D}$ we may assume that f is an isometry at $z = 0$ and that $f(0) = 0$, so that $|f'(0)| = 1$, and f is a rotation.

Lemma: *Let $U \subseteq \mathbb{C}$ be an open set. If a sequence (f_n) of analytic functions on U uniformly converges on the compact sets $K \subset U$ to a limit continuous function f, then f also is analytic, and the derivatives f'_n uniformly converge on any compact set to f'.*

Proof: Take an open disk D with center at $z_0 \in U$, such that $\bar{D} \subset U$. If $z \in D$, then

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\eta)}{\eta - z} \mathrm{d}\eta.$$

Since the functions f_n uniformly converge on the compact set ∂D, we also have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\eta)}{\eta - z} \mathrm{d}\eta,$$

and the argument of p. 276 shows that f is analytic on D. Hence f is analytic.

Now take a closed disk of smaller radius $\bar{D}'$, also with center at z_0. If $z \in \bar{D}'$, then

$$f'_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\eta)}{(\eta - z)^2} \mathrm{d}\eta,$$
$$\lim_{n\to\infty} f'_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\eta)}{(\eta - z)^2} \mathrm{d}\eta = f'(z),$$

and the limit clearly is uniform on the compact set $\bar{D}'$.

Since any compact set $K \subset U$ may be covered with a finite number of such closed disks, we conclude that $f'_n \to f'$ uniformly on any compact set K.

Weierstrass Theorem: *The ring $\mathcal{O}(X)$ of analytic functions on a Riemann surface X is closed in the ring $\mathcal{C}(X, \mathbb{C})$ of continuous complex-valued functions (with the compact convergence).*

Montel's Theorem: *Let $\mathcal{F}$ be a family of analytic functions on a Riemann surface X. If for any compact set $K \subseteq X$ there is a constant c such that $|f(x)| \leq c$, $\forall f \in \mathcal{F}$, $x \in K$, then $\mathcal{F}$ has compact closure in $\mathcal{O}(X)$.*

Proof: Since $\mathcal{O}(X)$ is closed in $\mathcal{C}(X, \mathbb{C})$, we have to show that $\mathcal{F}$ has compact closure in $\mathcal{C}(X, \mathbb{C})$.

Now, $\mathcal{F}(x) = \{f(x);\ f \in \mathcal{F}\}$ is a bounded set and, by Ascoli's theorem (p. 232) we only have to show that $\mathcal{F}$ is equicontinuous.

Let D be a relatively compact open disk in X. By hypothesis, there is an open disk $\mathbb{D}_r \subset \mathbb{C}$ such that $f(D) \subseteq \mathbb{D}_r$, $\forall f \in \mathcal{F}$. Now, $\mathrm{d}x^2 + \mathrm{d}y^2 \leq \frac{r^2}{4} g_r$, and by Ahlfors lemma,

$$f^*(\mathrm{d}x^2 + \mathrm{d}y^2) \leq \tfrac{r^2}{4} f^*(g_r) \leq \tfrac{r^2}{4} g_D.$$

Hence, $|f(x_1) - f(x_2)| \leq \frac{r^2}{4} \mathrm{d}_D(x_1, x_2)$ for any $x_1, x_2 \in D$, $f \in \mathcal{F}$, and $\mathcal{F}$ is equicontinuous.

Corollary: *The family* $\mathcal{F} = \mathrm{Hom}_{\mathrm{an}}(X, \mathbb{D})$ *has compact closure in* $\mathcal{O}(X)$*.*

By the maximum principle, any function in $\bar{\mathcal{F}} - \mathcal{F}$ is constant; therefore:

Corollary: *Fix* $p \in X$ *and* $\alpha \in \mathbb{D}$*. The family* $\{f \in \mathrm{Hom}_{\mathrm{an}}(X, \mathbb{D}) \colon f(p) = \alpha\}$ *is compact.*

Lemma: *Fix* $\alpha \in \mathbb{D} - 0$*. The family* $\mathcal{F} = \{f \in \mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, \mathbb{D} - 0) \colon f(0) = \alpha\}$ *is compact.*

Proof: Let $h \colon \mathbb{D} \simeq \{x < 0\} \xrightarrow{\exp} \mathbb{D} - 0$ be the universal covering and fix a point $\tilde{\alpha} \in \mathbb{D}$ over α. Since $\mathbb{D}$ is simply connected, any function $f \in \mathcal{F}$ admits a lifting $\tilde{f} \colon \mathbb{D} \to \mathbb{D}$ such that $\tilde{f}(0) = \tilde{\alpha}$. Hence, the continuous map (with the compact-open, hence the compact convergence, topologies)

$$\mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, \mathbb{D}) \xrightarrow{h\circ} \mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, \mathbb{D} - 0)$$

transforms the family $\tilde{\mathcal{F}} = \{\tilde{f} \in \mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, \mathbb{D}) \colon \tilde{f}(0) = \tilde{\alpha}\}$ onto $\mathcal{F}$. Since $\tilde{\mathcal{F}}$ is compact, so is $\mathcal{F}$.

Proposition: *Let* $U = \mathbb{C} - \bar{D}$ *be the complement of a closed disk, and* $\alpha \in U$*. Then the family* $\{f \in \mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, U) \colon f(0) = \alpha\}$ *is compact.*

Proof: We may assume that $D = \mathbb{D}$. Now the isomorphism $\frac{1}{z} \colon U = \mathbb{C} - \bar{\mathbb{D}} \to \mathbb{D} - 0$ induces a homeomorphism $\mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, U) \simeq \mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, \mathbb{D} - 0)$ and we conclude by the previous lemma.

Corollary: *If the complement of an open set* $U \subset \mathbb{C}$ *contains a disk and* $\alpha \in U$*, then the family* $\{f \in \mathrm{Hom}_{\mathrm{an}}(\mathbb{D}, U) \colon f(0) = \alpha\}$ *has compact closure in* $\mathcal{O}(\mathbb{D})$*.*

Koebe's Theorem: *The family of all injective analytic functions* f *on the unit disk* $\mathbb{D}$ *such that* $f(0) = 0$ *and* $f'(0) = 1$ *is compact.*

Proof: Let us see that in such family any sequence (f_n) admits a convergent subsequence.

For each index n, put $r_n = \sup\{r \in \mathbb{R}_+ \colon \mathbb{D}_r \subseteq f_n(\mathbb{D})\}$, so that $\mathbb{D}_{r_n} \subseteq f_n(\mathbb{D}$ and there exists a point $z_n \notin f_n(\mathbb{D})$ with $|z_n| = r_n$. Remark that $r_n \leq 1$, just applying Schwarz's lemma to the composition (f_n is injective)

$$\varphi \colon \mathbb{D} \xrightarrow{r_n \cdot} \mathbb{D}_{r_n} \xrightarrow{f_n^{-1}} \mathbb{D}\ , \quad \varphi(0) = 0,\ \varphi'(0) = r_n.$$

Now the functions $g_n = f_n / z_n$ fulfill that $1 \notin g_n(\mathbb{D})$ and $\mathbb{D} \subseteq g_n(\mathbb{D})$.

The covering $z^2 + 1 \colon \mathbb{C} - 0 \to \mathbb{C} - 1$ is trivial over the simply connected set $g_n(\mathbb{D})$; hence it admits a well-defined inverse function $u = \sqrt{z - 1}$ on $g_n(\mathbb{D})$ such that $u(0) = i$.

Since the two determinations u and $-u$ of $\sqrt{z - 1}$ have disjoint images, the functions $h_n = u \circ g_n = \sqrt{g_n - 1}$ don't attain values in the image of $-u$ and, by the previous corollary, considering a subsequence, we may assume that (h_n) converges to a limit function $h \colon \mathbb{D} \to \mathbb{C}$.

Then $g := h^2 + 1 = \lim(h_n^2 + 1) = \lim g_n$ and considering a subsequence, since $|z_n| \leq 1$, we may assume that $z_n \to z$. Then $f := zg = \lim(z_n g_n) = \lim f_n$ is injective (f is not constant because $f'(0) = \lim f_n'(0) = 1$):

Lemma: *If a sequence (f_n) of injective analytic functions on a connected open set $U \subseteq \mathbb{C}$ converges in $\mathcal{O}(U)$, then the limit function f is injective or constant.*

Proof: Assume that f is not constant and $f(z_1) = f(z_2) = a$, where $z_1 \neq z_2$.

Let us consider a compact manifold with boundary $\Omega \subset U$ containing z_1, z_2 such that f does not attain the value a on the boundary $\partial\Omega$. When $n \gg 0$, we have

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)}{f(z)-a}\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f_n'(z)}{f_n(z)-a}\mathrm{d}z,$$

because these integrals have integer values (p. 280). Now, the left integral is ≥ 2, while the right integral is ≤ 1, because f_n is injective. Absurd.

Lemma: *Let f be an analytic function without zeroes on a Riemann surface X. If $H^1_{DR}(X) = 0$, then there exist the analytic functions $\ln f$ and $\sqrt{f}$.*

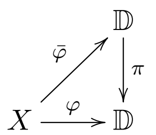
Proof: The closed 1-form $\frac{\mathrm{d}f}{f}$ is exact, so that $\ln f$ exists, and so does $\sqrt{f} = e^{(\ln f)/2}$.

Corollary: *Any open set $U \subset \mathbb{C}$ with $H^1_{DR}(U) = 0$ is isomorphic to an open subset of $\mathbb{D}$.*

Proof: We may assume that $0 \notin U$. The covering $z^2\colon \mathbb{C} - 0 \to \mathbb{C} - 0$ is trivial over U, because it admits the analytic sections $\pm h(z) := \pm\sqrt{z}$ by the lemma. Since both sections have disjoint images, U is isomorphic to an open set $h(U) \subset \mathbb{C}$ whose complement contains a disk, that we may assume to be $\mathbb{D}$. Now $f(z) = \frac{1}{z}$ transforms $h(U)$ onto an open subset of $\mathbb{D}$.

Lemma: *Let X be a Riemann surface with $H^1_{DR}(X) = 0$. If an injective morphism $\varphi\colon X \hookrightarrow \mathbb{D}$ is not surjective, there is an injective morphism $\bar{\varphi}\colon X \hookrightarrow \mathbb{D}$ such that $\bar{\varphi}^* g_{\mathbb{D}} > \varphi^* g_{\mathbb{D}}$.*

Proof: We may assume that $0 \notin \varphi(U)$. The function $\bar{\varphi} = \sqrt{\varphi}$ exists by the former lemma, so that we have a commutative diagram



where $\pi(z) = z^2$, and $\bar{\varphi}$ is injective because so is φ. By Ahlfors lemma, $\pi^* g_{\mathbb{D}} \leq g_{\mathbb{D}}$, and if we have $\pi^* g_{\mathbb{D}} = g_{\mathbb{D}}$ at a point of $\mathbb{D}$, then π is a global isometry (p. 282); absurd.

Hence $\bar{\varphi}^* g_{\mathbb{D}} > \bar{\varphi}^*(\pi^* g_{\mathbb{D}}) = \varphi^* g_{\mathbb{D}}$. q.e.d.

Now let X be a Riemann surface with a fixed vector $0 \neq D_p \in T_pU$, and let us consider pointed morphisms $\phi\colon X \hookrightarrow \mathbb{D}_r$; i.e. $\phi(p) = 0$ and $\phi_*(D_p) = \partial_x$. Any pointed morphism induces a hyperbolic metric $\phi^* g_r$ on X, that fully determines the radius r because $|D_p| = 2/r^2$. In particular, a bigger hyperbolic metric determines a smaller radius.

Given a morphism $\varphi\colon X \hookrightarrow \mathbb{D}$, composing it with a convenient isometry $\mathbb{D} \simeq \mathbb{D}_r$, we obtain a pointed morphism $\phi\colon X \hookrightarrow \mathbb{D}_r$ inducing the same hyperbolic metric, $\phi^* g_r = \varphi^* g_{\mathbb{D}}$. Hence, the above lemma may be restated as follows:

Lemma: *Let X be a Riemann surface with $H^1_{DR}(X) = 0$. If an injective morphism $\varphi\colon X \hookrightarrow \mathbb{D}_r$ is not surjective, there is an injective morphism $\bar{\varphi}\colon X \hookrightarrow \mathbb{D}_{\bar{r}}$ with $\bar{r} < r$.*

Riemann Mapping Theorem: *If $U \subset \mathbb{C}$ is a connected open set with $H^1_{DR}(U) = 0$, then U is isomorphic to the unit disk $\mathbb{D}$. In particular, U is simply connected.*

Proof: We may assume that $0 \in U$, and point U with $\partial_x \in T_0U$. The family of all injective analytic functions $f\colon U \hookrightarrow \mathbb{D}_r$ such that $f(0) = 0$ and $f'(0) = 1$ (i.e. pointed morphisms) is not empty. Let R be the infimum of all $r \in \mathbb{R}$ such that there is a pointed morphism $U \hookrightarrow \mathbb{D}_r$, and let $f_n\colon U \to \mathbb{D}_{r_n}$ be a sequence of pointed immersions such that (r_n) is a decreasing sequence with limit R. By Montel's theorem, we may assume that (f_n) converges to an analytic function $f \in \mathcal{O}(U)$, injective (p. 284) because so are the functions f_n and $f'(0) = \lim f'_n(0) = 1$.

We have $f(U) \subseteq \mathbb{D}_R$ because $f(U)$ is an open set contained in all the disks $\mathbb{D}_{r_n}$, and by the lemma $f\colon U \hookrightarrow \mathbb{D}_R$ is surjective. We conclude that $f\colon U \to \mathbb{D}_R$ is an isomorphism.

Chapter 10

Differential Geometry I

10.1 Smooth Manifolds

Definitions: To give a **sheaf** $\mathcal{O}_X$ of (real valued) continuous functions on a topological space X is to give a subalgebra $\mathcal{O}_X(U) \subseteq \mathcal{C}(U)$ for any open set $U \subseteq X$, so that for any open cover $U = \bigcup_i U_i$ we have that a continuous function $f \in \mathcal{C}(U)$ is in $\mathcal{O}_X(U)$ if and only if the restriction $f|_{U_i} \in \mathcal{O}_X(U_i)$ for any index i. That is to say, the following sequence is exact,

$$\mathcal{O}_X(U) \ \to \ \prod_i \mathcal{O}_X(U_i) \ \rightrightarrows \ \prod_{i,j} \mathcal{O}_X(U_i \cap U_j).$$

Given two topological spaces endowed with a sheaf of continuous functions $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a **morphism** $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is defined to be a continuous map $\phi\colon Y \to X$ transforming $\mathcal{O}_X(U)$ into $\mathcal{O}_Y(\phi^{-1}U)$,

$$f \in \mathcal{O}_X(U) \ \Rightarrow \ \phi^*(f) := f \circ \phi \in \mathcal{O}_Y(\phi^{-1}U).$$

Examples: A sheaf of continuous functions $\mathcal{O}_X$ on a topological space X induces a sheaf of continuous functions $\mathcal{O}_U$ on any open set $U \subseteq X$. Just put $(\mathcal{O}_U)(V) = \mathcal{O}_X(V) \subseteq \mathcal{C}(V)$, where $V \subseteq U$ is any open subset.

Given an open set $U \subseteq \mathbb{R}^n$, the $\mathcal{C}^\infty$-functions define a sheaf of continuous functions $\mathcal{C}^\infty_U$. If V is an open set in $\mathbb{R}^m$, morphisms $(U, \mathcal{C}^\infty_U) \to (V, \mathcal{C}^\infty_V)$ are just $\mathcal{C}^\infty$-maps $\phi\colon U \to V$.

Definitions: A **smooth manifold** is a topological space X endowed with a sheaf of continuous functions $\mathcal{O}_X$ such that any point of X has an open neighborhood U isomorphic to an open subset V of some $\mathbb{R}^n$ with the sheaf of $\mathcal{C}^\infty$functions, $(U, \mathcal{O}_U) \simeq (V, \mathcal{C}^\infty_V)$. In such case, the structural sheaf $\mathcal{O}_X$ will be denoted by $\mathcal{C}^\infty_X$ or $\mathcal{C}^\infty$, the **smooth functions** on an open set $U \subseteq X$ are defined to be the continuous functions in $\mathcal{C}^\infty_X(U) := \mathcal{O}_X(U)$, and the morphisms $(Y, \mathcal{C}^\infty_Y) \to (X, \mathcal{C}^\infty_X)$ between smooth manifolds are said to be the **smooth** maps. The smooth manifolds with the smooth maps define a category, and the isomorphisms are named **diffeomorphisms**.

In these notes all smooth manifolds are assumed to be σ-compact (hence separated).

The ring of germs at $p \in X$ of smooth functions is denoted by $\mathcal{O}_{X,p}$ or just $\mathcal{O}_p$,

$$\mathcal{O}_{X,p} = \varinjlim_{p \in U} \mathcal{C}^\infty_X(U).$$

A **coordinate open** set in X is an open set U diffeomorphic to an open set in $\mathbb{R}^n$. If a continuous map $(x_1, \ldots, x_n)\colon U \to V \subseteq \mathbb{R}^n$ is a diffeomorphism, we say that $(x_1, \ldots, x_n)$ is a **coordinate system** on U; then any smooth function on U is $f(x_1, \ldots, x_n)$ for a unique function $f \in \mathcal{C}^\infty(V)$. Some functions $x_1, \ldots, x_n \in \mathcal{C}^\infty(U)$ define a **local coordinate system** at a point $p \in U$ if they form a coordinate system on a neighborhood of p.

Lemma: *$\mathcal{O}_p$ is a local ring, the maximal ideal $\mathfrak{m}_p = \{f \in \mathcal{O}_p \colon f(p) = 0\}$ being generated by $x_1 - a_1, \ldots, x_n - a_n$; where $x_1, \ldots, x_n$ are local coordinates at $p = (a_1, \ldots, a_n)$.*

Proof: The epimorphism $\mathcal{O}_p \to \mathbb{R}$, $f \to f(p)$, shows that $\mathfrak{m}_p$ is a maximal ideal.

If $f \notin \mathfrak{m}_p$, then f has no zero on a neighborhood U of p; hence f is invertible in $\mathcal{C}^\infty(U)$, and the germ f_p is invertible in $\mathcal{O}_p$. The unique maximal ideal of $\mathcal{O}_p$ is $\mathfrak{m}_p$.

It is clear that $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}_p$. Conversely, if $f \in \mathfrak{m}_p$, by Hadamard's lemma in a neighborhood U of p we have $f = \sum_i f_i(x_i - a_i)$, where $f_i \in \mathcal{C}^\infty(U)$. Considering germs we see that $f \in (x_1 - a_1, \ldots, x_n - a_n)$.

Definition: $T_pX = \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_p, \mathcal{O}_p/\mathfrak{m}_p)$ is the **tangent space** to X at p. If $D \in T_pX$,

$$D(fg) = (Df) \cdot g(p) + f(p) \cdot (Dg); \;\; f, g \in \mathcal{O}_p.$$

If $(U; x_1, \ldots, x_n)$ is a coordinate neighborhood, $\left(\frac{\partial}{\partial x_i}\right)_p = (\partial_{x_i})_p = (\partial_i)_p$ denotes the vector

$$\Big(\frac{\partial}{\partial x_i}\Big)_p (f_p) = \frac{\partial f}{\partial x_i}(p).$$

Theorem: $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ *define a base of* T_pX.

Proof: They are linearly independent. If $\sum_i \lambda_i (\partial_i)_p = 0$, then

$$0 = \textstyle\sum_i \lambda_i (\partial_i)_p (x_j) = \sum_i \lambda_i \delta_{ij} = \lambda_j.$$

Moreover, if $D \in T_pX$ and $Dx_i = 0$ for any index i, then $D = 0$.

In fact, by the lemma any germ is $f = b + \sum_i h_i(x_i - a_i)$; hence

$$Df = \textstyle\sum_i (Dh_i)(x_i(p) - a_i) + \sum_i h_i(p) Dx_i = 0 + 0 = 0.$$

Now it is clear that for any vector $D \in T_pX$ we have

$$D = (Dx_1)\Big(\frac{\partial}{\partial x_1}\Big)_p + \ldots + (Dx_n)\Big(\frac{\partial}{\partial x_n}\Big)_p. \tag{10.1}$$

Definition: The **dimension** $\dim_p X$ of X at p is the dimension of the vector space T_pX, and by the above theorem it is locally constant. If it does not depend on the point p we say that it the dimension $\dim X$ of X.

Example: On any real affine space $(\mathbb{A}, E)$ of dimension n we may consider the initial topology of the affine functions $\mathbb{A} \to \mathbb{R}$ (the finest topology where the affine functions are continuous). If we consider the coordinates in an affine reference system, then the bijection $(x_1, \ldots, x_n) \colon \mathbb{A} \to \mathbb{R}^n$ is in fact a homeomorphism, since any affine function $a_1x_1 + \ldots + a_nx_n + b$ is continuous in $\mathbb{R}^n$ with the product topology (because the maps $\mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$ and $\mathbb{R} \xrightarrow{a\cdot} \mathbb{R}$ clearly are continuous). Now, the sheaf of functions $f(x_1, \ldots, x_n)$, where f is $\mathcal{C}^\infty$, does not depend on the affine reference system, and defines a structure of smooth manifold on $\mathbb{A}$.

Given a point $p \in \mathbb{A}$, the natural linear map $E \to T_p\mathbb{A}$, transforming any vector $e \in E$ into the directional derivative $\partial_{e,p}$,

$$\partial_{e,p} f = \lim_{t \to 0} \frac{f(p + te) - f(p)}{t} = \frac{\mathrm{d}(f \circ \sigma)}{\mathrm{d}t}\Big|_{t=0}, \quad \sigma(t) = p + te,$$

is an isomorphism: If $x_1, \ldots, x_n$ are coordinates in an affine reference $(p_0; e_1, \ldots, e_n)$; then $\partial_{e_i,p} = \left(\frac{\partial}{\partial x_i}\right)_p$.

Definition: Let $\varphi\colon X \to Y$ be a smooth map, and put $q = \varphi(p)$, where $p \in X$.
The **tangent linear map** of φ at p is the linear map

$$\varphi_*\colon T_pX \longrightarrow T_qY,\ (\varphi_* D)(f) = D(\varphi^* f) = D(f \circ \varphi).$$

The **chain rule** directly follows from the definition: $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

If the equations of φ on some coordinate neighborhoods of p and q are $y_i = f_i(x_1, \ldots, x_n)$, i.e. $\varphi^* y_i = f_i(x_1, \ldots, x_n)$, applying 10.1 we see that $\varphi_*((\partial_{x_j})_p) = \sum_i (\partial_{x_j} f_i)(p) \cdot (\partial_{y_i})_q$, so that the matrix of φ_* in the bases $\{(\partial_{x_j})_p\}$ and $\{(\partial_{y_i})_q\}$ is the **jacobian** matrix

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)$$

and the **inverse mapping theorem** states that *a smooth map $\varphi\colon X \to Y$ is a local diffeomorphism at p if and only if $\varphi_*\colon T_pX \to T_qY$ is an isomorphism.*

Definitions: The dual T_p^*X of the tangent space T_pX is the **cotangent space**.
If $f \in \mathcal{O}_p$, the **differential** of f at p is the 1-form

$$(\mathrm{d}_p f)(D) = Df,\ D \in T_pX.$$

If $\varphi\colon X \to Y$ is a smooth map and $q = \varphi(p)$, then the transpose map of $\varphi_*\colon T_pX \to T_qY$ is denoted by $\varphi^*\colon T_q^*Y \to T_p^*X$, $(\varphi^*\omega)(D) = \omega(\varphi_* D),\ D \in T_pX$.

1. *The differential is a derivation:*
   (a) $\mathrm{d}_p\lambda = 0$, $\lambda \in \mathbb{R}$.
   (b) $\mathrm{d}_p(f + g) = \mathrm{d}_p f + \mathrm{d}_p g$.
   (c) $\mathrm{d}_p(fg) = g(p)\mathrm{d}_p f + f(p)\mathrm{d}_p g$.
2. *If $(x_1, \ldots, x_n)$ is a local coordinate system at p, then the 1-forms $\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n$ define a base of T_p^*X, the dual base of $(\partial_{x_1})_p, \ldots, (\partial_{x_n})_p$, and*
$$\mathrm{d}_p f = \tfrac{\partial f}{\partial x_1}(p) \cdot \mathrm{d}_p x_1 + \ldots + \tfrac{\partial f}{\partial x_n}(p) \cdot \mathrm{d}_p x_n.$$
3. $\varphi^*(\mathrm{d}_q f) = \mathrm{d}_p(\varphi^* f)$, $f \in \mathcal{O}_{Y,q}$.

Proposition: *The differential defines an isomorphism $\mathrm{d}_p\colon \mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{\sim} T_p^*X$.*

Proof: The differential vanishes on $\mathfrak{m}_p^2$, since $\mathrm{d}_p(fg) = g(p)\mathrm{d}_p f + f(p)\mathrm{d}_p g$; hence it defines a linear map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to T_p^*X$, and it is an isomorphism since it transforms the generating system $[x_1 - a_1], \ldots, [x_n - a_n]$ into the base $\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n$ (see also p. 143).

Theorem: *If $\mathrm{d}_p u_1, \ldots, \mathrm{d}_p u_n$ form a base of T_p^*X, then the functions $u_1, \ldots, u_n$ define a local coordinate system at p.*

Proof: Let us consider the smooth map $\varphi = (u_1, \ldots, u_n)\colon X \to \mathbb{R}^n$, $q = \varphi(p)$.
Since $\varphi^*(\mathrm{d}_q x_i) = \mathrm{d}_p u_i$, then φ^* transforms a base into a base; hence $\varphi_*\colon T_pX \to T_q\mathbb{R}^n$ is an isomorphism, and φ is a local diffeomorphism by the inverse function theorem.

10.1.1 Tensor Fields

Definition: A **vector field** is a family of vectors $\{D_x\}_{x\in X}$, where $D_x \in T_xX$, and it is smooth (resp. of class $\mathcal{C}^m$) if for any $f \in \mathcal{C}^\infty(U)$ we have that the function $(Df)(x) = D_xf$ so is smooth (resp. of class $\mathcal{C}^m$), so that the vector field defines derivations $D\colon \mathcal{C}^\infty(U) \to \mathcal{C}^m(U)$ compatible with restrictions. Unless otherwise stated, we assume that all vector fields are smooth, and the $\mathcal{C}^\infty(X)$-module of all smooth vector fields on X is denoted by $\mathcal{D}(X)$.

Theorem: $\mathcal{D}(X) = \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty(X), \mathcal{C}^\infty(X))$.

Proof: The inverse map assigns to any derivation D the vector field $\{D_x\}_{x\in X}$ defined by the following vectors (any germ $f_x \in \mathcal{O}_x$ has a global representant f, p. 260)

$$D_xf_x \coloneqq (Df)(x),\ \ f_x \in \mathcal{O}_x.$$

We only have to prove that $(Df)(x)$ does not depend on the representant f.

If $f, g \in \mathcal{C}^\infty(X)$ have equal germ (coincide on a neighborhood U of x) we take (p. 260) a function $\phi \in \mathcal{C}^\infty(X)$, $\operatorname{supp}\phi \subseteq U$, $\phi(x) = 1$, and we have $\phi(f-g) = 0$,

$$0 = D(\phi(f-g)) = (D\phi)(f-g) + \phi(Df - Dg),$$

and taking values at x we see that $(Df)(x) = (Dg)(x)$.

Definition: A **1-form** on X is a family of 1-forms $\{\omega_x\}_{x\in X}$, where $\omega_x \in T_x^*X$, and it is smooth (resp. of class $\mathcal{C}^m$) if the function $\omega(D)(x) = \omega_x(D_x)$ is smooth (resp. of class $\mathcal{C}^m$) for any vector field $D \in \mathcal{D}(U)$, so that it defines $\mathcal{C}^\infty(U)$-linear morphisms $\omega\colon \mathcal{D}(U) \to \mathcal{C}^m(U)$ compatible with restrictions. Unless otherwise stated, we assume that all 1-forms are smooth, and the $\mathcal{C}^\infty(X)$-module of all smooth 1-forms on X is denoted $\Omega(X)$.

Theorem: $\Omega(X) = \mathrm{Hom}_{\mathcal{C}^\infty(X)}(\mathcal{D}(X), \mathcal{C}^\infty(X))$.

Proof: The inverse map assigns to any morphism of modules $\omega\colon \mathcal{D}(X) \to \mathcal{C}^\infty(X)$ the following family $\{\omega_x\}_{x\in X}$ of 1-forms,

$$\omega_x(D_x) \coloneqq \omega(D)(x),$$

where D is any vector field on X extending D_x (on a coordinate neighborhood U it clearly exists and, multiplying by a plateau function, it may be extended by 0 on $X - U$).

We have to show that $\omega(D)(x)$ does not depend on the vector field D.

If D' is another vector field and $D_x = D'_x$, just apply the following lemma to $D' - D$.

Lemma: *If D is a vector field on X and $D_x = 0$, then $\omega(D)(x) = 0$.*

Proof: We have $D = \sum_i f_i\partial_i$, $f_i(x) = 0$, on a coordinate neighborhood U.

Let $\phi \in \mathcal{C}^\infty(X)$ such that $\operatorname{supp}\phi \subseteq U$, $\phi(x) = 1$. Now $\phi^2 D = \sum_i(\phi f_i)(\phi\partial_i)$, where ϕf_i, $\phi\partial_i$ are extended by 0 on $X - U$. We conclude taking values at x in the equality

$$\phi^2\omega(D) = \textstyle\sum_i(\phi f_i)\omega(\phi\partial_i).$$

Definition: The **differential** of $f \in \mathcal{C}^\infty(X)$ is the 1-form $\{\mathrm{d}_xf\}_{x\in X}$. In the dual of $\mathcal{D}(X)$,

$$(\mathrm{d}f)(D) = Df.$$

Definition: A (p,q)-**tensor field** on X is a family of (p,q)-tensors $\{T_x\}_{x\in X}$ on T_xX, and it is smooth (resp. of class $\mathcal{C}^m$) if for any $(D^1,\dots,D^p,\omega^1,\dots,\omega^q) \in \mathcal{D}(U)^p \times \Omega(U)^q$, we have that

$f(x) = T_x(D_x^1, \ldots, D_x^p, \omega_x^1, \ldots, \omega_x^q)$ is a smooth (resp. of class $\mathcal{C}^m$) function on U, so that it defines $\mathcal{C}^\infty(U)$-multilinear maps $T\colon \mathcal{D}(U)^p \times \Omega(U)^q \to \mathcal{C}^\infty(U)$ compatible with restrictions.

Unless otherwise stated, any tensor field is assumed to be smooth, and the above argument shows that *the module $\mathcal{T}^{p,q}(X)$ of smooth (p,q)-tensor fields on X is isomorphic to the module of $\mathcal{C}^\infty(X)$-multilinear maps $\mathcal{D}(X)^p \times \Omega(X)^q \to \mathcal{C}^\infty(X)$, and the module $\Omega^p(X)$ of smooth p-forms is isomorphic to the module of alternate $\mathcal{C}^\infty(X)$-multilinear maps $\mathcal{D}(X)^p \to \mathcal{C}^\infty(X)$.*

Operations with tensors and p-forms (tensor and exterior products, contraction of indices,...) are pointwise extended to tensor fields and **differential p-forms**.

In a coordinate neighborhood $(U; x_1, \ldots, x_n)$, we have the vector fields $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$, and any vector field on U is $D = \sum_i f_i \partial_i$, where $f_i \in \mathcal{C}^\infty(U)$, since $f_i = Dx_i$ is smooth. Hence, *$\mathcal{D}(U)$ is a free module, with base $\partial_1, \ldots, \partial_n$, the module $\Omega(U)$ is free, with base $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$, the module $\mathcal{T}^{p,q}(U)$ is free, with base $\{\mathrm{d}x_{i_1} \otimes \ldots \otimes \mathrm{d}x_{i_p} \otimes \partial_{j_1} \otimes \ldots \otimes \partial_{j_q}\}$, and the module $\Omega^p(U)$ is free, with base $\{\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p}\}$.*

Definition: In a coordinate neighborhood, the **exterior differential** of p-forms is

$$\begin{aligned} \omega_p &= \textstyle\sum_\alpha f_\alpha \, \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p}, \ \alpha = (i_1 < \ldots < i_p) \\ \mathrm{d}\omega_p &= \textstyle\sum_\alpha (\mathrm{d}f_\alpha) \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_p}. \end{aligned}$$

It is easy to check that it is $\mathbb{R}$-linear, $\mathrm{d} \circ \mathrm{d} = 0$, and that it is an antiderivation,

$$\mathrm{d}(\omega_p \wedge \omega_q) = (\mathrm{d}\omega_p) \wedge \omega_q + (-1)^p \omega_p \wedge (\mathrm{d}\omega_q).$$

Moreover, we prove by induction on p that $\mathrm{d}(\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_p) = 0$.
If $p = 1$, it follows from $\mathrm{d}^2 = 0$, and in the general case we have:

$$\mathrm{d}(\mathrm{d}f_1 \wedge \ldots \mathrm{d}f_p) = (\mathrm{d}\mathrm{d}f_1) \wedge (\mathrm{d}f_2 \wedge \ldots \mathrm{d}f_p) - \mathrm{d}f_1 \wedge \mathrm{d}(\mathrm{d}f_2 \wedge \ldots \mathrm{d}f_p) = 0 + 0 = 0.$$

Now let us prove that the exterior differential is intrinsic, independent of the coordinate system. If $\omega_p = \sum_\alpha g_\alpha \, \mathrm{d}u_{i_1} \wedge \ldots \wedge \mathrm{d}u_{i_p}$, then, by the above properties,

$$\begin{aligned} \mathrm{d}\omega_p &= \textstyle\sum_\alpha \mathrm{d}g_\alpha \wedge \mathrm{d}u_{i_1} \wedge \ldots \wedge \mathrm{d}u_{i_p} + \sum_\alpha g_\alpha \mathrm{d}(\mathrm{d}u_{i_1} \wedge \ldots \wedge \mathrm{d}u_{i_p}) \\ &= \textstyle\sum_\alpha \mathrm{d}g_\alpha \wedge \mathrm{d}u_{i_1} \wedge \ldots \wedge \mathrm{d}u_{i_p} + 0. \end{aligned}$$

If we put $\Omega^\bullet(X) = \oplus_p \Omega^p(X)$, then $\mathrm{d}\colon \Omega^\bullet(X) \to \Omega^\bullet(X)$ is an antiderivation, and $\mathrm{d}(\mathrm{d}\omega) = 0$. Moreover, for any smooth map $\varphi\colon Y \to X$ we have $\varphi^*(\mathrm{d}\omega_p) = \mathrm{d}(\varphi^*\omega_p)$.

Lemma: $\mathrm{d} \circ i_D + i_D \circ \mathrm{d}$ *is a derivation of the algebra* $\Omega^\bullet(X)$.

Proof: $(\mathrm{d}i_D + i_D\mathrm{d})(\omega_p \wedge \omega_q) = \mathrm{d}((i_D\omega_p) \wedge \omega_q + (-1)^p \omega_p \wedge i_D\omega_q) + i_D((\mathrm{d}\omega_p) \wedge \omega_q + (-1)^p \omega_p \wedge \mathrm{d}\omega_q)$

$$\begin{aligned} &= (\mathrm{d}i_D\omega_p) \wedge \omega_q + (-1)^{p-1}(i_D\omega_p) \wedge \mathrm{d}\omega_q + (-1)^p(\mathrm{d}\omega_p) \wedge i_D\omega_q + \omega_p \wedge \mathrm{d}i_D\omega_q + \\ &\quad + (i_D\mathrm{d}\omega_p) \wedge \omega_q + (-1)^{p+1}(\mathrm{d}\omega_p) \wedge i_D\omega_q + (-1)^p(i_D\omega_p) \wedge \mathrm{d}\omega_q + \omega_p \wedge i_D\mathrm{d}\omega_q = \\ &= ((\mathrm{d}i_D + i_D\mathrm{d})(\omega_p) \wedge \omega_q + \omega_p \wedge (\mathrm{d}i_D + i_D\mathrm{d})(\omega_q). \end{aligned}$$

Cartan's Theorem: $D^L\omega_p = \mathrm{d}i_D\omega_p + i_D\mathrm{d}\omega_p$.

Proof: By the lemma, we may assume that $\omega_p = f$ or $\omega_p = \mathrm{d}f$,

$$(\mathrm{d}i_D + i_D\mathrm{d})f = \mathrm{d}0 + (\mathrm{d}f)(D) = Df = D^L f,$$
$$(\mathrm{d}i_D + i_D\mathrm{d})\mathrm{d}f = \mathrm{d}i_D\mathrm{d}f + i_D 0 = \mathrm{d}(Df) = D^L(\mathrm{d}f).$$

Corollary: *The Lie derivative commutes with the exterior differential,* $[D^L, \mathrm{d}] = 0$.

Cartan's Formula: $(\mathrm{d}\omega)(D, \bar{D}) = D(\omega(\bar{D})) - \bar{D}(\omega(D)) - \omega([D, \bar{D}])$.

Proof:
$$\begin{aligned}(\mathrm{d}\omega)(D, \bar{D}) &= (i_D\mathrm{d}\omega)(\bar{D}) = (D^L\omega - \mathrm{d}i_D\omega)(\bar{D}) = D(\omega(\bar{D})) - \omega([D, \bar{D}]) - \mathrm{d}(\omega(D))(\bar{D}) \\ &= D(\omega(\bar{D})) - \omega([D, \bar{D}]) - \bar{D}(\omega(D)).\end{aligned}$$

10.1.2 Smooth Submanifolds

Definitions: A smooth map $\varphi\colon Y \to X$ is a **local embedding** (resp. **regular projection**) at a point $q \in Y$ if $\varphi_*\colon T_qY \to T_pX$, $p = \varphi(q)$, is injective (resp. surjective).

Theorem: *If* $\varphi\colon Y \to X$ *is a local embedding at* q*, then there are coordinate neighborhoods of* p *and* q *where* $\varphi(y_1, \ldots, y_m) = (y_1, \ldots, y_m, 0 \ldots, 0)$.

Proof: Let $(U; x_1, \ldots x_n)$ be a coordinate neighborhood of p, and put $y_i = \varphi^*x_i$.

We may assume that $\mathrm{d}_qy_1, \ldots, \mathrm{d}_qy_m$ form a base of T^*_qY, since $\varphi^*\colon T^*_pX \to T^*_qY$ is surjective. Now $y_1, \ldots, y_m$ are coordinates on a neighborhood of q, where we have $y_{m+j} = f_j(y_1, \ldots, y_m)$, and all the functions $f_j(x_1, \ldots, x_m)$ are defined on a small neighborhood of p. Now

$$\begin{aligned}u_i &= x_i,\ i = 1, \ldots, m \\ u_{m+j} &= x_{m+j} - f_j(x_1, \ldots, x_m),\ j = 1, \ldots, r\end{aligned}$$

have linearly independent differentials at p, so that they are coordinates on a smaller neighborhood of p. In these coordinates $\varphi(y_1, \ldots, y_m) = (y_1, \ldots, y_m, 0 \ldots, 0)$.

Theorem: *If* $\varphi\colon Y \to X$ *is a regular projection at* q*, then there are coordinate neighborhoods of* p *and* q *where* $\varphi(y_1, \ldots, y_n) = (y_1, \ldots, y_m),\ m \leq n$.

Proof: Let $(U; x_1, \ldots x_m)$ be a coordinate neighborhood of p, and put $y_i = \varphi^*x_i$.

The differentials $\mathrm{d}_qy_1, \ldots, \mathrm{d}_qy_m$ are linearly independent, since $\varphi^*\colon T^*_pX \to T^*_qY$ is injective.

If we complete them so as to obtain a base $\mathrm{d}_qy_1, \ldots, \mathrm{d}_qy_n$ of T^*_qY, then $y_1, \ldots, y_n$ are coordinates on a neighborhood of q, and in these coordinates $\varphi(y_1, \ldots, y_n) = (y_1, \ldots, y_m)$.

Definition: Let Y be a subspace of a smooth manifold X. A continuous function $f\colon Y \to \mathbb{R}$ is smooth if it locally coincides with smooth functions on X: any point $y \in Y$ has an open neighborhood U in X such that $f|_{U\cap Y} = F|_{U\cap Y}$ for some function $F \in \mathcal{C}^\infty(U)$.

So we have a sheaf $\mathcal{C}^\infty_Y$ of continuous functions on Y

$$\mathcal{C}^\infty_Y(V) = \{f\colon V \to \mathbb{R} \text{ smooth}\},$$

and Y is a (smooth) **submanifold** of X when $(Y, \mathcal{C}^\infty_Y)$ is a smooth manifold.

For example, $Y = 0 \times \mathbb{R}^m$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$, since in this case the sheaf $\mathcal{C}^\infty_Y$ is just the sheaf of smooth functions on $\mathbb{R}^m$ in the usual sense.

Lemma: *If* Y *is a submanifold of* X*, the inclusion* $i\colon Y \to X$ *is a local embedding.*

Proof: The linear tangent map $i_*\colon \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{Y,q}, \mathbb{R}) \to \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,q}, \mathbb{R})$ is injective, since the restriction morphism $i^*\colon \mathcal{O}_{X,q} \to \mathcal{O}_{Y,q}$ is surjective by definition.

Theorem: *Let* $Y = \{x \in X\colon f_1(x) = \ldots = f_r(x) = 0\}$*; where* $f_1, \ldots, f_r \in \mathcal{C}^\infty(X)$*. If the differentials* $\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_r$ *are linearly independent at any point* p *of* Y*, then* Y *is a submanifold of* X *of codimension* r*, and the tangent space at* p *is*

$$T_p Y = \langle \mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_r \rangle^{\mathrm{o}}.$$

Proof: We complete the differentials so as to obtain a base $\mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_n$ of $T_p^* X$. Then

$$\varphi = (f_1, \ldots, f_n)\colon U \longrightarrow \mathbb{R}^n$$

defines a diffeomorphism of U onto an open set U' of $\mathbb{R}^n$, and an isomorphism of ringed spaces of $(U \cap Y, \mathcal{C}^\infty_{U\cap Y})$ onto the open set $\varphi(U \cap Y) = U' \cap (0 \times \mathbb{R}^{n-r})$ of $0 \times \mathbb{R}^{n-r}$.

Hence $(Y, \mathcal{C}^\infty_Y)$ is a manifold of codimension r.

Moreover, if $D \in T_p Y$, then $(\mathrm{d}f_j)(i_* D) = (i_* D) f_j = D(f_j \circ i) = D(0) = 0$.

Hence $T_p Y \subseteq \langle \mathrm{d}_p f_1, \ldots, \mathrm{d}_p f_r \rangle^{\mathrm{o}}$, and they coincide since both have dimension $n - r$.

Theorem: *Let* Y *be a submanifold of* X *of codimension* r*. Any point* $q \in Y$ *has a coordinate neighborhood* $(U; u_1, \ldots, u_n)$ *in* X *such that*

$$U \cap Y = \{x \in U\colon u_1(x) = \ldots = u_r(x) = 0\}.$$

Proof: Since the inclusion $i\colon Y \to X$ is a local embedding, there is a coordinate neighborhood $(V; y_1, \ldots, y_m)$ of q in Y and a coordinate neighborhood $(U; u_1, \ldots, u_n)$ of q in X, where $i(y_1, \ldots, y_m) = (y_1, \ldots, y_m, 0 \ldots, 0)$; hence $y_i = u_i|_Y$.

Since Y is a subspace of X, replacing U by a smaller neighborhood, $V = U \cap Y$.

We may assume that U is an open set of $\mathbb{R}^n$, and $U \cap Y = V \times 0$, where V is an open set of $\mathbb{R}^m$; now the statement is obvious.

Corollary: *Any submanifold is a* ***locally closed*** *subspace* (a closed set in an open subset).

10.2 Linear Connections

Definition: A **linear connection** on a smooth manifold X assigns to any pair of vector fields D_1, D_2 a vector field $D_1^\nabla D_2$, the **covariant derivative** of D_2 in the direction D_1, so that

1. $D^\nabla(D_1 + D_2) = D^\nabla D_1 + D^\nabla D_2$,
   $D^\nabla(f\bar{D}) = (Df)\bar{D} + f D^\nabla \bar{D}$.
2. $(D_1 + D_2)^\nabla D = D_1^\nabla D + D_2^\nabla D$,
   $(fD)^\nabla \bar{D} = f(D^\nabla \bar{D})$.

Lemma: ∇ *may be uniquely extended to a derivation of tensors preserving the type and*

1. $D^\nabla f = Df$.
2. $D^\nabla(C_i^j T) = C_i^j(D^\nabla T)$.

Proof: To prove the uniqueness, we derive $\omega(\bar{D}) = C_1^1(\omega \otimes \bar{D})$,

$$(D^\nabla \omega)(\bar{D}) = D(\omega(\bar{D})) - \omega(D^\nabla \bar{D})$$

and deriving $T(D_1, \ldots, \omega_q) = C^{1\ldots p+q}_{1\ldots p+q}(D_1 \otimes \ldots D_p \otimes T \otimes \omega_1 \otimes \ldots \otimes \omega_q)$, we see that

$$(D^\nabla T)(D_1, \ldots, \omega_q) = D(T(D_1, \ldots, \omega_q)) - T(D^\nabla D_1, \ldots, \omega_q) - \ldots - T(D_1, \ldots, D^\nabla \omega_q).$$

To prove the existence, just take these formulae as definitions.

Definition: The **covariant differential** ∇T of a (p,q)-tensor T is the $(p+1,q)$-tensor

$$(\nabla T)(D,D_1,\dots,D_p,\omega_1,\dots\omega_q) = (D^\nabla T)(D_1,\dots,D_p,\omega_1,\dots\omega_q)$$

and we say that T is **parallel** or constant when $\nabla T = 0$; i.e., $D^\nabla T = 0$ with any vector field D.

Lemma: *If D or $\bar{D}$ vanishes on an open set U, so does $D^\nabla \bar{D}$.*

Proof: If $x \in U$, take $\phi \in \mathcal{C}^\infty(X)$ such that $\operatorname{supp}\phi \subseteq U$, and $\phi(x) = 1$.

If $\phi\bar{D} = 0$, then $0 = D^\nabla(\phi\bar{D}) = (D\phi)\bar{D} + \phi(D^\nabla\bar{D})$, and $0 = \lambda\bar{D}_x + (D^\nabla\bar{D})_x = (D^\nabla\bar{D})_x$.

If $\phi D = 0$, then $0 = (\phi D)^\nabla\bar{D} = \phi(D^\nabla\bar{D})$, and $(D^\nabla\bar{D})_x = 0$. q.e.d.

This lemma shows that ∇ induces a linear connection on any open set U of X, so that

$$(D^\nabla\bar{D})|_U = (D|_U)^\nabla(\bar{D}|_U).$$

On a coordinate neighborhood $(U;x_1,\dots,x_n)$ the connection is given by the **Christoffel symbols** $\Gamma^k_{ij} \in \mathcal{C}^\infty(U)$,

$$\partial_i^\nabla\partial_j = \textstyle\sum_k \Gamma^k_{ij}\partial_k.$$

Example: *On any real vector space of finite dimension E there is a unique linear connection ∇ such that the fields D^e defined* (p. 288) *by the vectors $e \in E$ are parallel,* $D^\nabla(D^e) = 0$.

Uniqueness is obvious, and to prove the existence, just consider the connection with null Christoffel symbols in the coordinate system $(x_1,\dots,x_n)$ defined by a base of E,

$$D^\nabla(f_1\partial_1 + \ldots + f_n\partial_n) = (Df_1)\partial_1 + \ldots + (Df_n)\partial_n.$$

Definition: A **vector field** on X **with support** on a curve $\sigma\colon I \to X$ is a family of vectors $\{D_t\}_{t\in I}$, where $D_t \in T_{\sigma(t)}X$, and it is smooth (as we shall always assume) if the function $(Df)(t) := D_t f$ is smooth for any smooth function f on an open set $U \subseteq X$, so that it may be viewed as a derivation $D\colon \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(I)$.

The $\mathcal{C}^\infty(I)$-module of fields with support on σ is denoted by $\mathcal{D}_\sigma = \operatorname{Der}_\mathbb{R}(\mathcal{C}^\infty(X),\mathcal{C}^\infty(I))$.

In a coordinate neighborhood, a vector field with support is $D_t = \sum_i f_i(t)\partial_i$, $f_i \in \mathcal{C}^\infty(I)$.

The vector field ∂_t on I defines vector field T with support on σ (named **velocity** of σ)

$$T = \sigma_*(\partial_t)\colon \mathcal{C}^\infty(X) \xrightarrow{\sigma^*} \mathcal{C}^\infty(I) \xrightarrow{\partial_t} \mathcal{C}^\infty(I).$$

On the other hand, any vector field D on X also defines a vector field with support on σ

$$D|_\sigma\colon \mathcal{C}^\infty(X) \xrightarrow{D} \mathcal{C}^\infty(X) \xrightarrow{\sigma^*} \mathcal{C}^\infty(I),$$

and $\partial_t^\nabla(D|_\sigma) := T^\nabla D$ is a well defined vector field with support on σ by the following

Lemma: *Let $D,\bar{D}$ be vector fields on X. Given a linear connection ∇, the vector $(D^\nabla\bar{D})_p$ only depends on D_p and the value of $\bar{D}$ along a curve tangent to D_p.*

Proof: In a coordinate neighborhood, $D = \sum_i f_i\partial_i$, $\bar{D} = \sum_j g_j\partial_j$,

$$D^\nabla\bar{D} = \sum_j (Dg_j)\partial_j + \sum_j g_j D^\nabla\partial_j = \sum_j (Dg_j)\partial_j + \sum_{i,j,k} g_j f_i \Gamma^k_{ij}\partial_k,$$

$$(D^\nabla\bar{D})_p = \sum_j (D_p g_j)(\partial_j)_p + \sum_{i,j,k} g_j(p) f_i(p)\Gamma^k_{ij}(p)(\partial_k)_p,$$

and $(D^\nabla \bar{D})_p$ is determined by the values $f_i(p)$, i.e. D_p, the values $g_j(p)$, i.e. $\bar{D}_p$, and the values $D_p g_j$, fully determined by the values of g_j on any curve tangent to D_p.

Lemma: *This covariant derivative of vector fields $\partial_t^\nabla : \mathcal{D}(X) \to \mathcal{D}_\sigma$ may be uniquely extended to a covariant derivation of vector fields with support $\partial_t^\nabla : \mathcal{D}_\sigma \to \mathcal{D}_\sigma$, i.e. a map such that*

$$\partial_t^\nabla (D|_\sigma) = T^\nabla D; \quad D \in \mathcal{D}(X),$$
$$\partial_t^\nabla (D + D') = \partial_t^\nabla D + \partial_t^\nabla D'; \quad D, D' \in \mathcal{D}_\sigma,$$
$$\partial_t^\nabla (fD) = (\partial_t f)D + f(\partial_t^\nabla D); \quad f \in \mathcal{C}^\infty(I), D \in \mathcal{D}_\sigma.$$

Proof: Locally, if $\sigma(t) = (x_1(t), \ldots, x_n(t))$, then $T_t = \sum_i x_i'(t)\partial_j$ and the unique extension is

$$\partial_t^\nabla \Big(\sum_{j=1}^n f_j(t)\partial_j \Big) = \sum_{j=1}^n f_j'(t)\partial_j + \sum_{j=1}^n f_j(t)\partial_t^\nabla \partial_j = \sum_{j=1}^n f_j'(t)\partial_j + \sum_{i,j,k=1}^n x_i'(t) f_j(t) \Gamma_{ij}^k(\sigma(t))\partial_k.$$

Definition: The covariant derivative of vector field D with support on a curve $\sigma\colon I \to X$ with a vector field $\bar{D} = f(t)\partial_t$ on I is defined to be $\bar{D}^\nabla D := f(\partial_t^\nabla D)$, and D is **parallel** or constant when $\bar{D}^\nabla D = 0$ for any vector field $\bar{D}$ on I; i.e. $\partial_t^\nabla D = 0$.

The curve is a **geodesic** of ∇ if it has constant velocity, $\partial_t^\nabla T = 0$.

Theorem: *Given a curve $\sigma\colon I \to X$ and a vector $D_p \in T_pX$, $p = \sigma(t_0)$, there is a unique parallel vector field D with support on σ such that $D_{t_0} = D_p$, and this **parallel transport** of vectors $T_pX \to T_{\sigma(t)}X$, $D_p \mapsto D_{\sigma(t)}$ is a linear isomorphism (depending on the curve σ).*

Proof: In a coordinate neighborhood, we have $\partial_t^\nabla \partial_i = \sum_j h_{ij}(t)\partial_j$.

The condition of a vector field with support $D = \sum_i f_i(t)\partial_i$ being parallel,

$$0 = \partial_t^\nabla D = \textstyle\sum_i f_i'(t)\partial_i + \sum_{i,j} h_{ij}(t) f_i(t)\partial_j,$$

states that the functions $f_i(t)$ define a solution of the linear differential equation

$$f_i'(t) = -\textstyle\sum_j h_{ji}(t) f_j(t),$$

with the initial condition $f_i(t_0) = a_i$, when $D_p = \sum_i a_i\partial_i$. The parallel transport of D_p (hence of a base of T_pX) exists and it is unique (p. 262) on an interval $(t_0 - \varepsilon, t_0 - \varepsilon)$.

Since the parallel transport is linear, it is defined on $(t_0 - \varepsilon, t_0 - \varepsilon)$ for any vector of T_pX.

Now it is clear that the parallel transport is defined on the whole interval I.

Theorem: *Given $D_p \in T_pX$, there exists a geodesic $\sigma\colon I \to X$ passing through p at $t = 0$ with tangent D_p, and any two coincide on the common definition interval.*

Proof: Locally, $\sigma(t) = (x_1(t), \ldots, x_n(t))$ is a geodesic if

$$T = x_1'(t)\partial_1 + \ldots + x_n'(t)\partial_n,$$
$$\partial_t^\nabla T = \sum_i x_i''(t)\partial_i + \sum_{i,j,k} x_i'(t)x_j'(t)\Gamma_{ij}^k(\sigma(t))\,\partial_k = 0,$$
$$x_k''(t) = -\textstyle\sum_{i,j} x_i'(t)x_j'(t)\Gamma_{ij}^k(x_1(t), \ldots, x_n(t)),$$

and the theorem follows from the existence and uniqueness of the solution of a differential equation (p. 262).

10.2.1 Torsion and Curvature

Definition: The **torsion** Tor_∇ of a linear connection ∇ is the $(2,1)$-tensor

$$\mathrm{Tor}_\nabla(D_1, D_2) = D_1^\nabla D_2 - D_2^\nabla D_1 - [D_1, D_2]$$

and ∇ is **symmetric** if the torsion is null, $[D_1, D_2] = D_1^\nabla D_2 - D_2^\nabla D_1$, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Since $\mathrm{Tor}_\nabla(D, D) = 0$, we only have to show that Tor_∇ is $\mathcal{C}^\infty(X)$-linear in the first variable,

$$\begin{aligned}\mathrm{Tor}_\nabla(fD_1, D_2) &= fD_1^\nabla D_2 - (D_2 f)D_1 - fD_2^\nabla D_1 + [D_2, fD_1]\\ &= fD_1^\nabla D_2 - (D_2 f)D_1 - fD_2^\nabla D_1 + (D_2 f)D_1 + f[D_2, D_1]\\ &= fD_1^\nabla D_2 - fD_2^\nabla D_1 - f[D_1, D_2] = f\mathrm{Tor}_\nabla(D_1, D_2).\end{aligned}$$

Definition: The **curvature** R of ∇ is the $(3,1)$-tensor (alternate in the two first indices)

$$R(D_1, D_2, D_3) = D_1^\nabla(D_2^\nabla D_3) - D_2^\nabla(D_1^\nabla D_3) - [D_1, D_2]^\nabla D_3,$$

Let us see that R is $\mathcal{C}^\infty(X)$-linear in the third index. Given vector fields $\delta_1, \delta_2, \delta_3$,

$$\begin{aligned}R(\delta_1, \delta_2, f\delta_3) &= \delta_1^\nabla((\delta_2 f)\delta_3 + f\delta_2^\nabla\delta_3) - \delta_2^\nabla((\delta_1 f)\delta_3 + f\delta_1^\nabla\delta_3) - ([\delta_1, \delta_2]f)\delta_3 - f[\delta_1, \delta_2]^\nabla\delta_3 =\\ &= (\delta_1\delta_2 f)\delta_3 + (\delta_2 f)\delta_1^\nabla\delta_3 + (\delta_1 f)\delta_2^\nabla\delta_3 + f\delta_1^\nabla\delta_2^\nabla\delta_3 - -(\delta_2\delta_1 f)\delta_3 - (\delta_1 f)\delta_2^\nabla\delta_3 -\\ &\quad - (\delta_2 f)\delta_1^\nabla\delta_3 - f\delta_2^\nabla\delta_1^\nabla\delta_3 - (\delta_1\delta_2 f)\delta_3 + (\delta_2\delta_1 f)\delta_3 - f[\delta_1, \delta_2]^\nabla\delta_3 =\\ &= f\delta_1^\nabla\delta_2^\nabla\delta_3 - f\delta_2^\nabla\delta_1^\nabla\delta_3 - f[\delta_1, \delta_2]^\nabla\delta_3 = fR(\delta_1, \delta_2, \delta_3).\end{aligned}$$

Definition: A linear connection ∇ is **flat** if there are local bases $D_1, \dots, D_n$ of parallel vector fields, $\nabla D_i = 0$; and it is **locally Euclidean** if any point admits a local coordinate system $(x_1, \dots, x_n)$ with null Christoffel symbols, $\partial_i^\nabla \partial_j = 0$.

Theorem: *A linear connection ∇ is flat if and only if the curvature tensor R is null.*

Proof: If ∇ is flat, then $R(D_i, D_j, D_k) = D_i^\nabla D_j^\nabla D_k - D_j^\nabla D_i^\nabla D_k - [D_i, D_j]^\nabla D_k = 0$, since $D^\nabla D_k = 0$ for any vector field D; hence $R = 0$.

Conversely, if $R = 0$, we may assume that we are at the origin of $\mathbb{R}^n$.

Take a vector $D_0 \in T_0\mathbb{R}^n$, and extend it by parallel transport along the axis OX_1; then along the lines parallel to OX_2, so that we get a vector field on the plane OX_1X_2, and so on.

This field D is smooth (the solutions of a differential equation smoothly depend on the initial conditions, p. 264) and $\partial_1^\nabla D = 0$ on OX_1, $\partial_2^\nabla D = 0$ on OX_1X_2, ... , $\partial_n^\nabla D = 0$ on $\mathbb{R}^n$.

Let us show $\partial_r^\nabla D = 0$, by descending induction. If $\partial_{r+1}^\nabla D = \ldots = \partial_n^\nabla D = 0$, then

$$0 = R(\partial_{r+1}, \partial_r, D) = \partial_{r+1}^\nabla \partial_r^\nabla D - \partial_r^\nabla \partial_{r+1}^\nabla D = \partial_{r+1}^\nabla \partial_r^\nabla D,$$

so that $\partial_r^\nabla D$ is constant along the lines parallel to OX_{r+1}.

Since $\partial_r^\nabla D$ vanishes on the submanifold $OX_1 \dots X_r$, it is null on $OX_1 \dots X_{r+1}$.

Repeating the argument with the equality $R(\partial_{r+2}, \partial_r, D) = 0$ we obtain that $\partial_r^\nabla D$ vanishes on $OX_1 \dots X_{r+2}$, and finally we see that $\partial_r^\nabla D = 0$ on $\mathbb{R}^n$. The vector field D is parallel.

Repeating the argument with a base at the origin, we obtain a base of parallel vector fields.

Corollary: ∇ *is locally Euclidean if and only if $R = 0$ and $\mathrm{Tor}_\nabla = 0$.*

Proof: If ∇ is locally Euclidean, it is clear that $R = 0$ and $\mathrm{Tor}_\nabla = 0$.

Conversely, if $R = 0$, there are local bases of parallel vector fields $D_1, \dots, D_n$.

If moreover the torsion is null, $[D_i, D_j] = D_i^\nabla D_j - D_j^\nabla D_i = 0$, and there exist local coordinate systems where $D_i = \partial_i$ (p. 274). Now $\partial_i^\nabla \partial_j = D_i^\nabla D_j = 0$.

Theorem: *Any symmetric linear connection* ∇ *fulfills* **Bianchi's Identity** *and* **Bianchi's Differential Identity**:

$$R(D_1, D_2, D_3) + R(D_2, D_3, D_1) + R(D_3, D_1, D_2) = 0,$$
$$(\nabla R)(D_1, D_2, D_3, D_4) + (\nabla R)(D_2, D_3, D_1, D_4) + (\nabla R)(D_3, D_1, D_2, D_4) = 0.$$

Proof: Since R and ∇R are tensors, we may check these identities at a point; hence we may assume that $[D_i, D_j] = 0$, so that $D_i^\nabla D_j = D_j^\nabla D_i$.

$$\begin{aligned}
&R(D_1, D_2, D_3) + R(D_2, D_3, D_1) + R(D_3, D_1, D_2) = \\
&\quad = D_1^\nabla D_2^\nabla D_3 - D_2^\nabla D_1^\nabla D_3 + D_2^\nabla D_3^\nabla D_1 - D_3^\nabla D_2^\nabla D_1 + D_3^\nabla D_1^\nabla D_2 - D_1^\nabla D_3^\nabla D_2 \\
&\quad = D_1^\nabla (D_2^\nabla D_3 - D_3^\nabla D_2) + D_2^\nabla (D_3^\nabla D_1 - D_1^\nabla D_3) + D_3^\nabla (D_1^\nabla D_2 - D_2^\nabla D_1) = 0. \\
&(\nabla R)(D_1, D_2, D_3, D_4) + (\nabla R)(D_2, D_3, D_1, D_4) + (\nabla R)(D_3, D_1, D_2, D_4) = \\
&\quad = D_1^\nabla (R(D_2, D_3, D_4)) - R(D_1^\nabla D_2, D_3, D_4) - R(D_2, D_1^\nabla D_3, D_4) - R(D_2, D_3, D_1^\nabla D_4) \\
&\quad + D_2^\nabla (R(D_3, D_1, D_4)) - R(D_2^\nabla D_3, D_1, D_4) - R(D_3, D_2^\nabla D_1, D_4) - R(D_3, D_1, D_2^\nabla D_4) \\
&\quad + D_3^\nabla (R(D_1, D_2, D_4)) - R(D_3^\nabla D_1, D_2, D_4) - R(D_1, D_3^\nabla D_2, D_4) - R(D_1, D_2, D_3^\nabla D_4) \\
&\quad = D_1^\nabla (R(D_2, D_3, D_4)) - R(D_2, D_3, D_1^\nabla D_4) + D_2^\nabla (R(D_3, D_1, D_4)) - R(D_3, D_1, D_2^\nabla D_4) \\
&\quad + D_3^\nabla (R(D_1, D_2, D_4)) - R(D_1, D_2, D_3^\nabla D_4) = 0
\end{aligned}$$

Difference of Two Connections: The difference $T(D_1, D_2) = D_1^{\bar\nabla} D_2 - D_1^\nabla D_2$ of two linear connections $\bar\nabla, \nabla$ is a $(2,1)$-tensor. It is clearly $\mathcal{C}^\infty(X)$-linear in D_1, and

$$T(D_1, fD_2) = (D_1 f)D_2 + D_1^{\bar\nabla} D_2 - (D_1 f)D_2 - D_1^\nabla D_2 = fT(D_1, D_2).$$

1. $\bar\nabla$ *and* ∇ *have equal torsion if and only if* T *is symmetric.*
2. $\bar\nabla$ *and* ∇ *have the same geodesics if and only if* T *is alternate.*

   On a common geodesic, $T(D_p, D_p) = (D^{\bar\nabla} D)_p - (D^\nabla D)_p = 0 + 0$.

   Conversely, if T is alternate and $D^\nabla D = 0$, so is $D^{\bar\nabla} D = T(D, D) + D^\nabla D = 0$.
3. *If* $\bar\nabla$ *and* ∇ *have equal geodesics and torsion, they coincide.*

   If T is symmetric and alternate, it vanishes.
4. *The geodesics of* ∇ *are the geodesic lines of a unique symmetric connection.*

   The uniqueness follows from statement 3, and to prove the existence, we consider the connection $D_1^{\bar\nabla} D_2 = D_1^\nabla D_2 - \frac{1}{2}\mathrm{Tor}_\nabla(D_1, D_2)$. Since $T = -\frac{1}{2}\mathrm{Tor}_\nabla$ is alternate, ∇ and $\bar\nabla$ have the same geodesics. Finally, $\bar\nabla$ is symmetric:

   $$D_1^{\bar\nabla} D_2 - D_2^{\bar\nabla} D_1 - [D_1, D_2] = D_1^\nabla D_2 - D_2^\nabla D_1 - [D_1, D_2] - \mathrm{Tor}(D_1, D_2) = 0.$$

Newtonian Gravitation: A **Galilean spacetime** is a real affine space $\mathbb{A}_4$ whose space of free vectors V is endowed with a with a 1-dimensional vector space $\mathcal{Q}$ of symmetric metrics, generated by a metric g of type $(+, 0, 0, 0)$, so that $g = \omega \otimes \omega$ for a 1-form ω (and g is named the **time metric** because the **time interval** between two events p and $q = p + v$ is defined to be $\sqrt{v \cdot v} = \sqrt{g(v, v)} = |\omega(v)|$); and a Euclidean structure on the subspace $E = \mathrm{rad}\, g = \ker \omega$ of

spatial (or simultaneity) vectors, defined by a contravariant metric $h^*(\omega_1, \omega_2) := \langle \omega_1|_E, \omega_2|_E \rangle$ of type $(0, +, +, +)$, named **space metric**[1].

An **inertial** reference system is defined to be an event $p \in \mathbb{A}_4$, the origin, and a base $e_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$ of V such that $\omega(e_0) = 1$ and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is an orthonormal base of E. Now any event is $q = p + te_0 + x_1e_1 + x_2e_2 + x_3e_3$, and we say that (t, x_1, x_2, x_3) are the **inertial coordinates** of q. The time function t is well-defined up to the addition of a constant, because $\omega = \mathrm{d}t$, and the **trajectories** of punctual bodies are defined to be the smooth sections $\sigma\colon \mathbb{R} \to \mathbb{A}_4$ of t (curves where g is positive-definite), so that $\sigma(t) = \big(t, x_1(t), x_2(t), x_3(t)\big)$. The gravitational force intensity (force per mass unit) is given by a field of spatial vectors $\vec{F} = F_1\partial_1 + F_2\partial_2 + F_3\partial_3$, and freely falling trajectories are given by the newtonian law of motion $x_i''(t) = F_i$.

*There exists a unique symmetric linear connection ∇, the **Cartan connection**, such that the freely falling trajectories are just the geodesic trajectories of ∇:*

The difference tensor T of two such connections fulfills that $T(D, D) = 0$ when $\omega(D) = 1$; hence when $\omega(D) \neq 0$ and, by continuity, for any tangent vector D, so that T is alternate. Since T is symmetric (both connections are torsionless) we see that $T = 0$.

To prove the existence, just consider the following linear connection:

$$\partial_t^\nabla \partial_t = -\vec{F} \quad , \quad \partial_t^\nabla \partial_i = \partial_i^\nabla \partial_t = \partial_i^\nabla \partial_j = 0.$$

This connection ∇ (with the time and space metrics g, h) enables us to express all the elements of the newtonian theory of gravitation, so rendering superfluous the flat connection of $\mathbb{A}_4$ (the inertial reference systems):

1. Freely falling trajectories are just geodesic trajectories of ∇.

2. The curvature tensor R of ∇ defines a tensor $R_{2,2}$ on $\Lambda^2(TX)$, uniquely determined by the condition $R_{2,2}(D_1 \wedge D_2, \vec{V}_3 \wedge D_4) = \langle R(D_1, D_2, D_4), \vec{V}_3 \rangle$, because $R(D_1, D_2, D_4)$ always is spatial. The conservative character of gravitational forces, $\partial_i F_j = \partial_j F_i$, is just the symmetry condition[2] $R_{2,2}(D_1 \wedge D_2, D_3 \wedge D_4) = R_{2,2}(D_3 \wedge D_4, D_1 \wedge D_2)$.

3. When the matter distribution is given by a mass density function ρ on $\mathbb{A}_4$, Poisson's equation $\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 = -4\pi\rho$, the field equation encoding the inverse square gravitation law, states that the **Ricci** tensor R_2 of ∇ (the contraction $R_2 = C_1^1 R$ of the first indices of the curvature tensor) is[3] just $R_2 = 4\pi\rho\, \mathrm{d}t \otimes \mathrm{d}t$.

Physical Meaning of the Curvature: If D is a geodesic vector field, $D^\nabla D = 0$, the integral curves should be understand as trajectories of inertial bodies. The apparent position of nearby bodies is represented by a vector field E such that $D^L E = 0$, the relative velocity by $D^\nabla E$ and the relative acceleration by $D^\nabla D^\nabla E$. If ∇ is symmetric, the curvature appears as a non null relative acceleration of inertial bodies (in absence of forces!)

$$R(D, E, D) = D^\nabla E^\nabla D - E^\nabla D^\nabla D - [D, E]^\nabla D = D^\nabla E^\nabla D = D^\nabla D^\nabla E.$$

10.3 Riemannian Manifolds

Definition: A **semiriemannian metric** g is a non-singular symmetric 2-covariant (smooth) tensor field, and it is **riemannian** when it is positive-definite.

[1] Both metrics g and h^* are well defined up to a positive factor, and to fix g and h is just to fix time and length units. Then the 1-form ω is well-defined up to a sign, and to fix it is just to fix the time orientation.

[2] typical of the Levi-Civita connections, even if ∇ does not preserve a non-singular metric.

[3] so that the matter distribution is fully determined by the geometry (g, h, ∇) of spacetime.

A **riemannian manifold** is a smooth manifold X with a riemannian metric[4], and **isometries** are diffeomorphisms $\varphi\colon X \to Y$ preserving the metrics, $g_X = \varphi^*(g_Y)$.

In a riemannian manifold (X, g), we put $D \cdot \bar{D} = g(D, \bar{D})$.

Proposition: *Any smooth manifold X admits a riemannian metric.*

Proof: Let $\{\phi_i\}$ be a partition of unity subordinated to a cover $X = \bigcup_i U_i$ by coordinate open sets, and let us fix a riemannian metric g_i on each open set U_i.

Now the metric $g = \sum_i \phi_i g_i$ is positive definite: if $0 \neq D \in T_x X$, then $0 < \sum_i \phi_i(x) g_i(D, D)$ because some term is positive and no term is negative.

Fundamental Theorem of Riemannian Geometry: *If g is a semiriemannian metric, there exists a unique symmetric linear connection ∇ (the **Levi-Civita connection** of g) such that $\nabla g = 0$.*

Proof: The condition $\nabla g = 0$ states that the connection derives the product defined by g,

$$D(D_1 \cdot D_2) = (D^\nabla D_1) \cdot D_2 + D_1 \cdot (D^\nabla D_2),$$

so that for any three vector fields X, Y, Z we have

$$\begin{aligned} X(Y \cdot Z) &= (X^\nabla Y) \cdot Z + Y \cdot (X^\nabla Z), \\ Z(X \cdot Y) &= (Z^\nabla X) \cdot Y + X \cdot (Z^\nabla Y), \\ Y(X \cdot Z) &= (Y^\nabla X) \cdot Z + X \cdot (Y^\nabla Z). \end{aligned}$$

Adding them with alternate signs, and using that $[D_1, D_2] = D_1^\nabla D_2 - D_2^\nabla D_1$, we obtain

$$(*) \qquad 2(X^\nabla Y) \cdot Z = X(Y \cdot Z) + Y(X \cdot Z) - Z(X \cdot Y) - X \cdot [Y, Z] - Y \cdot [X, Z] + Z \cdot [X, Y].$$

Since g is non singular, this equality determines $X^\nabla Y$, so proving the uniqueness of ∇. To see the existence, just define ∇ by the above equality, and it is easy to check that it is a symmetric connection and $\nabla g = 0$. q.e.d.

In a local coordinate system, if we put $g_{ij} = \partial_i \cdot \partial_j$ and (g^{ij}) denotes the inverse matrix of (g_{ij}), then $(*)$ let us express the Christoffel symbols of ∇ in terms of the metric g,

$$\Gamma_{ij}^k = \frac{1}{2} \sum_h g^{kh} \left(\frac{\partial g_{jh}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_h} + \frac{\partial g_{hi}}{\partial x_j} \right).$$

Corollary: *The geodesic curves have tangent vector of constant module.*

Proof: $\partial_t(\partial_t \cdot \partial_t) = (\partial_t^\nabla \partial_t) \cdot \partial_t + \partial_t \cdot (\partial_t^\nabla \partial_t) = 0 \cdot \partial_t + \partial_t \cdot 0 = 0.$

Definition: A semiriemannian metric g is **locally Euclidean** if any point admits a coordinate neighborhood where $g = \sum_i \pm \mathrm{d}x_i \otimes \mathrm{d}x_i$ (i.e., $g_{ij} = \partial_i \cdot \partial_j = \pm\delta_{ij}$).

Theorem: *A metric g is locally Euclidean if and only if the curvature tensor R is null.*

Proof: If $g = \sum_i \pm \mathrm{d}x_i \otimes \mathrm{d}x_i$, then we have $\Gamma_{ij}^k = 0$, and $R = 0$.

Conversely, if $R = 0$, then (p. 296) each point p has a neighborhood with a base of parallel vector fields $D_1, \dots, D_n$, and we may assume that $D_i \cdot D_j = \pm\delta_{ij}$ at p.

[4] In fact, it is only well-defined up to a positive constant factor, and to fix it is just to fix the length unit.

We also have $D_i \cdot D_j = \pm\delta_{ij}$ on a neighborhood of p since $D_i \cdot D_j$ is locally constant,

$$D_k(D_i \cdot D_j) = (D_k^\nabla D_i) \cdot D_j + D_i \cdot (D_k^\nabla D_j) = 0 + 0 = 0.$$

Moreover $[D_i, D_j] = D_i^\nabla D_j - D_j^\nabla D_i = 0$, because ∇ is symmetric, and on a coordinate neighborhood we have $D_i = \partial_i$ (p. 274), so that $g_{ij} = \partial_i \cdot \partial_j = D_i \cdot D_j = \pm\delta_{ij}$.

Definition: The **Riemann-Christoffel tensor** $R_{2,2}$ of a semiriemannian metric g is

$$R_{2,2}(D_1, D_2; D_3, D_4) = R(D_1, D_2, D_4) \cdot D_3.$$

Theorem: 1. $R_{2,2}(D_2, D_1; D_3, D_4) = -R_{2,2}(D_1, D_2; D_3, D_4)$.
2. $R_{2,2}(D_1, D_2; D_4, D_3) = -R_{2,2}(D_1, D_2; D_3, D_4)$.
3. $R_{2,2}(D_3, D_4; D_1, D_2) = R_{2,2}(D_1, D_2; D_3, D_4)$.
4. *The cyclic sum on any three indices is null.*

Proof: (1) R is alternate in the first two indices.

(2) It is enough to show that $R_{2,2}(D_1, D_2; D_3, D_3) = 0$; and we may assume that $[D_i, D_j] = 0$,

$$\begin{aligned} 0 &= [D_1, D_2](D_3 \cdot D_3) = D_1(2D_2^\nabla D_3 \cdot D_3) - D_2(2D_1^\nabla D_3 \cdot D_3) = \\ &= 2\big((D_1^\nabla D_2^\nabla D_3) \cdot D_3 + (D_2^\nabla D_3) \cdot (D_1^\nabla D_3) - (D_2^\nabla D_1^\nabla D_3) \cdot D_3 - (D_1^\nabla D_3) \cdot (D_2^\nabla D_3)\big) = \\ &= 2R_{2,2}(D_1, D_2; D_3, D_3). \end{aligned}$$

(3) Bianchi's identity states that the cyclic sum, fixing the third index, is null, hence

$$\begin{aligned} R_{2,2}(D_1, D_3; D_2, D_4) + R_{2,2}(D_4, D_1; D_2, D_3) + R_{2,2}(D_3, D_4; D_2, D_1) &= 0 \\ R_{2,2}(D_2, D_4; D_3, D_1) + R_{2,2}(D_4, D_1; D_3, D_2) + R_{2,2}(D_1, D_2; D_3, D_4) &= 0 \\ R_{2,2}(D_2, D_4; D_1, D_3) + R_{2,2}(D_3, D_2; D_1, D_4) + R_{2,2}(D_4, D_3; D_1, D_2) &= 0 \\ R_{2,2}(D_1, D_3; D_4, D_2) + R_{2,2}(D_3, D_2; D_4, D_1) + R_{2,2}(D_2, D_1; D_4, D_3) &= 0 \end{aligned}$$

Adding them, and using 1 and 2, the two first columns cancel and we obtain

$$R_{2,2}(D_1, D_2; D_3, D_4) = -R_{2,2}(D_3, D_4; D_2, D_1) = R_{2,2}(D_3, D_4; D_1, D_2).$$

(4) The cyclic sum, fixing the third index, is null by Bianchi's identity; hence so is fixing any other index since $R_{2,2}$ is invariant under the Klein group according to 1, 2, 3.

General Relativity: A **lorentzian manifold** is a smooth manifold X of dimension 4 with a semiriemannian metric g of type $(+, -, -, -)$, the **time metric**. Trajectories of particles are just connected curves in X where g is definite-positive, the "length" of the curve being the proper time, and the freely falling trajectories being the geodesic trajectories. The impulse of a particle is defined to be the product of the 4-velocity by the rest mass, so that a continuous matter distribution is represented[5] by a vector valued 3-form Π_3, hence by the 2-contravariant tensor T^2 such that $\Pi_3 = C_1^1(\Omega_X \otimes T^2)$. The mass-energy conservation law and the law of geodesic motion state that T^2 is constant, $\nabla T^2 = 0$, and **Einstein's equation** $G_2 = 8\pi T_2$ relates matter and geometry, where G_2 is the Einstein tensor (p. 562).

[5] Intuitively $\Pi_3(D_1, D_2, D_3)$ is the sum of the impulses I_i of the particles crossing the infinitesimal parallelogram of edges D_1, D_2, D_3, with a sign depending on the orientation of I, D_1, D_2, D_3 with respect to a fixed orientation Ω_X of the spacetime X.

Definition: In a riemannian manifold (X, g), the **codifferential** $\delta\omega$ and the **laplacian** $\Delta\omega$ of a differential p-form ω are defined to be (the contraction of indices C_{12} with g)

$$\delta\omega = C_{12}(\nabla\omega),$$
$$\Delta\omega = (\mathrm{d}\delta + \delta\mathrm{d})\omega.$$

If g is a semiriemannian metric, $\Box = \mathrm{d}\delta + \delta\mathrm{d}$ is said to be the **lambertian** operator.

Sectional Curvatures: The tensor $R_{2,2}$ may be viewed as a symmetric metric on $\Lambda^2 T_pX$,

$$R_{2,2}(D_1 \wedge D_2, D_3 \wedge D_4) = R_{2,2}(D_1, D_2; D_3, D_4),$$

where we also have (p. 69) the metric $\Lambda^2 g$ induced by the riemannian metric g,

$$(\Lambda^2 g)(D_1 \wedge D_2, D_3 \wedge D_4) = (D_1 \cdot D_3)(D_2 \cdot D_4) - (D_1 \cdot D_4)(D_2 \cdot D_3).$$

Definition: If $\Pi \subseteq T_pX$ is a plane, then $\dim \Lambda^2\Pi = 1$ and both metrics are proportional, with a factor K_Π, the **sectional curvature** of Π. If D_1, D_2 is a base of Π,

$$K_\Pi = \frac{R_{2,2}(D_1, D_2; D_1, D_2)}{(D_1 \cdot D_1)(D_2 \cdot D_2) - (D_1 \cdot D_2)^2}.$$

and g is a metric of **constant curvature** if it does not depend on Π nor p.

1. $(K = 0)$. The Euclidean space $\mathbb{E}_n = (\mathbb{R}^n, \sum_i \mathrm{d}x_i^2)$ has null curvature. It is simply connected and the connection is **complete** (the geodesics are defined for any value $-\infty < t < \infty$).

2. $(K > 0)$. The sphere $\mathbb{S}_n$ of radius r in $\mathbb{E}_{n+1}$ has constant curvature $K = 1/r^2$. It is simply connected and complete, and the geodesic lines are just the intersections with the planes through the center of the sphere. The stereographic projection from a pole onto the equatorial hyperplane gives the metric

$$\frac{4(\mathrm{d}x_1^2 + \ldots + \mathrm{d}x_n^2)}{\big(1 + K(x_1^2 + \ldots + x_n^2)\big)^2} \quad , \quad (K = 1/r^2).$$

   on $\mathbb{R}^n$, so that the geodesics are the circles and straight lines intersecting at opposite points the sphere of radius r with center at the origin of $\mathbb{R}^n$.

3. $(K < 0)$. Let us consider on $\mathbb{R}^{n+1}$ the metric $g = \mathrm{d}t^2 - \mathrm{d}x_1^2 - \ldots - \mathrm{d}x_n^2$. The **hyperbolic space** $\mathbb{H}_n$ of curvature $K = -1/r^2$ is the hypersurface

$$t^2 - x_1^2 - \ldots - x_n^2 = r^2 \ , \quad t > 0,$$

   endowed with the restriction of $-g$. It is simply connected and complete, the geodesics being the intersections with the planes through the origin of $\mathbb{R}^{n+1}$.

   Klein's Projective Model: Via the projection $\mathbb{R}^{n+1} \to \mathbb{P}_n$, the hyperbolic space is identified with an open set in $\mathbb{P}_n$ bounded by a non-singular quadric of index 1, the geodesics being the lines. In affine coordinates the quadric is $\sum_i y_i^2 = 1$, and the metric is

$$r^2 \frac{(1 - \sum y_i^2)(\sum \mathrm{d}y_i^2) + (\sum y_i \mathrm{d}y_i)^2}{(1 - \sum y_i^2)^2}.$$

   Poincaré's Disk: If we project the hyperboloid $\mathbb{H}_n$ from the point $(-r, 0, \ldots, 0)$ onto the equatorial hyperplane $t = 0$, we may identify $\mathbb{H}_n$ with the open ball in $\mathbb{R}^n$ of radius r, the geodesics being the lines through the origin and the circles orthogonally intersecting the boundary of the ball. The metric is

$$\frac{4(\mathrm{d}x_1^2+\ldots+\mathrm{d}x_n^2)}{\big(1+K(x_1^2+\ldots+x_n^2)\big)^2} \quad , \quad (K=-1/r^2).$$

Poincaré's Half-plane: Apply to the Poincaré's disk an inversion and a translation, we may identify it with the semiplane $x_n>0$, the geodesics being the lines and the circles orthogonally intersecting the hyperplane $x_n=0$. The metric is

$$\frac{r^2}{x_n^2}(\mathrm{d}x_1^2+\ldots+\mathrm{d}x_n^2).$$

In the case of surfaces, the unique sectional curvature is the **curvature** K of the surface at the given point, and it determines the tensor $R_{2,2}$; hence the curvature tensor R. *A riemannian surface of null constant curvature is locally Euclidean.*

10.3.1 Normal Coordinates

Let X be a smooth manifold, and let us consider natural projection

$$\pi\colon TX:=\coprod_{x\in X} T_xX \longrightarrow X \ , \quad \pi(D_x)=x.$$

If $(x_1,\ldots,x_n)$ are coordinates on an open set $U\subseteq X$, then any vector $D_x\in\pi^{-1}(U)$ is fully determined by the coordinates $(x_1,\ldots,x_n)$ of x and the coordinates $y_i=D_x(x_i)$ of D_x in the base $(\partial_1)_x,\ldots,(\partial_n)_x$ of T_xX. On TX we have a unique structure of smooth manifold such that $\big(\pi^{-1}(U),x_1,\ldots x_n,y_1,\ldots,y_n\big)$ are local coordinate systems, and we say that the smooth map $\pi\colon TX\to X$ is the **tangent bundle** of X. The smooth vector fields D on any open set $U\subseteq X$ correspond to the smooth **sections** $s\colon U\to TX$ of π (i.e. $\pi\circ s=\mathrm{Id}_U$).

Analogously we may construct the **cotangent bundle** $T^*X\to X$ and the bundles of tensors $T_p^qX\to X$, so that the smooth sections are the smooth 1-forms and the smooth tensor fields of type (p,q) respectively.

Theorem: *Let p be a point of a riemannian manifold X. The tangent vectors $D\in T_pX$ such that the geodesic curve $\sigma_D(t)$, where $\sigma_D'(0)=D$, is defined at $t=1$ form an open neighborhood U_0 of 0 in T_pX, and the* ***exponential map***

$$\exp_p\colon U_0\longrightarrow X \ , \quad D\mapsto\sigma_D(1),$$

is a smooth map. Moreover, it is a local diffeomorphism at 0.

Proof: Any smooth curve $\gamma\colon I\to X$ has a canonical lifting $\tilde\gamma\colon I\to TX$, $\tilde\gamma(t)=T_t=\gamma_{*,t}(\partial_t)$, so that the geodesic curves define a vector field Z on TX, and it is smooth because in any local coordinate system $(x_1,\ldots x_n,y_1,\ldots,y_n)$ we have (p. 295)

$$Z=\sum_{i=1}^n y_i\frac{\partial}{\partial x_i}-\sum_{k=1}^n\left(\sum_{i,j=1}^n\Gamma_{ij}^k y_iy_j\right)\frac{\partial}{\partial y_k}.$$

If τ_t is the flow of Z, then $\pi\big(\tau_t(D)\big)=\sigma_D(t)$, so that $\exp_p=\pi\circ\tau_1$; hence (p. 264) U_0 is open and $\exp_p$ is smooth.

Moreover, since $\exp tD = \sigma_{tD}(1) = \sigma_D(t)$, the linear tangent map $T_pX = T_0(U_0) \to T_pX$ is just the identity, and we conclude by the inverse mapping theorem.

Proposition: *Let $\varphi, \phi\colon X \to Y$ be local isometries, X connected. If $\varphi(p) = \phi(p)$ and $\varphi_{*,p} = \phi_{*,p}$, then $\varphi = \phi$.*

Proof: The isometries commute with the exponential maps, so that we have commutative squares

$$\begin{array}{ccc} T_pX & \xrightarrow{\varphi_{*,p}} & T_{\varphi(p)}Y \\ {\scriptstyle \exp_p}\downarrow & & \downarrow{\scriptstyle \exp_{\varphi(p)}} \\ X & \xrightarrow{\varphi} & Y \end{array} \quad , \quad \begin{array}{ccc} T_pX & \xrightarrow{\phi_{*,p}} & T_{\phi(p)}Y \\ {\scriptstyle \exp_p}\downarrow & & \downarrow{\scriptstyle \exp_{\phi(p)}} \\ X & \xrightarrow{\phi} & Y \end{array}$$

Hence the set of points $x \in X$ where $\varphi(x) = \phi(x)$ and $\varphi_{*,x} = \phi_{*,x}$ is an open set. Since it is always closed, we conclude.

Corollary: *Let $\mathbb{M} = \mathbb{E}_n$, $\mathbb{S}_n$ or $\mathbb{H}_n$. Given $p, q \in X$ and a linear isometry $\phi\colon T_pX \to T_qX$, there exists a unique isometry $\varphi\colon \mathbb{M} \to \mathbb{M}$ such that $\varphi(p) = q$ and $\varphi_{*,p} = \phi$.*

Proof: We only have to prove the existence, the Euclidean case being obvious.

In the case of the sphere $\mathbb{S}_n$ of radius r in $\mathbb{R}^{n+1}$, just consider the restriction φ of the isometry

$$\mathbb{R}^{n+1} = \mathbb{R}p \perp T_p\mathbb{S}_n \xrightarrow{1 \perp \phi} \mathbb{R}q \perp T_q\mathbb{S}_n = \mathbb{R}^{n+1},$$

and the case $\mathbb{H}_n$ is totally analogous: $\mathbb{R}^{n+1} = \mathbb{R}p \perp T_p\mathbb{H}_n \to 1 \perp \phi\mathbb{R}q \perp T_q\mathbb{H}_n = \mathbb{R}^{n+1}$.

Definition: Let p be a point of a riemannian manifold X. The exponential map defines a diffeomorphism of an open ball $B(0, r)$ in T_pX onto a neighborhood U of p in X. Once we fix an orthonormal base in T_pX, we may identify it with $\mathbb{R}^n$, and so we obtain a local coordinate system $(x_1, \ldots, x_n)$ in U, named **normal coordinates** at p, such that

1. The coordinates of the point p are just $(0, \ldots, 0)$.
2. The curves $x_i = a_i t$ are geodesic lines.
3. The vectors $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ define an orthonormal base of T_pX.

Gauss Lemma: *Let $(U; x_1, \ldots, x_n)$ be normal coordinates at p and put $r = \sqrt{\sum_i x_i^2}$. On $U - p$ the gradient of r is the tangent vector field $N = \sum_i \frac{x_i}{r}\partial_i$ defined by the geodesic curves with unitary tangent vector at p. In particular $|\mathrm{grad}\, r| = 1$ and $\sum_i x_i g_{ij} = x_j$.*

Proof: By definition $N^\nabla N = 0$, $|N| = 1$, and a direct calculation shows that $Nr = 1$.

For any vector $D \in T_xX$, we have to prove that

$$N \cdot D = Dr$$

and it is clear when $D = N_x$, because $N \cdot N = 1 = Nr$. When D is tangent to a "sphere" $r = a$, so that $Dr = 0$ and we have to prove that $N \cdot D = 0$, we may extend it so as to obtain a vector field D such that $[N, D] = 0$. Now, $N \cdot D$ is constant along the geodesics through p:

$$N(N \cdot D) = (N^\nabla N) \cdot D + N \cdot (N^\nabla D) = 0 + (D^\nabla N) \cdot N = \tfrac{1}{2}D(N \cdot N) = \tfrac{1}{2}D(1) = 0.$$

Since the uniparametric group of N shrinks the spheres as we approach p, and it leaves D invariant, we have that $D \to 0$ as we approach p along the geodesic joining x with p. We conclude that $N \cdot D = 0$.

Finally, $|\mathrm{grad}\, r| = |N| = 1$, and the equality $N \cdot \partial_j = \partial_j r$ gives that $\sum_i x_i g_{ij} = x_j$.

Corollary: *In normal coordinates,* $\dfrac{\partial g_{ij}}{\partial x_k}(p) = 0$ *and* $\Gamma^k_{ij}(p) = 0$.

Proof: Deriving $\sum_h x_h g_{hj} = x_j$ with $\partial_k \partial_i$ at $x = p$, we obtain $(\partial_k g_{ij})(p) = -(\partial_i g_{kj})(p)$.

Since $(\partial_k g_{ij})(p)$ is symmetric in i, j and alternate in i, k, it is also alternate in j, k; hence in i, j, and it vanishes.

Definition: A map $\gamma\colon [a, b] \to X$ is $\mathcal{C}^\infty$ when it is the restriction of a map $(a - \varepsilon, b + \varepsilon) \to X$ of class $\mathcal{C}^\infty$, and it is **piecewise** $\mathcal{C}^\infty$ when there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that the restriction $\gamma\colon [x_{i-1}, x_i] \to X$ is $\mathcal{C}^\infty$ for any index i. Then the tangent vector $T_t = \gamma_*(\partial_t)$ is well-defined up to a finite number of points, and the **length** of γ is

$$\int_a^b |T_t| \mathrm{d}t.$$

Theorem: *Let $(U; x_1, \ldots, x_n)$ be normal coordinates at p. For any point $x \in X$ there is a unique (up to reparametrizations) piecewise $\mathcal{C}^\infty$ curve of minimal length joining p and x, namely the geodesic in U joining them, whose length is just $\sqrt{x_1^2 + \ldots + x_n^2}$.*

Proof: Since $|N| = 1$, we have $(\mathrm{grad}\, r) \cdot T = N \cdot T \leq |T|$ for any tangent vector T, and the equality holds if and only if $T = \lambda N$, $\lambda \geq 0$.

Now, let $\gamma\colon [a, b] \to U$ be a curve joining p and $x \in U$. Then

$$\mathrm{length}\, \gamma = \int_a^b |T_t|\; t \leq \int_a^b \big((\mathrm{grad}\, r) \cdot T_t\big) \mathrm{d}t = \int_\gamma \gamma^*(\mathrm{d}r) = r(x) - r(p) = \sqrt{x_1^2 + \ldots + x_n^2},$$

and the equality holds just when $T_t = \lambda(t) N_{\gamma(t)}$, $\lambda(t) \geq 0$; i.e. γ is a reparametrization of the geodesic joining p and x in U.

Finally, if a curve γ joining p and x is not contained in U, it intersects some "sphere" $x_1^2 + \ldots + x_n^2 = r^2$ of radius $r > r(p)$, so that the length is $\geq r$.

Corollary: *Any riemannian manifold is a metric space (with the same underlying topology):*

$$\mathrm{d}(p, q) = \inf\{\mathrm{length}\, \gamma;\ \gamma \text{ a piecewise } \mathcal{C}^\infty \text{ curve joining } p \text{ and } q\}.$$

Note: The normal neighborhood U of p is the image, by the exponential map, of an open ball $B(0, R)$, and the theorem shows that U is just the open ball $B(p, r)$ of the metric d.

The theorem also shows that the function $f(x) = \mathrm{d}(p, x)^2$ is $\mathcal{C}^\infty$ in a neighborhood of p, has null differential and the hessian is just $2g_p$. Hence the metric d (and the differentiable structure) determine the riemannian metric g.

Proposition: *Any local isometry $\varphi\colon X \to Y$ between complete and connected riemannian manifolds is a covering.*

Proof: Let $B(y, r)$ be a normal neighborhood of $y \in Y$. It is enough to show that $\varphi^{-1}\big(B(y, r)\big)$ is just the disjoint union of the balls (they exist because X is complete) $B(x, r)$, where $\varphi(x) = y$, and that the maps $\varphi\colon B(x, r) \to B(y, r)$ are diffeomorphisms.

When $\varphi(x) = y$, the commutative square

$$\begin{array}{ccccc} T_xX \supset & B(0,r) & = & B(0,r) \subseteq & T_yY \\ & {\scriptstyle \exp_x}\downarrow & & \| & \downarrow{\scriptstyle \exp_y} \\ X \supseteq & \exp_x B(0,r) & \xrightarrow{\varphi} & B(y,r) & \subseteq Y \end{array}$$

shows that the surjective maps $\exp_x\colon B(0,r) \to \exp_x B(0,r)$ and $\varphi\colon \exp_x B(0,r) \to B(y,r)$ are in fact homeomorphisms.

Since $\varphi\colon X \to Y$ is a local homeomorphism, it follows that $\exp_x B(0,r)$ is open in X.

Now the commutative square shows that both homeomorphisms, being smooth maps, are diffeomorphisms, so that $\exp_x B(0,r) = B(x,r)$, and $\varphi\colon B(x,r) \to B(y,r)$ is a diffeomorphism.

Finally, let us show that $\varphi^{-1}\big(B(y,r)\big) = \coprod B(x,r)$. If $y' = \varphi(x') \in B(y,r)$, then the geodesic joining y' with y in $B(y,r)$, of length $< r$, may be lifted to a geodesic in X, of length $< r$, joining x' with some point $x \in \varphi^{-1}(y)$. Hence $x' \in B(x,r)$.

10.3.2 Surfaces of Constant Curvature

Minding's Theorem: *All surfaces of constant curvature K are locally isometric.*

Proof: Let (X,g) be a surface, fix a point $p \in X$, and consider the geodesic curve γ, $\gamma(0) = p$, tangent to a unitary vector $D_p \in T_pX$. Let N_p, D_p be an orthonormal base in T_pX and consider the parallel transport N of N_p along γ, so that $N \cdot T = 0$. Let σ_y be the geodesic curve tangent to $N_{\gamma(y)}$ and such that $\sigma_y(0) = \gamma(y)$.

Then $\sigma\colon \mathbb{R}^2 \to X$, $\sigma(x,y) = \sigma_y(x)$, is a smooth map defined on a neighborhood of 0, and it is a local diffeomorphism at 0 because $\sigma_*\colon T_0\mathbb{R}^2 \to T_pX$ transforms ∂_x, ∂_y into the base N_p, D_p. Hence σ defines a local coordinate system (x,y) on a neighborhood of $p = (0,0)$.

By construction, $\partial_x^\nabla \partial_x = 0$ and $|\partial_x| = 1$, because ∂_x is tangent to the geodesics σ_y. Moreover, on the curve $x = 0$ we have

$$\partial_y^\nabla \partial_y = \partial_y^\nabla \partial_x = 0 \ , \ \ |\partial_y| = 1 \ , \ \ \partial_x \cdot \partial_y = 0.$$

Now, $\partial_y \cdot \partial_x = 0$ because it vanishes on $x = 0$ and, deriving $\partial_x \cdot \partial_x = 1$, we have

$$0 = (\partial_y^\nabla \partial_x) \cdot \partial_x = (\partial_x^\nabla \partial_y) \cdot \partial_x = \partial_x(\partial_y \cdot \partial_x).$$

Hence the riemannian metric is $g = \mathrm{d}x^2 + f^2\mathrm{d}y^2$, where $f = |\partial_y|$.

We have $f(0,y) = 1$, because $|\partial_y| = 1$ on $x = 0$, and deriving $f^2 = \partial_y \cdot \partial_y$ with ∂_x we obtain that $2f(\partial_x f) = 2(\partial_x^\nabla \partial_y) \cdot \partial_y$ vanishes on $x = 0$, because so does $\partial_x^\nabla \partial_y$; hence $(\partial_x f)(0,y) = 0$.

On the other hand, a direct calculation shows that the curvature K of $\mathrm{d}x^2 + f^2\mathrm{d}y^2$ is

$$K = -\frac{\partial_x^2 f}{f} \quad , \quad f'' + Kf = 0.$$

When K is constant, integrating this ordinary differential equation, with the initial conditions $f(0,y) = 1$, $f'(0,y) = 0$, we conclude

$$f(x,y) = \begin{cases} 1 & \text{when } K = 0, \\ \cos\sqrt{K}\,x & \text{when } K > 0, \\ \cosh\sqrt{-K}\,x & \text{when } K < 0. \end{cases}$$

Theorem: *Any simply connected riemannian surface X of constant curvature K admits a local isometry $X \to \mathbb{M}$, unique up to an isometry of $\mathbb{M}$ (where $\mathbb{M} = \mathbb{E}_2$, $\mathbb{S}_2$ or $\mathbb{H}_2$, depending on K).*

Proof: Let I_x be the non empty set (by Minding's theorem) of germs φ_x at $x \in X$ of isometries $\varphi\colon U \hookrightarrow \mathbb{M}$ of a connected neighborhood of x onto an open set of the model $\mathbb{M}$. Now we put (as in p. 197 and p. 275)

$$\pi\colon \widetilde{I} = \textstyle\coprod_x I_x \longrightarrow X,\ \pi(\varphi_x) = x.$$

Any local isometry $U \to \mathbb{M}$ defines a local section $U \to \widetilde{I}$ of π. The images of these sections generate a topology on $\widetilde{I}$, so that the continuous sections $U \to \widetilde{I}$ of π correspond to the local isometries $U \to \mathbb{M}$.

Moreover, π is in fact a covering of X, and even a principal bundle when endowed with the natural left action of the group G of all isometries of $\mathbb{M}$, because any germ φ_x is fully determined (p. 303) by $\varphi(x)$ and $\varphi_{*,x}$, so that any two such germs differ in a unique isometry of $\mathbb{M}$.

If X is simply connected, it is a trivial principal bundle, $\widetilde{I} \simeq G \times X$, and it admits a continuous section $X \to \widetilde{I}$, unique up to the action of G.

Theorem: *Any simply connected complete riemannian surface of constant curvature is isometric to* $\mathbb{E}_2$, $\mathbb{S}_2$ *or* $\mathbb{H}_2$.

Proof: The local isometry $X \to \mathbb{M}$ is a covering, because X is complete.

Hence it is a trivial covering, because $\mathbb{M}$ is simply connected.

10.4 Riemannian Embeddings

Let $(X; g, \nabla)$ be a riemannian manifold, $\bar{X}$ a submanifold of X, and $\bar{g}$ the restriction of g to $\bar{X}$.

If $p \in \bar{X}$, we put $T_pX = T_p\bar{X} \perp N_p$. If D_1, D_2 are vector fields on $\bar{X}$, in general $D_1^{\nabla} D_2$ has a tangential component $D_1^{\overline{\nabla}} D_2$, and a normal component $\Phi(D_1, D_2) = (D_1^{\nabla} D_2)^{\perp}$,

$$D_1^{\nabla} D_2 = D_1^{\overline{\nabla}} D_2 + \Phi(D_1, D_2).$$

Theorem: $\overline{\nabla}$ *is the Levi-Civita connection of* $\bar{g}$*, and* Φ *is a symmetric* $\mathcal{C}^\infty(\bar{X})$*-bilinear map,* (hence it defines symmetric bilinear maps $\Phi_p\colon T_p\bar{X} \times T_p\bar{X} \to N_p$).

Proof: To see that $\overline{\nabla}$ is a symmetric connection and $D(D_1 \cdot D_2) = (D^{\overline{\nabla}} D_1) \cdot D_2 + D_1 \cdot (D^{\overline{\nabla}} D_2)$, just project orthogonally onto $T_p\bar{X}$ the corresponding properties of ∇, using that $[D_1, D_2]$ is tangent to $\bar{X}$ when so are D_1, D_2.

Moreover Φ is symmetric and bilinear,

$$\begin{aligned} 0 &= D_1^{\nabla} D_2 - D_2^{\nabla} D_1 - [D_1, D_2] \\ &= D_1^{\overline{\nabla}} D_2 + \Phi(D_1, D_2) - D_2^{\overline{\nabla}} D_1 - \Phi(D_2, D_1) - [D_1, D_2] \\ &= \Phi(D_1, D_2) - \Phi(D_2, D_1), \\ \Phi(fD_1, D_2) &= (fD_1^{\nabla} D_2)^{\perp} = f(D_1^{\nabla} D_2)^{\perp} = f\Phi(D_1, D_2). \end{aligned}$$

Weingarten Formula: $\Phi(D_1, D_2) \cdot N = -(D_1^{\nabla} N) \cdot D_2$; *where* N *is normal to* $\bar{X}$.

Proof: $\Phi(D_1, D_2) \cdot N = (D_1^{\nabla} D_2) \cdot N = D_1(D_2 \cdot N) - D_2 \cdot (D_1^{\nabla} N)$.

Gauss Equation: *If* R *and* $\bar{R}$ *are the Riemann-Christoffel tensors of* X *and* $\bar{X}$*,*

$$\bar{R}(D_1, D_2; D_3, D_4) = R(D_1, D_2; D_3, D_4) + \Phi(D_1, D_3) \cdot \Phi(D_2, D_4) - \Phi(D_1, D_4) \cdot \Phi(D_2, D_3).$$

Proof: Since $R, \bar{R}$ and $\Phi \cdot \Phi$ are tensors, we may assume that $[D_i, D_j] = 0$.

$$\begin{aligned} R(D_1, D_2; D_3, D_4) &= (D_1^\nabla D_2^\nabla D_4 - D_2^\nabla D_1^\nabla D_4) \cdot D_3 \\ &= \left[D_1^\nabla D_2^{\bar\nabla} D_4 - D_2^\nabla D_1^{\bar\nabla} D_4 + D_1^\nabla(\Phi(D_2, D_4)) - D_2^\nabla(\Phi(D_1, D_4)) \right] \cdot D_3 \\ &= \bar{R}(D_1, D_2; D_3, D_4) - \Phi(D_1, D_3) \cdot \Phi(D_2, D_4) + \Phi(D_2, D_3) \cdot \Phi(D_1, D_4). \end{aligned}$$

Curves

Definitions: In the case of a curve of unitary tangent vector T, the **curvature** of the curve is $\kappa = |T^\nabla T|$, and it is null just when the curve is a geodesic. If the curve lies in a submanifold $\bar{X}$, it also has a **geodesic curvature** $\kappa_g = |T^{\bar\nabla} T|$, and a **normal curvature** $\kappa_n = |\Phi(T, T)|$,

$$\kappa^2 = \kappa_g^2 + \kappa_n^2.$$

If the curve lies in an oriented Euclidean space of dimension 3, and the curvature does not vanish at any point, the **principal normal** $N = \frac{1}{\kappa} T^\nabla T$ is orthogonal to T,

$$0 = T(T \cdot T) = (T^\nabla T) \cdot T + T \cdot (T^\nabla T) = 2(T^\nabla T) \cdot T = 2\kappa(N \cdot T),$$

and at any point of the curve we have the **binormal** B, so that (T, N, B) is a direct orthonormal base. The **torsion** of the curve is $\tau = (T^\nabla N) \cdot B$.

Frénet Formulae: $$\begin{cases} T^\nabla T = \quad \kappa N \\ T^\nabla N = -\kappa T + \tau B \\ T^\nabla B = -\tau N \end{cases}$$

Proof: The first equality is just the definition of N, and the second one follows from the equality

$$(T^\nabla N) \cdot T = T(N \cdot T) - N \cdot (T^\nabla T) = -N \cdot (\kappa N) = -\kappa.$$

Now, deriving with T the equality $B \cdot T = 0$, we see that $(T^\nabla B) \cdot T = 0$; deriving $B \cdot N = 0$, that $(T^\nabla B) \cdot N = -\tau$, and deriving $B \cdot B = 1$, that $(T^\nabla B) \cdot B = 0$.

Theorem: *Fix a direct orthonormal base (T_p, N_p, B_p) at a point $p \in \mathbb{R}^3$. Given smooth functions $\kappa(t) > 0$, $\tau(t)$ on an interval I, then there exists a unique curve $I \to \mathbb{R}^3$ of curvature κ and torsion τ, such that the Frénet frame at $t_0 \in I$ is (T_p, N_p, B_p).*

Proof: The Frénet frame $T = (f_1, f_2, f_3)$, $N = (g_1, g_2, g_3)$, $B = (h_1, h_2, h_3)$ of the sought curve $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ would define a solution of Frénet formulae and equations $\sigma_i' = f_i$; which form a system of 12 linear differential equations.

With the initial conditions $\sigma(t_0) = p$, $(T_{t_0}, N_{t_0}, B_{t_0}) = (T_p, N_p, B_p)$, there exists a unique solution on I (pp. 262, 295).

We only have to show that this solution defines an orthonormal base at each point, since (T_p, N_p, B_p) is direct. If $A = (T, N, B)$, we have to prove that $A^t A = I$, or $AA^t = I$.

That is to say $f_i f_j + g_i g_j + h_i h_j = \delta_{ij}$. Now,

$$\begin{aligned} (f_i f_j + g_i g_j + h_i h_j)' &= f_i' f_j + f_i f_j' + g_i' g_j + g_i g_j' + h_i' h_j + h_i h_j' \\ &= \kappa g_i f_j + \kappa g_j f_i + (\tau h_i - \kappa f_i) g_j + (\tau h_j - \kappa f_j) g_i - \tau g_i h_j - \tau g_j h_i = 0, \end{aligned}$$

and the function $f_i f_j + g_i g_j + h_i h_j$ is constant. Since the value at p is δ_{ij}, we conclude.

Hypersurfaces

Definition: Let $(\bar{X}; \bar{g}, \overline{\nabla})$ be a hypersurface of an Euclidean space. We may fix a unitary normal vector field N on a neighborhood of any point of $\bar{X}$, so that we have a true metric ϕ_2, the **second fundamental form** of $\bar{X}$, such that (p. 306)

$$D^{\nabla}D' = D^{\overline{\nabla}}D' + \phi_2(D, D')N,$$
$$\phi_2(D, D') = (D^{\nabla}D') \cdot N = \Phi(D, D') \cdot N.$$

The **Weingarten endomorphism** is the endomorphism ϕ attached to $(\bar{g}, \phi_2)$,

$$\phi_2(D, D') = \phi(D) \cdot D'.$$

Weingarten Formula: $\phi(D) = -D^{\nabla}N$.

Proof: $(D^{\nabla}N) \cdot N = \frac{1}{2}D(N \cdot N) = 0$, and $-(D^{\nabla}N) \cdot D' = \phi_2(D, D')$, (p. 306).

Gauss Equation: *The Riemann-Christoffel tensor* $\bar{R}_{2,2}$ *of* $\bar{X}$ *is*

$$\bar{R}_{2,2}(D_1, D_2; D_3, D_4) = \phi_2(D_1, D_3)\phi_2(D_2, D_4) - \phi_2(D_1, D_4)\phi_2(D_2, D_3).$$

Proof: $R_{2.2} = 0$, and $\Phi(D_i, D_j) \cdot \Phi(D_i', D_j') = \phi_2(D_i, D_j)\phi_2(D_i', D_j')$, (p. 306).

Codazzi-Mainardi Equation: $D_1^{\overline{\nabla}}(\phi(D_2)) - D_2^{\overline{\nabla}}(\phi(D_1)) - \phi([D_1, D_2]) = 0$.

Proof: Just take the tangential component in the following equality (due to the vanishing of the curvature tensor of the Euclidean space),

$$0 = D_1^{\nabla}D_2^{\nabla}N - D_2^{\nabla}D_1^{\nabla}N - [D_1, D_2]^{\nabla}N = D_1^{\nabla}(\phi(D_2)) - D_2^{\nabla}(\phi(D_1)) - \phi([D_1, D_2]).$$

Definitions: The endomorphism ϕ diagonalizes in an orthonormal base (p. 168), and the **principal curvatures** of $\bar{X}$ are the eigenvalues κ_i of ϕ (the sign changes if we replace N by $-N$). The **curvature lines** are the curves tangent to eigenvectors of ϕ (hence non null).

Euler Theorem: *The principal curvatures* κ_i *are the normal curvatures of curvature lines.*

Proof: If T is unitary and $\phi(T) = \kappa_i T$, then $\kappa_i = \phi(T) \cdot T = \phi_2(T, T) = \kappa_n$.

Proposition: *Any closed connected hypersurface of the Euclidean space where every point is* ***umbilical***, $\phi = \lambda\mathrm{Id}$, *is a hyperplane or a hypersphere.*

Proof: Put $n = \dim \bar{X}$. If $\phi = \lambda\mathrm{Id}$, then $\lambda = \frac{1}{n}\,\mathrm{tr}\phi$ is a smooth function, and by the Codazzi-Mainardi equation,

$$0 = D_1^{\overline{\nabla}}(\lambda D_2) - D_2^{\overline{\nabla}}(\lambda D_1) - \lambda[D_1, D_2] = (D_1\lambda)D_2 - (D_2\lambda)D_1.$$

Taking D_1, D_2 linearly independent we see that $D\lambda = 0$ for any vector field D on $\bar{X}$. Hence λ is constant, since $\bar{X}$ is connected.

If $\lambda = 0$, then $\phi = 0$, $D^{\nabla}N = 0$, and N is locally constant, $N = \sum a_i\partial_i$. Let us consider the vector field $H = \sum_i x_i\partial_i$, so that $D^{\nabla}H = D$. When D is tangent to the hypersurface,

$$D(H \cdot N) = (D^{\nabla}H) \cdot N + H \cdot D^{\nabla}N = D \cdot N + H \cdot 0 = 0.$$

That is to say, $H \cdot N = \sum_i a_i x_i$ is constant, and any point has a neighborhood contained in a hyperplane. Since it is connected, the hypersurface is contained in a hyperplane, and since it is closed, it is the whole hyperplane.

If $\lambda \neq 0$, we have $D^{\nabla}(H + \frac{1}{\lambda}N) = D + \frac{1}{\lambda}D^{\nabla}N = 0$; hence $H + \frac{1}{\lambda}N = \sum_i a_i\partial_i$, and $|H - \sum_i a_i\partial_i| = \frac{1}{|\lambda|}$ on the hypersurface.

It is locally contained in the hypersphere $\sum_i (x_i - a_i)^2 = \lambda^{-2}$, and we conclude as above.

Surfaces

If $\bar{X}$ is a surface in an Euclidean space of dimension 3, at any point we have two principal curvatures κ_1, κ_2 (coincident at an umbilical point).

Let D_1, D_2 be an orthonormal base such that $\phi(D_i) = \kappa_i D_i$.

If $T = D_1 \cos\vartheta + D_2 \sin\vartheta$ is a unitary vector, the normal curvature of curves tangent to T is

$$\kappa_n = \phi_2(T,T) = \phi(T)\cdot T = (\kappa_1 D_1 \cos\vartheta + \kappa_2 D_2 \sin\vartheta)\cdot T = \kappa_1 \cos^2\vartheta + \kappa_2 \sin^2\vartheta.$$

The extremal values of the normal curvature are the principal curvatures.

Gauss Theorema Egregium: *The product $\kappa_1\kappa_2$ of the two principal curvatures is an intrinsic invariant of the riemannian surface $(\bar{X}, \bar{g})$.*

Proof: By Gauss equation, $\kappa_1\kappa_2$ is the curvature K of the surface,

$$K = \bar{R}_{2,2}(D_1, D_2; D_1, D_2) = \phi_2(D_1, D_1)\phi_2(D_2, D_2) - \phi_2(D_1, D_2)^2 = \kappa_1\kappa_2 - 0.$$

10.5 Lie Groups

Definitions: A **Lie group** is a smooth manifold G, with a smooth group law $G \times G \to G$, $(x, y) \mapsto xy$, such that $G \to G$, $x \mapsto x^{-1}$ is smooth.

The neutral or identity element is denoted by e.

A **morphism of Lie groups** is a smooth morphism of groups.

Any element x of a Lie group G defines a diffeomorphism

$$L_x \colon G \longrightarrow G,\ \ L_x(g) = xg,$$

and a vector field D is (**left**) **invariant** when $L_x D = D$, $\forall x \in G$; i.e., $L_{x,*}D_y = D_{xy}$.

The invariant vector fields are stable under the Lie bracket, since $L_x[D, D'] = [L_x D, L_x D']$, and they form the **Lie algebra** $\mathfrak{g}$ of G.

Theorem: *The map $\mathfrak{g} \to T_e G$, $D \mapsto D_e$, is an isomorphism.*

Proof: It is injective because any invariant vector field D is determined by the value at a point,

$$D_x = L_x(D_e).$$

Now, given $D_e \in T_e G$, we have to show that the vector field $\{D_x = L_x D_e\}$ is smooth, i.e., that the following function is smooth for any $f \in \mathcal{C}^\infty(G)$,

$$h(x) = D_x f = D_e(f \circ L_x).$$

Let D be a vector field on G extending D_e, and let us consider on $G \times G$ the vector field $\widetilde{D} = (0, D)$ and the function $\tilde{f}(x, y) = xy$. We conclude since $h = (\widetilde{D}\tilde{f}\,)\big|_{G\times e}$,

$$h(a) = D_e(f(ay)) = \widetilde{D}_{(a,e)}\tilde{f}.$$

Corollary: *Any Lie group admits a global base of vector fields, and it is orientable.*

Theorem: *If $\psi \colon G \to G'$ is a morphism of Lie groups, then $\psi_* \colon \mathfrak{g} = T_e G \to T_e G' = \mathfrak{g}'$ preserves the Lie bracket, and $\psi_*(D_x) = (\psi_* D)_{\psi(x)}$, where $D \in \mathfrak{g}$.*

Proof: Let us show that $D' = \psi_* D$ when D' is invariant and $D'_e = \psi_*(D_e)$.

Now, since $\psi L_x = L_{\psi(x)}\psi$, we have

$$\psi_*(D_x) = \psi_*(L_x D_e) = L_{\psi(x)}(\psi_* D_e) = L_{\psi(x)} D'_e = D'_{\psi(x)}.$$

That is to say, $\psi^*(D'f) = D(\psi^* f)$, and therefore $\psi^*([D'_1, D'_2]f) = [D_1, D_2](\psi^* f)$.

Hence $\psi_*[D_1, D_2]_x = [D'_1, D'_2]_{\psi(x)}$, and we conclude when $x = e$.

Lemma: *Any invariant vector field is complete (the flow is defined on $\mathbb{R} \times G$).*

Proof: The integral curve of an invariant vector field D passing through x at $t = 0$ is just $\sigma_x(t) = L_x(\sigma_e(t)) = x\sigma_e(t)$.

If $\sigma_e(t)$ is defined on $(-\varepsilon, \varepsilon)$, so is any integral curve $\sigma_x(t)$, and D is complete. q.e.d.

A morphism of Lie groups $g_t \colon \mathbb{R} \to G$ defines a uniparametric group on G,

$$\tau(t, x) = xg_t,\ \ \tau_{s+t}(x) = xg_{t+s} = xg_t g_s = \tau_s(\tau_t x),$$

and L_x transforms integral curves yg_t into integral curves xyg_t; hence the infinitesimal generator D is invariant (and D_e is the tangent vector at $t = 0$ to the curve g_t).

Theorem: $\mathrm{Hom}(\mathbb{R}, G) = \mathfrak{g}$.

Proof: It is enough to show that the integral curve $g_t \colon \mathbb{R} \to G$ of $D \in \mathfrak{g}$ passing through e is a morphism of groups. Since the integral curve passing through x is $\tau_t(x) = xg_t$,

$$g_{s+t} = \tau_{s+t}(e) = \tau_t(\tau_s e) = \tau_s(e) g_t = g_s g_t.$$

Definition: The **exponential map** of G is $\exp \colon \mathfrak{g} \to G$, $\exp(D) = g_1$, where g_t is the morphism corresponding to D. The tangent vector at $t = 0$ to the morphism $\phi(t) = g_{\lambda t}$ is λD_e, so that it corresponds to the vector field λD,

$$g_t = \exp(tD).$$

Theorem: *The exponential map is a smooth map, and the tangent linear map at the origin $\mathfrak{g} = T_0\mathfrak{g} \to T_e G = \mathfrak{g}$ is the identity.*

Proof: $\widetilde{D}_{(x,D)} = (D_x, 0) \in T_x G \times T_D \mathfrak{g}$ is a vector field on $G \times \mathfrak{g}$, and the integral curve through (x, D) is $t \mapsto (x \cdot \exp(tD), D)$; hence the flow is

$$\tau_t(x, D) = (x \cdot \exp(tD), D),$$

and we see that the map $\tau_1(e, D) = (\exp(D), D)$ is smooth.

Now, in $\mathfrak{g}$, the tangent vector at $t = 0$ to the curve tD is D; hence the tangent linear map tangent sends it to the tangent vector at $t = 0$ to the curve $\exp(tD)$, which is D_e.

Theorem: *If $\psi \colon G \to G'$ is a morphism od Lie groups, the following square commutes,*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & G \\ {\scriptstyle \psi_*}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathfrak{g}' & \xrightarrow{\exp} & G' \end{array}$$

Proof: The tangent vector at $t = 0$ to the morphism $\psi(g_t) \colon \mathbb{R} \to G'$ is $\psi_*(D_e)$; hence it corresponds to the invariant vector field $\psi_* D$, and $\exp(\psi_* D) = \psi(g_1) = \psi(\exp(D))$.

Corollary: *Let G be a connected Lie group. If two morphisms of Lie groups $\psi, \varphi \colon G \to G'$ coincide on the Lie algebra, $\psi_* = \varphi_* \colon \mathfrak{g} \to \mathfrak{g}'$, then $\psi = \varphi$.*

Proof: Since the exponential map is a local diffeomorphism at the origin, by the above theorem we have $\psi = \varphi$ on a neighborhood of the identity. Now we conclude by

Lemma: *If G is connected, and U is a neighborhood of the identity, then $G = \bigcup_n U^n$. Hence, any open subgroup coincides with G.*

Proof: Replacing U by $U^{-1} \cap U$ we may assume that $U = U^{-1}$.

Now $H = \bigcup_n U^n$ is a subgroup, and it is open since $hU \subseteq H$ when $h \in H$.

Hence all the classes gH are open, and therefore closed; $H = G$.

Example: The Lie algebra of the linear group $Gl(n, \mathbb{R})$ is $M_n(\mathbb{R})$, each matrix A corresponds to the morphism of groups $t \mapsto e^{At}$, the Lie bracket is $[A, B] = AB - BA$, and the exponential map is $\exp(A) = e^A$.

Lemma: *If G is abelian and connected,* $\exp \colon \mathfrak{g} \to G$ *is a surjective morphism of groups.*

Proof: Let us consider the product $\mu \colon G \times G \to G$, $\mu(x, y) = xy$. The tangent linear map $\mu_* \colon T_eG \times T_eG \to T_eG$ is $\mu_*(D_e, D'_e) = D_e + D'_e$, since it is the identity on each factor.

Take $D, D' \in \mathfrak{g}$, and let $g_t, g'_t \colon \mathbb{R} \to G$ be the corresponding morphisms. Since G is abelian, $g_t g'_t = \mu(g_t, g'_t) \colon \mathbb{R} \to G$ is a morphism, corresponding to $\mu_*(D_e, D'_e) = D_e + D'_e$; hence

$$\exp(D + D') = g_1 h_1 = \exp(D)\exp(D').$$

Now, since it is a morphism of groups, and it is a local diffeomorphism at the origin, so it is at any point, and the image is an open subgroup of G.

Lemma: *Any discrete subgroup H of a real vector space E of finite dimension is generated by a family of linearly independent vectors,*

$$H = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_r.$$

Proof: Any discrete subgroup is closed, because any Cauchy sequence stabilizes.

Replacing E by the vector subspace that H spans, we may assume that H contains a base $v_1, \ldots, v_r$ of E. Let us consider the open projection $\pi \colon E \to E/(\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_r) \simeq S_1^r$.

The subgroup $\pi(H) \subset S_1^r$ is closed, because so is $\pi^{-1}(\pi(H)) = H + \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_r = H$, and it is discrete: if U is a neighborhood of the origin intersecting H at no other point, $\pi(U) \cap \pi(H) = 0$.

Since S_1^r is compact, $\pi(H)$ is finite, and H is a finitely generated group of rank r.

Now, H is torsion free, since so is E, then (p. 176) H is a free group, $H = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_r$, and $e_1, \ldots, e_r$ are $\mathbb{R}$-linearly independent since they span E.

Theorem: *Any abelian connected Lie group G is isomorphic to $(\mathbb{R}/\mathbb{Z})^r \times \mathbb{R}^n$.*

Proof: The kernel H of the exponential $\exp \colon \mathfrak{g} \to G$ is a discrete subgroup because the exponential is a local diffeomorphism; hence the exponential factors through an isomorphism of groups $\phi \colon E/\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_r \to G$, and it is smooth since so is $\phi \circ \pi = \exp$ and π is a local diffeomorphism. Moreover, $E/\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_r \simeq (\mathbb{R}/\mathbb{Z})^r \times \mathbb{R}^n$.

Corollary: *Any abelian compact connected Lie group is a torus $(\mathbb{R}/\mathbb{Z})^r$.*

Corollary: *Any abelian Lie group is $G \simeq (\mathbb{R}/\mathbb{Z})^r \times \mathbb{R}^n \times D$, where D is discrete.*

Proof: Let us consider the connected component G_e of the identity, and the exact sequence

$$0 \longrightarrow G_e \longrightarrow G \longrightarrow G/G_e \longrightarrow 0$$

Since $G_e \simeq (\mathbb{R}/\mathbb{Z})^r \times \mathbb{R}^n$ is divisible, it is an injective $\mathbb{Z}$-module (p. 127) and the sequence admits a section $s\colon G/G_e \to G$ (smooth since G/G_e is discrete) defining an isomorphism

$$G \simeq G_e \times (G/G_e), \;\; g \mapsto (g - s(\bar{g}), \bar{g}).$$

Part IV

Fourth Year

Chapter 11

Algebraic Geometry I

11.1 Sheaves and Presheaves

Definition: A **presheaf** $\mathcal{P}$ of sets on a topological space X is given by a set $\mathcal{P}(U)$ for any open set $U \subseteq X$, and a map $\rho_V^U \colon \mathcal{P}(U) \to \mathcal{P}(V)$ whenever $V \subseteq U$, so that

1. $\rho_U^U = \mathrm{Id}$.
2. $\rho_W^U = \rho_W^V \circ \rho_V^U$, when $W \subseteq V \subseteq U$.

(a contravariant functor from the category of open sets in X and inclusion morphisms, to the category of sets). The elements $s \in \mathcal{P}(U)$ are named **sections** of $\mathcal{P}$ on U, and we say that $s|_V := \rho_V^U(s)$ is the **restriction** of s to V.

Given two presheaves $\mathcal{P}, \mathcal{P}'$ on X, a **morphism of presheaves** $f \colon \mathcal{P} \to \mathcal{P}'$ is a morphism of functors; i.e., a family of maps $f_U \colon \mathcal{P}(U) \to \mathcal{P}'(U)$ compatible with the restriction morphisms, in the sense that the following squares commute

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{f_U} & \mathcal{P}'(U) \\ \downarrow{\scriptstyle \rho_V^U} & & \downarrow{\scriptstyle \rho_V^U} \\ \mathcal{P}(V) & \xrightarrow{f_V} & \mathcal{P}'(V) \end{array}$$

Definition: A presheaf $\mathcal{F}$ is a **sheaf** when, for any open cover $U = \bigcup_{i \in I} U_i$ of an open set $U \subseteq X$, we have that

1. If two sections $s, s' \in \mathcal{F}(U)$ coincide on the cover, $s|_{U_i} = s'|_{U_i}$, then $s = s'$.
2. Given sections $s_i \in \mathcal{F}(U_i)$ coinciding on the intersections, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for any index $i \in I$.

That is to say, for any open cover $U = \bigcup_{i \in I} U_i$ we have an exact sequence of sets[1]

$$(*) \qquad \mathcal{F}(U) \xrightarrow{s \longmapsto (s_{|U_i})} \prod_{i \in I} \mathcal{F}(U_i) \underset{(s_i) \longmapsto (s_{j\,|U_i \cap U_j})}{\overset{(s_i) \longmapsto (s_{i\,|U_i \cap U_j})}{\rightrightarrows}} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

Morphisms of sheaves are just morphisms of presheaves.

A sheaf $\mathcal{F}'$ is a **subsheaf** of $\mathcal{F}$ if $\mathcal{F}'(U)$ is a subset of $\mathcal{F}(U)$ for any open set U, and the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$ is a morphism of sheaves.

[1] In particular $\mathcal{F}(\emptyset) = *$, because an open cover of $\emptyset$ is the empty family of open sets, and $\prod_{i \in \emptyset} U_i = *$.

The **restriction** of $\mathcal{F}$ to an open set $U \subseteq X$ is the sheaf $(\mathcal{F}|_U)(V) = \mathcal{F}(V)$, $V \subseteq U$.

Analogously we may define presheaves of groups, rings (or with values in a category $\mathbf{C}$), and the corresponding morphisms; and sheaves by requiring that the above sequence (*) is exact in $\mathbf{C}$ for any open cover $U = \bigcup_{i \in I} U_i$.

Definition: Two sections s', s of a presheaf of sets $\mathcal{P}$, defined on neighborhoods of a point x, have equal **germ** at x if they coincide on a neighborhood of x. The germ of s is denoted s_x, and the **stalk** or **fibre** of $\mathcal{P}$ at x is the set $\mathcal{P}_x = \varinjlim_{x \in U} \mathcal{P}(U)$ of all germs. Put (pp. 197, 275, 306)

$$\pi \colon \mathcal{P}^{\mathrm{et}} = \coprod_{x \in X} \mathcal{P}_x \longrightarrow X,\ \pi(s_x) = x.$$

When $x \in U$, we have a map $\mathcal{P}(U) \to \mathcal{P}_x$, so that $s \in \mathcal{P}(U)$ defines a section $\tilde{s}$ of π,

$$\tilde{s} \colon U \longrightarrow \mathcal{P}^{et},\ \tilde{s}(x) = s_x.$$

If two sections $\tilde{s}', \tilde{s}$ coincide at a point, $s_x = s'_x$, by definition they coincide on a neighborhood of x, so that the images of these sections $\tilde{s}$ form a base of a topology on $\mathcal{P}^{\mathrm{et}}$, such that π is a local homeomorphism, and these sections $\tilde{s}$ are continuous.

Moreover, any continuous section of π locally coincides with some of these sections $\tilde{s}$, so that $\mathcal{P}_x$ is also the set of germs at x of continuous sections of π.

This map $\pi \colon \mathcal{P}^{\mathrm{et}} \to X$ is the **étalé space** of the presheaf $\mathcal{P}$, and the sheaf $\mathcal{P}^\sharp$ of continuous sections of π is the **sheafification** of $\mathcal{P}$ or the **associated sheaf** to $\mathcal{P}$. The maps $\mathcal{P}(U) \to \mathcal{P}^\sharp(U)$, $s \mapsto \tilde{s}$, define a canonical morphism $\mathcal{P} \to \mathcal{P}^\sharp$ inducing bijections on stalks, $\mathcal{P}_x = \mathcal{P}^\sharp_x$.

Any morphism of presheaves $f \colon \mathcal{P}_1 \to \mathcal{P}_2$ induces maps $f_x \colon \mathcal{P}_{1,x} \to \mathcal{P}_{2,x}$ and a continuous map $f^{\mathrm{et}} \colon \mathcal{P}_1^{\mathrm{et}} \to \mathcal{P}_2^{\mathrm{et}}$, hence a morphism of sheaves $f^\sharp \colon \mathcal{P}_1^\sharp \to \mathcal{P}_2^\sharp$.

Universal Property: *If $\mathcal{F}$ is a sheaf of sets, then the morphism $\mathcal{F} \to \mathcal{F}^\sharp$ is an isomorphism, and for any presheaf of sets $\mathcal{P}$,*

$$\mathrm{Hom}(\mathcal{P}, \mathcal{F}) = \mathrm{Hom}(\mathcal{P}^\sharp, \mathcal{F}).$$

Proof: The maps $\mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ are injective: if $s', s \in \mathcal{F}(U)$ have equal germ at any point of U, then they coincide on a cover of U; hence $s' = s$.

They are surjective: any continuous section $\sigma \colon U \to \mathcal{F}^{\mathrm{et}}$ coincides on an open cover $U = \bigcup U_i$ with some sections $\tilde{s}_i$, $s_i \in \mathcal{F}(U_i)$. Since $\mathcal{F}(U_i \cap U_j) \to \mathcal{F}^\sharp(U_i \cap U_j)$ is injective, $s_i = s_j$ on $U_i \cap U_j$, and there is $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$, hence $\sigma = \tilde{s}$.

Now any morphism $f \colon \mathcal{P} \to \mathcal{F}$ factors through the canonical morphism $\mathcal{P} \to \mathcal{P}^\sharp$,

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & \mathcal{F} \\ \downarrow & & \downarrow \wr \\ \mathcal{P}^\sharp & \xrightarrow{f^\sharp} & \mathcal{F}^\sharp \end{array}$$

uniquely since sections of $\mathcal{P}^\sharp$ locally coincide with sections $\tilde{s}$, where $s \in \mathcal{P}(U)$.

Corollary: $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_X(\mathcal{F}^{\mathrm{et}}, \mathcal{G}^{\mathrm{et}})$, *when $\mathcal{F}$ and $\mathcal{G}$ are sheaves of sets.*

Corollary: *A morphism $f \colon \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves of sets is an isomorphism if and only if the map $f_x \colon \mathcal{F}_{1,x} \to \mathcal{F}_{2,x}$ is bijective at any point $x \in X$.*

Proof: If the maps f_x are bijective, then the continuous map $f^{\mathrm{et}} \colon \mathcal{F}_1^{\mathrm{et}} \to \mathcal{F}_2^{\mathrm{et}}$ is bijective; hence an homeomorphism since the projections $\mathcal{F}_i^{\mathrm{et}} \to X$ are local homeomorphisms. q.e.d.

1. If F is a set, the **constant sheaf** F is the sheafification of the **constant presheaf** $U \rightsquigarrow F$. The fibres are $F_x = F$, and the étalé space is the trivial covering $F \times X \to X$, where F is discrete: the constant sheaf F is the sheaf of continuous maps $U \to F$.

   A sheaf of sets $\mathcal{F}$ is **locally constant** if it is constant on an open neighborhood of any point $x \in X$; i.e., when the étalé space $\mathcal{F}^{\text{et}} \to X$ is a covering.

2. If ω is a closed 1-form on a smooth manifold X, then $\mathcal{F}(U) = \{f \in \mathcal{C}^\infty(U) \colon \omega = \mathrm{d}f\}$ is a sheaf on X. It is locally constant, of fibre $\mathbb{R}$, by Poincaré's lemma, and the covering $\mathcal{F}^{\text{et}} \to X$ was used in p. 275 to prove that ω is exact when X is simply connected.

When $\mathcal{P}$ is a presheaf of groups (rings, A-modules, ...) the fibres $\mathcal{P}_x$ inherit a group structure, so that $\mathcal{P}^\sharp$ is a sheaf of groups, with the obvious universal property in the category of sheaves of groups (resp. ...). Unless otherwise stated, from now on *in these notes all presheaves and sheaves are assumed to be of abelian groups*, and we put $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$.

Definition: A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is **injective** (resp. **surjective**) if so are the morphisms $\mathcal{F}_x \to \mathcal{G}_x$, and in general we say that a sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is **exact** if so is the sequence of stalks $\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$ at any point $x \in X$.

Definitions: If $\mathcal{O}_X$ is a sheaf of rings on a topological space X, an $\mathcal{O}_X$**-module** is a sheaf of abelian groups $\mathcal{M}$ such that $\mathcal{M}(U)$ has a structure of $\mathcal{O}_X(U)$-module compatible with the restriction morphisms, in the sense that, for any open set $V \subset U$,

$$(fm)|_V = f|_V m|_V; \ f \in \mathcal{O}_X(U), \ m \in \mathcal{M}(U),$$

and it is **locally free** of rank r if it is locally isomorphic to $\mathcal{O}_X^r$ (we may cover X with open sets U where $\mathcal{M}|_U \simeq \mathcal{O}_X^r|_U$) and it is a **line sheaf** when $r = 1$.

For example, if $\mathcal{C}_X^\infty$ is the sheaf of smooth functions on a smooth manifold X of dimension n, then the sheaf of smooth vector fields is a locally free $\mathcal{C}_X^\infty$-module of rank n, and the sheaf Ω_X^n of smooth n-forms is a line sheaf.

Operations with Sheaves: If $\mathcal{F}'$ is a subsheaf of a sheaf $\mathcal{F}$ on X, $\{\mathcal{F}_i\}$ is a family of sheaves on X, and $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves,

1. $\mathcal{F}/\mathcal{F}'$ is the sheafification of $U \rightsquigarrow \mathcal{F}(U)/\mathcal{F}'(U)$, and $(\mathcal{F}/\mathcal{F}')_x = \mathcal{F}_x/\mathcal{F}'_x$.
2. $\operatorname{Ker} f$ is the subsheaf $(\operatorname{Ker} f)(U) = \operatorname{Ker}[\mathcal{F}(U) \to \mathcal{G}(U)]$, and $(\operatorname{Ker} f)_x = \operatorname{Ker} f_x$.
3. $\operatorname{Im} f$ is the sheafification of $U \rightsquigarrow \operatorname{Im}[\mathcal{F}(U) \to \mathcal{G}(U)]$, and $(\operatorname{Im} f)_x = \operatorname{Im} f_x$.
4. $\bigoplus_i \mathcal{F}_i$ is the sheafification of $U \rightsquigarrow \bigoplus_i \mathcal{F}_i(U)$, and $(\bigoplus_i \mathcal{F}_i)_x = \bigoplus_i (\mathcal{F}_i)_x$.
5. $\prod_i \mathcal{F}_i$ is the sheaf $(\prod_i \mathcal{F}_i)(U) = \prod_i \mathcal{F}_i(U)$, and the stalk may be not $\prod_i (\mathcal{F}_i)_x$.
6. $\varinjlim \mathcal{F}_i$ is the sheafification of $U \rightsquigarrow \varinjlim \mathcal{F}_i(U)$, and $(\varinjlim \mathcal{F}_i)_x = \varinjlim (\mathcal{F}_i)_x$.
7. $\varprojlim \mathcal{F}_i$ is the sheaf $(\varprojlim \mathcal{F}_i)(U) = \varprojlim \mathcal{F}_i(U)$, and the stalk may be not $\varprojlim (\mathcal{F}_i)_x$.
8. If $\mathcal{M}, \mathcal{N}$ are $\mathcal{O}$-modules, $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ is the sheafification of $U \rightsquigarrow \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$, and $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})_x = \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{N}_x$.
9. The sheaf of morphisms $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})$ is the sheaf $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$.
10. The **direct image** $\phi_*\mathcal{F}$ by a continuous map $\phi \colon X \to Y$ is the following sheaf on Y

$$(\phi_*\mathcal{F})(V) = \mathcal{F}(\phi^{-1}V), \quad V \subseteq Y.$$

11.2 Sheaf Cohomology

Definitions: A **differential** A-module is a module K with a null square endomorphism d,

$$\mathrm{d}\colon K \longrightarrow K,\ \mathrm{d}^2 = 0,\ \mathrm{Im}\,\mathrm{d} \subseteq \mathrm{Ker}\,\mathrm{d},$$

and the **cohomology** of (K, d) is the quotient module $H(K) = (\mathrm{Ker}\,\mathrm{d})/(\mathrm{Im}\,\mathrm{d})$.

A **morphism of differential modules** $f\colon K \to L$ is a morphism of modules commuting with the differentials, $f\mathrm{d} = \mathrm{d}f$, so that f induces a morphism in cohomology,

$$f\colon H(K) \longrightarrow H(L),\ f[c] = [f(c)].$$

Theorem: *If* $0 \to K' \xrightarrow{i} K \xrightarrow{p} K'' \to 0$ *is an exact sequence of differential modules, we have a connecting morphism* δ *and an* ***exact triangle*** (the image of each morphism coincides with the kernel of the following one)

$$\begin{array}{ccc} H(K') & \xrightarrow{\ i\ } & H(K) \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow{\scriptstyle p} & \\ & H(K'') & \end{array}$$

Proof: Once we define the *connecting* δ, the exactness is easy to check.

$$\begin{aligned} [m''] &\in H(K''),\ \text{where } \mathrm{d}m'' = 0, \\ m'' &= p(m),\ \text{hence } p(\mathrm{d}m) = \mathrm{d}(p(m)) = \mathrm{d}m'' = 0, \\ \mathrm{d}m &\in K',\ \text{and } \mathrm{d}(\mathrm{d}m) = 0, \\ \delta([m'']) &:= [\mathrm{d}m] \in H(K'). \end{aligned}$$

It does not depend on the representant m:

If $p(m) = p(n)$, then $n - m \in K'$, $\mathrm{d}n - \mathrm{d}m \in \mathrm{d}K'$, and $[\mathrm{d}n] = [\mathrm{d}m]$ in $H(K')$.

Definitions: If the module is graded, $K = \oplus_{n\in\mathbb{Z}} K^n$, and $\mathrm{d}(K^n) \subseteq K^{n+1}$, it may be identified with a **complex** $K^\bullet$ of modules; i.e. a sequence of A-modules $\{K^n\}_{n\in\mathbb{Z}}$ with A-linear morphisms $\mathrm{d}^n\colon K^n \to K^{n+1}$ such that $\mathrm{d}^n \circ \mathrm{d}^{n-1} = 0$, so that the cohomology is $H(K) = \bigoplus_{n\in\mathbb{Z}} H^n(K^\bullet)$, where $H^n(K^\bullet) := \mathrm{Ker}\,\mathrm{d}^n/\mathrm{Im}\,\mathrm{d}^{n-1}$. A **morphism** of complexes $f\colon K^\bullet \to L^\bullet$ is a sequence of A-linear morphisms $f^n\colon K^n \to L^n$ commuting with the differentials (a homogeneous differential morphisms $f\colon \oplus_n K^n \to \oplus_n L^n$),

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\mathrm{d}} & K^{n-1} & \xrightarrow{\mathrm{d}} & K^n & \xrightarrow{\mathrm{d}} & K^{n+1} & \xrightarrow{\mathrm{d}} & \ldots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \ldots & \xrightarrow{\mathrm{d}} & L^{n-1} & \xrightarrow{\mathrm{d}} & L^n & \xrightarrow{\mathrm{d}} & L^{n+1} & \xrightarrow{\mathrm{d}} & \ldots \end{array}$$

so that f induces morphisms $f\colon H^n(K^\bullet) \to H^n(L^\bullet)$. We say that f is a **quasi-isomorphism** when $f\colon H^n(K^\bullet) \to H^n(L^\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$, and we put $f\colon K^\bullet \xrightarrow{\sim} L^\bullet$.

Hence, a **resolution** $0 \to M \to R^\bullet$ (an infinite exact sequence $0 \to M \to R^0 \to R^1 \to \ldots$) defines a quasi-isomorphism $M \xrightarrow{\sim} R^\bullet$ when M is viewed as a complex with null differential: $K^0 = M$ and $K^n = 0$ when $n \neq 0$.

A sequence of morphisms of complexes $K^\bullet \to L^\bullet \to M^\bullet$ is **exact** when $K^n \to L^n \to M^n$ is an exact sequence of A-modules for any $n \in \mathbb{Z}$; hence, by the above theorem, any exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ induces a cohomology exact sequence

$$\ldots \longrightarrow H^{n-1}(M^\bullet) \xrightarrow{\delta} H^n(K^\bullet) \longrightarrow H^n(L^\bullet) \longrightarrow H^n(M^\bullet) \xrightarrow{\delta} H^{n+1}(K^\bullet) \longrightarrow \ldots$$

When the complex has components of negative degree, sometimes it is convenient to put $K_n = K^{-n}$, $\mathrm{d}_n = \mathrm{d}^{-n} \colon K_n \to K_{n-1}$, and $H_n(K_\bullet) = H^{-n}(K^\bullet)$.

Snake's Lemma: Any morphism $f \colon M \to N$ may be viewed as a complex $K^\bullet$, where $K^0 = M$, $K^1 = N$ and $\mathrm{d}_0 = f$, so that $H^0(K^\bullet) = \operatorname{Ker} f$ and $H^1(K^\bullet) = \operatorname{Coker} f$.

Hence any commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

induces a cohomology exact sequence

$$0 \longrightarrow \operatorname{Ker} f' \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Coker} f' \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} f'' \longrightarrow 0$$

Example: If a sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact, it is easy to check that so is

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$$

for any open set U; but $\mathcal{F}(U) \to \mathcal{F}''(U)$ may be not surjective even if so is the morphism of sheaves $\mathcal{F} \to \mathcal{F}''$. For example, if $\mathcal{O}$ is the sheaf of smooth functions on S_1 and Ω is the sheaf of 1-forms, the differential $\mathrm{d} \colon \mathcal{O} \to \Omega$ is surjective (any 1-form is locally exact); but $\mathrm{d} \colon \mathcal{O}(S_1) \to \Omega(S_1)$ is not surjective: exact 1-forms have null integral,

$$\int_{S_1} \mathrm{d}f = \int_{\partial S_1} f = \int_{\emptyset} f = 0.$$

Definition: A sheaf of abelian groups $\mathcal{F}$ on a topological space X is **flasque** when the restriction morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for any open set U in X.

Lemma: *If $0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{p} \mathcal{F}'' \to 0$ is an exact sequence of sheaves and $\mathcal{F}'$ is flasque, then for any open set U, the following sequence is also exact*

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{i} \mathcal{F}(U) \xrightarrow{p} \mathcal{F}''(U) \longrightarrow 0.$$

If moreover $\mathcal{F}$ is flasque, then so is $\mathcal{F}''$.

Proof: It is enough to show that p is surjective. If $s'' \in \mathcal{F}''(U)$, we consider pairs (V, s), where $s \in \mathcal{F}(V)$, $p(s) = s''|_V$, and by Zorn's lemma, there is a maximal pair (V, s).

If $V \neq U$, and $x \in U - V$, since $\mathcal{F}_x \to \mathcal{F}''_x$ is surjective, there is a neighborhood W of x, and $w \in \mathcal{F}(W)$, such that $p(w) = s''$. Since $\mathcal{F}'$ is flasque, and $p(s - w) = 0$ on $V \cap W$, there is a section $s' \in \mathcal{F}'(W)$ coinciding with $s - w$ on $V \cap W$.

Now $w + s' \in \mathcal{F}(W)$ and $s \in \mathcal{F}(V)$ coincide on $V \cap W$, and define a section of $\mathcal{F}$ on $U \cup W$ projecting onto s'', against the maximal character of (V, s).

Finally, if $\mathcal{F}$ is flasque, we have a commutative square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}''(X) \\ {\scriptstyle \text{epi}}\downarrow & & \downarrow \\ \mathcal{F}(U) & \xrightarrow{\text{epi}} & \mathcal{F}''(U) \end{array}$$

q.e.d.

Definitions: If $\mathcal{F}$ is a sheaf, the **Godement sheaf** $C^0\mathcal{F}$ is the flasque sheaf

$$(C^0\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x.$$

$\mathcal{F}$ is a subsheaf of $C^0\mathcal{F}$, and we put $\mathcal{F}_1 := (C^0\mathcal{F})/\mathcal{F}$, so that we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0\mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

and we repeat the process, $0 \to \mathcal{F}_1 \to C^0\mathcal{F}_1 \to \mathcal{F}_2 \to 0$, and so on. We put $C^n\mathcal{F} := C^0\mathcal{F}_n$, so that we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0\mathcal{F} \longrightarrow C^1\mathcal{F} \longrightarrow C^2\mathcal{F} \longrightarrow \ldots \longrightarrow C^n\mathcal{F} \longrightarrow \ldots$$

The complex of sheaves $C^\bullet\mathcal{F} = \{C^0\mathcal{F} \to C^1\mathcal{F} \to C^2\mathcal{F} \to \ldots\}$ is the **Godement canonical resolution** of $\mathcal{F}$, and taking global sections we obtain a complex of abelian groups

$$\Gamma(X, C^\bullet\mathcal{F}) = \{\Gamma(X, C^0\mathcal{F}) \xrightarrow{\mathrm{d}_0} \Gamma(X, C^1\mathcal{F}) \xrightarrow{\mathrm{d}_1} \Gamma(X, C^2\mathcal{F}) \xrightarrow{\mathrm{d}_2} \ldots\}$$

The n-th **cohomology group** of X with coefficients in the sheaf $\mathcal{F}$ is

$$H^n(X, \mathcal{F}) = H^n\big[\Gamma(X, C^\bullet\mathcal{F})\big] = \mathrm{Ker}\,\mathrm{d}_n/\mathrm{Im}\,\mathrm{d}_{n-1}$$

and we say that $\mathcal{F}$ is an **acyclic** sheaf if $H^n(X, \mathcal{F}) = 0$, $n \geq 1$.

Any morphism of sheaves $f\colon \mathcal{F} \to \mathcal{G}$ induces morphisms $f_x\colon \mathcal{F}_x \to \mathcal{G}_x$, hence a morphism $f_0\colon C^0\mathcal{F} \to C^0\mathcal{G}$, inducing a morphism $\mathcal{F}_1 = (C^0\mathcal{F})/\mathcal{F} \to (C^0\mathcal{G})/\mathcal{G} = \mathcal{G}_1$, and so on,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & C^0\mathcal{F} & \longrightarrow & C^1\mathcal{F} & \longrightarrow & C^2\mathcal{F} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & C^0\mathcal{G} & \longrightarrow & C^1\mathcal{G} & \longrightarrow & C^2\mathcal{G} & \longrightarrow & \ldots \end{array}$$

Taking global sections we have a morphism of complexes $\Gamma(X, C^\bullet\mathcal{F}) \to \Gamma(X, C^\bullet\mathcal{G})$ inducing morphisms on the cohomology groups $f\colon H^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{G})$.

Theorem: $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Proof: Since $0 \to \mathcal{F} \to C^0\mathcal{F} \to C^1\mathcal{F}$ is exact, so is $0 \to \mathcal{F}(X) \to (C^0\mathcal{F})(X) \to (C^1\mathcal{F})(X)$.

Theorem: *Any flasque sheaf is acyclic.*

Proof: $0 \longrightarrow \mathcal{F} \longrightarrow C^0\mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow 0$ is exact, and $\mathcal{F}$ is flasque; hence,

$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^0\mathcal{F}(X) \longrightarrow \mathcal{F}_1(X) \longrightarrow 0$ is exact, and $\mathcal{F}_1$ is flasque,

$0 \longrightarrow \mathcal{F}_1(X) \longrightarrow C^1\mathcal{F}(X) \longrightarrow \mathcal{F}_2(X) \longrightarrow 0$ is exact, and $\mathcal{F}_2$ is flasque,...

$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^0\mathcal{F}(X) \longrightarrow C^1\mathcal{F}(X) \longrightarrow C^2\mathcal{F}(X) \longrightarrow \ldots$ is exact.

Theorem: *Any exact sequence of sheaves* $0 \to \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{p} \mathcal{F}'' \to 0$ *induces a long cohomology exact sequence,*

$$0 \to H^0(X, \mathcal{F}') \xrightarrow{i} H^0(X, \mathcal{F}) \xrightarrow{p} H^0(X, \mathcal{F}'') \xrightarrow{\delta} H^1(X, \mathcal{F}') \xrightarrow{i} H^1(X, \mathcal{F}) \xrightarrow{p} H^1(X, \mathcal{F}'') \xrightarrow{\delta} \ldots$$

Proof: The sequences $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ are exact; hence so are the sequences

$$0 \longrightarrow \Gamma(U, C^0\mathcal{F}') \longrightarrow \Gamma(U, C^0\mathcal{F}) \longrightarrow \Gamma(U, C^0\mathcal{F}'') \longrightarrow 0,$$

so that $0 \to C^0\mathcal{F}' \to C^0\mathcal{F} \to C^0\mathcal{F}'' \to 0$ is exact, and so is $0 \to \mathcal{F}_1' \to \mathcal{F}_1 \to \mathcal{F}_1'' \to 0$ by the snake's lemma. We obtain an exact sequence $0 \to C^\bullet\mathcal{F}' \to C^\bullet\mathcal{F} \to C^\bullet\mathcal{F}'' \to 0$ of complexes of flasque sheaves; hence an exact sequence of complexes of abelian groups

$$0 \longrightarrow \Gamma(X, C^\bullet\mathcal{F}') \longrightarrow \Gamma(X, C^\bullet\mathcal{F}) \longrightarrow \Gamma(X, C^\bullet\mathcal{F}'') \longrightarrow 0$$

inducing the required exact sequence.

De Rham's Theorem: *If* $0 \to \mathcal{F} \to \mathcal{R}^0 \to \mathcal{R}^1 \to \mathcal{R}^2 \to \dots$ *is an acyclic* ***resolution*** *of* $\mathcal{F}$ *(the sequence is exact and the sheaves* $\mathcal{R}^n$ *are acyclic), then we have natural isomorphisms*

$$H^n(X, \mathcal{F}) \simeq H^n\big[\Gamma(X, \mathcal{R}^\bullet)\big].$$

Proof: We have exact sequences $0 \to \mathcal{F} \to \mathcal{R}^0 \to \mathcal{C}_1 \to 0$ and $0 \to \mathcal{C}_1 \to \mathcal{R}^1 \to \mathcal{C}_2 \to 0$.

The corresponding cohomology long exact sequences show that

$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{R}^0(X) \xrightarrow{\mathrm{d}_0} \mathcal{C}_1(X) \xrightarrow{\delta} H^1(X, \mathcal{F}) \longrightarrow 0$ is exact,

$\delta \colon H^{n-1}(X, \mathcal{C}_1) \xrightarrow{\sim} H^n(X, \mathcal{F})$ is an isomorphism,

$0 \longrightarrow \mathcal{C}_1(X) \longrightarrow \mathcal{R}^1(X) \longrightarrow \mathcal{C}_2(X)$ is exact,

$H^0\big[\mathcal{R}^\bullet(X)\big] = \operatorname{Ker} \mathrm{d}_0 = \mathcal{F}(X)$,

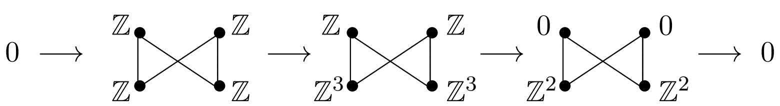
$H^1\big[\mathcal{R}^\bullet(X)\big] = \mathcal{C}_1(X)/\operatorname{Im} \mathrm{d}_0 = H^1(X, \mathcal{F})$,

Finally, $H^{n-1}(X, \mathcal{C}_1) \simeq H^n\big[\mathcal{R}^\bullet(X)\big]$ by induction on n. q.e.d.

1. $H^n(X, \mathcal{F} \times \mathcal{G}) = H^n(X, \mathcal{F}) \times H^n(X, \mathcal{G})$ because $\mathcal{C}^\bullet(\mathcal{F} \times \mathcal{G}) = (\mathcal{C}^\bullet\mathcal{F}) \times (C^\bullet\mathcal{G})$.

2. If a sheaf $\mathcal{F}$ is supported at a finite number of closed points $x_1, \dots, x_n$ (that is to say, $\mathcal{F}_x = 0$ when $x \neq x_i$), then $\mathcal{F}(U) = \oplus_{x_i \in U} \mathcal{F}_{x_i}$, and the sheaf $\mathcal{F}$ is flasque.

3. Let $\mathbb{Z}$ be the constant sheaf on the finite space $X = \{x_1, x_2, y_1, y_2\}$, $x_i < y_j$, representing a circle (p. 238). The stalks of $0 \to \mathbb{Z} \to C^0\mathbb{Z} \to \mathcal{F}_1 \to 0$ are



and $\mathcal{F}_1$ is flasque since it is supported in two closed points.

Hence $H^n(X, \mathbb{Z}) = 0$, $n \geq 2$, and $H^0(X, \mathbb{Z}) = H^1(X, \mathbb{Z}) = \mathbb{Z}$ since these groups are the kernel and cokernel of the morphism

$$\mathbb{Z}^4 = (C^0\mathbb{Z})(X) \xrightarrow{\mathrm{d}} \mathcal{F}_1(X) = \mathbb{Z}^4,\ \mathrm{d}(x_1, x_2, y_1, y_2) = (y_1 - x_1, y_2 - x_1, y_1 - x_2, y_2 - x_2).$$

Theorem: *Let* A *be a noetherian*[2] *lattice semiring and let* $\mathcal{F}$ *be a sheaf on* $\operatorname{Spec} A$*. If* $\mathcal{F}_x = 0$ *at any point* $x \in \operatorname{Spec} A$ *of dimension* $> d$*, then* $H^p(\operatorname{Spec} A, \mathcal{F}) = 0$ *for any* $p > d$*. In particular,*

$$H^p(\operatorname{Spec} A, \mathcal{F}) = 0,\ p > \dim A.$$

Proof: We proceed by induction on d, and it is obvious when $d = -1$.

Let $\{x_i\}$ be the family of all points of $X = \operatorname{Spec} A$ of dimension d.

[2] See p. 345 for the case of an arbitrary lattice semiring.

The presheaf $\mathcal{F}_d(U) = \oplus_{x_i \in U} \mathcal{F}_{x_i}$ is a (flasque) sheaf since any open set is compact, and we have a natural morphism $\phi\colon \mathcal{F} \to \mathcal{F}_d$ since the support of a section of $\mathcal{F}$, being closed, has a finite number of irreducible components, hence a finite number of points of dimension d.

Moreover $\mathcal{F}_d$ has null stalk at any point of dimension $> d$, and $(\mathcal{F}_d)_{x_i} = \mathcal{F}_{x_i}$, so that $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ have null stalk at any point of dimension $> d-1$, and

$$H^p(X, \operatorname{Ker}\phi) = H^p(X, \operatorname{Coker}\phi) = 0,\ p \geq d$$

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow \mathcal{F} \longrightarrow \operatorname{Im}\phi \longrightarrow 0 \text{ is exact} \tag{11.1}$$

$$H^p(X, \mathcal{F}) = H^p(X, \operatorname{Im}\phi),\ p \geq d,$$

$$0 \longrightarrow \operatorname{Im}\phi \longrightarrow \mathcal{F}_d \longrightarrow \operatorname{Coker}\phi \longrightarrow 0 \text{ is exact} \tag{11.2}$$

$$H^p(X, \operatorname{Im}\phi) = 0,\ p > d.$$

Theorem: If A is a ring, any A-module M defines a sheaf $\widetilde{M}$ on $\operatorname{Spec} A$, attached to the localization presheaf $U \rightsquigarrow M_U$, and $\Gamma(\operatorname{Spec} A, \widetilde{M}) = M$, (p. 197). *The sheaves $\widetilde{M}$ are acyclic,*

$$H^n(\operatorname{Spec} A, \widetilde{M}\,) = 0,\ n \geq 1.$$

Proof: Proof: If $j\colon U_f \to X = \operatorname{Spec} A$ is a basic open set, we put $\mathcal{F}_f = j_* j^* \mathcal{F}$, so that we have $\mathcal{F}_f(U) = \mathcal{F}(U_f \cap U)$, and $\mathcal{F}_f$ is flasque when so is the sheaf $\mathcal{F}$.

Moreover, $(\widetilde{M})_f$ is the sheaf associated to the A-module M_f.

We proceed by induction on n and, given a cohomology class $c \in H^n(X, \widetilde{M})$, first we see that any point x has a basic neighborhood U_f such that $c = 0$ in $H^n(X, \widetilde{M}_f)$.

We truncate a flasque resolution $0 \to \widetilde{M} \to \mathcal{C}^\bullet$ at the n-th step, so that we have a commutative diagram with exact rows, where $\mathcal{K}'$ is the image of $\mathcal{C}_f^{n-1} \to \mathcal{K}_f$,

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \widetilde{M} & \longrightarrow & \mathcal{C}^0 & \longrightarrow & \dots & \longrightarrow & \mathcal{C}^{n-1} & \longrightarrow & \mathcal{K} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{M}_f & \longrightarrow & \mathcal{C}_f^0 & \longrightarrow & \dots & \longrightarrow & \mathcal{C}_f^{n-1} & \longrightarrow & \mathcal{K}' & \longrightarrow & 0 \end{array} \qquad \mathcal{K}' \subseteq \mathcal{K}_f$$

In fact the bottom row is exact because, when $0 < p < n$, the cohomology groups of the complex of sections over any basic open set U_g are $H^p(U_f \cap U_g, \widetilde{M}) = H^p(\operatorname{Spec} A_{fg}, \widetilde{M}) = 0$.

Hence $c \in H^n(X, \widetilde{M}) = \mathcal{K}(X)/\mathcal{C}^{n-1}(X)$ is represented by a section $s \in \mathcal{K}(X)$, and on some neighborhood U_f of x it comes from a section in $\mathcal{C}^{n-1}(U_f) = \mathcal{C}_f^{n-1}(X)$.

We conclude that $c = 0$ in $H^n(X, \widetilde{M}_f) = \mathcal{K}'(X)/\mathcal{C}_f^{n-1}(X)$, as claimed.

Now, X being compact, $X = U_{f_1} \cup \ldots \cup U_{f_r}$, where $c = 0$ in $H^n(X, \widetilde{M}_{f_i})$, and the long cohomology exact sequence exact given by (the sheafification of) the exact sequence

$$0 \longrightarrow M \longrightarrow M_{f_1} \oplus \ldots \oplus M_{f_r} \longrightarrow N \longrightarrow 0$$

shows that the morphism $H^n(X, \widetilde{M}) \to \oplus_i H^n(X, \widetilde{M}_{f_i})$ is injective (and we see that $c = 0$). When $n = 1$, because the morphism $\oplus_i M_{f_i} = H^0(X, \oplus_i \widetilde{M}_{f_i}) \to H^0(X, \widetilde{N}) = N$ is surjective, and when $n > 1$, because $H^{n-1}(X, \widetilde{N}) = 0$ by induction.

Note: In the case of curves, the above results are much easier to prove.

Let X be a topological space with a dense point x whose open neighborhoods are just sets with finite complement. Constant sheaves on X are flasque, as well as sheaves $\mathcal{F}$ with null fibre at x because the support of any local section $s \in \mathcal{F}(U)$, being a closed set, is a finite set of closed points, and s may be extended by 0 out of U.

In general, the exact sequences 11.1 and 11.2 of the natural morphism $\phi\colon \mathcal{F} \to \mathcal{F}_x$ let us conclude that $H^p(X,\mathcal{F}) = 0$, $p > 1$, since $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ have null fibre at x.

Moreover, if $X = \operatorname{Spec} A$, where A is a noetherian domain of dimension 1, let us prove that $H^p(X,\widetilde{M}) = 0$, $p \geq 1$. The torsion submodule T defines a flasque sheaf, because it has null stalk at the generic point x, and the long cohomology exact sequence given by (the sheafification of) the exact sequence $0 \to T \to M \to M/T \to 0$ shows that we may assume that M is torsion free. Now, the long cohomology exact sequence given by (the sheafification of) the exact sequence

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{M_x} \longrightarrow (M_x/M)^{\sim} \longrightarrow 0$$

let us conclude because the sheaf $\widetilde{M_x}$ is flasque, the sheaf $(M_x/M)^{\sim}$ has null fibre at x, and the morphism $M_x = H^0(X,\widetilde{M_x}) \to H^0(X,(M_x/M)^{\sim}) = M_x/M$ is surjective.

Definition: If $f\colon Y \to X$ is a continuous map and $\mathcal{F}$ is a sheaf on Y, the **higher direct images** $R^n f_*\mathcal{F} = \mathcal{H}^n\big[f_*(C^\bullet\mathcal{F})\big]$ are the cohomology sheaves of the complex of sheaves $f_*(C^\bullet\mathcal{F})$.

That is to say, $R^n f_*\mathcal{F}$ is the associated sheaf of the presheaf

$$U \ \rightsquigarrow\ H^n(f^{-1}U,\mathcal{F}).$$

Since the direct image of any flasque sheaf is flasque, when $R^n f_*\mathcal{F} = 0$, $n \geq 1$, we have that $f_*(C^\bullet\mathcal{F})$ is a flasque resolution of $f_*\mathcal{F}$, and

$$H^n(Y,\mathcal{F}) = H^n(X,f_*\mathcal{F}).$$

Theorem: *If Y admits a base of open neighborhoods V such that $\mathcal{F}$ is acyclic on $f^{-1}V$, then*

$$H^n(Y,f_*\mathcal{F}) = H^n(X,\mathcal{F}).$$

Theorem: *$H^n(X,i_*\mathcal{F}) = H^n(Y,\mathcal{F})$, when $i\colon Y \to X$ is a closed embedding.*

Proof: The functor i_* is exact since $(i_*\mathcal{F})_x = \mathcal{F}_x$ when $x \in Y$, and 0 otherwise.

11.3 Schemes and Coherent Sheaves

Definitions: A **ringed space** $(X,\mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$, and it is a **locally ringed** space if the fibres $\mathcal{O}_{X,x}$ are local rings.

The elements of the ring $\mathcal{O}_X(U)$ are said to be the **functions** on the open set U.

A **morphism** of ringed spaces $(Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ is a continuous map $\phi\colon Y \to X$ with a morphism of sheaves of rings $\psi\colon \mathcal{O}_X \to \phi_*\mathcal{O}_Y$ (morphisms of rings $\mathcal{O}_X(U) \to \mathcal{O}_Y(\phi^{-1}U)$ compatible with the restriction morphisms), and it is a morphism of locally ringed spaces if moreover the morphisms $\mathcal{O}_{X,\phi(y)} \to \mathcal{O}_{Y,y}$ are local, $\mathfrak{m}_y \cap \mathcal{O}_{X,\phi(y)} = \mathfrak{m}_{\phi(y)}$ (i.e. a function $f \in \mathcal{O}_X(U)$ vanishes at a point $x = \phi(y) \in U$ if and only if $\psi(f) \in \mathcal{O}_Y(\phi^{-1}U)$ vanishes at $y \in \phi^{-1}U$).

If A is a ring, then $(\operatorname{Spec} A,\tilde{A})$ is a locally ringed space, and any ring morphism $\phi\colon A \to B$ defines a morphism of locally ringed spaces

$$(f,\phi)\colon (\operatorname{Spec} B,\tilde{B}) \longrightarrow (\operatorname{Spec} A,\tilde{A}),$$

where $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is the continuous map induced by ϕ, and $\phi\colon \tilde{A} \to f_*\tilde{B}$ is the morphism of sheaves induced by the morphism of presheaves

$$A_U \xrightarrow{\ \phi\ } B_{f^{-1}(U)} \longrightarrow \tilde{B}(f^{-1}U) \longrightarrow (f_*\tilde{B})(U).$$

Theorem: $\mathrm{Hom}_{\mathrm{rings}}(A,B) = \mathrm{Hom}_{\mathrm{loc.\,ring.\,sp.}}(\mathrm{Spec}\,B, \mathrm{Spec}\,A)$.

Proof: Any morphism of locally ringed spaces $(f,\phi)\colon (\mathrm{Spec}\,B, \tilde{B}) \to (\mathrm{Spec}\,A, \tilde{A})$ induces a ring morphism $\phi\colon A = \Gamma(\mathrm{Spec}\,A, \tilde{A}) \to \Gamma(\mathrm{Spec}\,B, \tilde{B}) = B$ and, $A_{f(y)} \to B_y$ being local, the following commutative square shows that f is the map induced by ϕ,

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ \downarrow & & \downarrow \\ A_{f(y)} & \xrightarrow{\ \phi\ } & B_y \end{array}$$

Definition: A ringed space $(X, \mathcal{O}_X)$ is an **affine scheme** if it is isomorphic to $(\mathrm{Spec}\,A, \tilde{A})$ for some ring, necessarily $A = \mathcal{O}_X(X)$, and it is a **scheme** if any point has an open neighborhood U (hence a base of open neighborhoods) such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Morphisms of schemes are morphisms of locally ringed spaces.

If you fix a base scheme S, then a S-**scheme** is a scheme X endowed with a morphism of schemes $\pi_X\colon X \to S$ and a morphism of S-schemes is a morphism of schemes $f\colon X \to Y$ such that $\pi_X = \pi_Y \circ f$.

1. The above theorem shows that the functors $A \rightsquigarrow (\mathrm{Spec}\,A, \tilde{A})$ and $X \rightsquigarrow \mathcal{O}_X(X)$ define an antiequivalence of the category o rings with the category of affine schemes; hence an antiequivalence of the category of k-algebras with the category of affine k-schemes.

2. Smooth manifolds and Riemann surfaces are locally ringed spaces, and the argument given in the proof of the above theorem shows that smooth maps and analytic morphisms are just the morphisms of locally ringed spaces.

3. If A is a domain, then the local rings A_x are subrings of the field of fractions Σ, and the morphism of presheaves $A_U \to \bigcap_{x\in U} A_x$ is an isomorphism on stalks; hence
$$\tilde{A}(U) = \bigcap_{x\in U} A_x .$$

Proposition: *Any irreducible closed set C in a scheme is the closure of a unique point, named* ***generic point*** *point of C.*

Proof: If we consider an affine open set U intersecting C, then $U \cap C$ is irreducible, so that it has a dense point x (p. 140). Now $C = \overline{x}$, because C is irreducible.

Definition: A scheme X is **noetherian** if it is a finite union of affine open sets $U_i = \mathrm{Spec}\,A_i$, where the rings A_i are noetherian.

In such a case any affine open set is $U = \mathrm{Spec}\,A$, with A noetherian. In fact, we may assume that $X = \mathrm{Spec}\,A$, and a chain of ideals of A stabilizes just when it stabilizes in any ring A_i.

Moreover, any noetherian scheme X is a noetherian space where closed irreducible sets have a dense point; hence $X = \mathrm{Spec}\,A_X$ (p. 236) and $H^p(X, \mathcal{F}) = 0$ when $p > \dim X$ (p. 321).

Definitions: If X is a scheme, an $\mathcal{O}_X$-module $\mathcal{M}$ is **quasi-coherent** if we may cover X by affine open sets $U = \mathrm{Spec}\,A$ where $\mathcal{M}|_U = \tilde{M}$ for some A-module M (hence on any affine open set $V \subset U$). When the scheme X is noetherian, $\mathcal{M}$ is **coherent** if moreover M is a finitely generated A-module.

Locally free sheaves are quasi-coherent, and isomorphism classes of line sheaves form an abelian group with the tensor product $\otimes_{\mathcal{O}_X}$, the **Picard group** $\mathrm{Pic}(X)$.

The unity is $\mathcal{O}_X$, and the inverse of a line sheaf $\mathcal{L}$ is $\mathcal{L}^{-1} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Theorem: *Any quasi-coherent sheaf* $\mathcal{M}$ *on* $\operatorname{Spec} A$ *is* $\mathcal{M} = \widetilde{M}$ *for some* A*-module* M.

Proof: Put $M = \mathcal{M}(\operatorname{Spec} A)$.

Then the natural morphisms $M_U \to \mathcal{M}(U)$ induce an isomorphism $\widetilde{M} \to \mathcal{M}$.

In fact, we may cover $\operatorname{Spec} A$ by basic open sets U_i where $\mathcal{M}|_{U_i} = \widetilde{M}_i$, and the following commutative diagram with exact rows shows that $M_f = \mathcal{M}(U_f)$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_f & \longrightarrow & \bigoplus_i \mathcal{M}(U_i)_f & \rightrightarrows & \bigoplus_{i,j} \mathcal{M}(U_i \cap U_j)_f \\ & & \downarrow & & \| & & \| \\ 0 & \longrightarrow & \mathcal{M}(U_f) & \longrightarrow & \bigoplus_i \mathcal{M}(U_i \cap U_f) & \rightrightarrows & \bigoplus_{i,j} \mathcal{M}(U_i \cap U_j \cap U_f) \end{array}$$

Corollary: *Let* X *be an affine scheme. The functor* $\mathcal{M} \rightsquigarrow \mathcal{M}(X)$ *defines an equivalence of the category of quasi-coherent* $\mathcal{O}_X$*-modules with the category of* $\mathcal{O}_X(X)$*-modules.*

Definitions: If U is an open set in X, we say that the scheme $(U, \mathcal{O}_X|_U)$ is a **open subscheme** of X. If $\mathcal{I}$ is a quasi-coherent sheaf of ideals, the support Y of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$ is closed in X, and we say that the scheme $(Y, \mathcal{O}_X/\mathcal{I})$ is the **closed subscheme** of X defined by $\mathcal{I}$.

A morphism $Y \to X$ is an open or closed **embedding** if it defines an isomorphism of Y onto an open or closed subscheme of X.

Definitions: A scheme over a field k is a morphism $X \to \operatorname{Spec} k$ (a structure of k-algebra on $\mathcal{O}_X(U)$, so that restriction morphisms are morphisms of k-algebras), and it is a scheme of **finite type** over k if it is a finite union of affine open sets $U_i = \operatorname{Spec} k[\xi_1, \ldots, \xi_n]$.

If moreover it is of dimension 1, it is a **curve**, and it is a **non singular** curve if any local ring $\mathcal{O}_{X,x}$ is regular (a discrete valuation ring).

A scheme X is **reduced** when so is the ring $\mathcal{O}_X(U)$ for any open set U, and it **integral** when so is the ring $\mathcal{O}_X(U)$ for any open set U, so that X is irreducible (with generic point p_g) and $\mathcal{O}_X(U) \subseteq \Sigma$, where the generic fibre $\Sigma = \mathcal{O}_{X,p_g}$ is a field. A integral scheme X of finite type over a field k is **complete** if any valuation ring $\mathcal{V}$ of Σ, containing k, centers at a unique point $x \in X$ (i.e., $\mathcal{V}$ dominates the local ring $\mathcal{O}_{X,x} \subset \Sigma$).

The Riemann variety

Definition: The **Riemann variety** of a finite extension Σ of $k(t)$ is the ringed space

$$X = \{\text{discrete valuation rings of } \Sigma \text{ containing } k\}$$
$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_x$$

where $\mathcal{O}_x$ is the valuation ring of $x \in X$, and closed sets $\neq X$ are finite subsets not containing the generic point p_g defined by the trivial valuation ring Σ. Any complete non singular curve C over k is the Riemann variety of the field of rational functions $\Sigma_C = \mathcal{O}_{C,p_g}$.

A **projective line** over k is any k-scheme isomorphic to the Riemann variety $\mathbb{P}_1$ of $k(t)$.

Theorem: *The Riemann variety* $(X, \mathcal{O}_X)$ *is a non singular complete curve.*

Proof: If B (resp. B') is the integral closure of $k[t]$ (resp. $k[\frac{1}{t}]$) in Σ, the morphisms $k[t] \to B$ and $k[\frac{1}{t}] \to B'$ are finite (p. 212) and

$$X = U \cup U', \quad \begin{cases} U = \{x \in X \colon v_x(t) \geq 0\} = \operatorname{Spec} B \\ U' = \{x \in X \colon v_x(t) \leq 0\} = \operatorname{Spec} B' \end{cases}$$

where U, U' are open sets in X, since $X - U = (\frac{1}{t})_0$, and $X - U' = (t)_0$.

We conclude since B and B' are Dedekind domains (p. 212).

Definitions: The group $\mathrm{Div}(X)$ of **divisors** of X is the free abelian group on all closed points of X. The degree $\deg x = [\kappa(x)\colon k]$ of a closed point $x \in X$ is finite by the nullstellensatz, and the **degree** of a divisor $D = \sum_x n_x \cdot x$ is

$$\deg D = \textstyle\sum_x n_x(\deg x).$$

Any rational function $0 \neq f \in \Sigma$ has a finite number of zeros and poles (in a Dedekind domain, any non null function has a finite number of zeros) and the divisor of f is

$$D(f) = \textstyle\sum_x v_x(f) \cdot x,$$
$$D(fh) = D(f) + D(h).$$

Two divisors D, D' are **linearly equivalent**, and we put $D' \sim D$, when $D' = D + D(f)$ for some $f \in \Sigma$, and linear equivalence classes of divisors form a group $\mathrm{Div}(X)/\!\sim$.

Theorem: $\deg D(f) = 0$.

Proof: If f is algebraic over k, then $D(f) = 0$. Otherwise, since f is algebraic over $k(t)$, then t is algebraic over $k(f)$, and Σ is a finite extension of $k(f)$.

The integral closure B of $k[f]$ in Σ is a torsion free finite $k[f]$-module; hence it is free,

$$k[f]^n \simeq B,$$

and localizing at the generic point we see that $n = [\Sigma : k(f)]$.

Moreover, the ring B/fB, being of dimension 0, decomposes as a direct sum (p. 198)

$$k^n \simeq B/fB = (B_{x_1}/fB_{x_1}) \oplus \ldots \oplus (B_{x_r}/fB_{x_r})$$
$$v_{x_i}(f) = l(B_{x_i}/fB_{x_i})$$

and we see that $n = \sum v_{x_i}(f)(\deg x_i)$, where x_i runs over the zeros of f.

The number of zeros of f is $n = [\Sigma : k(f)]$, and the number of poles of f (the number of zeros of f^{-1}) is $[\Sigma : k(f^{-1})]$. Both coincide since $k(f^{-1}) = k(f)$. q.e.d.

Any divisor D defines a line subsheaf L_D of the (constant) sheaf of rational functions Σ,

$$L_D(U) := \{f \in \Sigma\colon D + D(f) \geq 0 \text{ on } U\}.$$

In fact, at any point x we may fix a parameter $t \in \mathcal{O}_x$ and a neighborhood U where t has no other zeros or poles, nor the divisor D has other points, so that if n is the coefficient of x in D, we have that L_D is free on U,

$$L_D|_U = t^{-n}\mathcal{O}_X|_U.$$

If $D' = D + D(h)$, then $\phi\colon L_{D'} \to L_D$, $\phi(f) = hf$, is an isomorphism, so that the sheaf L_D only depends on the class $[D]$.

Moreover, the morphism $\phi\colon L_{D_1} \otimes_{\mathcal{O}_X} L_{D_2} \to L_{D_1+D_2}$, $\phi(f_1 \otimes f_2) = f_1 f_2$, is an isomorphism since so is on stalks,

$$L_{D_1+D_2} = L_{D_1} \otimes_{\mathcal{O}_X} L_{D_2}.$$

Theorem: $\mathrm{Pic}(X) = \mathrm{Div}(X)/\!\sim$.

Proof: An isomorphism $L_{D'} \xrightarrow{\sim} L_D$ defines an isomorphism $\Sigma \simeq \Sigma$ on the generic stalks, hence a homothety of ratio h.

On any open set we have $0 \leq D' + D(f)$ if and only if $0 \leq D + D(hf) = D + D(h) + D(f)$; hence $D' = D + D(h)$, and $D' \sim D$.

Conversely, given a line sheaf $\mathcal{L}$, if we fix an isomorphism $\mathcal{L}_{p_g} \xrightarrow{\sim} \Sigma$, then we may view inside Σ the sections of $\mathcal{L}$ and stalks $\mathcal{L}_x$, which are free $\mathcal{O}_x$-modules of rank 1.

Hence $\mathcal{L}_x = \mathfrak{m}_x^{n_x}$ for some integer n_x.

We have $n_x = 0$ at any point, up to a finite number, since $\mathcal{L}(U) = f\mathcal{O}_X(U)$ on an affine open set U, and f has a finite number of zeros and poles.

Now, if we put $D = -\sum_x n_x \cdot x$, the natural morphism $\mathcal{L} \to L_D$ is an isomorphism.

Corollary: $\mathrm{Pic}(\mathbb{P}_1) = \mathbb{Z}$, and we put $\mathcal{O}_{\mathbb{P}_1}(n) = L_{np_\infty}$.

Proof: We have to show that any divisor D of degree 0 is $D = D(f)$, $f \in k(t)$.

Since both have degree 0, it is enough to see that they coincide in $\mathrm{Spec}\, k[t]$.

If $D = n_1x_1 + \ldots + n_rx_r$ in the affine part, we take polynomials $p_i(t)$ generating the maximal ideals $\mathfrak{m}_{x_i}$ of $k[t]$, and D coincides with the divisor of $f(t) = p_1(t)^{n_1} \ldots p_r(t)^{n_r}$.

Morphisms: Any k-morphism $\Sigma' \to \Sigma$ induces a morphism of k-schemes $\pi\colon X \to X'$ between the Riemann varieties. In fact, if a discrete valuation ring $\mathcal{O}_x$ of Σ contains k, then $\mathcal{O}_{x'} = \mathcal{O}_x \cap \Sigma'$ is a discrete valuation ring of Σ' containing k, and we put $x' = \pi(x)$.

Since any fibre of π is finite, π is continuous, and the morphism of sheaves $\mathcal{O}_{X'} \to \pi_*\mathcal{O}_X$ is defined by the inclusions

$$\mathcal{O}_{X'}(U) = \bigcap_{x' \in U} \mathcal{O}_{x'} \longrightarrow \mathcal{O}_X(\pi^{-1}U) = \bigcap_{\pi(x) \in U} \mathcal{O}_x.$$

If $U = \mathrm{Spec}\, A$ is affine, then $\pi^{-1}(U) = \mathrm{Spec}\, B$, where the integral closure B of A in Σ is a locally free A-module of rank $d = [\Sigma : \Sigma']$, the **degree** of the morphism.

Therefore, any fibre $\mathrm{Spec}\, B/\mathfrak{m}_{x'}B$ of π defines a divisor of degree $d(\deg x')$,

$$\pi^*(x') = \sum_{\pi(x) = x'} l(B_x/\mathfrak{m}_{x'}B_x) \cdot x$$

so that we have an inverse image π^*D' of divisors, and $L_{\pi^*D'} = \pi^*(L_{D'})$.

So is any non constant k-morphism of schemes $X \to X'$.

11.4 Riemann-Roch Theorem

Theorem: $\dim_k H^p(\mathbb{P}_1, \mathcal{O}(n)) = \begin{cases} n+1 & p = 0 \\ 0 & p \neq 0 \end{cases}$

$$\dim_k H^p(\mathbb{P}_1, \mathcal{O}(-n)) = \begin{cases} n-1 & p = 1 \\ 0 & p \neq 1 \end{cases}$$

Proof: Once we prove that the structural sheaf $\mathcal{O}_{\mathbb{P}_1}$ is acyclic, the exact sequences

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(n+1) \longrightarrow k_\infty \longrightarrow 0$$

where k_∞ is the sheaf k supported at infinity, let us conclude.

Now, the projection $\pi\colon \mathbb{P}_1 \to$ , transforming the origin and the infinity onto the two closed points, and any other point onto the dense point, preserves the cohomology of quasi-coherent sheaves (pp. 322, 323), and $\pi_*\mathcal{O}_{\mathbb{P}_1}$ admits the flasque resolution,

$$0 \longrightarrow \begin{array}{c} k[t,\frac{1}{t}] \\ k[t] \bullet\!\!\wedge\!\!\bullet k[\frac{1}{t}] \end{array} \longrightarrow \begin{array}{c} k[t,\frac{1}{t}] \\ k[t,\frac{1}{t}] \bullet\!\!\wedge\!\!\bullet k[t,\frac{1}{t}] \end{array} \longrightarrow \begin{array}{c} 0 \\ k[\frac{1}{t}]/k \bullet\!\!\wedge\!\!\bullet k[t]/k \end{array} \longrightarrow 0$$

$$0 \longrightarrow H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) \longrightarrow k[t,\tfrac{1}{t}] \longrightarrow \big(k[\tfrac{1}{t}]/k\big) \oplus \big(k[t]/k\big) \longrightarrow H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) \longrightarrow 0$$

We conclude since the morphism $k[t,\frac{1}{t}] \to (k[\frac{1}{t}]/k) \oplus (k[t]/k)$ is surjective.

Theorem: *The cohomology groups of any coherent sheaf $\mathcal{M}$ on a non singular complete curve X are finite dimensional vector spaces.*

Proof: Any morphism $k(t) \to \Sigma$ defines a projection $\pi\colon X \to \mathbb{P}_1$ preserving the cohomology of coherent sheaves, and $\pi_*\mathcal{M}$ is coherent when so is $\mathcal{M}$, because $\pi^{-1}(\operatorname{Spec} k[t]) = \operatorname{Spec} B$, where B is a finite $k[t]$-module (p. 212).

Hence we may assume that $X = \mathbb{P}_1$.

The torsion of $\mathcal{M}$ is supported at a finite number of closed points, where the stalk has finite dimension (in particular it is a flasque sheaf); hence we may assume that it is null, so that $\mathcal{M}$ is a subsheaf of the constant sheaf $\mathcal{M}_g$.

Now a non null section $s \in \mathcal{M}_g$ defines a coherent sheaf of ideals $\mathcal{I}$

$$\mathcal{I}(U) = \{f \in \mathcal{O}_{\mathbb{P}_1}(U)\colon fs \in \mathcal{M}(U)\}.$$

The theorem holds for $\mathcal{I} \simeq \mathcal{O}(-n)$, and for $\mathcal{M}/\mathcal{I}s$ by induction on the rank. The following exact sequence let us conclude:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{I}s \longrightarrow 0.$$

Definitions: We put $h^p(\mathcal{M}) = \dim_k H^p(X, \mathcal{M})$, and the **Euler-Poincaré characteristic**

$$\chi(\mathcal{M}) = \chi(X, \mathcal{M}) = \textstyle\sum_p (-1)^p h^p(\mathcal{M}) = h^0(\mathcal{M}) - h^1(\mathcal{M})$$

is an additive function on the category of coherent sheaves.

The **genus** of a non singular complete curve X is $g = h^1(\mathcal{O}_X)$.

According to the above theorem, $\chi(\mathbb{P}_1, \mathcal{O}(n)) = n + 1$, $n \in \mathbb{Z}$.

The algebraic closure of k in Σ is $H^0(X, \mathcal{O}_X)$. It is a finite extension $\bar{k}$ of k, and *from now on, we always assume that* $\bar{k} = k$ (or replace k by $\bar{k}$), so that $\chi(\mathcal{O}_X) = 1 - g$.

Riemann-Roch Theorem (weak): $\chi(L_D) = \chi(\mathcal{O}_X) + \deg D$.

Proof: The cohomology exact sequence of the exact sequence

$$0 \longrightarrow L_D \longrightarrow L_{D+x} \longrightarrow \kappa(x) \longrightarrow 0$$

shows that $\chi(L_{D+x}) = \chi(L_D) + \deg x$.

Hence the theorem holds for a divisor D if and only if it holds for $D + x$.

Since it obviously holds when $D = 0$, we conclude.

Corollary: *A non singular complete curve X is* ***rational****, $\Sigma_X \simeq k(t)$, if only if the genus is 0 and it has some rational point.*

Proof: If $g = 0$ and X has a rational point x, then $\chi(L_x) = 1 - 0 + 1 = 2$, so that $h^0(L_x) \geq 2$, and there is a non constant rational function $f \in \Sigma$ with a unique simple pole at x.

Hence $[\Sigma : k(f)] =$ number of poles of $f = 1$, and $\Sigma = k(f)$.

Corollary: *Given a closed point $x \in X$, there is $f \in \Sigma$ with a unique pole at x, so that the open set $U = X - x$ is affine.*

Proof: We have $h^0(L_{nx}) \geq 1 - g + n$.

When $n \geq g + 1$, there exists a non constant morphism $f\colon X \to \mathbb{P}_1$ with a unique pole at x; hence $U = f^{-1}(\mathbb{A}_1)$ is affine.

Definition: Let X be a curve over a field k. The k-linear functor $F(\mathcal{M}) = H^1(X, \mathcal{M})^*$ is left exact on the category of coherent sheaves, because $H^2(X, \mathcal{M}) = 0$, and any pair is dominated by a minimal pair. In fact, any sequence of epimorphisms

$$\mathcal{M}_\xi \xrightarrow{p_1} \mathcal{M}'_{\xi'} \xrightarrow{p_2} \mathcal{M}''_{\xi''} \xrightarrow{p_3} \dots$$

stabilizes since $\operatorname{Ker} p_1 \subseteq \operatorname{Ker}(p_2 p_1) \subseteq \operatorname{Ker}(p_3 p_2 p_1) \dots$ and X is noetherian. By the representability theorem F is an inductive limit of representable functors: there is an inductive system of coherent sheaves $\{\mathcal{M}_i\}$ such that, for any coherent $\mathcal{O}_X$-module $\mathcal{M}$, we have natural linear isomorphisms (where the quasi-coherent sheaf $\omega_{X/k} := \varinjlim \mathcal{M}_i$ is the **dualizing** or **canonical** sheaf of X)

$$H^1(X, \mathcal{M})^* = \varinjlim \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}_i) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \omega_{X/k}),$$

Theorem: $\omega_{\mathbb{P}_1} = \mathcal{O}_{\mathbb{P}_1}(-2)$.

Proof: We have $H^1(\mathbb{P}_1, \mathcal{O}(-2)) \neq 0$, so that $\mathcal{O}(-2)$, with any non null $\xi \in H^1\big(\mathbb{P}_1, \mathcal{O}(-2)\big)^*$, is a minimal pair (quotients of a line sheaf are of torsion sheaves, hence acyclic).

Hence $\mathcal{O}(-2)$ is a submodule of $\omega_{\mathbb{P}_1}$, and if we put $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}_{\mathbb{P}_1}} \mathcal{O}(n)$, $n \geq 1$,

$$\begin{gathered}
0 \longrightarrow \mathcal{O}(-2) \longrightarrow \omega_{\mathbb{P}_1} \longrightarrow \mathcal{K} \longrightarrow 0 \text{ is exact} \\
0 \longrightarrow \mathcal{O}(n-2) \longrightarrow \omega_{\mathbb{P}_1}(n) \longrightarrow \mathcal{K}(n) \longrightarrow 0 \text{ is exact} \\
0 \longrightarrow \Gamma\big(\mathbb{P}_1, \mathcal{O}(n-2)\big) \longrightarrow \Gamma\big(\mathbb{P}_1, \omega_{\mathbb{P}_1}(n)\big) \longrightarrow \Gamma\big(\mathbb{P}_1, \mathcal{K}(n)\big) \longrightarrow 0 \text{ is exact} \\
\Gamma(\mathbb{P}_1, \omega_{\mathbb{P}_1}(n)) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}(-n), \omega_{\mathbb{P}_1}) = H^1(\mathbb{P}_1, \mathcal{O}(-n))^* \\
\dim \Gamma(\mathbb{P}_1, \mathcal{O}(n-2)) = n - 1 = \dim H^1(\mathbb{P}_1, \mathcal{O}(-n))^* \\
\Gamma(\mathbb{P}_1, \mathcal{K}(n)) = 0
\end{gathered}$$

so that $\mathcal{K} = 0$, because torsion elements of stalks define global sections and elements of the generic stalk define (p. 328) morphisms $\mathcal{O}(-n) \to \mathcal{K}$; hence sections of $\mathcal{K}(n)$.

Theorem: *The dualizing sheaf ω_X of any non singular complete curve X is a line sheaf.*

Proof: Let us consider a morphism $k(t) \to \Sigma$ and the corresponding projection $\pi\colon X \to \mathbb{P}_1$.

The direct image π_* preserves the cohomology of coherent sheaves, and $\pi_*\mathcal{M}$ is coherent when so is $\mathcal{M}$, so that

$$\begin{aligned}
H^1(X, \mathcal{M})^* &= H^1(\mathbb{P}_1, \pi_*\mathcal{M})^* = \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{M}, \omega_{\mathbb{P}_1}) \\
&= \operatorname{Hom}_{\pi_*\mathcal{O}_X}\big(\pi_*\mathcal{M}, \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{O}_X, \omega_{\mathbb{P}_1})\big), \\
H^1(X, \mathcal{M})^* &= \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \omega_X) = \operatorname{Hom}_{\pi_*\mathcal{O}_X}(\pi_*\mathcal{M}, \pi_*\omega_X), \qquad (11.3) \\
\pi_*\omega_X &= \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{O}_X, \omega_{\mathbb{P}_1}), \qquad (11.4)
\end{aligned}$$

hence ω_X is torsion free and it has rank 1. It is a line sheaf.

Definition: A divisor K of the curve X is **canonical** if $\omega_X = L_K$.

Riemann-Roch Theorem: *If K is a canonical divisor of a non singular complete curve X of genus g, then for any divisor D,*

$$\dim H^0(X, L_D) = \chi(\mathcal{O}_X) + \deg D + \dim H^0(X, L_{K-D}).$$

Proof: We have $H^1(X, L_D)^* = \mathrm{Hom}_{\mathcal{O}_X}(L_D, L_K) = H^0(X, L_{K-D})$, and we conclude by the weak Riemann-Roch theorem.

Corollary: $h^1(L_K) = 1$, $h^0(L_K) = g$, $\deg K = 2g-2$.

Proof: $H^1(X, L_K)^* = \mathrm{Hom}_{\mathcal{O}_X}(L_K, L_K) = H^0(X, \mathcal{O}_X)$ has dimension 1.

Moreover $H^0(X, L_K) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L_K) = H^1(X, \mathcal{O}_X)^*$ has dimension g, and

$$g = h^0(L_K) = 1 - g + \deg K + h^0(\mathcal{O}_X) = 2 - g + \deg K.$$

11.4.1 Calculation of the Canonical Sheaf

Definition: If X is a k-scheme, the sheafification of $U \rightsquigarrow \Omega_{\mathcal{O}_X(U)/k}$ is the **sheaf of differentials** $\Omega_{X/k}$. Since the module of differentials localizes (p. 144), on any affine open set $U = \mathrm{Spec}\, A$ it is the sheaf defined by $\Omega_{A/k}$, and it is coherent when X is of finite type.

Proposition: *If X is a non singular curve over a perfect field, $\Omega_{X/k}$ is a line sheaf.*

Proof: If t is a local parameter, $\mathfrak{m}_x = t\mathcal{O}_x$, the exact sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \Omega_{\mathcal{O}_x} \otimes_{\mathcal{O}_x} \kappa(x) \longrightarrow \Omega_{\kappa(x)/k} = 0$$

shows that $\Omega_{\mathcal{O}_x} \otimes_{\mathcal{O}_x} \kappa(x) = \langle \mathrm{d}t \rangle$, and by Nakayama's lemma $\Omega_{\mathcal{O}_x} = \mathcal{O}_x \mathrm{d}t$.

If it is not torsion free, then $\Omega_{\Sigma/k} = 0$, and Σ is (p. 148) a separable extension of $k(t)$ such that no k-derivation of $k(t)$ may be extended to Σ. Absurd.

Theorem: *The dualizing sheaf of a non singular complete curve X over an algebraically closed field is the sheaf of differentials, $\omega_{X/k} = \Omega_{X/k}$.*

First Proof: The theorem holds in $\mathbb{P}_1$ since $\Omega_{\mathbb{P}_1} = \mathcal{O}(-2)$.

In fact, $\mathrm{d}t$ has no zeros nor poles in the affine part, and it has has two poles at infinity,

$$\mathrm{d}t = \mathrm{d}(u^{-1}) = -u^{-2}\mathrm{d}u.$$

In general, we take a separable morphism $k(t) \to \Sigma$, so that the trace metric (p. 149) is non singular, and it defines an exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_X \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}_1}) \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{C}$ is supported at a finite number of closed points, since the generic stalk is null.

Hence $\mathcal{C} \otimes_{\mathcal{O}_{\mathbb{P}_1}} \mathcal{L} = \mathcal{C}$ for any line sheaf $\mathcal{L}$, and we have exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Omega_{\mathbb{P}_1} \otimes_{\mathcal{O}_{\mathbb{P}_1}} \pi_*\mathcal{O}_X & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}_1}) \otimes_{\mathcal{O}_{\mathbb{P}_1}} \Omega_{\mathbb{P}_1} & \longrightarrow & \mathcal{C} & \longrightarrow 0 \\ 0 \longrightarrow & \Omega_{\mathbb{P}_1} \otimes_{\mathcal{O}_{\mathbb{P}_1}} \pi_*\mathcal{O}_X & \longrightarrow & \pi_*\Omega_X & \longrightarrow & \Omega_{X/\mathbb{P}_1} & \longrightarrow 0 \end{array}$$

where $\Omega_{X/\mathbb{P}_1}(\operatorname{Spec} A) = \Omega_{B/A}$, and B is the integral closure of A in Σ. By 11.4 we have

$$\underline{\operatorname{Hom}}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}_1}) \otimes_{\mathcal{O}_{\mathbb{P}_1}} \Omega_{\mathbb{P}_1} = \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathbb{P}_1}}(\pi_*\mathcal{O}_X, \Omega_{\mathbb{P}_1}) = \pi_*\omega_X,$$

and to conclude that $\omega_X \simeq \Omega_X$ we have to show that B^*/B and $\Omega_{B/A}$ have equal length at any point of $\operatorname{Spec} B$, where we view B inside $B^* = \operatorname{Hom}_A(B,A)$ via the trace metric.

The trace metric and the module of differentials are stable under base changes $A \to A'$.

Hence we may prove the theorem after localization and completion at a point $y \in \operatorname{Spec} A$,

$$\widehat{A}_y \longrightarrow \varprojlim B_y/\mathfrak{m}_y^n B_y = \varprojlim B_y/(\mathfrak{m}_{x_1}^{n_1} \dots \mathfrak{m}_{x_r}^{n_r})^n = \widehat{B}_{x_1} \oplus \dots \oplus \widehat{B}_{x_r}$$

where $\pi^{-1}(y) = \{x_1, \dots, x_r\}$, and it is enough to prove it for the morphisms $\widehat{A}_y \to \widehat{B}_x$.

Since k is algebraically closed, both are rings of formal series (p. 206),

$$\begin{aligned} k[[t]] = \widehat{A} &\longrightarrow \widehat{B} = k[[x]] \\ t\widehat{B} &= (x^n) \\ \widehat{B}/t\widehat{B} &= k[x]/(x^n) \end{aligned}$$

and Nakayama's lemma shows that $(1, x, \dots, x^{n-1})$ is a base of the $\widehat{A}$-module $\widehat{B}$,

$$\widehat{B} = \widehat{A} \oplus \widehat{A}x \oplus \dots \oplus \widehat{A}x^{n-1} = \widehat{A}[x]/(x^n + \dots) = \widehat{A}[x]/(P).$$

Lemma: *Let A be a Dedekind domain and B the integral closure of A in a finite separable extension of the field of fractions. If $B = A[\xi] = A[x]/(P) = A[x]/(x^n + \dots)$, then*

$$B^* = A\tfrac{1}{P'(\xi)} \oplus A\tfrac{\xi}{P'(\xi)} \oplus \ \dots \ \oplus A\tfrac{\xi^{n-1}}{P'(\xi)}$$

and therefore $B^*/B \simeq B/(P'(\xi)) \simeq \Omega_{B/A}$.

Proof: If $\xi = \alpha_1, \dots, \alpha_n$ are the roots of the separable polynomial $P(x)$, decomposing $\frac{1}{P(x)}$ into simple fractions, and expanding as a power series of x^{-1}, we see that

$$\sum_{i=1}^{n} \frac{1}{P'(\alpha_i)(x-\alpha_i)} = \frac{1}{P(x)} = x^{-n}(1 + a_1x^{-1} + \dots)$$

$$\operatorname{tr}\left(\frac{\xi^i}{P'(\xi)}\right) = \begin{cases} 0 & 0 \le i \le n-2 \\ 1 & i = n-1 \end{cases}$$

Hence $\frac{\xi^i}{P'(\xi)} \in B^*$, since $\xi^{i+j} \in A \oplus A\xi \dots \oplus A\xi^{n-1}$, and the matrix

$$\left(\operatorname{tr}\frac{\xi^i\xi^j}{P'(\xi)}\right) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \bullet \\ 0 & \iddots & \iddots & \vdots \\ 1 & \bullet & \cdots & \bullet \end{pmatrix}$$

is invertible: $\frac{1}{P'(\xi)}, \frac{\xi}{P'(\xi)}, \dots, \frac{\xi^{n-1}}{P'(\xi)}$ form a base of B^*, and $B^*/B \simeq B/(P'(\xi))$. q.e.d.

Second Proof: Let us show that $\underline{\operatorname{Hom}}_{\mathcal{O}}(\Omega_X, \omega_X)$ *is a trivial line sheaf*:

Let $X = U \cup U' = (\operatorname{Spec} B) \cup (\operatorname{Spec} B')$ be the open cover of p. 325.

For any k-algebra A, the base change X_A is the scheme

$$X_A = X \times_k (\operatorname{Spec} A) = U_A \cup U'_A = \operatorname{Spec}(B \otimes_k A) \cup \operatorname{Spec}(B' \otimes_k A)$$

and the direct product $X \times_k X$ is the scheme (integral since k is algebraically closed)

$$X \times_k X = X_B \cup X_{B'} = (X \times_k \operatorname{Spec} B) \cup (X \times_k \operatorname{Spec} B').$$

The diagonal morphism $X \to X \times_k X$ is a closed embedding since the natural morphism $B \otimes_k B' \longrightarrow \mathcal{O}_X(U \cap U') = B\big[\frac{1}{t}\big] = B'[\,t\,]$ is surjective, and the sheaf of differentials $\Omega_X = \Delta/\Delta^2$ is a line sheaf; hence the sheaf of ideals of the diagonal Δ is locally principal.

Now, given a closed point $x \in X$, we restrict to $X \times_k x$ the following exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_X \otimes_k \mathcal{O}_X & \longrightarrow & \underline{\operatorname{Hom}}(\Delta, \omega_X \otimes_k \mathcal{O}_X) & \longrightarrow & \underline{\operatorname{Hom}}(\Delta/\Delta^2, \omega_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega_X & \longrightarrow & \underline{\operatorname{Hom}}(\mathfrak{m}_x, \omega_X) & \longrightarrow & \operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \omega_X/\mathfrak{m}_x\omega_X) & \longrightarrow & 0
\end{array}$$

and we take the direct image by the second projection $\pi\colon X \times_k X \to X$,

$$\begin{array}{ccccc}
\underline{\operatorname{Hom}}_{\mathcal{O}}(\Omega_X, \omega_X) & \xrightarrow{\ \delta\ } & R^1\pi_*(\omega_X \otimes_k \mathcal{O}_X) & & \\
\downarrow & & \downarrow & & \\
\operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \omega_X/\mathfrak{m}_x\omega_X) & \xrightarrow{\ \delta_x\ } & H^1(X, \omega_X) & \longrightarrow & H^1(X, L_{K+x}) = 0
\end{array} \tag{11.5}$$

where $\underline{\operatorname{Hom}}_{\mathcal{O}}(\Omega_X, \omega_X)$ is a line sheaf, and $R^1\pi_*(\omega_X \otimes_k \mathcal{O}_X) = H^1(X, \omega_X) \otimes_k \mathcal{O}_X$ is a trivial line sheaf (according to the following lemma). This commutative diagram shows that δ is surjective at any closed point $x \in X$; hence it is an isomorphism.

Lemma: *Let X be a non singular complete curve over a field k. If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module and A is a k-algebra, we have natural isomorphisms*

$$H^p(X_A, \mathcal{M} \otimes_k A) = H^p(X, \mathcal{M}) \otimes_k A.$$

Proof: Comparing the Mayer-Vietoris sequences of $\mathcal{M}$ and $\mathcal{M}_A = \mathcal{M} \otimes_k A$,

$$0 \longrightarrow H^0(X, \mathcal{M}) \longrightarrow \Gamma(U, \mathcal{M}) \oplus \Gamma(U', \mathcal{M}) \longrightarrow \Gamma(U \cap U', \mathcal{M}) \longrightarrow H^1(X, \mathcal{M}) \longrightarrow 0$$
$$0 \longrightarrow H^0(X_A, \mathcal{M}_A) \longrightarrow \Gamma(U, \mathcal{M})_A \oplus \Gamma(U', \mathcal{M})_A \longrightarrow \Gamma(U \cap U', \mathcal{M})_A \longrightarrow H^1(X_A, \mathcal{M}_A) \longrightarrow 0$$

we see, the functor $\otimes_k A$ being exact, that $H^p(X, \mathcal{M}) \otimes_k A \xrightarrow{\ \sim\ } H^p(X_A, \mathcal{M} \otimes_k A)$.

Residues: Let X be a complete non singular curve over an algebraically closed field k, and Σ the function field. The sheaf of differentials $\Omega_{X/k}$ admits the flasque resolution

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \Omega_{\Sigma/k} \longrightarrow \Omega_\Sigma/\Omega_X \longrightarrow 0, \tag{11.6}$$

where Ω_Σ is the constant sheaf and Ω_Σ/Ω_X the sheaf of principal parts of meromorphic forms:

$$(\Omega_\Sigma/\Omega_X)(U) = \bigoplus_{x \in U} (\Omega_\Sigma/\Omega_{\mathcal{O}_x}) = \bigoplus_{x \in U} \{a_n \tfrac{\mathrm{d}t}{t^n} + \ldots + a_1 \tfrac{\mathrm{d}t}{t}\}\,, \qquad t\mathcal{O}_x = \mathfrak{m}_x.$$

We have $H^0_x(X, \Omega_\Sigma) = H^1_x(X, \Omega_\Sigma) = 0$; hence $\Omega_\Sigma/\Omega_{\mathcal{O}_x} = H^0_x(X, \Omega_\Sigma/\Omega_X) \xrightarrow{\delta_x} H^1_x(X, \Omega_X)$ is an isomorphism. On the other hand,

$$\operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, \Omega_X/\mathfrak{m}_x\Omega_X) = (\mathfrak{m}_x^{-1}\Omega_{\mathcal{O}_x})/\Omega_{\mathcal{O}_x} \subset \Omega_\Sigma/\Omega_{\mathcal{O}_x},$$

where the identity $\mathfrak{m}_x/\mathfrak{m}_x^2 = \Omega_X/\mathfrak{m}_x\Omega_X$ corresponds to $\frac{\mathrm{d}t}{t} \in (\mathfrak{m}_x^{-1}\Omega_{\mathcal{O}_x})/\Omega_{\mathcal{O}_x}$.

Now take global sections in 11.5, with $\omega_X = \Omega_X$, and with support in x in the exact sequence $0 \to \Omega_X \to \underline{\mathrm{Hom}}(\mathfrak{m}_x, \Omega_X) \to (\mathfrak{m}_x^{-1}\Omega_{\mathcal{O}_x})/\Omega_{\mathcal{O}_x} \to 0$. We obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(\Omega_X,\Omega_X) & \longrightarrow & (\mathfrak{m}_x^{-1}\Omega_{\mathcal{O}_x})/\Omega_{\mathcal{O}_x} & \xleftarrow[\mathrm{Id}]{} & (\mathfrak{m}_x^{-1}\Omega_{\mathcal{O}_x})/\Omega_{\mathcal{O}_x} \\ \delta \downarrow \wr & & \delta_x \downarrow \wr & & \downarrow \delta_x \\ H^1(X,\Omega_X) & \xrightarrow{\mathrm{Id}} & H^1(X,\Omega_X) & \longleftarrow & H^1_x(X,\Omega_X) \end{array}$$

and we see that $\delta_x\left(\frac{\mathrm{d}t}{t}\right) \in H^1(X,\Omega_X)$ does not depend on the point x, since it is just $\delta(\mathrm{Id})$. Therefore, we have a global **residue** $\mathrm{Res}\colon H^1(X,\Omega_X) \xrightarrow{\sim} k$, hence a local residue

$$\mathrm{Res}_x\colon \Omega_\Sigma/\Omega_{\mathcal{O}_x} = H^1_x(X,\Omega_X) \longrightarrow H^1(X,\Omega_X) \xrightarrow{\mathrm{Res}} k$$

at any closed point x, such that $\mathrm{Res}_x\left(\frac{\mathrm{d}t}{t}\right) = 1$ when $t\mathcal{O}_x = \mathfrak{m}_x$. But the fully determination of the local residue, $\mathrm{Res}_x\left(\frac{\mathrm{d}t}{t^n}\right) = 0$, $n > 1$, is postponed to p. 433.

Theorem: *A finite family* $(\theta_x) \in \bigoplus_{x\in X}\{a_n\frac{\mathrm{d}t}{t^n} + \ldots + a_1\frac{\mathrm{d}t}{t}\}$ *of principal parts of meromorphic forms is defined by a global meromorphic form* $\omega \in \Omega_\Sigma$ *if and only if* $\sum_{x\in X}\mathrm{Res}_x(\theta_x) = 0$.

Proof: By 11.6, we have an exact sequence $\Omega_\Sigma \to \bigoplus_{x\in X}(\Omega_\Sigma/\Omega_{\mathcal{O}_x}) \xrightarrow{\delta} H^1(X,\Omega_X) \to 0$.

Hurwitz's Formula: *If* $\pi\colon X \to X'$ *is a morphism between non singular complete curves over an algebraically closed field, defined by a separable extension* $\Sigma' \to \Sigma$ *of degree* d*, then*

$$2g - 2 = d(2g' - 2) + \sum_{x\in X} e_x\ , \quad e_x = l\big(\Omega_{\mathcal{O}_x/\mathcal{O}_{x'}}\big).$$

Proof: We have $\deg \pi^*(K') = d(\deg K') = d(2g'-2)$, and the stalk of $\Omega_{X/X'}$ at the generic point of X is $\Omega_{\Sigma/\Sigma'} = 0$, so that we have an exact sequence

$$0 \longrightarrow \pi^*\Omega_{X'} = L_{\pi^*K'} \longrightarrow \Omega_X = L_K \longrightarrow \Omega_{X/X'} \longrightarrow 0$$

Definition: A curve X over a field k is **smooth** if $\Omega_{X/k}$ is a line sheaf.

Examples: When k is algebraically closed, $\Omega_{\mathcal{O}_x}/\mathfrak{m}_x\Omega_{\mathcal{O}_x} = \mathfrak{m}_x/\mathfrak{m}_x^2$ (p. 145), and smooth curves are just non singular curves.

The curve $y^2 = x^p - t$ over $k = \mathbb{F}_p(t)$ is non singular; but it is not smooth, since it is singular over the algebraic closure $\bar{k}$.

Proposition: *Let* X *be a smooth complete curve over a field* k*. For any extension* $k \to K$ *we have that the dualizing sheaf is stable under base change,*

$$\omega_{X/k} \otimes_k K = \omega_{X_K/K}.$$

Proof: The cohomology group $H^1(X_K, \omega_X \otimes_k K) = H^1(X,\omega_X) \otimes_k K$ is non null, so that there exists a non null morphism $\mathcal{D}_X \otimes_k K \to \omega_{X_K}$, and both are line sheaves.

Now we have exact sequences

$$0 \longrightarrow \omega_X \otimes_k K \longrightarrow \omega_{X_K} \longrightarrow \mathcal{C} \longrightarrow 0$$

$$\begin{array}{ccccccccccc} 0 \to & H^0(X,\omega_X)_K & \to & H^0(X_K,\omega_{X_K}) & \to & H^0(X_K,\mathcal{C}) & \to & H^1(X,\omega_X)_K & \to & H^1(X_K,\mathcal{D}_{X_K}) & \to 0 \\ & \| & & \| & & & & \| & & \| & \\ & H^1(X,\mathcal{O}_X)^*_K & & H^1(X_K,\mathcal{O}_{X_K})^* & & & & H^0(X,\mathcal{O}_X)^*_K & & H^0(X_K,\mathcal{O}^*_{X_K}) & \end{array}$$

and we see that $H^0(X_K, \mathcal{C}) = 0$. Since it is flasque, $\mathcal{C} = 0$ and $\omega_X \otimes_k K = \omega_{X_K}$.

Theorem: *The dualizing sheaf of a smooth complete curve X over a field k is the sheaf of differentials $\Omega_{X/k}$.*

Proof: The dualizing sheaf and the sheaf of differentials are stable under base changes $k \to K$, and the theorem holds over the algebraic closure $\bar{k}$ (p. 330).

11.5 The Projective Spectrum

Definitions: The **projective spectrum** of a graded ring $R = R_0 \oplus R_1 \oplus \ldots$ is the subspace $X = \operatorname{Proj} R$ of $\operatorname{Spec} R$ formed by all homogeneous prime ideals $\mathfrak{p} = \oplus_n \mathfrak{p}_n$ not containing the **irrelevant ideal** $R_+ = \oplus_{n \geq 1} R_n$ (the closed sets are $(I)_0 = \{x \in X \colon I \subseteq \mathfrak{p}_x\}$, where $I = \oplus_n I_n$ is a homogeneous ideal, and the open sets $U_f = X - (f)_0$, f homogeneous, form a base) with the sheaf of rings $\mathcal{O}_X$ attached to the presheaf of homogeneous localization

$$U \rightsquigarrow R_{(U)} = \left\{ \tfrac{a_n}{f_n} \colon a_n, f_n \in R_n;\ f_n \text{ without zeros on } U \right\}$$

where $R_{(U)}$ is the degree 0 component of the localization of R by all homogeneous elements without zeros on U. The stalk $\mathcal{O}_{X,x}$ is the homogeneous localization $R_{(x)}$ of R by all homogeneous elements not vanishing at x.

We always assume that R_1 generates R_+, so that the open sets $U_f = \operatorname{Proj} R - (f)_0$, with $\deg f = 1$, cover X.

A **projective space** over A is any A-scheme isomorphic to some $\mathbb{P}_{d,A} = \operatorname{Proj} A[x_0, \ldots, x_d]$, and **projective schemes** over A are closed subschemes of projective spaces.

Theorem: $U_f = \operatorname{Spec} R_{(f)}$*, when* $\deg f = 1$*, so that* $\operatorname{Proj} R$ *is a scheme.*

Proof: Homogeneous prime ideals of R not containing f correspond to homogeneous primes $\mathfrak{q} = \oplus_{-\infty}^{\infty} f^n \mathfrak{q}_0$ of $R_f = \oplus_n f^n R_{(f)}$, where $\mathfrak{q}_0$ is a prime ideal of $R_{(f)}$.

This bijection $U_f = \operatorname{Spec} R_{(f)}$ is a homeomorphism since $(a_n)_0 \cap U_f = \left(\frac{a_n}{f^n}\right)_0$.

It is an isomorphism of ringed spaces since we have an isomorphism of presheaves

$$U \subseteq U_f \qquad (R_{(f)})_U \longrightarrow R_{(U)},\ \frac{a_n/f^n}{b_m/f^m} \mapsto \frac{f^m a_n}{f^n b_m}.$$

Proposition: *Any integral projective scheme* $\operatorname{Proj} k[\xi_0, \ldots, \xi_n]$ *over a field is complete.*

Proof: Let $\Sigma = \left\{ \frac{P_m(\xi_0, \ldots, \xi_n)}{Q_m(\xi_0, \ldots, \xi_n)} \right\}$ be the field of rational functions.

We have to show that any valuation ring $\mathcal{V}$ of Σ containing k centers at a unique point.

Take a quotient $\frac{\xi_s}{\xi_i}$ of maximal valuation, so that all the fractions $\frac{\xi_r}{\xi_i} = \frac{\xi_r}{\xi_s} \cdot \frac{\xi_s}{\xi_i} \in \mathcal{V}$,

$$A_i = k\left[\tfrac{\xi_0}{\xi_i}, \ldots, \tfrac{\xi_n}{\xi_i}\right] \subset \mathcal{V}$$

and $\mathcal{V}$ centers at the point of U_{ξ_i} defined by $A_i \cap \mathfrak{m}_{\mathcal{V}}$.

If $\mathcal{V}$ centers at a point of $U_{\xi_j} = \operatorname{Spec} A_j$, then $A_j \subset \mathcal{V}$, and $\frac{\xi_j}{\xi_i}$ is invertible in $\mathcal{V}$, so that

$$A_{ij} = A_i\left[\tfrac{\xi_i}{\xi_j}\right] = A_j\left[\tfrac{\xi_j}{\xi_i}\right] \subset \mathcal{V}.$$

Both centers of $\mathcal{V}$ are in $U_i \cap U_j = \operatorname{Spec} A_{ij}$; hence they coincide since $\mathcal{V}$ may not center at two different points of an affine open set.

Definitions: If $M = \oplus_{-\infty}^{\infty} M_n$ is a graded R-module, then $\widetilde{M}$ denotes the sheaf on $\operatorname{Proj} R$ induced by the presheaf of homogeneous localization

$$U \rightsquigarrow M_{(U)} = \left\{ \tfrac{m_n}{f_n} \colon m_n \in M_n,\ f_n \in R_n \text{ without zeros on } U \right\}$$

and, as in the case of the sheaf of local rings, it coincides on $U_f = \operatorname{Spec} R_{(f)}$ with the sheaf defined by $M_{(f)}$, when $\deg f = 1$.

$M(n)$ is the graded R-module $M(n)_d = M_{n+d}$, and we put $\mathcal{O}_X(n) = R(n)^\sim$.

$\mathcal{O}_X(n)$ is a line sheaf since on U_f we have isomorphisms $f^n \colon \mathcal{O}_X|_{U_f} \to \mathcal{O}_X(n)|_{U_f}$.

If $\mathcal{M}$ is an $\mathcal{O}_X$-module, we put

$$\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Since $M_f = \oplus_n f^n M_{(f)} = M_{(f)} \otimes_{R_{(f)}} R_f$, we have $M_{(f)} \otimes_{R_{(f)}} N_{(f)} = (M \otimes_R N)_{(f)}$.

Hence, the natural morphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to (M \otimes_R N)^\sim$ is an isomorphism,

$$M(n)^\sim = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M}(n),$$
$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m).$$

Example: Since $k[x_0, \ldots, x_d]$ is a unique factorization domain, the presheaf is a sheaf

$$\mathcal{O}_{\mathbb{P}_d}(n)(U) = \left\{ \tfrac{P_{n+m}(x_0,\ldots,x_d)}{Q_m(x_0,\ldots,x_d)} \colon Q_m \text{ without zeros on } U \right\}$$

On $\mathbb{P}_{d,A}$ we have that x_i defines a global section of $\mathcal{O}_{\mathbb{P}_{d,A}}(1)$ not vanishing at any point of $U_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \ldots, \frac{x_d}{x_i}\right]$. Hence the sections $x_0, \ldots, x_d$ generate the stalk of $\mathcal{O}_{\mathbb{P}_{d,A}}(1)$ at any point, and they define an epimorphism $\mathcal{O}_{\mathbb{P}_{d,A}}^{d+1} \to \mathcal{O}_{\mathbb{P}_{d,A}}(1)$ that, after transposing it, should be viewed as a line sub-bundle of the trivial bundle of rank $d+1$.

This is the universal line sub-bundle: for any A-scheme X, the X-points of the projective space are families, parameterized by X, of lines of the trivial bundle of rank $d+1$.

Universal Property: $\operatorname{Hom}_A(X, \operatorname{Proj} A[x_0, \ldots, x_d]) = \begin{bmatrix} \text{Line quotients} \\ \text{of } \mathcal{O}_X^{d+1} \end{bmatrix}$.

Proof: Let $s_0, \ldots, s_d$ be sections of a line sheaf $\mathcal{L}$ on X generating the stalk at any point.

On the open set V_i where s_i generates, we have $s_j = \frac{s_j}{s_i} s_i$, where $\frac{s_j}{s_i} \in \mathcal{O}_X(V_i)$.

By the following lemma, the morphism of A-algebras

$$A\left[\tfrac{x_0}{x_i}, \ldots, \tfrac{x_d}{x_i}\right] \longrightarrow \mathcal{O}_X(V_i),\ \tfrac{x_j}{x_i} \mapsto \tfrac{s_j}{s_i},$$

induces a morphism of A-schemes $\phi_i \colon V_i \to U_i$, and we have $\phi_i^* \mathcal{O}(1) = \mathcal{L}|_{V_i}$, $\phi_i^*(x_j) = s_j$.

These morphisms ϕ_i coincide on intersections, hence they define a morphism $\phi \colon X \to \mathbb{P}_{d,A}$ and an isomorphism $\phi^* \mathcal{O}(1) = \mathcal{L}$, $\phi^*(x_j) = s_j$. The uniqueness is obvious.

Lemma: $\operatorname{Hom}(X, \operatorname{Spec} A) = \operatorname{Hom}(A, \mathcal{O}_X(X))$, *for any scheme* X.

Proof: The natural map $\operatorname{Hom}(U, \operatorname{Spec} A) \to \operatorname{Hom}(A, \mathcal{O}_X(U))$ is bijective when U is affine (p. 324). Since both terms are sheaves of sets, it is bijective when $U = X$.

Corollary: *Let E be a finite dimensional k-vector space. The group of all automorphisms of the scheme $\mathbb{P}(E) = \operatorname{Proj} S^\bullet E^*$ is the group $PSl(E)$ of projectivizations of semilinear automorphisms of E, and the subgroup of all k-automorphisms is the subgroup $PGl(E)$ of projectivities.*

Proof: Any automorphism $\sigma\colon \mathbb{P}(E) \to \mathbb{P}(E)$ defines an automorphism σ^* of the ring of global sections $\Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}) = k$. Hence we are reduced to the case of a k-automorphism, which directly follows from the universal property.

Definition: Let X be a non singular complete curve over an algebraically closed field. The global sections of a line sheaf L_D **separate points and infinitely near points** when for any pair of closed points p, q there is a section of L_{D-p} which is not a section of L_{D-p-q}; i.e., a global section of L_D vanishing at p and not at q (if $p = q$, not vanishing twice at p).

If $X \to \mathbb{P}_d$ is the morphism defined by a base $s_0, \ldots, s_d$ of $\Gamma(X, L_D)$, and B_i is the integral closure of $A_i = k\big[\frac{s_0}{s_i}, \ldots, \frac{s_d}{s_i}\big]$ in Σ_X, the global sections of L_D separate points when the rings $B_i/\mathfrak{m}B_i$ of the fibres are local, and separate infinitely near points when they are reduced; hence $B_i/\mathfrak{m}B_i = A_i/\mathfrak{m}$ since k is algebraically closed, and Nakayama's lemma shows that $A_i = B_i$.

That is to say, the morphism $X \to \mathbb{P}_d$ is a closed embedding.

Corollary: *If $\deg D > 2g$, the global sections of L_D separate points and infinitely near points. Any non singular complete curve over an algebraically closed field is projective.*

Proof: The divisors $K - (D - p)$ and $K - (D - p - q)$ are of negative degree and, by the Riemann-Roch theorem, $h^0(L_{D-p}) = h^0(L_{D-p-q}) + 1$.

11.5.1 Projective Morphisms

Let $R = A[\xi_0, \ldots, \xi_d]$, with A a noetherian ring.

We put $X = \operatorname{Proj} R$, and we consider the open embedding $j\colon U_i = X - (\xi_i)_0 \hookrightarrow X$.

The morphisms $\xi_i\colon R(n) \to R(n+1)$ induce morphisms $\mathcal{O}_X(n) \to \mathcal{O}_X(n+1)$.

If $\mathcal{M}$ is a quasi-coherent sheaf on X, we have morphisms $\mathcal{M}(n) \to \mathcal{M}(n+1)$, and

$$\varinjlim \mathcal{M}(n) \longrightarrow j_*(\mathcal{M}|_{U_i}),\ m \otimes \tfrac{f_{r+n}}{h_r} \mapsto \tfrac{f_{r+n}}{\xi_i^n h_r} m$$

is an isomorphism. In fact, on the affine open sets U_i we have that $\varinjlim M_n = M_{\xi_i}$ when $M_n = M$ and the transition morphisms are $\xi_i\colon M = M_n \to M_{n+1} = M$.

Theorem: $\varinjlim H^p(X, \mathcal{F}_i) = H^p(X, \varinjlim \mathcal{F}_i)$*, when X is a noetherian topological space.*

Proof: The natural morphism $\varinjlim \mathcal{F}_i(X) \to (\varinjlim \mathcal{F}_i)(X)$ is injective since X is compact (p. 345).

Any section $s \in (\varinjlim \mathcal{F}_i)(X)$ is locally defined by some sections $s_i \in \mathcal{F}_i(U)$.

These sections s_i do not coincide on intersections but, the intersections being compact, they coincide as sections of $\mathcal{F}_k$ for a big index k, and they define a global section of $\mathcal{F}_k$ inducing s.

Therefore $\varinjlim C^\bullet \mathcal{F}_i$ is a flasque resolution of $\varinjlim \mathcal{F}_i$, and

$$H^p(X, \varinjlim \mathcal{F}_i) = H^p\big[\Gamma(X, \varinjlim C^\bullet \mathcal{F}_i)\big] = \varinjlim H^p\big[\Gamma(X, C^\bullet \mathcal{F}_i)\big] = \varinjlim H^p(X, \mathcal{F}_i).$$

Theorem: *Any quasi-coherent sheaf $\mathcal{M}$ on X is the homogeneous localization $\mathcal{M} = \widetilde{M}$ of a graded R-module M, finitely generated if $\mathcal{M}$ is coherent.*

Proof: We put $M = \oplus_n \Gamma\big(X, \mathcal{M}(n)\big)$. The natural morphism $\widetilde{M} \to \mathcal{M}$ is an isomorphism

$$\widetilde{M}(U_i) = \bigcup_n \tfrac{\Gamma(X,\mathcal{M}(n))}{\xi_i^n} = \varinjlim \Gamma\big(X, \mathcal{M}(n)\big) = \Gamma\big(X, \varinjlim \mathcal{M}(n)\big) = \Gamma\big(X, j_*\mathcal{M}|_{U_i}\big) = \mathcal{M}(U_i).$$

When $\mathcal{M}$ is coherent, the modules $\mathcal{M}(U_i)$ are finitely generated, and there is a finitely generated submodule $N = \bigoplus_n N_n \subseteq M$ such that $\widetilde{N} \to \mathcal{M}$ is surjective.

Hence an isomorphism since $\widetilde{N} \subseteq \widetilde{M} = \mathcal{M}$.

Theorem: *Any coherent sheaf $\mathcal{M}$ on X admits a finite presentation*

$$\oplus_j \mathcal{O}_X(n_j) \longrightarrow \oplus_i \mathcal{O}_X(n_i) \longrightarrow \mathcal{M} \longrightarrow 0$$

Proof: The homogeneous localization preserves exact sequences, and any finitely generated graded R-module M admits a finite presentation $\oplus_j R(n_j) \to \oplus_i R(n_i) \to M \to 0$.

Theorem: $H^p(\mathbb{P}_{d,A}, \mathcal{O}_{\mathbb{P}_{d,A}}(n)) = \begin{cases} \text{free } A\text{-module of rank } \binom{n+d}{d} & n \geq 0,\ p = 0 \\ \text{free } A\text{-module of rank } \binom{-n-1}{d} & n < -d,\ p = d \\ 0 & \text{otherwise} \end{cases}$

Proof: Once we prove the statement for the sheaf of local rings $\mathcal{O}_{\mathbb{P}_d}$, it follows, by induction on n and d, for $\mathcal{O}_{\mathbb{P}_d}(n)$ and $\mathcal{O}_{\mathbb{P}_d}(-n)$ according to the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_d}(n-1) \longrightarrow \mathcal{O}_{\mathbb{P}_d}(n) \longrightarrow \mathcal{O}_{\mathbb{P}_{d-1}}(n) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_d}(-n) \longrightarrow \mathcal{O}_{\mathbb{P}_d}(-(n-1)) \longrightarrow \mathcal{O}_{\mathbb{P}_{d-1}}(-(n-1)) \longrightarrow 0$$

The first one also shows that $H^p(\mathbb{P}_d, \mathcal{O}(n)) = H^p(\mathbb{P}_d, \mathcal{O}(n+1))$ for all $p \geq 1$, $n \geq 0$. Therefore, when $p \geq 1$, we have

$$H^p(\mathbb{P}_d, \mathcal{O}) = \varinjlim H^p(\mathbb{P}_d, \mathcal{O}(n)) = H^p(\mathbb{P}_d, \varinjlim \mathcal{O}(n)) = H^p(\mathbb{P}_d, i_*\mathcal{O}_{U_0}) = H^p(U_0, \mathcal{O}_{U_0}) = 0.$$

Finally, since $\mathcal{O}_{\mathbb{P}_d}(U_i) = A\big[\frac{x_0}{x_i}, \ldots, \frac{x_d}{x_i}\big]$, it is clear that $H^0(\mathbb{P}_d, \mathcal{O}_{\mathbb{P}_d}) = R_0 = A$.

Example: If $\mathfrak{p}_C$ is the sheaf of ideals of a conic C of equation $q(x_0, x_1, x_2) = 0$, then we have an isomorphism $q\colon \mathcal{O}_{\mathbb{P}_2}(-2) \to \mathfrak{p}_C$, and the exact sequence

$$0 \longrightarrow \mathfrak{p}_C = \mathcal{O}_{\mathbb{P}_2}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

shows that $H^0(C, \mathcal{O}_C) = k$, $H^1(C, \mathcal{O}_C) = 0$; hence (p. 328) *any non-singular conic with a rational point is isomorphic to the projective line* (**Steiner's Theorem**).

Serre's Theorem: *Let $\mathcal{M}$ be a coherent sheaf on $X = \operatorname{Proj} A[\xi_0, \ldots, \xi_d]$.*

1. *$H^p(X, \mathcal{M})$ is a finitely generated A-module, null when $p > d$.*

2. *There is an integer n_0 such that, for any $n > n_0$, the sheaves $\mathcal{M}(n)$ are acyclic, generated by the global sections,*
$$\Gamma(X, \mathcal{M}(n)) \otimes_A \mathcal{O}_X \longrightarrow \mathcal{M}(n) \longrightarrow 0$$

Proof: We may assume that $X = \mathbb{P}_{d,A}$. The semiring of closed sets generated by the complements of $U_0, \ldots, U_d$ defines a projection $\pi\colon X \to \beta\Delta_d$ onto a finite space of dimension d, and π_* preserves the cohomology of quasi-coherent sheaves (pp. 322, 323), hence (p. 321)

$$H^p(X, \mathcal{M}) = H^p(\beta\Delta_d, \pi_*\mathcal{M}) = 0,\ p > d.$$

On the other hand, if we consider a presentation

$$0 \longrightarrow \mathcal{K} \longrightarrow \oplus_i \mathcal{O}_X(n_i) \longrightarrow \mathcal{M} \longrightarrow 0$$

the kernels and cokernels of the connecting morphisms $H^p(X, \mathcal{M}) \to H^{p+1}(X, \mathcal{K})$ are finitely generated A-modules and, by descending induction, $H^p(X, \mathcal{M})$ is a finitely generated A-module.

$$0 \longrightarrow \mathcal{K}(n) \longrightarrow \oplus_i \mathcal{O}_X(n_i + n) \longrightarrow \mathcal{M}(n) \longrightarrow 0$$

$\oplus_i \mathcal{O}_X(n_i+n)$ is acyclic when $n_i+n \gg 0$; hence $H^p(X,\mathcal{M}(n)) = H^{p+1}(X,\mathcal{K}(n))$; $p \geq 1$.
By descending induction we conclude that $H^p(X,\mathcal{M}(n)) = 0$, $n \gg 0$.
Moreover $\oplus_i \mathcal{O}_X(n_i+n)$ is generated by global sections; hence so is $\mathcal{M}(n)$.

Theorem: *If M is a finitely generated graded R-module, then $M_n = \Gamma(X, \widetilde{M}(n))$ when $n \gg 0$.*

Proof: We have exact sequences $0 \to K \to \oplus_i R(n_i) \to M \to 0$, $0 \to K' \to \oplus_j R(n_j) \to K \to 0$,

$$0 \longrightarrow \mathcal{K}(n) \longrightarrow \oplus_i\, \mathcal{O}_X(n_i+n) \longrightarrow \mathcal{M}(n) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{K}'(n) \longrightarrow \oplus_j\, \mathcal{O}_X(n_j+n) \longrightarrow \mathcal{K}(n) \longrightarrow 0$$

When $n \gg 0$, they remain exact on global sections, and the following commutative diagram with exact rows let us conclude,

$$\begin{array}{ccccccc} \oplus_j R(n_j)_n & \longrightarrow & \oplus_i R(n_i)_n & \longrightarrow & M_n & \longrightarrow & 0 \\ \| & & \| & & \downarrow & & \\ H^0\big(X, \oplus_j \mathcal{O}_X(n_j+n)\big) & \longrightarrow & H^0\big(X, \oplus_i \mathcal{O}_X(n_i+n)\big) & \longrightarrow & H^0(X,\mathcal{M}) & \longrightarrow & 0 \end{array}$$

Bézout's Theorem: Let $R = k[x_0,x_1,x_2]$, and let C, C' be two plane projective curves of degrees n, m and equations $P_n = 0$, $P'_m = 0$, without common irreducible components.

If $\mathfrak{p}$, $\mathfrak{p}'$ are the respective sheaves of ideals, the **intersection multiplicity** at a point z is the length of the $\mathcal{O}_{\mathbb{P}_2,z}$-module $\mathcal{O}_{\mathbb{P}_2,z}/\mathfrak{p}_z + \mathfrak{p}'_z$,

$$(C' \cap C)_z = l\big(\mathcal{O}_{\mathbb{P}_2,z}/\mathfrak{p}_z + \mathfrak{p}'_z\big),$$

so that the dimension of the k-vector space $\Gamma(\mathbb{P}_2, \mathcal{O}/\mathfrak{p}+\mathfrak{p}') = \oplus_z \mathcal{O}_z/\mathfrak{p}_z + \mathfrak{p}'_z$ is the intersection number, counting points with degree and multiplicity.

Since R is a unique factorization domain, and P_n, P'_m have no common irreducible factor, we have exact sequences

$$0 \longrightarrow R(-n-m) \xrightarrow{\ \phi\ } R(-n) \oplus R(-m) \xrightarrow{\ \varphi\ } R \longrightarrow R/(P_n, P'_m) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-n-m) \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-n) \oplus \mathcal{O}_{\mathbb{P}_2}(-m) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_{\mathbb{P}_2}/\mathfrak{p}+\mathfrak{p}' \longrightarrow 0$$

where $\phi(Q) = (P'_m Q, P_n Q)$ and $\varphi(A,B) = P_n A - P'_m B$. Now, the Euler-Poincaré characteristic is additive, so that *the intersection number is the product of the degrees:*

$$\begin{aligned} \dim_k \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}/\mathfrak{p}+\mathfrak{p}') &= \chi(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}/\mathfrak{p}+\mathfrak{p}') \\ &= \chi(\mathcal{O}_{\mathbb{P}_2}) - \chi(\mathcal{O}_{\mathbb{P}_2}(-n)) - \chi(\mathcal{O}_{\mathbb{P}_2}(-m)) + \chi(\mathcal{O}_{\mathbb{P}_2}(-n-m)) \\ &= 1 - \tbinom{n-1}{2} - \tbinom{m-1}{2} + \tbinom{n+m-1}{2} = nm. \end{aligned}$$

11.6 Complete Curves

Let Σ be the field of rational functions on a complete curve X over a field k, and let $\bar{X}$ be the Riemann variety of Σ.

Since any discrete valuation centers at a unique point of X, we have a natural morphism

$$p\colon \bar{X} \longrightarrow X$$

and $p^{-1}(\operatorname{Spec} A) = \operatorname{Spec} \bar{A}$, where $\bar{A}$ is the integral closure of A in Σ, so that the direct image p_* preserves (p. 323) the cohomology of quasi-coherent sheaves.

(∗) $$0 \longrightarrow \mathcal{O}_X \longrightarrow p_*\mathcal{O}_{\bar{X}} \longrightarrow \mathfrak{C} \longrightarrow 0$$

where $\mathfrak{C} = (p_*\mathcal{O}_{\bar{X}})/\mathcal{O}_X$ is supported at the singular points of X.

Definition: The **arithmetic genus** of X is $\pi = \dim_k H^1(X, \mathcal{O}_X)$ and the **geometric genus** of X is $g = \dim_k H^1(\bar{X}, \mathcal{O}_{\bar{X}})$.

When k is algebraically closed in Σ, we have $k = H^0(X, \mathcal{O}_X) = H^0(\bar{X}, \mathcal{O}_{\bar{X}})$, and the exact sequence $(*)$ shows that $g = \pi - \sum_{x \in X} \dim_k(\bar{\mathcal{O}}_x/\mathcal{O}_x)$.

Theorem: *If $\mathcal{M}$ is a coherent sheaf on a complete curve X, the k-vector spaces $H^n(X, \mathcal{M})$ have finite dimension.*

Proof: Let $\phi\colon \mathcal{M} \to p_*p^*\mathcal{M}$ be the natural morphism. The theorem holds for $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ since both are coherent sheaves supported at the singular points of X.

It also holds for $p_*p^*\mathcal{M}$ since $H^n(X, p_*p^*\mathcal{M}) = H^n(\bar{X}, p^*\mathcal{M})$, and we conclude,

$$0 \longrightarrow \operatorname{Im}\phi \longrightarrow p_*p^*\mathcal{M} \longrightarrow \operatorname{Coker}\phi \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow \mathcal{M} \longrightarrow \operatorname{Im}\phi \longrightarrow 0$$

Proposition: *The dualizing sheaf of a plane curve X of degree n is*

$$\omega_X = \mathcal{O}_X(n-3).$$

Proof: We have an exact sequence $0 \to \mathcal{O}_{\mathbb{P}_2}(m-n) \to \mathcal{O}_{\mathbb{P}_2}(m) \to \mathcal{O}_X(m) \to 0$.

Hence $h^1(\mathcal{O}_X(n-3)) = h^2(\mathcal{O}_{\mathbb{P}_2}(-3)) = 1$, and $\mathcal{O}_X(n-3)$ is a minimal pair with any non null element of $H^1(X, \mathcal{O}_X(n-3))^*$, since torsion sheaves are acyclic.

If it is dominated by another minimal pair $\mathcal{M}_\xi$, when $m \gg 0$,

$$0 \longrightarrow \mathcal{O}_X(n-3) \longrightarrow \mathcal{M} \longrightarrow \mathcal{K} \longrightarrow 0 \text{ is exact}$$
$$0 \longrightarrow \Gamma(X, \mathcal{O}_X(m+n-3)) \longrightarrow \Gamma(X, \mathcal{M}(m)) \longrightarrow \Gamma(X, \mathcal{K}(m)) \longrightarrow 0 \text{ is exact}$$
$$h^0(\mathcal{M}(m)) \leq \dim \operatorname{Hom}(\mathcal{O}_X(-m), \omega_X) = h^1(\mathcal{O}_X(m))$$
$$= h^2(\mathcal{O}(-m-m)) - h^2(\mathcal{O}(-m)) = \binom{m+n-1}{2} - \binom{m-1}{2}$$
$$h^0(\mathcal{O}_X(m+n-3)) = h^0(\mathcal{O}(m+n-3)) - h^0(\mathcal{O}(m-3)) = \binom{m+n-1}{2} - \binom{m-1}{2}$$
$$h^0(\mathcal{K}(m)) = 0$$

and we conclude that $\mathcal{K} = 0$; hence $\mathcal{O}_X(n-3) = \mathcal{M}$, and $\mathcal{O}_X(n-3) = \omega_X$.

Corollary: *If X is a plane curve of degree n and L_K is the canonical line sheaf of the non singular model $\bar{X}$, we have an isomorphism of $p_*\mathcal{O}_{\bar{X}}$-modules*

$$p_*(L_K) = \underline{\operatorname{Hom}}_{\mathcal{O}_X}(p_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n-3).$$

Proof: The argument of p. 329 shows that $p_*\omega_{\bar{X}} = \underline{\operatorname{Hom}}_{\mathcal{O}_X}(p_*\mathcal{O}_{\bar{X}}, \omega_X)$.

Since ω_X is a line sheaf, we see that $\underline{\operatorname{Hom}}(p_*\mathcal{O}_{\bar{X}}, \omega_X) = \underline{\operatorname{Hom}}(p_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \omega_X$.

Chapter 12

Algebraic Topology I

12.1 Cohomology with Supports

Definitions: Let $\mathcal{F}$ be a sheaf on a topological space X. The **support** of a section $s \in \mathcal{F}(U)$ is

$$\operatorname{supp}(s) = \{x \in U \colon s_x \neq 0\}$$

and it is closed in U. We put $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$, the subgroup of sections with compact support is denoted by $\Gamma_c(U, \mathcal{F})$ and, given a closed set Y in X, the subgroup of sections with support in Y is denoted by $\Gamma_Y(X, \mathcal{F})$. The **cohomology groups with compact supports** and the **local cohomology** groups with support in Y are defined with the Godement resolution

$$H^n_c(X, \mathcal{F}) = H^n\big[\Gamma_c(X, C^\bullet\mathcal{F})\big]$$
$$H^n_Y(X, \mathcal{F}) = H^n\big[\Gamma_Y(X, C^\bullet\mathcal{F})\big]$$

and the proofs of p. 320 remain valid because flasque sheaves are Γ_c-acyclic and Γ_Y-acyclic since, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence and $\mathcal{F}'$ is flasque, any section $s'' \in \mathcal{F}''(U)$ comes from a section $s \in \mathcal{F}(U)$, and we may fix s so that $\operatorname{supp}(s) = \operatorname{supp}(s'')$.

In fact, s defines on $V = U - \operatorname{supp}(s'')$ a section of $\mathcal{F}'$, which may be extended to a section s' on U, and now $s - s'$ vanishes on V, so that $s - s'$ maps to s'' and $\operatorname{supp}(s - s') = \operatorname{supp}(s'')$.

Definition: A sheaf of rings $\mathcal{O}$ admits **partitions of unity** if for any open cover $X = \bigcup_i U_i$ there are $f_i \in \mathcal{O}(X)$ such that the family $\{\operatorname{supp} f_i\}$ is locally finite, $\operatorname{supp} f_i \subseteq U_i$, and $\sum_i f_i = 1$.

The sheaf of smooth functions $\mathcal{C}^\infty_X$ on a smooth manifold, and the sheaf of continuous functions $\mathcal{C}_X$ on a σ-compact space admit partitions of unity (pp. 260, 259).

On any σ-compact space X, the sheaf of integer discontinuous functions $C^0\mathbb{Z}$ admits partitions of unity, since any open cover $X = \bigcup_i U_i$ admits a partition of unity $\{\phi_i\}$ by continuous functions. If for any point $x \in X$ we fix an index i such that $\phi_i(x) \neq 0$, and we take $f_i \colon X \to \mathbb{Z}$ with value 1 at any point where we choose the index i, and 0 at any other point, then $\sum_i f_i = 1$ and the family $\operatorname{supp} f_i \subseteq \operatorname{supp} \phi_i \subseteq U_i$ is locally finite.

Lemma: *If a sheaf of rings $\mathcal{O}$ admits partitions of unity, any $\mathcal{O}$-module $\mathcal{M}$ is acyclic (and Γ_c-acyclic when X is σ-compact).*

Proof: $H^p(X, \mathcal{M})$ is the p-th cohomology group of the complex of $\mathcal{O}(X)$-modules

$$\Gamma(X, C^0\mathcal{M}) \xrightarrow{\mathrm{d}^0} \Gamma(X, C^1\mathcal{M}) \xrightarrow{\mathrm{d}^1} \Gamma(X, C^2\mathcal{M}) \xrightarrow{\mathrm{d}^2} \dots$$

If $\mathrm{d}^{p+1} z = 0$, there is an open cover $X = \bigcup_i U_i$ and $z_i \in \Gamma(U_i, C^p\mathcal{M})$ such that $z|_{U_i} = \mathrm{d}^p z_i$.

Fix a partition of unity $\{f_i\}$ subordinated to $\{U_i\}$, so that $\sum_i f_i z_i$ is a well defined global section of $C^p\mathcal{M}$, and at any point $x \in X$ we have

$$\mathrm{d}^p\big(\textstyle\sum_i (f_i z_i)_x\big) = \sum_i f_{i,x}\mathrm{d}^p z_{i,x} = \sum_i f_{i,x} z_x = z_x.$$

If X is σ-compact and $\operatorname{supp} z$ is compact, we may take $U_0 = (\operatorname{supp} z)^c, U_1, \dots, U_n$, with $\overline{U}_1, \dots, \overline{U}_n$ compact, and $z_0 = 0$, so that the support of $\sum_i f_i z_i$ is compact.

Theorem: *The De Rham cohomology groups of a smooth manifold X are topological invariants,*

$$H^p_{DR}(X) = H^p(X, \mathbb{R}).$$

Proof: Let Ω^p be the sheaf of differential p-forms on X.

Any closed form is locally exact by Poincaré's lemma, so that we have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty_X \xrightarrow{\ \mathrm{d}\ } \Omega^1 \xrightarrow{\ \mathrm{d}\ } \Omega^2 \xrightarrow{\ \mathrm{d}\ } \dots$$

which is an acyclic resolution of the constant sheaf $\mathbb{R}$ since Ω^p is a $\mathcal{C}^\infty_X$-module.

By De Rham's theorem, $H^p(X, \mathbb{R}) = H^p\big[\Gamma(X, \Omega^\bullet)\big] = H^p_{DR}(X)$. q.e.d.

We have an analogous result for the cohomology groups with compact support,

$$H^p_c(X, \mathbb{R}) = \frac{\Gamma_c(X, \Omega^p)}{\mathrm{d}(\Gamma_c(X, \Omega^{p-1}))}.$$

Definition: If Y is a subspace of X, the **restriction** $\mathcal{F}|_Y$ of a sheaf $\mathcal{F}$ on X is the sheaf of continuous sections of the local homeomorphism $(\mathcal{F}^{et})|_Y \to Y$.

The stalk at any point coincides with the stalk of $\mathcal{F}$, and we put $H^n(Y, \mathcal{F}) = H^n(Y, \mathcal{F}|_Y)$.

When $i\colon Y \to X$ is a closed subset, the direct image i_* preserves the cohomology groups (p. 323), and if we put $\mathcal{F}_Y = i_*(\mathcal{F}|_Y)$, then we have

$$H^n(X, \mathcal{F}_Y) = H^n(Y, \mathcal{F}).$$

Restriction of continuous sections defines an epimorphism $\mathcal{F} \to \mathcal{F}_Y$, since $(\mathcal{F}_Y)_x = \mathcal{F}_x$ or 0, according to $x \in Y$ or not, and the kernel is denoted by $\mathcal{F}_U$, where $U = X - Y$,

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0$$

Taking sections on an open set V we see that

$$\begin{aligned} \mathcal{F}_U(V) &= \{s \in \mathcal{F}(V)\colon \operatorname{supp} s \subseteq U \cap V\} \\ &= \{s \in \mathcal{F}(U \cap V)\colon \operatorname{supp} s \text{ is closed in } V\} \\ \Gamma_c(X, \mathcal{F}_U) &= \Gamma_c(U, \mathcal{F}) \quad \text{when } X \text{ is separated.} \end{aligned}$$

For example, $\mathbb{Z}^{et}_U$, apart from the zero section, has a copy of U for each non null integer.

$\operatorname{Hom}(\mathbb{Z}_U, \mathcal{F}) = \mathcal{F}(U)$, and any sheaf $\mathcal{F}$ admits an epimorphism $\oplus_i \mathbb{Z}_{U_i} \to \mathcal{F} \to 0$.

Lemma: $H^n_c(X, \mathcal{F}_U) = H^n_c(U, \mathcal{F})$, *when X is σ-compact.*

Proof: If $\mathcal{M}$ is an $\mathcal{O}$-module, so is $\mathcal{M}_U$ since $\operatorname{supp}(fm) \subseteq \operatorname{supp} m$.

Hence $(C^n\mathcal{F})_U$ is a $C^0\mathbb{Z}$-module and $(C^\bullet\mathcal{F})_U$ is a Γ_c-acyclic resolution of $\mathcal{F}_U$,

$$H^n_c(X, \mathcal{F}_U) = H^n\big[\Gamma_c(X, (C^\bullet\mathcal{F})_U)\big] = H^n\big[\Gamma_c(U, C^\bullet\mathcal{F})\big] = H^n_c(U, \mathcal{F}).$$

Mayer-Vietoris Exact Sequences: *Let $\mathcal{F}$ be a sheaf on a topological space X. If U_1, U_2 are open subsets of X, we have an exact sequence*

$$\ldots \xrightarrow{\delta} H^n(U_1 \cup U_2, \mathcal{F}) \to H^n(U_1, \mathcal{F}) \oplus H^n(U_2, \mathcal{F}) \to H^n(U_1 \cap U_2, \mathcal{F}) \xrightarrow{\delta} H^{n+1}(U_1 \cup U_2, \mathcal{F}) \to \ldots$$

and if moreover X is σ-compact, we also have an exact sequence

$$\ldots \xrightarrow{\delta} H^n_c(U_1 \cap U_2, \mathcal{F}) \to H^n_c(U_1, \mathcal{F}) \oplus H^n_c(U_2, \mathcal{F}) \to H^n_c(U_1 \cup U_2, \mathcal{F}) \xrightarrow{\delta} H^{n+1}_c(U_1 \cap U_2, \mathcal{F}) \to \ldots$$

If Y_1, Y_2 are closed subsets of X, we have an exact sequence

$$\ldots \xrightarrow{\delta} H^n(Y_1 \cup Y_2, \mathcal{F}) \to H^n(Y_1, \mathcal{F}) \oplus H^n(Y_2, \mathcal{F}) \to H^n(Y_1 \cap Y_2, \mathcal{F}) \xrightarrow{\delta} H^{n+1}(Y_1 \cup Y_2, \mathcal{F}) \to \ldots$$

Proof: We have exact sequences

$0 \longrightarrow \Gamma(U_1 \cap U_2, C^\bullet\mathcal{F}) \longrightarrow \Gamma(U_1, C^\bullet\mathcal{F}) \oplus \Gamma(U_2, C^\bullet\mathcal{F}) \longrightarrow \Gamma(U_1 \cap U_2, C^\bullet\mathcal{F}) \longrightarrow 0.$

$0 \longrightarrow \mathcal{F}_{U_1 \cap U_2} \longrightarrow \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2} \longrightarrow \mathcal{F}_{U_1 \cup U_2} \longrightarrow 0$, and the above lemma.

$0 \longrightarrow \mathcal{F}_{Y_1 \cup Y_2} \longrightarrow \mathcal{F}_{Y_1} \oplus \mathcal{F}_{Y_2} \longrightarrow \mathcal{F}_{Y_1 \cap Y_2} \longrightarrow 0$, and $H^n(X, \mathcal{F}_Y) = H^n(Y, \mathcal{F})$.

Closed Subspace Exact Sequence: *If Y is a closed set in a σ-compact space X, and we put $U = X - Y$, we have an exact sequence*

$$\ldots \xrightarrow{\delta} H^n_c(U, \mathcal{F}) \longrightarrow H^n_c(X, \mathcal{F}) \longrightarrow H^n_c(Y, \mathcal{F}) \xrightarrow{\delta} H^{n+1}_c(U, \mathcal{F}) \longrightarrow \ldots$$

Proof: The cohomology exact sequence of $0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0$ and the above lemma.

Local Cohomology Exact Sequence: *If Y is a closed set in X, and we put $U = X - Y$, we have an exact sequence*

$$\ldots \xrightarrow{\delta} H^n_Y(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}) \xrightarrow{\delta} H^{n+1}_Y(X, \mathcal{F}) \longrightarrow \ldots$$

Proof: The sequence $0 \longrightarrow \Gamma_Y(X, C^\bullet\mathcal{F}) \longrightarrow \Gamma(X, C^\bullet\mathcal{F}) \longrightarrow \Gamma(U, C^\bullet\mathcal{F}) \longrightarrow 0$ is exact.

Excision: $H^n_Y(X, \mathcal{F}) = H^n_Y(U, \mathcal{F})$, *when U is an open neighborhood of Y.*

Proof: We have $\Gamma_Y(X, \mathcal{F}) = \Gamma_Y(U, \mathcal{F})$ for any sheaf $\mathcal{F}$ on X.

Theorem: *Any constant sheaf G on the cube $C = [0,1]^n$ is acyclic,*

$$H^p(C, G) = 0, \ p \geq 1.$$

Proof: By induction on n, and it is obvious when C is a point.

If $n \geq 1$, we decompose $C = P_1 \cup P_2$ as a union of two parallelograms intersecting at a cube of dimension $n - 1$. Since the morphism

$$G \oplus G = H^0(P_1, G) \oplus H^0(P_2, G) \longrightarrow H^0(P_1 \cap P_2, G) = G, \ (g_1, g_2) \mapsto g_2 - g_1,$$

is surjective, by Mayer-Vietoris $H^p(C, G) = H^p(P_1, G) \oplus H^p(P_2, G)$, $p \geq 1$.

Iterating the subdivisions, $C = P_1 \cup \ldots \cup P_r$, we obtain that

$$H^p(C, G) = H^p(P_1, G) \oplus \ldots \oplus H^p(P_r, G), \ p \geq 1.$$

Since any cohomology class vanishes on a neighborhood of each point, and C is compact, we conclude that $H^p(C, G) = 0$, $p \geq 1$. q.e.d.

1. **Cohomology of Spheres:** $H^p(S_n, G) = \begin{cases} G & p = 0, n \\ 0 & p \neq 0, n \end{cases}$

   The Mayer-Vietoris exact sequence for two hemispheres intersecting at the equator.

2. $H^p_c(\mathbb{R}^n, G) = \begin{cases} G & p = n \\ 0 & p \neq n \end{cases}$

   The closed subspace exact sequence when Y is a point of S_n, and $U = \mathbb{R}^n$.

3. *$\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic when $n \neq m$.*

4. *Let X be a compact Hausdorff space, and let $\emptyset = X_{-1} \subset X_0 \subseteq X_1 \ldots \subseteq X_d = X$ be closed sets such that $X_i - X_{i-1} = \amalg_{n_i} \mathbb{R}^i$. The cohomology groups $H^p(X, k)$ with coefficients in a field k have finite dimension, they vanish when $p > d$, and the* ***Euler-Poincaré characteristic*** *$\chi(X) := \sum_p (-1)^p \dim_k H^p(X, k)$ of X is $\chi(X) = \sum_i (-1)^i n_i$.*

   Let $U = \mathbb{R}^d$ be a connected component of $X_d - X_{d-1}$. The exact sequence of the closed subspace $Y = X - U$ shows that $H^p(X, k) = H^p(Y, k)$ when $p \neq d, d-1$, and that we have an exact sequence (and we conclude by induction on $\sum_i n_i$)

   $$0 \longrightarrow H^{d-1}(X, k) \longrightarrow H^{d-1}(Y, k) \longrightarrow k \longrightarrow H^d(X, k) \longrightarrow H^d(Y, k) \longrightarrow 0.$$

5. **Cohomology of Complex Projective Spaces:** $H^p(\mathbb{P}_{n,\mathbb{C}}, G) = \begin{cases} G & p = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$

   Induction on n and the exact sequence of the closed subspace $Y = \mathbb{P}_{n-1}$, $U = \mathbb{C}^n$.

6. *If $\pi\colon \bar{X} \to X$ is a covering of a topological manifold, $H^p(\bar{X}, G) = H^p(X, \pi_* G)$.*

   We must show that the sequence $0 \to \pi_* G \to \pi_*(C^\bullet G)$ is exact (p. 323). Now, any point of X admits a base of neighborhoods C_i which are cubes and

   $$H^p(\pi^{-1} C_i, G) = H^p\big(\amalg C_i, G\big) = \prod H^p(C_i, G) = 0, \ p \geq 1.$$

7. **Cohomology of Real Projective Spaces:** $H^p(\mathbb{P}_{n,\mathbb{R}}, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & 0 \leq p \leq n \\ 0 & p > n \end{cases}$

   By induction on n, the exact sequence of the closed subspace $Y = \mathbb{P}_{n-1}$, $U = \mathbb{R}^n$, shows that it is enough to check that $H^n(\mathbb{P}_n, \mathbb{F}_2) \neq 0$. Now, if $\pi\colon S_n \to \mathbb{P}_n$ is the universal covering, we have an exact sequence of sheaves on $\mathbb{P}_n$

   $$0 \longrightarrow \mathbb{F}_2 \longrightarrow \pi_* \mathbb{F}_2 \xrightarrow{\text{tr}} \mathbb{F}_2 \longrightarrow 0$$

   where $\text{tr} f = f + \tau f$, and $\{\text{Id}, \tau\}$ is the Galois group of π. Hence $H^n(\mathbb{P}_n, \mathbb{F}_2) \neq 0$ by the cohomology exact sequence $H^n(\mathbb{P}_n, \mathbb{F}_2) \to H^n(\mathbb{P}_n, \pi_* \mathbb{F}_2) = \mathbb{F}_2 \to H^n(\mathbb{P}_n, \mathbb{F}_2)$.

12.1.1 Cohomology and Dimension

Lemma: *Let A be a lattice semiring, and let $\mathcal{C}$ be a sheaf on $X = \operatorname{Spec} A$. If the restriction morphism $\mathcal{C}(X) \to \mathcal{C}(U)$ is surjective for any basic open set U, then $\mathcal{C}$ is acyclic.*

Proof: If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence and $\mathcal{F}'(X) \to \mathcal{F}'(U)$ is surjective for any basic open set U, let us see that we have exact sequences

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{i} \mathcal{F}(U) \xrightarrow{p} \mathcal{F}''(U) \longrightarrow 0$$

If $s'' \in \mathcal{F}''(U)$, since U is compact, it admits a finite basic cover $U = U_1 \cup \ldots \cup U_n$ where $s''|_{U_i}$ comes from of a section of $\mathcal{F}$, and the argument of p. 319 proves that s'' comes from a section of $\mathcal{F}$ on the basic open set $U_1 \cup U_2$ (in the spectrum of a semiring, basic open sets form a lattice!). Hence s'' comes from a section of $\mathcal{F}$ on $U = U_1 \cup \ldots \cup U_n$.

Now, if $\mathcal{F}(X) \to \mathcal{F}(U)$ also is surjective, so is $\mathcal{F}''(X) \to \mathcal{F}''(U)$, and the argument of p. 320 let us conclude (replacing flasque sheaves by sheaves $\mathcal{C}$ such that $\mathcal{C}(X) \to \mathcal{C}(U)$ is surjective for any basic open set U).

Theorem: *Let A be a lattice semiring, and let $\mathcal{F}$ be a sheaf on $X = \operatorname{Spec} A$. If $\mathcal{F}_x = 0$ at any point $x \in X$ of dimension $> d$, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$. In particular,*

$$H^p(X, \mathcal{F}) = 0, \ p > \dim X.$$

Proof: Let $i \colon X_d \to X$ be the subspace of points of dimension $\geq d$. The natural morphism

$$\phi \colon \mathcal{F} \longrightarrow i_*(\mathcal{F}|_{X_d})$$

induces an isomorphism on stalks at any point of X_d and, if $i_*(\mathcal{F}|_{X_d})$ is acyclic, the argument of p. 322 let us conclude. By the lemma, it is enough to show that the morphism

$$(\mathcal{F}|_{X_d})(X_d) = (i_*\mathcal{F}|_{X_d})(X) \longrightarrow (i_*\mathcal{F}|_{X_d})(U) = (\mathcal{F}|_{X_d})(X_d \cap U)$$

is surjective on any basic open set U.

If the support $|s|$ of $s \in (\mathcal{F}|_{X_d})(X_d \cap U)$ is closed in X_d, then s may be extended by 0.

Now, $|s|$ is closed in $X_d \cap U$; hence the closure Y of $|s|$ in U has no point of dimension $> d$, and $|s|$ is closed in X_d by the following lemma.

Lemma: *If a point $x \in X - U_f$ is adherent to a closed set Y of a basic open set U_f, then the dimension of x is smaller than that of some point of Y.*

Proof: Since any closed set in $\operatorname{Spec} A$ is the spectrum of a semiring, we may assume that $Y = U_f$.

If $\mathfrak{p}$ is the prime ideal of x, there is no $h \in A - \mathfrak{p}$ such that $U_h \cap U_f = \emptyset$.

Hence $\mathfrak{p}$ contains the annihilator of f, and f is not zero in $A_\mathfrak{p}$.

Since the intersection of all prime ideals of a semiring is null, f is not in some prime ideal $\mathfrak{q}$ of $A_\mathfrak{p}$, different from $\mathfrak{p}$ since $x \in (f)_0$, and $\mathfrak{q}$ defines a point of U_f of bigger dimension than the dimension of x.

Theorem: *If X is a compact separated space, we have $H^p(X, \mathcal{F}) = 0$, $p > \dim X$.*

Proof: If B is a base of the topology of X of dimension d, then $X = \operatorname{Spec}_m B$ and the inclusion $j \colon X \to \S B$ admits a continuous retract $r \colon \operatorname{Spec} B \to X$ (p. 235).

Moreover, $r^{-1}(U)$ is contained in any open set in $\operatorname{Spec} B$ intersecting X at U.

Hence $(r_*\mathcal{C})_x = \mathcal{C}_x$, and r_* preserves cohomology. Since $\mathcal{F} = r_* j_* \mathcal{F}$,

$$H^p(X, \mathcal{F}) = H^p(\operatorname{Spec} B, j_*\mathcal{F}) = 0, \ p > d.$$

Fundamental Theorem of Dimension Theory: $\dim \mathbb{R}^n = n$.

Proof: On a closed ball B_n we have $H^n(B_n, \mathbb{Z}_U) = H^n_c(U, \mathbb{Z}) = \mathbb{Z}$ when $U \simeq \mathbb{R}^n$.

Hence $\dim \mathbb{R}^n \geq \dim B_n \geq n$, and we already know that $\dim \mathbb{R}^n \leq n$ (p. 243).

Lemma: $\varinjlim \Gamma(X, \mathcal{F}_i) = \Gamma(X, \varinjlim \mathcal{F}_i)$, *when X is a compact Hausdorff space.*

Proof: The natural morphism $\varinjlim \Gamma(X, \mathcal{F}_i) \to \Gamma(X, \varinjlim \mathcal{F}_i)$ is injective:

If $s \in \mathcal{F}_i(X)$ vanishes as a section of the inductive limit, then it vanishes on a neighborhood of any point x as a section of $\mathcal{F}_j$ for some index $j \geq i$, independent of x since X is compact. Hence $s = 0$ in $\mathcal{F}_j(X)$.

The morphism is surjective:

Given a global section s of the inductive limit, on some finite open cover $X = \bigcup_r U_r$ it comes from some sections $s_r \in \mathcal{F}_i(U_r)$, which do not coincide on intersections; but any point x has an open neighborhood W_x where all the sections s_r (with $x \in U_r$) define the same section $s^x \in \mathcal{F}_j(W_x)$, $j \gg i$.

If we consider an open cover $X = \bigcup_r V_r$, where $\overline{V}_r \subseteq U_r$, we may assume that $W_x \subseteq V_r$ when $x \in V_r$, and $W_x \cap \overline{V}_r = \emptyset$ when $x \notin \overline{V}_r$; so that, when W_x and W_y intersect, we have $x, y \in U_r$ for some index r, and the sections s^x, s^y coincide on $W_x \cap W_y$, when $j \gg i$. A finite number of such open sets W_x cover X, and the sections s^x define a global section of $\mathcal{F}_j$ inducing s.

Theorem: $H^n_c(X, \varinjlim \mathcal{F}_i) = \varinjlim H^n_c(X, \mathcal{F}_i)$, *when X is σ-compact.*

Proof: The above argument shows that the morphism $\varinjlim \Gamma_c(X, \mathcal{F}_i) \to \Gamma_c(X, \varinjlim \mathcal{F}_i)$ is injective.

Now, given any global section s of the inductive limit with compact support, contained in the interior of a compact K', contained in the interior of a compact K'', by the former lemma it comes from a section s'' of some sheaf $\mathcal{F}_j$ on K'', vanishing outside of the interior of K' when j is big enough. Extending s'' by 0 outside from K', we obtain a global section of $\mathcal{F}_j$, with support contained in K', inducing s.

Any $C^0\mathbb{Z}$-module is Γ_c-acyclic, because $C^0\mathbb{Z}$ admits partitions of unity (p. 341); hence

$$\begin{aligned} H^n_c(X, \varinjlim \mathcal{F}_i) &= H^n\big[\Gamma_c(X, C^\bullet\mathbb{Z} \otimes_{\mathbb{Z}} (\varinjlim \mathcal{F}_i))\big] = H^n\big[\varinjlim \Gamma_c(X, C^\bullet\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{F}_i)\big] \\ &= \varinjlim H^n\big[\Gamma_c(X, C^\bullet\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{F}_i)\big] = \varinjlim H^n_c(X, \mathcal{F}_i). \end{aligned}$$

Theorem: *If X is a topological manifold of dimension n, we have $H^p_c(X, \mathcal{F}) = 0,\ p > n$.*

Proof:
$$H^p_c(X, \mathcal{F}) = H^p_c(X, \varinjlim \mathcal{F}_U) = \varinjlim H^p_c(X, \mathcal{F}_U) = \varinjlim H^p_c(U, \mathcal{F}),$$

where U runs over finite unions $U = V_1 \cup \ldots \cup V_m$ of open sets contained in compact sets of dimension n, so that $H^p_c(V, \mathcal{F}) = H^p(K, \mathcal{F}_V) = 0$, $p > n$, for any open set $V \subseteq V_i$.

Now, if $H^p_c(V_1 \cup \ldots \cup V_r, \mathcal{F}) = 0$, $p > n$, the Mayer-Vietoris exact sequence shows that also $H^p_c(V_1 \cup \ldots \cup V_{r+1}, \mathcal{F}) = 0$, $p > n$. Hence $H^p_c(U, \mathcal{F}) = 0$, $p > n$, and we conclude.

12.2 Derived Functors

Definition: A covariant functor $F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ from the category of A-modules into the category of B-modules is **additive** when the maps $F\colon \mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_B(F(M), F(N))$ are group morphisms, $F(f + g) = F(f) + F(g)$.

Additive functors preserve finite direct sums because they preserve split exact sequences and $F(0) = 0$. Hence F is left exact (resp. right exact) when it preserves exact sequences of the type $0 \to M' \to M \to M''$ (resp. $M' \to M \to M'' \to 0$).

Moreover, if $K^\bullet = \{K^n, \mathrm{d}_n\}$ is a complex of A-modules, then $F(K^\bullet) = \{F(K^n), F(\mathrm{d}_n)\}$ is a complex of B-modules, since $F(\mathrm{d})^2 = F(\mathrm{d}^2) = F(0) = 0$.

In this section F is always assumed to be an additive covariant functor.

Any A-module M is a quotient, $P_0M \to M \to 0$, of the free module generated by the elements of M, modulo the submodule generated by the 0 of M (so that the functor P_0 preserves the morphism 0 and complexes, even if it is not additive); hence M *admits a functorial projective* (in fact free) *resolution* $P_\bullet M \to M \to 0$ preserving complexes.

Now (p. 127) M is a submodule of the injective module $I^0 M = (P_0 M^*)^*$, and we see that M also *admits a functorial injective resolution* $0 \to M \to I^\bullet M$ preserving complexes (even if I^0 fails to be additive).

Definitions: When F is left exact, the right **derived functor** $R^n F \colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is

$$R^n F(M) = H^n \left[F(I^\bullet M) \right],$$

so that $F(M) = R^0 F(M)$, and any injective module I is F**-acyclic**; i.e. $R^n F(I) = 0$ for all $n > 0$.

When F is right exact, the left **derived functor** $L_n F \colon \mathbf{A} \rightsquigarrow \mathbf{B}$ is

$$L_n F(M) = H_n \left[F(P_\bullet M) \right],$$

so that $F(M) = L_0 F(M)$, and any projective module P is F**-acyclic**; i.e. $L_n F(P) = 0$ for all $n > 0$.

1. The right derived functors of $\mathrm{Hom}_A(M, -)$ are the functors $\mathrm{Ext}^n_A(M, -)$.

2. The left derived functors of $M \otimes_A (-)$ are the functors $\mathrm{Tor}^A_n(M, -)$.

3. Left exact contravariant additive functors may be derived on the right using projective resolutions instead of injective resolutions. We shall see that the right derived functors of $\mathrm{Hom}_A(-, M)$ are just the functors $\mathrm{Ext}^n_A(-, M)$.

Definitions: A **bicomplex** $(K^{\bullet\bullet}, \mathrm{d}_1, \mathrm{d}_2)$ is a family of A-modules $\{K^{p,q}\}_{p,q \in \mathbb{Z}}$ with morphisms of A-modules $\mathrm{d}_1^{pq} \colon K^{p,q} \to K^{p+1,q}$, $\mathrm{d}_2^{pq} \colon K^{p,q} \to K^{p,q+1}$ such that the following diagram is commutative, and all the rows and columns are complexes:

$$\begin{array}{ccccccc}
 & & \uparrow \mathrm{d}_2 & & \uparrow \mathrm{d}_2 & & \\
\ldots & \xrightarrow{\mathrm{d}_1} & K^{p,q+1} & \xrightarrow{\mathrm{d}_1} & K^{p+1,q+1} & \xrightarrow{\mathrm{d}_1} & \ldots \\
 & & \uparrow \mathrm{d}_2 & & \uparrow \mathrm{d}_2 & & \\
\ldots & \xrightarrow{\mathrm{d}_1} & K^{p,q} & \xrightarrow{\mathrm{d}_1} & K^{p+1,q} & \xrightarrow{\mathrm{d}_1} & \ldots \\
 & & \uparrow \mathrm{d}_2 & & \uparrow \mathrm{d}_2 & &
\end{array}$$

A **morphism** of bicomplexes $f \colon K^{\bullet\bullet} \to L^{\bullet\bullet}$ is a family of morphisms $f^{pq} \colon K^{pq} \to L^{pq}$ commuting with the differentials, $f\mathrm{d}_1 = \mathrm{d}_1 f$, $f\mathrm{d}_2 = \mathrm{d}_2 f$.

We always see a bicomplex $K^{\bullet\bullet}$ as a complex (we introduce the sign so that $\mathrm{d} \circ \mathrm{d} = 0$)

$$(K^{\bullet\bullet})^n = \bigoplus_{p+q=n} K^{p,q}, \ \mathrm{d}(m_{pq}) = \mathrm{d}_1 m_{pq} + (-1)^p \mathrm{d}_2(m_{pq})$$

and we say that $K^{\bullet\bullet}$ has **bounded diagonals** if all the direct sums $\oplus_{p+q=n} K^{p,q}$ are finite.

A complex $K^\bullet$ is viewed as a bicomplex with $K^{0n} = K^n$ and $K^{pn} = 0$ when $p \neq 0$.

If $r \in \mathbb{Z}$, then $K^\bullet[r]$ denotes the complex $K^\bullet[r]^n = K^{n+r}$ with the differential $(-1)^r \mathrm{d}$, so that $H^n(K^\bullet[r]) = H^{n+r}(K^\bullet)$.

A morphism of complexes $f \colon K^\bullet \to L^\bullet$ may be viewed as a bicomplex with K^n at degree $(-1, n)$, L^n at degree $(0, n)$, and $\mathrm{d}_1 = f$. The **cone** $\mathrm{Cone}^\bullet f$ of f is the associated simple complex

$$\mathrm{Cone}^n f = K^{n+1} \oplus L^n, \ \mathrm{d}(a, b) = (-\mathrm{d}a, f(a) + \mathrm{d}b).$$

We have an exact sequence of complexes

$$0 \longrightarrow L^\bullet \longrightarrow \text{Cone}^\bullet f \longrightarrow K^\bullet[1] \longrightarrow 0$$

and the connecting induced on cohomology is just f. Hence, *f is a quasi-isomorphism if and only if* $\text{Cone}^\bullet f$ *is an acyclic complex*, $H^n(\text{Cone}^\bullet f) = 0$.

Lemma: *Let $K^{\bullet\bullet}$ be a bicomplex with bounded diagonals. If the columns $K^{p\bullet}$ (resp. the rows $K^{\bullet q}$) are exact sequences, then $H^n(K^{\bullet\bullet}) = 0$, $\forall n \in \mathbb{Z}$.*

Proof: If $[m] \in H^n(K^{\bullet\bullet})$ and $m = m_{p,q} + m_{p-1,q+1} + m_{p-2,q+2} + \ldots$, where $m_{p,q} \neq 0$, then the cycle condition $\mathrm{d}m = 0$ implies that $\mathrm{d}_1 m_{p,q} = 0$.

Hence $m_{p,q} = \mathrm{d}_1 n_{p-1,q}$, with $n_{p-1,q} \in K^{p-1,q}$, since the columns are exact, and $[m] = [m']$, where $m' = m - (-1)^p \mathrm{d}n_{p,q-1}$ has lower height than m in the bicomplex.

Hence $[m]$ may be represented by elements of arbitrarily low height.

The diagonals being bounded, we conclude that $[m] = 0$.

Bicomplex Theorem: *Let $f\colon K^{\bullet\bullet} \to L^{\bullet\bullet}$ be a morphism of bicomplexes with bounded diagonals. If $f\colon K^{p\bullet} \xrightarrow{\sim} L^{p\bullet}$ (resp. $f\colon K^{\bullet q} \xrightarrow{\sim} L^{\bullet q}$) is a quasi-isomorphism for all $p \in \mathbb{Z}$ (resp. all $q \in \mathbb{Z}$), then $f\colon K^{\bullet\bullet} \xrightarrow{\sim} L^{\bullet\bullet}$ is a quasi-isomorphism.*

Proof: By hypothesis the cone $C^{p\bullet}$ of $f\colon K^{p\bullet} \to L^{p\bullet}$ is an acyclic complex; hence so is the complex associated to the bicomplex $C^{\bullet\bullet}$, which is the cone of $f\colon K^{\bullet\bullet} \xrightarrow{\sim} L^{\bullet\bullet}$, so that $f\colon K^{\bullet\bullet} \xrightarrow{\sim} L^{\bullet\bullet}$ is a quasi-isomorphism.

Theorem: *The group* $\text{Tor}_n^A(M,N)$ *may be computed with any projective resolution $P_\bullet \to M \to 0$ or any projective resolution $P'_\bullet \to N \to 0$,*

$$\text{Tor}_n^A(M,N) = H_n(P_\bullet \otimes_A N) = H_n(M \otimes_A P'_\bullet).$$

Proof: We have quasi-isomorphisms $P_n \otimes_A P'_\bullet \xrightarrow{\sim} P_n \otimes_A N$, $P_\bullet \otimes_A P'_n \xrightarrow{\sim} M \otimes_A P'_n$ because projective modules are flat. By the bicomplex theorem we conclude that

$$H_n(P_\bullet \otimes_A N) = H_n(P_\bullet \otimes_A P'_\bullet) = H_n(M \otimes_A P'_\bullet).$$

Theorem: *The group* $\text{Ext}_A^n(M,N)$ *may be computed with any projective resolution $P_\bullet \to M \to 0$ or any injective resolution $0 \to N \to I^\bullet$,*

$$\text{Ext}_A^n(M,N) = H^n\big[\text{Hom}_A(P_\bullet, N)\big] = H^n\big[\text{Hom}_A(M, I^\bullet)\big].$$

Proof: The natural maps $\text{Hom}_A(P_n, N) \to om_A(P_n, I^\bullet)$ and $\text{Hom}_A(M, I^n) \to \text{Hom}_A(P_\bullet, I^n)$ are quasi-isomorphisms. By the bicomplex theorem we conclude that

$$H^n\big[\text{Hom}_A(P_\bullet, N)\big] = H^n\big[\text{Hom}_A(P_\bullet, I^\bullet)\big] = H^n\big[\text{Hom}_A(M, I^\bullet)\big].$$

Definition: The derived functors R^nF (resp. L_nF) may be extended to bounded below (resp. above) complexes. By the bicomplex theorem $K^\bullet \to I^\bullet K^\bullet$ (resp. $P_\bullet K^\bullet \to K^\bullet$) is a quasi-isomorphism, and we put

$$\mathbf{R}F(K^\bullet) = F(I^\bullet K^\bullet) \quad , \quad \mathbf{R}^n F(K^\bullet) = H^n\left[F(I^\bullet K^\bullet)\right]$$
$$\mathbf{L}F(K^\bullet) = F(P_\bullet K^\bullet) \quad , \quad \mathbf{L}_n F(K^\bullet) = H_n\left[F(P_\bullet K^\bullet)\right],$$

and we say that a complex $K^\bullet$ is injective, F-acyclic,...., when so are all the modules K^n.

We shall develop the theory for a left exact additive covariant functor F and bounded below complexes, the other cases being totally analogous (eventually using projective resolutions and bounded above complexes).

Theorem: *If $f : I^\bullet \xrightarrow{\sim} J^\bullet$ is a quasi-isomorphism of bounded below injective complexes, then $F(f) : F(I^\bullet) \xrightarrow{\sim} F(J^\bullet)$ is a quasi-isomorphism.*

Hence if $f\colon K^\bullet \xrightarrow{\sim} L^\bullet$ is a quasi-isomorphism of bounded below complexes, then so is $f\colon \mathbf{R}F(K^\bullet) \xrightarrow{\sim} \mathbf{R}F(L^\bullet)$, so that $f\colon \mathbf{R}^nF(K^\bullet) \xrightarrow{\sim} \mathbf{R}^nF(L^\bullet)$ is an isomorphism.

Proof: If $J^\bullet = 0$, then $I^\bullet = 0 \to I_n \to I_{n+1} \to \dots$ is an exact sequence of injective modules; hence it splits and $F(I^\bullet)$ is also an exact sequence. In the general case, $\mathrm{Cone}^\bullet f \xrightarrow{\sim} 0$; hence $\mathrm{Cone}^\bullet F(f) = F(\mathrm{Cone}^\bullet f) \xrightarrow{\sim} 0$, and $F(f)$ is a quasi-isomorphism.

Theorem: *If $A^\bullet$ is a bounded below F-acyclic complex, then $F(A^\bullet) \xrightarrow{\sim} \mathbf{R}F(A^\bullet)$.*

Proof: The sequence $0 \to F(A^q) \to F(I^\bullet A^q)$ is exact because A^q is F-acyclic, and the bicomplex theorem shows that $F(A^\bullet) \xrightarrow{\sim} F(I^\bullet A^\bullet) = \mathbf{R}F(A^\bullet)$.

Definition: The natural morphism $K^\bullet \to I^\bullet K^\bullet$ induces a morphism $F(K^\bullet) \to \mathbf{R}F(K^\bullet)$, hence morphisms $H^n\,[F(K^\bullet)] \to \mathbf{R}^nF(K^\bullet)$. So, any quasi-isomorphism $K^\bullet \xrightarrow{\sim} L^\bullet$ induces natural morphisms[1] $\mathrm{DR}\colon H^n\,[F(L^\bullet)] \to \mathbf{R}^nF(L^\bullet) = \mathbf{R}^nF(K^\bullet)$, and any commutative square

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\sim} & L^\bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t} \\ \bar{K}^\bullet & \xrightarrow{\sim} & \bar{L}^\bullet \end{array}$$

induces commutative squares

$$\begin{array}{ccc} H^n[F(L^\bullet)] & \xrightarrow{\mathrm{DR}} & \mathbf{R}^nF(K^\bullet) \\ {\scriptstyle F(t)}\downarrow & & \downarrow{\scriptstyle f} \\ H^n[F(\bar{L}^\bullet)] & \xrightarrow{\mathrm{DR}} & \mathbf{R}^nF(\bar{K}^\bullet) \end{array}$$

De Rham's Theorem: *Let $K^\bullet, A^\bullet$ be bounded below complexes. If $A^\bullet$ is F-acyclic, any quasi-isomorphism $K^\bullet \xrightarrow{\sim} A^\bullet$ induces isomorphisms $\mathrm{DR}\colon H^n[F(A^\bullet)] \to \mathbf{R}^nF(K^\bullet)$.*

Proof: We have quasi-isomorphisms $F(A^\bullet) \xrightarrow{\sim} \mathbf{R}F(A^\bullet) \xleftarrow{\sim} \mathbf{R}F(K^\bullet)$.

Derived Functors Exact Sequence: *Any exact sequence $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ of bounded below complexes induces a long exact sequence*

$$\dots \xrightarrow{\delta} \mathbf{R}^nF(K^\bullet) \longrightarrow \mathbf{R}^nF(L^\bullet) \longrightarrow \mathbf{R}^nF(M^\bullet) \xrightarrow{\delta} \mathbf{R}^{n+1}F(K^\bullet) \longrightarrow \dots$$

Proof: We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & M^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\bullet K^\bullet & \longrightarrow & I^\bullet L^\bullet & \longrightarrow & I^\bullet L^\bullet / I^\bullet K^\bullet & \longrightarrow & 0 \end{array}$$

[1]Coinciding up to a sign with the connecting iteration of p. 321; but, in the case of the quasi-isomorphism $M \to I^\bullet M$, the morphism $\mathrm{DR}\colon H^n\,[F(I^\bullet M)] \to R^nF(M)$ is the identity (p. 554).

where $I^\bullet L^\bullet / I^\bullet K^\bullet$ is an injective resolution of $M^\bullet$. We conclude considering the cohomology exact sequence induced by the exact sequence of complexes

$$0 \longrightarrow F(I^\bullet K^\bullet) \longrightarrow F(I^\bullet L^\bullet) \longrightarrow F(I^\bullet L^\bullet / I^\bullet K^\bullet) \longrightarrow 0.$$

Corollary: *Let $0 \to M \to A^0 \to A^1 \to \ldots \to A^{n-1} \to M_n \to 0$ be an exact sequence. If the modules A^i are F-acyclic, we have natural isomorphisms*

$$R^p F(M_n) = R^{p+n} F(M), \; p \geq 1.$$

Proof: Any F-acyclic resolution $0 \to M_n \to B^0 \to B^1 \ldots$ defines an F-acyclic resolution of M,

$$0 \longrightarrow M \longrightarrow A^0 \longrightarrow \ldots \longrightarrow A^{n-1} \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \ldots$$

and by De Rham's theorem we have isomorphisms $R^p F(M_n) = H^p[F(B^\bullet)] = R^{p+n} F(M)$, $p \geq 1$.

Corollary: *Any module M over a principal ideal domain A admits a projective resolution*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Proof: Quotients of injective (= divisible, p. 127) modules are injective; hence any module N admits an injective resolution $0 \to N \to I \to I/N \to 0$, and $\mathrm{Ext}^n(-, N) = 0$, $n \geq 2$.

Now, if $0 \to K \to L \to M \to 0$ is an exact sequence, where L is a free module, then

$$\mathrm{Ext}^1(K, N) = \mathrm{Ext}^2(M, N) = 0$$

for any module N. Hence the functor $\mathrm{Hom}_A(K, -)$ is exact and K is projective.

Existence of Injective Sheaves: Let $\mathcal{O}$ be a sheaf of rings on X. If we fix at any point $x \in X$ an $\mathcal{O}_x$-module M_x, the corresponding **Godement sheaf** $\mathcal{C}$ is the flasque sheaf

$$\mathcal{C}(U) = \prod_{x \in U} M_x,$$

and *for any $\mathcal{O}$-module $\mathcal{N}$ we have a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, \mathcal{C}) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{N}_x, M_x).$$

Hence, *$\mathcal{C}$ is an injective $\mathcal{O}$-module when M_x is an injective $\mathcal{O}_x$-module for any $x \in X$.*

In fact, M_x defines an $\mathcal{O}$-module with support in the closure of x,

$$M_x(U) = \begin{cases} M_x & x \in U \\ 0 & x \notin U \end{cases}$$

and the Godement sheaf $\mathcal{C}$ is the direct product sheaf, $\mathcal{C} = \prod_x M_x$.

Moreover, it is easy to check that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, M_x) = \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{N}_x, M_x)$, so that

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, \mathcal{C}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, \prod_{x \in X} M_x) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}}(\mathcal{N}, M_x) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{N}_x, M_x).$$

Now, if $\mathcal{M}$ is an $\mathcal{O}$-module, and for any stalk $\mathcal{M}_x$ we consider the injective $\mathcal{O}_x$-module $I^0 \mathcal{M}_x$, the corresponding Godement sheaf $\mathcal{I}^0 \mathcal{M}$ is an injective $\mathcal{O}$-module, and we have an injective morphism of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{I}^0 \mathcal{M}$.

Therefore, *any $\mathcal{O}$-module $\mathcal{M}$ admits an injective functorial resolution* $0 \to \mathcal{M} \to \mathcal{I}^\bullet \mathcal{M}$ preserving complexes.

Hence we also have the right derived functors of Γ, Γ_c, Γ_Y and the direct image f_*, and they coincide with the cohomology groups H^n, H^n_c, H^n_Y and the higher direct images R^nf_* defined with the Godement resolutions, since any injective sheaf $\mathcal{I}$ is flasque,

$$\mathcal{I}(X) = \text{Hom}(\mathbb{Z}_X, \mathcal{I}) \longrightarrow \text{Hom}(\mathbb{Z}_U, \mathcal{I}) = \mathcal{I}(U) \longrightarrow 0.$$

For any bounded below complex of abelian sheaves $\mathcal{K}^\bullet$ we have **hypercohomology** groups $\mathbf{H}^n(X, \mathcal{K}^\bullet)$, $\mathbf{H}^n_c(X, \mathcal{K}^\bullet)$, $\mathbf{H}^n_Y(X, \mathcal{K}^\bullet)$, and it is immediate to generalize to this case the forthcoming inverse image and cup product.

12.3 Inverse Image

Definition: If $f\colon Y \to X$ is a continuous map, the **inverse image** of a sheaf $\mathcal{F}$ on X is the following sheaf $f^{-1}\mathcal{F}$ on Y

$$(f^{-1}\mathcal{F})(V) = \text{Hom}_X(V, \mathcal{F}^{\text{et}}) = \text{Hom}_Y(V, \mathcal{F}^{\text{et}} \times_X Y).$$

The étalé space of $f^{-1}\mathcal{F}$ is $\mathcal{F}^{\text{et}} \times_X Y \to Y$, and the stalks are $(f^{-1}\mathcal{F})_y = \mathcal{F}_{f(x)}$; hence f^{-1} is an exact functor, it preserves inductive limits and $f^{-1}(\mathbb{Z}_U) = \mathbb{Z}_{f^{-1}U}$.

If $s \in \mathcal{F}(U)$, then $f^*s = s \circ f \in (f^{-1}\mathcal{F})(f^{-1}U)$, and we have an inverse image of sections

$$f^*\colon \mathcal{F}(U) \longrightarrow (f^{-1}\mathcal{F})(f^{-1}U).$$

Now, if $\mathcal{G}$ is a sheaf on Y, any morphism of sheaves $f^{-1}\mathcal{F} \to \mathcal{G}$ induces a morphisms $\mathcal{F}(U) \to (f^*\mathcal{F})(f^{-1}U) \to \mathcal{G}(f^{-1}U)$, so defining a morphism of sheaves $\mathcal{F} \to f_*\mathcal{G}$,

Adjunction Formula: $\text{Hom}(f^{-1}\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, f_*\mathcal{G})$.

Proof: Since $\mathcal{F}$ admits (p. 342) a presentation $\oplus_j \mathbb{Z}_{U_j} \to \oplus_i \mathbb{Z}_{U_i} \to \mathcal{F} \to 0$ and both functors are left exact, we may assume that $\mathcal{F} = \mathbb{Z}_U$. Now,

$$\text{Hom}(f^{-1}\mathbb{Z}_U, \mathcal{G}) = \text{Hom}(\mathbb{Z}_{f^{-1}U}, \mathcal{G}) = \mathcal{G}(f^{-1}U) = (f_*\mathcal{G})(U) = \text{Hom}(\mathbb{Z}_U, f_*\mathcal{G}).$$

Note: In the case of a morphism of ringed spaces $(f, \phi)\colon Y \to X$, the morphism of sheaves of rings $\phi\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ corresponds to a morphism of sheaves of rings $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$, and the inverse image of $\mathcal{O}_X$-modules is defined to be

$$f^*\mathcal{M} = (f^{-1}\mathcal{M}) \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y,$$

so that the adjunction formula $\text{Hom}_{\mathcal{O}_Y}(f^*\mathcal{M}, \mathcal{N}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N})$ also holds.

In this course topological spaces are assumed to be ringed with the constant sheaf $\mathbb{Z}$, and we put $f^*\mathcal{F}$ or $f^{-1}\mathcal{F}$ indistinctively (but on schemes $f^*\mathcal{M} \neq f^{-1}\mathcal{M}$).

Definition: The sequence $0 \to f^*\mathcal{F} \to f^*(C^\bullet\mathcal{F})$ is exact, since the functor f^* is exact. Hence the inverse image of sections induces an inverse image of cohomology classes (compatible with morphisms of sheaves in an obvious sense)

$$f^*\colon H^n(X, \mathcal{F}) = H^n\big[\Gamma(X, C^\bullet\mathcal{F})\big] \xrightarrow{f^*} H^n\big[\Gamma(Y, f^*C^\bullet\mathcal{F})\big] \xrightarrow{\text{DR}} H^n(Y, f^*\mathcal{F})$$

Lemma: *If $0 \to \mathcal{F} \to \mathcal{R}^\bullet$ is a resolution, the following square is commutative,*

$$\begin{array}{ccc} H^n[\Gamma(X, \mathcal{R}^\bullet)] & \xrightarrow{f^*} & H^n[\Gamma(Y, f^*\mathcal{R}^\bullet)] \\ \downarrow{\scriptstyle \text{DR}} & & \downarrow{\scriptstyle \text{DR}} \\ H^n(X, \mathcal{F}) & \xrightarrow{f^*} & H^n(Y, f^*\mathcal{F}) \end{array}$$

Proof: The following diagram is commutative (p. 349)

$$\begin{array}{ccccc} H^n[\Gamma(X,C^\bullet\mathcal{F})] & \xrightarrow{\sim} & H^n[\Gamma(X,C^\bullet\mathcal{R}^\bullet)] & \longleftarrow & H^n[\Gamma(X,\mathcal{R}^\bullet)] \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^n[\Gamma(Y,f^*C^\bullet\mathcal{F})] & \longrightarrow & H^n[\Gamma(Y,f^*C^\bullet\mathcal{R}^\bullet)] & \longleftarrow & H^n[\Gamma(Y,f^*\mathcal{R}^\bullet)] \end{array}$$

Corollary: *If $f\colon Y\to X$ is a smooth map, the morphism $f^*\colon H^p(X,\mathbb{R})\to H^p(Y,\mathbb{R})$ is defined by the inverse image of differential forms, $f^*[\,\omega_p\,]=[\,f^*\omega_p\,]$.*

Theorem: *If $Z\xrightarrow{g}Y\xrightarrow{f}X$ are continuous maps, $(fg)^*=g^*f^*$.*

Proof: Since $f^*(C^\bullet\mathcal{F})$ is a resolution of $f^*\mathcal{F}$, the following diagram commutes

$$\begin{array}{ccccc} H^n[\Gamma(X,C^\bullet\mathcal{F})] & \xrightarrow{f^*} & H^n[\Gamma(Y,f^*C^\bullet\mathcal{F})] & \xrightarrow{g^*} & H^n[\Gamma(Z,(fg)^*C^\bullet\mathcal{F})] \\ \| & & \downarrow{\scriptstyle \mathrm{DR}} & & \downarrow{\scriptstyle \mathrm{DR}} \\ H^n(X,\mathcal{F}) & \xrightarrow{f^*} & H^n(Y,f^*\mathcal{F}) & \xrightarrow{g^*} & H^n(Z,g^*(f^*\mathcal{F})) \end{array}$$

Proposition: *The inverse image preserves the connecting, $f^*(\delta c_n)=\delta(f^*c_n)$.*

Proof: If $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ is an exact sequence of sheaves, just take cohomology in the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X,C^\bullet\mathcal{F}') & \longrightarrow & \Gamma(X,C^\bullet\mathcal{F}) & \longrightarrow & \Gamma(X,C^\bullet\mathcal{F}'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \\ 0 & \longrightarrow & \Gamma(Y,C^\bullet(f^*C^\bullet\mathcal{F}')) & \longrightarrow & \Gamma(Y,C^\bullet(f^*C^\bullet\mathcal{F})) & \longrightarrow & \Gamma(Y,C^\bullet(f^*C^\bullet\mathcal{F}'')) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Gamma(Y,C^\bullet f^*\mathcal{F}') & \longrightarrow & \Gamma(Y,C^\bullet f^*\mathcal{F}) & \longrightarrow & \Gamma(Y,C^\bullet f^*\mathcal{F}'') & \longrightarrow & 0 \end{array}$$

Lemma: *If a sheaf of rings $\mathcal{O}$ on X admits partitions of unity, then the restriction $\mathcal{O}|_Y$ to any closed set Y also admits partitions of unity.*

Proof: If $\{V_i\}$ is an open cover of Y, and $\{f_0,\,f_i\}$ is a partition of unity of $\mathcal{O}$ subordinated to the cover $\{U_0=X-Y,\,U_i\}$, where $V_i=U_i\cap Y$, then $\{f_i|_Y\}$ is a partition of unity of $\mathcal{O}|_Y$ subordinated to $\{V_i\}$.

Cohomology of the Fibre: *Let $f\colon Y\to X$ be a proper morphism between σ-compact spaces. If $\mathcal{F}$ is a sheaf on Y, and $x\in X$, we have natural isomorphisms*

$$(R^nf_*\mathcal{F})_x=H^n(f^{-1}x,\mathcal{F}).$$

Proof: Put $\mathcal{C}=C^p\mathcal{F}$. The restriction of sections defines an isomorphism

$$(f_*\mathcal{C})_x=\varinjlim_{x\in U}\Gamma(f^{-1}U,\mathcal{C})\xrightarrow{\sim}\Gamma(f^{-1}x,\mathcal{C}).$$

It is injective: if the support of a section $s\in\Gamma(f^{-1}U,\mathcal{C})$ does not intersect the (compact) fibre of x, then $f(\mathrm{supp}\ s)$ is closed in U, and s vanishes on $f^{-1}(U-f(\mathrm{supp}\,s))$.

It is surjective: if $\bar{s}\in\Gamma(f^{-1}x,\mathcal{C})$, then for any point $y\in f^{-1}x$ there is a neighborhood U_i of compact closure and $s_i\in\Gamma(U_i,\mathcal{C})$ extending $\bar{s}$, and a finite number $U_1,\ldots,U_r$ cover the fibre $f^{-1}(x)$. If $f_0, f_1, \ldots, f_r$ is a partition of unity of $C^0\mathbb{Z}_Y$ subordinated to the open cover $\{U_0 = Y - f^{-1}(x), U_1, \ldots, U_r\}$, then $f_1 s_1 + \ldots + f_r s_r \in \Gamma(Y, \mathcal{C})$ extends $\bar{s}$.

Now, by the lemma, $(C^\bullet \mathcal{F})|_{f^{-1}x}$ is an acyclic resolution of $\mathcal{F}|_{f^{-1}x}$, and

$$(R^n f_* \mathcal{F})_x = H^n\big[(f_* C^\bullet \mathcal{F})_x\big] = H^n\big[\Gamma(f^{-1}x, C^\bullet \mathcal{F})\big] = H^n(f^{-1}x, \mathcal{F}).$$

Base Change Theorem: *Let us consider a fibred product of continuous maps between σ-compact spaces,*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{\ \bar{\phi}\ } & X \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\ \phi\ } & S \end{array}$$

If f is proper, then we have isomorphisms $\phi^*(R^n f_* \mathcal{F}) = R^n \bar{f}_*(\bar{\phi}^* \mathcal{F})$.

Proof: The inverse image $\bar{\phi}^*$ induces morphisms

$$H^n(f^{-1}U, \mathcal{F}) \longrightarrow H^n(\bar{\phi}^{-1} f^{-1} U, \bar{\phi}^* \mathcal{F}) = H^n(\bar{f}^{-1} \phi^{-1} U, \bar{\phi}^* \mathcal{F}) \longrightarrow (R^n \bar{f}_* \bar{\phi}^* \mathcal{F})(\phi^{-1} U)$$

defining a morphism of sheaves $R^n f_* \mathcal{F} \to \phi_*(R^n \bar{f}_* \bar{\phi}^* \mathcal{F})$, and the corresponding morphism $\phi^*(R^n f_* \mathcal{F}) \to R^n \bar{f}_*(\bar{\phi}^* \mathcal{F})$ is an isomorphism. In fact, if $s = f(t)$, then

$$(\phi^* R^n f_* \mathcal{F})_t = (R^n f_* \mathcal{F})_s = H^n(f^{-1}s, \mathcal{F}) \xrightarrow{\ \bar{\phi}^*\ } H^n(\bar{f}^{-1}t, \bar{\phi}^* \mathcal{F}) = (R^n \bar{f}_* \bar{\phi}^* \mathcal{F})_t$$

is an isomorphism because $\bar{\phi}\colon \bar{f}^{-1}(t) \to f^{-1}(s)$ is a homeomorphism.

Corollary: $\pi^*\colon H^n(X, G) \xrightarrow{\ \sim\ } H^n(X \times [0,1], G)$, *when X is σ-compact.*

Proof: Let $\mathcal{C}^\bullet$ be the Godement resolution of the constant sheaf G on $X \times [0,1]$.

The cohomology of the fibre shows that $\pi_* \mathcal{C}^\bullet$ is a (flasque) resolution of $\pi_* G = G$; hence it calculates the inverse image. We conclude since we have a canonical morphism $\pi^* \pi_* \mathcal{C}^\bullet \to \mathcal{C}^\bullet$ and the following composition is the identity,

$$\Gamma(X, \pi_* \mathcal{C}^\bullet) \xrightarrow{\ \pi^*\ } \Gamma(X \times [0,1], \pi^* \pi_* \mathcal{C}^\bullet) \longrightarrow \Gamma(X \times [0,1], \mathcal{C}^\bullet).$$

Corollary: *If two continuous maps $\phi, \psi\colon X \to Y$ between σ-compact spaces are homotopic, then $\phi^* = \psi^*\colon H^n(Y, G) \to H^n(X, G)$. Therefore, if $\phi\colon X \to Y$ is a homotopical equivalence, then $\phi^*\colon H^n(Y, G) \to H^n(X, G)$ is an isomorphism.*

Proof: If $H\colon X \times [0,1] \to Y$ is a continuous map and $\phi = H i_0$, $\psi = H i_1$, where $i_t(x) = (x, t)$, then $\phi^* = i_0^* H^*$, $\psi^* = i_1^* H^*$. Now $i_0^* = i_1^*$, both being the inverse of π^*. q.e.d.

1. **Cohomology of Affine Spaces:** $H^p(\mathbb{R}^n, G) = 0$, $p \geq 1$.

2. *If X is a n-dimensional topological manifold,* $H^p_x(X, G) = \begin{cases} G & p = n \\ 0 & p \neq n \end{cases}$

   Excision and the local cohomology sequence, since $\mathbb{R}^n - x$ is homotopic to S_{n-1}.

3. *The boundary of a* ***manifold with boundary*** (any point has a neighborhood homeomorphic to an open set in $[0, \infty) \times \mathbb{R}^{n-1}$) *is a topological concept.*

   If x is in the boundary of $P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$, then P and $P - x$ are contractible, and $H^p_x(P, G) = 0$, $p \geq 0$, by the local cohomology sequence.

4. *If the spheres S_n and S_m are homotopically equivalent, then $n = m$.*

5. *The inclusion $i\colon S_{n-1} \hookrightarrow B_n$ of the sphere in the ball has no continuous retract r.*

   Otherwise r^* would be a section of $i^*\colon 0 = H^{n-1}(B_n, \mathbb{Z}) \to H^{n-1}(S_{n-1}, \mathbb{Z}) = \mathbb{Z}$.

6. *Any continuous map $\phi\colon B_n \to B_n$ has some fixed point* (**Brouwer's Theorem**).

   Otherwise we have a continuous retract $r\colon B_n \to S_{n-1}$, where $r(x)$ is the intersection point of S_{n-1} and the half-line with origin at $\phi(x)$ passing through x.

7. *Any continuous tangent vector field to S_n vanishes at a a point when n is even.*

   If a continuous vector field D on B_n does not vanish at any point, then it points outside at some point $x \in S_{n-1}$.

   We may repeat the proofs of the differentiable case (p. 275).

Finiteness Theorem: *Let A be a noetherian ring and $\mathcal{F}$ a sheaf of A-modules on a separated locally compact space X. If any point has a base of neighborhoods U in X such that the modules $H^p(U, \mathcal{F})$ are finitely generated, then for any compact set K contained in the interior of a compact set L we have that the images of the restriction morphisms $H^p(L, \mathcal{F}) \to H^p(K, \mathcal{F})$ also are finitely generated A-modules.*

Proof: By induction on p. We fix L and we consider the family of all compact sets K with some compact neighborhood $\bar{K}$ contained in the interior of L such that $H^p(L, \mathcal{F}) \to H^p(\bar{K}, \mathcal{F})$ has finitely generated image (so that any compact set contained in K is also in the family).

By hypothesis any point in the interior of L has a compact neighborhood in the family; hence we only have to show that the family is stable under finite unions.

If K_1, K_2 are in the family, by definition K_i has a compact neighborhood $\bar{K}_i$ contained in the interior of L such that the images of $H^p(L, \mathcal{F}) \to H^p(\bar{K}_i, \mathcal{F})$are finitely generated.

We may choose a compact neighborhood K'_i of K_i contained in the interior of $\bar{K}_i$.

We have a commutative diagram

$$\begin{array}{ccccc}
 & & H^p(L,\mathcal{F}) & \longrightarrow & H^p(L,\mathcal{F}) \oplus H^p(L,\mathcal{F}) \\
 & & \downarrow & & \downarrow \rho \\
H^{p-1}(\bar{K}_1 \cap \bar{K}_2, \mathcal{F}) & \longrightarrow & H^p(\bar{K}_1 \cup \bar{K}_2, \mathcal{F}) & \longrightarrow & H^p(\bar{K}_1,\mathcal{F}) \oplus H^p(\bar{K}_2,\mathcal{F}) \\
\downarrow \gamma & & \downarrow & & \\
H^{p-1}(K'_1 \cap K'_2, \mathcal{F}) & \longrightarrow & H^p(K'_1 \cup K'_2, \mathcal{F}) & &
\end{array}$$

where the central row is exact by Mayer-Vietoris, and the image of ρ is finitely generated, and the image of γ by induction. Now it is easy to see that the image of $H^p(L, \mathcal{F}) \to H^p(K'_1 \cup K'_2, \mathcal{F})$ is finitely generated, so that $K_1 \cup K_2$ is in the considered family.

Corollary: *If $\mathcal{F}$ is a locally constant sheaf of stalk A on a compact manifold with boundary X, then the A-modules $H^p(X, \mathcal{F})$ are finitely generated. In particular so are $H^p(X, A)$.*

Proof: The A-modules $H^p(\mathbb{R}^d, A)$ and $H^p(\mathbb{R}^d, A)$ are finitely generated.

Hence, any point $x \in X$ has a base of neighborhoods U such that the A-modules $H^p(U, \mathcal{F})$ are finitely generated. Now just put $K = L = X$ in the above theorem.

12.4 Cup Product

Definition: If $(K^\bullet, \mathrm{d})$, $(L^\bullet, \mathrm{d})$ are complexes of A-modules (or sheaves of A-modules), we have a bicomplex $K^\bullet \otimes_A L^\bullet = \{K^p \otimes_A L^q\}$ with the differentials $\mathrm{d}_1 = \mathrm{d} \otimes 1$, $\mathrm{d}_2 = 1 \otimes \mathrm{d}$, and the natural isomorphisms $K^p \otimes_A L^q \xrightarrow{\sim} L^q \otimes_A K^p$, with a factor $(-1)^{pq}$, define an isomorphism of complexes $K^\bullet \otimes_A L^\bullet \xrightarrow{\sim} L^\bullet \otimes_A K^\bullet$,

$$(\mathrm{d} \otimes 1 + (-1)^q 1 \otimes \mathrm{d})((-1)^{pq} b_q \otimes a_p) = (-1)^{(p+1)q} b_q \otimes \mathrm{d}a_p + (-1)^{p(q+1)}(-1)^p \mathrm{d}b_q \otimes a_p.$$

The tensor product of two cycles is a cycle of $K^\bullet \otimes_A L^\bullet$, so that we have morphisms

$$H^p(K^\bullet) \otimes_A H^q(L^\bullet) \xrightarrow{\otimes} H^{p+q}(K^\bullet \otimes_A L^\bullet).$$

Definition: If $\mathcal{M}$ is a sheaf of A-modules on a topological space X, the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow C^0\mathcal{M} \longrightarrow \mathcal{M}_1 \longrightarrow 0$$

splits on stalks, a retract $(C^0\mathcal{M})_x \to \mathcal{M}_x$ transforming any germ of discontinuous section into the value at $x \in X$. Hence, for any sheaf of A-modules $\mathcal{N}$, the sequence

$$0 \longrightarrow \mathcal{M} \otimes_A \mathcal{N} \longrightarrow C^0\mathcal{M} \otimes_A \mathcal{N} \longrightarrow C^1\mathcal{M} \otimes_A \mathcal{N} \longrightarrow C^2\mathcal{M} \otimes_A \mathcal{N} \longrightarrow \dots$$

is exact, $\mathcal{M} \otimes_A \mathcal{N} \xrightarrow{\sim} C^\bullet\mathcal{M} \otimes_A \mathcal{N}$, so that hence $\mathcal{M} \otimes_A C^q\mathcal{N} \xrightarrow{\sim} C^\bullet\mathcal{M} \otimes_A C^q\mathcal{N}$.

By the bicomplex theorem $\mathcal{M} \otimes_A C^\bullet\mathcal{N} \xrightarrow{\sim} C^\bullet\mathcal{M} \otimes_A C^\bullet\mathcal{N}$; hence $\mathcal{M} \otimes_A \mathcal{N} \xrightarrow{\sim} C^\bullet\mathcal{M} \otimes_A C^\bullet\mathcal{N}$, and we get a **cup product** $\cup$: $H^p(X, \mathcal{M}) \otimes_A H^q(X, \mathcal{N}) \to H^{p+q}(X, \mathcal{M} \otimes_A \mathcal{N})$,

$$H^p\big[\Gamma(X, C^\bullet\mathcal{M})\big] \otimes H^q\big[\Gamma(X, C^\bullet\mathcal{N})\big] \xrightarrow{\otimes} H^{p+q}\big[\Gamma(X, C^\bullet\mathcal{M} \otimes C^\bullet\mathcal{N})\big] \xrightarrow{\mathrm{DR}} H^{p+q}(X, \mathcal{M} \otimes \mathcal{N}).$$

Proposition: *If $0 \to \mathcal{M} \to \mathcal{R}^\bullet$ and $0 \to \mathcal{N} \to \mathcal{S}^\bullet$ are resolutions such that $\mathcal{R}^\bullet \otimes_A \mathcal{S}^\bullet$ is a resolution of $\mathcal{M} \otimes_A \mathcal{N}$, then the following square is commutative*

$$\begin{array}{ccc}
H^p[\Gamma(X, \mathcal{R}^\bullet)] \otimes_A H^q[\Gamma(X, \mathcal{S}^\bullet)] & \xrightarrow{\otimes} & H^{p+q}[\Gamma(X, \mathcal{R}^\bullet \otimes_A \mathcal{S}^\bullet)] \\
\downarrow \mathrm{DR} & & \downarrow \mathrm{DR} \\
H^p(X, \mathcal{M}) \otimes_A H^q(X, \mathcal{N}) & \xrightarrow{\cup} & H^{p+q}(X, \mathcal{M} \otimes_A \mathcal{N})
\end{array}$$

Proof: By the bicomplex theorem we have quasi-isomorphisms

$$\mathcal{M} \otimes_A \mathcal{N} \xrightarrow{\sim} \mathcal{R}^\bullet \otimes_A \mathcal{S}^\bullet \xrightarrow{\sim} C^\bullet\mathcal{R}^\bullet \otimes_A \mathcal{S}^\bullet \xrightarrow{\sim} C^\bullet\mathcal{R}^\bullet \otimes_A C^\bullet\mathcal{S}^\bullet$$

and the following commutative diagram let us conclude (p. 349)

$$\begin{array}{ccc}
H^p[\Gamma(X, C^\bullet\mathcal{M})] \otimes H^q[\Gamma(X, C^\bullet\mathcal{N})] & \longrightarrow & H^{p+q}[\Gamma(X, C^\bullet\mathcal{M} \otimes C^\bullet\mathcal{N})] \\
\| & & \downarrow \\
H^p[\Gamma(X, C^\bullet\mathcal{R}^\bullet) \otimes H^q[\Gamma(X, C^\bullet\mathcal{S}^\bullet)] & \longrightarrow & H^{p+q}[\Gamma(X, C^\bullet\mathcal{R}^\bullet \otimes C^\bullet\mathcal{S}^\bullet)] \\
\uparrow & & \uparrow \\
H^p[\Gamma(X, \mathcal{R}^\bullet)] \otimes H^q[\Gamma(X, \mathcal{S}^\bullet)] & \longrightarrow & H^{p+q}[\Gamma(X, \mathcal{R}^\bullet \otimes \mathcal{S}^\bullet)]
\end{array}$$

Corollary: $(c_p \cup c_q) \cup c_r = c_p \cup (c_q \cup c_r)$.
$c_p \cup c_q = (-1)^{pq} c_q \cup c_p$.
$f^*(c_p \cup c_q) = (f^*c_p) \cup (f^*c_q)$.

Corollary: *If X is a smooth manifold, the cup product of $H^\bullet_{DR}(X) = H^\bullet(X, \mathbb{R})$ is defined by the exterior product of differential forms, $[\omega_p] \cup [\omega_q] = [\omega_p \wedge \omega_q]$.*

12.4.1 Universal Coefficients and Künneth's Theorem

Now A will denote a principal ideal domain, so that torsion free A-modules are flat.

Definition: A sheaf of A-modules $\mathcal{M}$ is **flat** if so is any stalk $\mathcal{M}_x$, or equivalently any module $\mathcal{M}(U)$ since $\mathcal{M}(U)$ is a submodule of $\prod_{x\in U}\mathcal{M}_x$, and $\mathcal{M}_x = \varinjlim \mathcal{M}(U)$.

Lemma: *If $\mathcal{M}$ is a sheaf of A-modules on a σ-compact space X, and P is projective,*

$$H^n_c(X,\mathcal{M})\otimes_A P = H^n_c(X,\mathcal{M}\otimes_A P).$$

Proof: It holds when P is free (p. 346) and P is a direct summand of a free module.

Universal Coefficients Formula: *If $\mathcal{M}$ is a flat sheaf of A-modules on a σ-compact space X, for any A-module N we have exact sequences*

$$0 \longrightarrow H^n_c(X,\mathcal{M})\otimes_A N \longrightarrow H^n_c(X,\mathcal{M}\otimes_A N) \longrightarrow \mathrm{Tor}^A_1(H^{n+1}_c(X,\mathcal{M}),N) \longrightarrow 0,$$

and if moreover N is finitely generated, we have exact sequences

$$0 \longrightarrow H^n(X,\mathcal{M})\otimes_A N \longrightarrow H^n(X,\mathcal{M}\otimes_A N) \longrightarrow \mathrm{Tor}^A_1(H^{n+1}(X,\mathcal{M}),N) \longrightarrow 0$$

Proof: Let $0\to P_1\to P_0\to N\to 0$ be a projective resolution (p. 350). The kernel and cokernel of $\phi_n\colon H^n_c(X,\mathcal{M})\otimes_A P_1 \to H^n_c(X,\mathcal{M})\otimes_A P_0$ are $\mathrm{Tor}_1(H^n_c(X,\mathcal{M}),N)$, $H^n_c(X,\mathcal{M})\otimes_A N$.

Since $\mathcal{M}$ is flat, the sequence $0\to\mathcal{M}\otimes_A P_1\to\mathcal{M}\otimes_A P_0\to\mathcal{M}\otimes_A N\to 0$ is exact, and we obtain exact sequences

$$H^n_c(X,\mathcal{M})\otimes P_1 \xrightarrow{\phi} H^n_c(X,\mathcal{M})\otimes P_0 \to H^n_c(X,\mathcal{M}\otimes N) \to H^{n+1}_c(X,\mathcal{M})\otimes P_1 \xrightarrow{\phi} H^{n+1}_c(X,\mathcal{M})\otimes P_0$$
$$0 \longrightarrow \mathrm{Coker}\,\phi_n \longrightarrow H^n_c(X,\mathcal{M}\otimes_A N) \longrightarrow \mathrm{Ker}\,\phi_{n+1} \longrightarrow 0\,.$$

If N is finite, it admits a resolution $0\to A^m\to A^n\to N\to 0$. When $P=A^r$, the equality $H^n(X,\mathcal{M}\otimes_A P)=H^n(X,\mathcal{M})\otimes_A P$ is clear and we repeat the above proof.

Corollary: $0\longrightarrow H^n_c(X,\mathbb{Z})\otimes_{\mathbb{Z}} G \longrightarrow H^n_c(X,G)\longrightarrow \mathrm{Tor}^{\mathbb{Z}}_1(H^{n+1}_c(X,\mathbb{Z}),G)\longrightarrow 0$ *is exact.*

Projection Formula: *Let $f\colon X\to Y$ be a proper morphism of σ-compact spaces. If $\mathcal{M}$ is a flat sheaf of A-modules on X, for any sheaf of A-modules $\mathcal{N}$ on Y we have isomorphisms*

$$(R^n f_*\mathcal{M})\otimes_A\mathcal{N} = R^n f_*(\mathcal{M}\otimes_A f^*\mathcal{N}).$$

Proof: The morphism $Fdegree(R^n f_*\mathcal{M})\otimes_A\mathcal{N}\to R^n f_*(\mathcal{M}\otimes_A \phi^*\mathcal{N})$ given by the universal coefficients formula is an isomorphism according to the cohomology of the fibre,

$$\begin{aligned}((R^n f_*\mathcal{M})\otimes_A\mathcal{N})_y &= (R^n f_*\mathcal{M})_y\otimes_A\mathcal{N}_y = H^n(f^{-1}y,\mathcal{M})\otimes_A\mathcal{N}_y\\ &= H^n(f^{-1}y,\mathcal{M}\otimes_A\mathcal{N}_y) = (R^n f_*(\mathcal{M}\otimes_A f^*\mathcal{N}))_y.\end{aligned}$$

Lemma: *If $K^\bullet$, $L^\bullet$ are complexes of A-modules and $L^\bullet$ is flat, we have exact sequences*

$$0\to\bigoplus_{p+q=n} H^p(K^\bullet)\otimes_A H^q(L^\bullet)\to H^n(K^\bullet\otimes_A L^\bullet)\to\bigoplus_{p+q=n+1}\mathrm{Tor}^A_1(H^p(K^\bullet),H^q(L^\bullet))\to 0$$

Proof: Since $L^\bullet$ is flat, cycles Z^q and boundaries B^q are torsion free, and

$$0\longrightarrow B^q\longrightarrow Z^q\longrightarrow H^q(L^\bullet)\longrightarrow 0$$

is a flat resolution of $H^q(L^\bullet)$; hence the cokernel and kernel of $\phi_{pq}\colon H^p(K^\bullet)\otimes B^q \to H^p(K^\bullet)\otimes Z^q$ are $H^p(K^\bullet)\otimes H^q(L^\bullet)$ and $\mathrm{Tor}_1(H^p(K^\bullet),H^q(L^\bullet))$.

On the other hand, we have an exact sequence

$$0 \longrightarrow Z^\bullet \longrightarrow L^\bullet \xrightarrow{\ \mathrm{d}\ } B^\bullet[1] \longrightarrow 0$$

where the differentials of $Z^\bullet$ and $B^\bullet[1]$ are null, and the connecting

$$\delta\colon B^q = H^{q-1}(B^\bullet[1]) \longrightarrow H^q(Z^\bullet) = Z^q$$

is just the inclusion. Since the modules B^q are flat, we have an exact sequence

$$0 \longrightarrow K^\bullet \otimes_A Z^\bullet \longrightarrow K^\bullet \otimes_A L^\bullet \xrightarrow{1\otimes \mathrm{d}} K^\bullet \otimes_A (B^\bullet[1]) \longrightarrow 0$$

inducing exact sequences

$$\ldots \xrightarrow{\delta_n} H^n(K^\bullet\otimes Z^\bullet) \longrightarrow H^n(K^\bullet\otimes L^\bullet) \longrightarrow H^n(K^\bullet\otimes(B^\bullet[1])) \xrightarrow{\delta_{n+1}} H^{n+1}(K^\bullet\otimes Z^\bullet) \longrightarrow \ldots$$

$$0 \longrightarrow \mathrm{Coker}\,\delta_n \longrightarrow H^n(K^\bullet\otimes_A L^\bullet) \longrightarrow \mathrm{Ker}\,\delta_{n+1} \longrightarrow 0$$

Moreover, since Z^q and B^q are flat modules, we have that

$$H^n(K^\bullet\otimes_A Z^\bullet) = \bigoplus_{p+q=n} H^p(K^\bullet\otimes_A Z^q) = \bigoplus_{p+q=n} H^p(K^\bullet)\otimes_A Z^q$$

$$H^{n-1}(K^\bullet\otimes_A B^\bullet[1]) = \bigoplus_{p+q=n-1} H^p(K^\bullet\otimes_A B^\bullet[1]^q) = \bigoplus_{p+q=n} H^p(K^\bullet)\otimes_A B^q$$

so that $\delta_n = \bigoplus_{p+q=n}\phi_{pq}$, and we conclude,

$$\mathrm{Coker}\,\delta_n = \bigoplus_{p+q=n}\mathrm{Coker}\,\phi_{pq} = \bigoplus_{p+q=n} H^p(K^\bullet)\otimes_A H^q(L^\bullet),$$

$$\mathrm{Ker}\,\delta_{n+1} = \bigoplus_{p+q=n+1}\mathrm{Ker}\,\phi_{pq} = \bigoplus_{p+q=n+1}\mathrm{Tor}_1^A(H^p(K^\bullet),H^q(L^\bullet)).$$

Note: On σ-compact spaces we also consider the **direct image with proper supports**

$$(f_!\mathcal{F})(U) = \{s\in\mathcal{F}(f^{-1}U)\colon \text{ the support of } s \text{ is proper over } U\ \}$$

so that $\Gamma_c(X,f_!\mathcal{F}) = \Gamma_c(Y,\mathcal{F})$, and we put $R^nf_!\mathcal{F} = \mathcal{H}^n\big[f_!(C^\bullet\mathcal{F})\big]$.

If $R^nf_!\mathcal{F}=0$, $n\geq 1$, then $f_!(C^\bullet\mathcal{F})$ is a Γ_c-acyclic resolution of $f_!\mathcal{F}$, because the sheaves $f_!(C^p\mathcal{F})$ are $C^0\mathbb{Z}$-modules, so that $H^n_c(Y,\mathcal{F}) = H^n_c(X,f_!\mathcal{F})$.

The Cohomology of the Fibre, Base Change and Projection formulas, and their proofs, remain valid for (non proper) continuous maps between σ-compact spaces if we replace the direct images R^nf_* by $R^nf_!$, and the cohomology groups H^n by H^n_c.

If a sheaf $\mathcal{M}$ is flat, so is $C^0\mathcal{M}$ since $(C^0\mathcal{M})(U) = \prod_{x\in U}\mathcal{M}_x$ is torsion free.

Since the exact sequence $0\to\mathcal{M}\to C^0\mathcal{M}\to\mathcal{M}_1\to 0$ splits on stalks, the sheaf $\mathcal{M}_1$ is flat, and we conclude that the sheaves $C^p\mathcal{M}$ are also flat.

Künneth's Theorem: *Let X,Y be σ-compact spaces and $p_1\colon X\times Y\to X$, $p_2\colon X\times Y\to Y$ the natural maps. If $\mathcal{M},\mathcal{N}$ are flat sheaves of A-modules on X,Y, we have exact sequences*

$$0\to\bigoplus_{p+q=n} H^p_c(X,\mathcal{M})\otimes H^q_c(Y,\mathcal{N}) \to H^n_c(X\times Y, p_1^*\mathcal{M}\otimes p_2^*\mathcal{N}) \to \bigoplus_{p+q=n+1}\mathrm{Tor}_1(H^p_c(X,\mathcal{M}),H^q_c(Y,\mathcal{N}))\to 0$$

Proof: According to the former lemma, it is enough to prove the following two statements.

1. $p_1^* \otimes p_2^* \colon \Gamma_c(X, C^\bullet\mathcal{M}) \otimes_A \Gamma_c(Y, C^\bullet\mathcal{N}) = \Gamma_c(X \times Y, p_1^* C^\bullet\mathcal{M} \otimes_A p_2^* C^\bullet\mathcal{N})$.

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q_1} \\ Y & \xrightarrow{q_2} & \bullet \end{array}$$

$$\begin{aligned} \Gamma_c(X \times Y, p_1^* C^i\mathcal{M} \otimes p_2^* C^j\mathcal{N}) &= q_{1!}p_{1!}(p_1^*\mathcal{C}^i \otimes p_2^*\mathcal{C}^j) = q_{1!}(\mathcal{C}^i \otimes p_{1!}(p_2^*\mathcal{C}^j)) \\ &= q_{1!}(\mathcal{C}^i \otimes q_1^* q_{2!}(\mathcal{C}^j)) = (q_{1!}\mathcal{C}^i) \otimes (q_{2!}\mathcal{C}^j) \\ &= \Gamma_c(X, C^i\mathcal{M}) \otimes \Gamma_c(Y, C^j\mathcal{N}). \end{aligned}$$

2. *$p_1^* C^\bullet\mathcal{M} \otimes_A p_2^* C^\bullet\mathcal{N}$ is a Γ_c-acyclic resolution of $p_1^*\mathcal{M} \otimes_A p_2^*\mathcal{N}$.*

$0 \to p_1^*\mathcal{M} \to p_1^* C^\bullet\mathcal{M}$ and $0 \to p_2^*\mathcal{N} \to p_2^* C^\bullet\mathcal{N}$ split on stalks; hence $p_1^* C^\bullet\mathcal{M} \otimes p_2^* C^\bullet\mathcal{N}$ is a resolution of $p_1^*\mathcal{M} \otimes p_2^*\mathcal{N}$, and the sheaves $p_1^* C^i\mathcal{M} \otimes p_2^* C^j\mathcal{N}$ are Γ_c-acyclic,

$$R^n p_{1!}(p_1^*\mathcal{C}^i \otimes p_2^*\mathcal{C}^j) = \mathcal{C}^i \otimes R^n p_{1!}(p_2^*\mathcal{C}^j) = \mathcal{C}^i \otimes q_1^*(R^n q_{2!}\mathcal{C}^j) = 0, \ n \geq 1,$$

and the sheaves $p_{1!}(p_1^*\mathcal{C}^i \otimes p_2^*\mathcal{C}^j) = \mathcal{C}^i \otimes q_1^*(q_{2!}\mathcal{C}^j)$ areΓ_c-acyclic,

$$H_c^n(X, \mathcal{C}^i \otimes q_1^*(q_{2!}\mathcal{C}^j)) = R^n q_{1!}(\mathcal{C}^i \otimes q_1^*(q_{2!}\mathcal{C}^j)) = (R^n q_{1!}\mathcal{C}^i) \otimes (q_{2!}\mathcal{C}^j) = 0, \ n \geq 1.$$

Example: *No continuous map $\mu \colon S_n \times S_n \to S_n$ defines a group structure when n is even.*

In fact, the cohomology ring of the sphere is $H^\bullet(S_n, \mathbb{Z}) = \mathbb{Z}[t_n]/(t_n^2)$, so that

$$H^\bullet(S_n \times S_m, \mathbb{Z}) = H^\bullet(S_n, \mathbb{Z}) \otimes_{\mathbb{Z}} H^\bullet(S_m, \mathbb{Z}) = \mathbb{Z}[x_n, y_m]/(x_n^2, y_m^2),$$

where $p_1^*(t_n) = t_n \otimes 1 = x_n$, $p_2^*(t_m) = 1 \otimes t_m = y_m$. Put $\mu^*(t_n) = ax_n + by_n$.

If e is the identity of S_n, the continuous map $j \colon S_n \to S_n \times S_n$, $j(p) = (p, e)$, fulfills that μj and $p_1 j$ are the identity, and that $p_2 j$ is constant,

$$t_n = j^*\mu^*(t_n) = j^*(ax_n + by_n) = j^*(ap_1^*(t_n) + bp_2^*(t_n)) = at_n + 0,$$

and we obtain $a = 1$. Analogously $b = 1$, and we obtain a contradiction if n is even

$$0 = \mu^*(t_n^2) = (\mu^* t_n)^2 = (x_n + y_n)^2 = x_n y_n + y_n x_n = 2x_n y_n.$$

12.5 Locally Trivial Structures

Lemma: *If G is an abelian group, the sheaf of automorphisms $\mathcal{G}$ of the trivial principal covering $X \times G \to X$ is the constant sheaf G.*

Proof: Any element $h \in G$ defines an automorphism $\tau_h(y) = hy$ since G is abelian,

$$\tau_h(gy) = hgy = ghy = g\tau_h(y),$$

and this injective morphism of sheaves $G \to \mathcal{G}$ is surjective.

In fact, if $\tau \colon U \times G \to U \times G$ is an automorphism and $\tau(x, 1) = (x, h(x))$, then $\tau = \tau_h$,

$$\tau_h(x, g) = (x, h(x)g) = g(x, h(x)) = g\tau(x, 1) = \tau(x, g).$$

Theorem: $H^1(X,G) = \begin{bmatrix}\text{Principal coverings of } X \\ \text{of abelian group } G\end{bmatrix}$ *(up to isomorphisms)*.

Proof: Let us fix an open cover $\mathcal{R} = \{U_i\}_{i\in I}$ of X.

If a principal covering $P \to X$ is trivial on it, $\phi_i \colon P_{U_i} \xrightarrow{\sim} U_i \times G$, we have automorphisms $g_{ij} = \phi_i\phi_j^{-1} \colon U_{ij} \times G \xrightarrow{\sim} U_{ij} \times G$, $U_{ij} = U_i \cap U_j$, and on any open set $U_{ijk} = U_i \cap U_j \cap U_k$,

$$g_{ij}g_{jk} = g_{ik}.$$

Families $\{g_{ij} \in G(U_{ij})\}_{i,j\in I}$ satisfying such condition are named construction data, since they let us reconstruct the principal covering as a quotient

$$P \simeq \left[\textstyle\coprod_{i\in I} U_i \times G\right] / \equiv$$

by the following equivalence relation: $(x,g_i) \equiv (x,g_j)$ when $x \in U_{ij}$, $g_i = g_{ij}(x)g_j$.

The construction datum depends on the trivialization ϕ_i. If we fix other isomorphisms $\bar\phi_i \colon P_{U_i} \xrightarrow{\sim} G \times U_i$, we obtain another construction datum $\{\bar g_{ij}\}$.

Now, $g_i = \bar\phi_i\phi_i^{-1} \colon U_i \times G \xrightarrow{\sim} U_i \times G$ is an automorphism, so that $g_i \in G(U_i)$ and

$$\bar g_{ij} = g_i g_{ij} g_j^{-1}.$$

Therefore, if two construction data $\{g_{ij}\}$ and $\{\bar g_{ij}\}$ are said to be equivalent when there are sections $g_i \in G(U_i)$ such that $\bar g_{ij} = g_i g_{ij} g_j^{-1}$, we have a natural bijection

$$\begin{bmatrix}\text{Principal coverings} \\ \text{of } X \text{ trivial over } \mathcal{R}\end{bmatrix} = \frac{\{\text{Construction data}\}}{\text{Data equivalence}}$$

Now let us consider the Godement resolution of G,

$$0 \longrightarrow G \longrightarrow C^0G \longrightarrow G_1 \longrightarrow 0$$
$$0 \longrightarrow G_1 \longrightarrow C^1G \longrightarrow G_2 \longrightarrow 0$$
$$H^1(X,G) = \frac{\Gamma(X,G_1)}{\Gamma(X,C^0G)}$$

If $H^1(X,G)_{\mathcal{R}}$ denotes the subgroup of all cohomology classes in $H^1(X,G)$ vanishing on the cover $\mathcal{R}$, and $\Gamma(X,G_1)_{\mathcal{R}}$ denotes the subgroup of all global sections of G_1 coming from sections of C^0G on the cover $\mathcal{R}$, we have

$$H^1(X,G)_{\mathcal{R}} = \frac{\Gamma(X,G_1)_{\mathcal{R}}}{\Gamma(X,C^0G)}.$$

If $f' \in \Gamma(X,G_1)_{\mathcal{R}}$, there are $f_i \in \Gamma(U_i, C^0G)$ such that $f'|_{U_i} = f_i$; hence $f_i = g_{ij}f_j$ on U_{ij} for some construction datum $g_{ij} \in G(U_{ij})$. If we consider other sections $\bar f_i$ representing f', and defining a datum $\{\bar g_{ij}\}$, then $\bar f_i = g_i f_i$, where $g_i \in G(U_i)$, and

$$\bar g_{ij} = \bar f_i/\bar f_j = g_i f_i / g_j f_j = g_i g_{ij} g_j^{-1}.$$
$$\Gamma(X,G_1)_{\mathcal{R}} \longrightarrow \frac{\{\text{Construction data}\}}{\text{Data equivalence}}$$

This group morphism is not injective, the kernel being the image of $\Gamma(X,C^0G)$ since condition $g_{ij} = 1$ signifies that sections f_i coincide on intersections U_{ij}.

Finally let us see that this morphism is surjective. Given a construction datum $\{g_{ij}\}$, we fix at any point $x \in X$ an index $k_x \in I$ such that $x \in U_{k_x}$ and we put

$$f_i(x) = g_{ik_x}(x),\ \ f_i \in \Gamma(U_i, C^0G),$$

so that on U_{ij} we have $f_i = g_{ik_x} = g_{ij}g_{jk_x} = g_{ij}f_j$.

These sections of C^0G coincide as sections of G_1, so that they define a global section f' of G_1 inducing the construction datum $\{g_{ij}\}$. In summary,

$$H^1(X,G)_{\mathcal{R}} = \frac{\{\text{Construction data}\}}{\text{Data equivalencia}} = \begin{bmatrix}\text{Principal coverings}\\ \text{of } X \text{ trivial over } \mathcal{R}\end{bmatrix}$$

and we conclude since any cohomology class vanishes on some open cover and any principal covering is trivial over some open cover.

Note: This argument shows that any kind of locally trivial structure is classified, modulo isomorphisms, by the cohomology group $H^1(X,\mathcal{G})$, where $\mathcal{G}$ is the sheaf of automorphisms of the corresponding trivial structure, provided that $\mathcal{G}$ is abelian.

Hurewicz's Theorem: *If X is a connected and locally simply connected space, then for any abelian group G we have a group isomorphism*

$$H^1(X,G) = \mathrm{Hom}_{\mathrm{gr}}(\pi_1(X,x),G).$$

Proof: Principal G-coverings are classified (p. 255) by $\mathrm{Hom}_{\mathrm{gr}}(\pi_1(X,x),G)$, and we must show that the bijection $H^1(X,G) = \mathrm{Hom}_{\mathrm{gr}}(\pi_1(X,x),G)$ is in fact a group morphism.

Now, $\pi_1 = \pi_1(X,x)$ is the group of automorphisms of the universal covering $\tilde{X} \to X$, and any group morphism $\phi\colon \pi_1 \to G$ correspond to the principal covering

$$P = (G \times \tilde{X})/\pi_1 \longrightarrow \tilde{X}/\pi_1 = X,$$

where the action of π_1 on P is $\sigma(g,\tilde{x}) = (g\cdot\phi(\sigma^{-1}),\sigma\tilde{x})$.

Let us fix an open cover of X by simply connected open sets U_i.

We have π_1-isomorphisms $\tilde{X}_{U_i} \simeq \pi_1 \times U_i$ inducing trivializations

$$\phi_i\colon P_{U_i} \xrightarrow{\ \sim\ } G \times U_i,\ \ \phi_i[(g,\sigma,x)] = (g\cdot\phi(\sigma),x)$$

and on U_{ij} the isomorphism $(\pi_1 \times U_j)_{U_{ij}} \simeq \tilde{X}_{U_{ij}} \simeq (\pi_1 \times U_i)_{U_{ij}}$ transforms $(1,x)$ into $(\sigma_{ij}(x),x)$. Hence a construction datum of P is precisely $\phi(\sigma_{ij})$ because

$$(\phi_i\phi_j^{-1})(g,x) = \phi_j[(g,\sigma_{ij},x)] = (g\phi(\sigma_{ij}),x).$$

Now it is clear that the product of morphisms corresponds to the data product.

1. $H^n(\tau_g,\mathbb{Z}) = \begin{cases}\mathbb{Z} & n = 0,2\\ \mathbb{Z}^{2g} & n = 1\\ 0 & n > 2\end{cases}$ where τ_g is the connected sum of g toruses.

$$H^1(\tau_g,\mathbb{Z}) = \mathrm{Hom}(\pi_1(\tau_g)_{ab},\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}^{2g},\mathbb{Z}) = \mathbb{Z}^{2g}.$$

The complement of an open disc is homotopic to $2g$ circles identified at a point, and the closed subspace exact sequence let us conclude,

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{Z} \longrightarrow H^2(\tau_g,\mathbb{Z}) \longrightarrow 0$$

2. $H^n(\pi_g, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & n = 0, 2 \\ \mathbb{F}_2^g & n = 1 \\ 0 & n > 2 \end{cases}$ where π_g is the connected sum of g projective planes.

$$H^1(\pi_g, \mathbb{F}_2) = \mathrm{Hom}(\pi_1(\pi_g)_{ab}, \mathbb{F}_2) = \mathrm{Hom}(\mathbb{Z}^{g-1} \oplus \mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2^g.$$

The complement of an open disc is homotopic to g circles identified at a point, and the closed subspace exact sequence let us conclude,

$$0 \longrightarrow \mathbb{F}_2^g \longrightarrow \mathbb{F}_2^g \longrightarrow \mathbb{F}_2 \longrightarrow H^2(\pi_g, \mathbb{F}_2) \longrightarrow 0$$

12.5.1 Vector Bundles

Definitions: A real vector space over a topological space X is a continuous map $\pi\colon E \to X$ endowed with continuous operations $E \times_X E \xrightarrow{+} E$ and $\mathbb{R} \times E \xrightarrow{\cdot} E$ satisfying the vector space axioms, including the existence of a continuous zero section $0\colon X \to E$ (any fibre $E_x = \pi^{-1}(x)$ inherits a structure of real vector space). A morphism between two vector spaces $E \to X$ and $E' \to X$ over X is a continuous map $\phi\colon E \to E'$ over X such that all maps $\phi_x\colon E_x \to E'_x$ are linear. The trivial vector space of rank n over X is $\mathbb{R}^n \times X \to X$, and a **real vector bundle** of rank n is a locally trivial vector space $E \to X$ of rank n; i.e., locally $E|_U \simeq \mathbb{R}^n \times U$.

It is a **line** bundle when $n = 1$.

Analogously we define complex vector bundles replacing $\mathbb{R}$ by $\mathbb{C}$, and smooth vector bundles replacing topological spaces and continuous maps by smooth manifolds and smooth maps.

1. If X is a smooth manifold, the tangent and cotangent bundles are smooth vector bundles.

2. The sheaf of continuous sections of a real vector bundle of rank n is a locally free sheaf of rank n over the sheaf of real continuous functions $\mathcal{C}_X$, and so we obtain an equivalence of categories, the inverse functor transforming any locally free $\mathcal{C}_X$-module $\mathcal{E}$ into the vector bundle $E = \coprod_{x \in X} (\mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x) \to X$ (with the obvious topology).

   Analogously complex vector bundles correspond to locally free modules over the sheaf of complex continuous functions, and smooth vector bundles correspond to locally free modules over the sheaf of smooth functions.

3. The graph $\xi \subset \mathbb{P}_n \times E$ of the incidence (i.e., $(p, e) \in \xi$ when $e \in E$ is in the line represented by $p \in \mathbb{P}_n$) defines a line bundle $\pi\colon \xi \to \mathbb{P}_n$, the **tautological bundle** of $\mathbb{P}_n$. It is not trivial since any continuous section $s\colon \mathbb{P}_n \to \xi$ vanishes at some point: otherwise, considering a scalar product on E and dividing $s(p)$ by the module, we obtain a continuous section of the covering $S_n \to \mathbb{P}_n$, absurd since the sphere S_n is connected.

Lemma: *The sheaf of automorphisms of a sheaf of rings $\mathcal{O}$ is the sheaf $\mathcal{O}^*$ of invertible sections.*

Proof: $\mathcal{O}(U) = \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{O}|_U, \mathcal{O}|_U)$.

Classification of Line Sheaves: $H^1(X, \mathcal{O}^*) = [\,\text{Line } \mathcal{O}\text{-modules}\,]$ (*up to isomorphisms*).

Corollary: $H^1(X, \mathbb{F}_2) = \begin{bmatrix} \text{Locally constant} \\ \text{sheaves of stalk } \mathbb{Z} \end{bmatrix}$.

Corollary: *If X is a σ-compact space (or a smooth manifold and the line bundles are $\mathcal{C}^\infty$),*

$$H^1(X, \mathbb{F}_2) = \begin{bmatrix} \text{Real line} \\ \text{bundles over } X \end{bmatrix} \quad , \quad H^2(X, \mathbb{Z}) = \begin{bmatrix} \text{Complex line} \\ \text{bundles over } X \end{bmatrix}$$

Proof: The sheaf of real continuous (resp. smooth) functions $\mathcal{O}$ is Γ-acyclic, and the following exact sequence shows that the natural morphism $H^1(X,\mathcal{O}^*) \to H^1(X,\mathbb{F}_2)$ is an isomorphism,

$$0 \longrightarrow \mathcal{O} \xrightarrow{e^f} \mathcal{O}^* \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

The sheaf of complex continuous (resp. smooth) functions $\mathcal{O}$ is also Γ-acyclic, and we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{e^f} \mathcal{O}^* \longrightarrow 0$$

Hence (p.350) we have a canonical isomorphism $H^1(X,\mathcal{O}^*) = H^2(X,\mathbb{Z})$.

Definition: If X is σ-compact, real line bundles $L \to X$ are classified, up to isomorphisms, by the **obstruction class** $\delta(L) \in H^1(X,\mathbb{F}_2)$, and complex line bundles by $\delta(L) \in H^2(X,\mathbb{Z})$.

Proposition: *If $\phi\colon Y \to X$ is a continuous map, $\delta(\phi^*L) = \phi^*(\delta(L))$.*

Proof: If $\{g_{ij}\}$ is a construction datum of L on an open cover $\{U_i\}$, then $\{g_{ij}\circ\phi\}$ is a construction datum of ϕ^*L on the cover $\{\phi^{-1}U_i\}$; hence if $c \in H^1(X,\mathcal{O}_X^*)$ corresponds to L, then ϕ^*L corresponds to the image of c by the morphism

$$H^1(X,\mathcal{O}_X^*) \xrightarrow{\phi^*} H^1(Y,\phi^*\mathcal{O}_X^*) \longrightarrow H^1(Y,\mathcal{O}_Y^*).$$

We conclude by the compatibility of ϕ^* with the morphism $H^1(X,\mathcal{O}^*) \to H^1(X,\mathbb{F}_2)$ in the real case, and with $\delta\colon H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})$ in the complex case. q.e.d.

1. A smooth manifold X of dimension n is orientable if and only if the line sheaf Ω_X^n is trivial. Hence, if $H^1(X,\mathbb{F}_2) = 0$, then X is orientable.

2. If Y is a closed hypersurface of a smooth manifold X, then the sheaf of ideals $\mathfrak{p}_Y(U) = \{f \in \mathcal{C}_X^\infty(U)\colon f|_{U\cap Y} = 0\}$ is a line sheaf, and it is trivial if and only if Y admits a global equation $f = 0$ with non null differential at any point of Y.

   If $H^1(X,\mathbb{F}_2) = 0$, any closed hypersurface Y of X is orientable and admits a global equation. In fact, if $f = 0$ is a global equation of Y, then $N = \operatorname{grad} f$ does not vanish at any point of Y, and an orientation $[\omega_X]$ of X defines an orientation $[i_N\omega_X]$ of Y.

Definition: Let X be a smooth manifold, and $\pi\colon E \to X$ a smooth vector bundle. A smooth section $s\colon X \to E$ is **transversal** to the null section s_0 at a point $p \in X$ when $s(p) \neq 0$ or

$$T_{s(p)}E = T_{s(p)}\big(s_0(X)\big) + T_{s(p)}\big(s(X)\big),$$

and s is transversal to S_0 when so it is at any point of X, so that the zero set $s^{-1}\big(s_0(X)\big)$ is empty, or a smooth submanifold of codimension the rank of E.

Any trivialization $E|_U = U \times \mathbb{R}^r$ defines a linear topology on the vector space of smooth sections $\Gamma(U,E) = \Gamma(U,U\times\mathbb{R}^r) = \mathcal{C}^\infty(U)^r$, and we equip the vector space of smooth global sections with the initial topology of the restriction maps $\Gamma(X,E) \to \Gamma(U,E)$, so that $\Gamma(X,E)$ is metrizable, complete and with a countable base of open sets (p. 120).

Theorem: *The smooth sections transversal to 0 are dense in $\Gamma(X,E)$.*

Proof: When X is an open set in $\mathbb{R}^d$ and $E = X \times \mathbb{R}^r$, the condition of $s\colon X \to X\times\mathbb{R}^r$, $s(x) = \big(x, f(x)\big)$, not being transversal to 0 means that $0 \in \mathbb{R}^r$ is a critical value of $f\colon X \to \mathbb{R}^r$. Given $\varepsilon > 0$, by Sard's theorem there exists $e \in \mathbb{R}^r$, $\|e\| < \varepsilon$, such that e is not a critical value of f, so that $s'(x) = \big(x, f(x) + e\big)$ is transversal to 0 and the theorem is proved.

In the general case, given a coordinate open set $U \subseteq X$, a trivialization $E|_U = U \times \mathbb{R}^r$, and a compact set $K \subset U$, we pick $\phi \in \mathcal{C}^\infty(X)$ with compact support $\subset U$ and $\phi = 1$ on a neighborhood of K. If $s \in \Gamma(X, E)$, there is a sequence σ_n in $\Gamma(U, E)$ such that $\lim(s|_U + \sigma_n) = s|_U$ in $\Gamma(U, E)$ and $s|_U + \sigma_n$ is transversal to 0.

Then $s + \phi\sigma_n \in \Gamma(X, E)$ is transversal to 0 on K and $\lim(s + \phi\sigma_n) = s$ in $\Gamma(X, E)$. Hence the open set $U_K \subset \Gamma(X, E)$ of all global sections transversal to 0 at any point of K is dense.

Put $X = \bigcup K_n$ where K_n is a compact set contained in a coordinate open set where E is trivial. The set $\bigcap_n U_{K_n}$ of all sections transversal to 0 is dense in $\Gamma(X, E)$ by Baire's theorem.

Lemma: *Let s be a global section of E, and $U \subseteq X$ a connected open set. If s has a unique zero x on U, then for any point $x' \in U$ there is a section $s' \in \Gamma(X, E)$ such that $s' = s$ on $X - U$ and x' is the unique zero of s' on U.*

Proof: When $E = X \times \mathbb{R}^r$ is trivial, so that $s(x) = \big(x, f(x)\big)$, consider a neighborhood $x \in V \subset U$ such that for any point $x' \in V$ there is a vector field D with compact support $\subset V$ whose flow $\{\tau_t\}$ fulfills $\tau_1(x') = x$. Then $s'(x) = \big(x, f(\tau_1 x)\big)$ coincides with s on $X - U$, because $\operatorname{supp} D \subseteq U$, and x' is the unique zero of s' on U.

Hence, in general, the points $x' \in U$ where the lemma holds (resp. is false) is an open set. Since U is connected and the lemma holds when $x' = x$, we conclude.

Theorem: *Let X be a connected non-compact smooth manifold of dimension n. Any smooth real vector bundle $E \to X$ of rank n admits a smooth global section without zeros.*

Proof: Put $X = \bigcup_n K_n$, where K_n is compact, $K_n \subseteq \overset{\circ}{K}_{n+1}$, and $X - K_n$ has no relatively compact connected component (p. 100). Pick a global section s transversal to 0, so that the zeros are discrete (hence finite on each K_n).

Inductively, we construct global sections s_n with a discrete zero-set and no zero in K_n, such that $s_{n+1} = s_n$ on K_n. Let $x \in K_{n+1}$ be a zero of s_n, and V the connected component of $X - K_n$ containing x. Since V is not relatively compact, we may pick $x' \in V - K_{n+1}$, and we may assume that $s_n(x') \neq 0$, because s_n has discrete zeros.

Let $U \subseteq V$ be a connected open set containing x', x and no other zero of s_n.

By the lemma, there is a global section s' coinciding with s_n on $X - U$ (hence on K_n) and with the same zeros as s_n, except that $x \in K_{n+1}$ is replaced by $x' \notin K_{n+1}$.

Since the zeros of s_n on K_{n+1} are finite, iterating we obtain the required section s_{n+1}. Now the section s such that $s|_{K_n} = s_n$ is smooth (so it is on the interior of K_n) and has no zero.

Corollary: *Any connected non-compact smooth manifold admits a vector field without zeros.*

Corollary: *Any complex line bundle L on a connected non-compact smooth surface is trivial.*

Proof: As a real vector bundle, L has rank 2; hence it has a global section without zeros.

Corollary: $H^2(X, \mathbb{Z}) = 0$ *when X is a connected non-compact smooth surface.*

12.6 Local Cohomology

Definition: Let $Y \subset X$ be a closed set. The sheaf of sections of $\mathcal{F}$ with support in Y is $\underline{\Gamma}_Y \mathcal{F} = \underline{\mathrm{Hom}}(\mathbb{Z}_Y, \mathcal{F})$, and $\mathcal{H}^p_Y \mathcal{F} = R^p \underline{\Gamma}_Y(\mathcal{F}) = \underline{\mathrm{Ext}}^p(\mathbb{Z}_Y, \mathcal{F})$ is the p-th **local cohomology sheaf**; i.e., $(\underline{\Gamma}_Y \mathcal{F})(U) = \Gamma_{Y \cap U}(U, \mathcal{F})$, and $\mathcal{H}^p_Y \mathcal{F}$ is the associated sheaf of the presheaf $U \rightsquigarrow H^p_{Y \cap U}(U, \mathcal{F})$,

$$(\mathcal{H}^p_Y \mathcal{F})(U) = \mathcal{H}^p\left[\underline{\Gamma}_Y(C^\bullet \mathcal{F})\right].$$

Lemma: *If $\mathcal{C}$ is a Godement sheaf, then $\underline{\Gamma}_Y\mathcal{C}$ is a flasque sheaf.*

Proof: If $\mathcal{C}(U) = \prod_{x\in U} M_x$, then $\Gamma_{Y\cap U}(U,\mathcal{C}) = \prod_{x\in Y\cap U} M_x$.

Theorem: *If Y is a closed submanifold[2] of codimension d in a topological manifold X, then the local cohomology sheaves $\mathcal{H}^p_Y\mathbb{Z}$ are null, except the **normal orientation sheaf** $\mathbb{T}_{Y/X} = \mathcal{H}^d_Y\mathbb{Z}$, which is a sheaf concentred on Y, locally constant of stalk $\mathbb{Z}$.*

Proof: The statement is local, and we may assume that $X = \mathbb{R}^m \times \mathbb{R}^d$, $Y = \mathbb{R}^m \times 0$, and in such a case we conclude by the local cohomology exact sequence, $X - Y = \mathbb{R}^m \times (\mathbb{R}^d - 0)$ being homotopic to the sphere S_{d-1}.

Corollary: $H^p_Y(X,\mathbb{Z}) = H^{p-d}(Y,\mathbb{T}_{Y/X})$.

Proof: If $\mathbb{Z} \xrightarrow{\sim} \mathcal{C}^\bullet$ is the Godement resolution of the constant sheaf $\mathbb{Z}$ on X, we have quasi-isomorphisms $\mathbb{T}_{Y/X}[-d] \xleftarrow{\sim} \mathcal{Z}^\bullet \xrightarrow{\sim} \underline{\Gamma}_Y\mathcal{C}^\bullet$, where $\mathcal{Z}^\bullet$ is the complex

$$\underline{\Gamma}_Y\mathcal{C}^0 \longrightarrow \underline{\Gamma}_Y\mathcal{C}^1 \longrightarrow \ldots \longrightarrow \underline{\Gamma}_Y\mathcal{C}^{d-1} \longrightarrow \mathcal{Z}^d \longrightarrow 0 \longrightarrow \ldots$$

Since $\mathbb{T}_{Y/X}$ is concentred on Y, and $\underline{\Gamma}_Y\mathcal{C}^\bullet$ is a complex of flasque sheaves,

$$H^{p-d}(Y,\mathbb{T}_{Y/X}) = \mathbf{H}^p(X,\mathbb{T}_{Y/X}[-d]) = \mathbf{H}^p(X,\mathcal{Z}^\bullet) = \mathbf{H}^p(X,\underline{\Gamma}_Y\mathcal{C}^\bullet) = H^p_Y(X,\mathbb{Z}).$$

Definitions: A closed submanifold Y is **normally orientable** in X if $\mathbb{T}_{Y/X}$ is isomorphic to the sheaf $\mathbb{Z}_Y$, and the normal orientations of Y in X are the isomorphisms $\mathbb{Z}_Y \xrightarrow{\sim} \mathbb{T}_{Y/X}$.

Once we fix a normal orientation, the morphism $H^{p-d}(Y,\mathbb{Z}) \xrightarrow{\sim} H^p_Y(X,\mathbb{Z}) \to H^p(X,\mathbb{Z})$ is denoted by i_*, and the local cohomology exact sequence gives the **Gysin Exact Sequence**

$$\ldots \longrightarrow H^{p-d}(Y,\mathbb{Z}) \xrightarrow{i_*} H^p(X,\mathbb{Z}) \longrightarrow H^p(X-Y,\mathbb{Z}) \xrightarrow{\delta} H^{p-d+1}(Y,\mathbb{Z}) \longrightarrow \ldots$$

and the **cohomology class** of Y in X is $[Y] := i_*(1) \in H^d(X,\mathbb{Z})$.

Projection Formula: $i_*(i^*(a)\cup b) = a \cup i_*(b);\ a \in H^\bullet(X,\mathbb{Z}),\ b \in H^\bullet(Y,\mathbb{Z})$.

Proof: That is to say, $i_*\colon H^\bullet(Y,\mathbb{Z}) \to H^\bullet(X,\mathbb{Z})$ is a morphism of $H^\bullet(X,\mathbb{Z})$-modules, where the structure of module on $H^\bullet(Y,\mathbb{Z})$ is induced by the ring morphism $i^*\colon H^\bullet(X,\mathbb{Z}) \to H^\bullet(Y,\mathbb{Z})$.

Now, up to a degree change, the direct image i_* is a composition of morphisms of modules (all are compatible with the cup product)

$$\mathbf{H}^\bullet(Y,\mathbb{Z}[-d]) \xleftarrow{i^*} \mathbf{H}^\bullet(X,\mathbb{Z}_Y[-d]) \xleftarrow{\sim} \mathbf{H}^\bullet(X,\mathcal{Z}^\bullet) \xrightarrow{\sim} \mathbf{H}^\bullet(X,\underline{\Gamma}_Y\mathcal{C}^\bullet) \longrightarrow \mathbf{H}^\bullet(X,\mathcal{C}^\bullet).$$

Finally, the degree change $H^\bullet(Y,\mathbb{Z}) \xrightarrow{\sim} \mathbf{H}^\bullet(Y,\mathbb{Z}[-d])$ is an isomorphism of modules, when we consider the left module structure, since the total differential of a bicomplex is not affected when we change the second degree. q.e.d.

1. If $H^1(Y,\mathbb{F}_2) = 0$, any line sheaf over the constant sheaf $\mathbb{Z}$ is trivial; hence Y is normally orientable regardless the ambient manifold X and the codimension d.

[2] In the sense that any point of Y has an open neighborhood U in X such that the inclusion $Y\cap U \to U$ is homeomorphic to the obvious inclusion $\mathbb{R}^m \times 0 \to \mathbb{R}^m \times \mathbb{R}^d$.

2. If Y is connected and $H^d_Y(X,\mathbb{Z}) \neq 0$, then $\mathbb{T}_{Y/X}|_Y$ has a non null section, hence it is trivial and Y is normally orientable.

   If a connected hypersurface Y admits a connected open neighborhood U such that $U - Y$ disconnects, then $H^1_Y(X,\mathbb{Z}) = H^1_Y(U,\mathbb{Z}) \neq 0$ by the local cohomology exact sequence, and Y is normally orientable.

3. Any line sheaf over $\mathbb{F}_2$ is trivial since $\mathbb{F}_2^* = 1$. If we use cohomology with coefficients in $\mathbb{F}_2$, any closed submanifold is normally orientable, with a unique normal orientation, and the cohomology class always is well defined.

4. *If Y is a connected closed submanifold in $\mathbb{R}^n$ of codimension 1, then the complement $U = \mathbb{R}^n - Y$ has two connected components.*

   Gysin's exact sequence shows directly that $\dim H^0(U,\mathbb{F}_2) = 2$,
   $$0 \longrightarrow \mathbb{F}_2 = H^0(\mathbb{R}^n,\mathbb{F}_2) \longrightarrow H^0(U,\mathbb{F}_2) \longrightarrow H^0(Y,\mathbb{F}_2) = \mathbb{F}_2 \longrightarrow H^1(\mathbb{R}^n,\mathbb{F}_2) = 0.$$

12.6.1 Topological Intersection Theory

Let Y_1, Y_2 be closed submanifolds of codimension d_1, d_2 of a topological manifold X. If $Y_1 \cap Y_2$ is a submanifold of codimension $d_1 + d_2$, on any open set $U \subseteq X$ we have a group morphism

$$\cup\colon H^{d_1}_{Y_1\cap U}(U,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d_2}_{Y_2\cap U}(U,\mathbb{Z}) \longrightarrow H^{d_1+d_2}_{Y_1\cap Y_2\cap U}(U,\mathbb{Z}),$$

so that the cup product defines a morphism of sheaves

$$\cup\colon \mathbb{T}_{Y_1/X} \otimes_{\mathbb{Z}} \mathbb{T}_{Y_2/X} \longrightarrow \mathbb{T}_{Y_1\cap Y_2/X}.$$

Definition: If we fix normal orientations ξ_1, ξ_2, ξ of Y_1, Y_2 and $Y_1 \cap Y_2$ at a point $y \in Y_1 \cap Y_2$, the **intersection multiplicity** of Y_1 and Y_2 at y is the integer number m such that

$$\xi_1 \cup \xi_2 = m\xi \in \mathbb{T}_{Y_1\cap Y_2/X,y}.$$

It is locally constant, and the sign changes if we change some normal orientation.

Theorem: *Let Y_1, Y_2 be normally oriented closed submanifolds of codimension d_1, d_2. If $Y_1 \cap Y_2$ is a normally oriented submanifold of codimension $d_1 + d_2$ with a finite number of connected components $C_1, \ldots, C_r$, and m_i is the intersection multiplicity of Y_1 with Y_2 along C_i, then*

$$[Y_1] \cup [Y_2] = m_1[C_1] + \ldots + m_r[C_r].$$

Proof: $H^{d_1+d_2}_{Y_1\cap Y_2}(X,\mathbb{Z}) = \oplus_j H^{d_1+d_2}_{C_j}(X,\mathbb{Z})$. Once we fix a normal orientation of $Y_1 \cap Y_2$, we have $\xi_{Y_1} \cup \xi_{Y_2} = \sum_j m_j \xi_{C_i}$, and the following commutative square let us conclude,

$$\begin{array}{ccc} H^{d_1}_{Y_1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d_2}_{Y_2}(X,\mathbb{Z}) & \xrightarrow{\cup} & H^{d_1+d_2}_{Y_1\cap Y_2}(X,\mathbb{Z}) = \oplus_j H^{d_1+d_2}_{C_j}(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^{d_1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d_2}(X,\mathbb{Z}) & \xrightarrow{\cup} & H^{d_1+d_2}(X,\mathbb{Z}) \end{array}$$

Definition: We say that Y_1 and Y_2 **transversally** intersect at a point $y \in Y_1 \cap Y_2$ when there is an open neighborhood U of y in X and a homeomorphism $\phi\colon U \xrightarrow{\sim} \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^m$ such that $\phi(Y_1 \cap U) = 0 \times \mathbb{R}^{d_2} \times \mathbb{R}^m$ and $\phi(Y_2 \cap U) = \mathbb{R}^{d_1} \times 0 \times \mathbb{R}^m$.

Theorem: *The morphism $\mathbb{T}_{Y_1/X} \otimes_{\mathbb{Z}} \mathbb{T}_{Y_2/X} \to \mathbb{T}_{Y/X}$ defined by the cup product is an isomorphism when Y_1, Y_2 transversally intersect; i.e., the intersection multiplicity is ± 1.*

Proof: Since the intersection multiplicity is a local topological invariant we may assume that $X = S_{d_1} \times S_{d_2} \times S_{d_3}$, $Y_1 = p_1 \times S_{d_2} \times S_{d_3}$, $Y_2 = S_{d_1} \times p_2 \times S_{d_3}$.

By Künneth's theorem we have an isomorphism of graded rings

$$\mathbb{Z}[x_1]/(x_1^2) \otimes_{\mathbb{Z}} \mathbb{Z}[x_2]/(x_2^2) \otimes_{\mathbb{Z}} \mathbb{Z}[x_3^2]/(x_3^2) = H^\bullet(X, \mathbb{Z}),$$

where $[Y_1] = x_1 \otimes 1 \otimes 1$ and $[Y_2] = 1 \otimes x_2 \otimes 1$.

Now, $Y_1 \cap Y_2$ is a connected submanifold of codimension $d_1 + d_2$ and

$$x_1 \otimes x_2 \otimes 1 \ = \ [Y_1] \cup [Y_2] \ = \ m[Y_1 \cap Y_2]$$

is not divisible in $H^{d_1+d_2}(X, \mathbb{Z})$ by a natural number $m \neq \pm 1$. Hence $m = \pm 1$.

Corollary: *The group $H^2(\mathbb{P}_{n,\mathbb{C}}, \mathbb{Z})$ is generated by the cohomology class x of any hyperplane and we have an isomorphism of graded rings*

$$H^\bullet(\mathbb{P}_{n,\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}), \ \deg x = 2.$$

Proof: $\mathbb{P}_{n-1}$ is normally orientable in $\mathbb{P}_n$ since $H^2_{\mathbb{P}_{n-1}}(\mathbb{P}_n, \mathbb{Z}) \neq 0$, and by the Gysin exact sequence we have isomorphisms

$$i_* \colon H^{p-2}(\mathbb{P}_{n-1}, \mathbb{Z}) \xrightarrow{\sim} H^p(\mathbb{P}_n, \mathbb{Z}), \ p \geq 1.$$

In particular $x = i_*(1)$ generates $H^2(\mathbb{P}_n, \mathbb{Z})$. If y is a generator of $H^2(\mathbb{P}_{n-1}, \mathbb{Z})$, then $i_*(y^{p-1})$ generates $H^p(\mathbb{P}_n, \mathbb{Z})$ by induction on n. The closed subspace exact sequence

$$H^2(\mathbb{P}_n, \mathbb{Z}) \xrightarrow{i^*} H^2(\mathbb{P}_{n-1}, \mathbb{Z}) \xrightarrow{\delta} H^3(\mathbb{C}^n, \mathbb{Z}) = 0$$

shows that $y = i^*(x)$, and by the projection formula $H^p(\mathbb{P}_n, \mathbb{Z})$ is generated by

$$i_*(y^{p-1}) = i_*((i^*x)^{p-1}) = i_*(1)x^{p-1} = x^p.$$

Corollary: *The group $H^1(\mathbb{P}_{n,\mathbb{R}}, \mathbb{F}_2)$ is generated by the cohomology class x of any hyperplane, and we have an isomorphism of graded rings*

$$H^\bullet(\mathbb{P}_{n,\mathbb{R}}, \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1}), \ \deg x = 1.$$

Proof: The same argument of the complex case holds. q.e.d.

1. *The inclusion $i \colon \mathbb{P}_m \hookrightarrow \mathbb{P}_n$, $m < n$, has no continuous retract.*

   The epimorphism of rings $i^* \colon A[x]/(x^n) \to A[x]/(x^m)$ has no section ($A = \mathbb{Z}$ or $\mathbb{F}_2$).

2. *Any diffeomorphism $\phi \colon \mathbb{P}_{2n,\mathbb{C}} \to \mathbb{P}_{2n,\mathbb{C}}$ preserves orientations.*

   Since $\phi^*(x) = \pm x$, then $\phi^*(x^{2n}) = (\pm x)^{2n} = x^{2n}$.

3. *$\mathbb{P}_{n,\mathbb{R}}$ can not be covered with n open sets homeomorphic to affine spaces.*

   If a topological space admits an open cover $X = U_1 \cup \ldots \cup U_n$, $U_i \simeq \mathbb{R}^n$, and we put $Y_i = X - U_i$, then the morphisms $H^p_{Y_i}(X, A) \to H^p(X, A)$ are surjective for all $p \geq 1$, and the following commutative square shows that, in the ring $H^\bullet(X, A)$, any product of n elements of positive degree is null,

$$\begin{array}{ccc} H^\bullet_{Y_1}(X,A) \otimes \ldots \otimes H^\bullet_{Y_n}(X,A) & \xrightarrow{\cup} & H^\bullet_{Y_1 \cap \ldots \cap Y_n}(X,A) \ = 0 \\ \downarrow & & \downarrow \\ H^\bullet(X,A) \otimes \ldots \otimes H^\bullet(X,A) & \xrightarrow{\cup} & H^\bullet(X,A) \end{array}$$

4. **Borsuk-Ulam Theorem:** *Any continuous map $\phi\colon S_n \to \mathbb{R}^n$ identifies some pair of antipodal points.*

   Otherwise $\varphi\colon S_n \to S_{n-1}$, $\varphi(x) = \frac{\phi(x)-\phi(-x)}{\|\phi(x)-\phi(-x)\|}$, is continuous and $\varphi(-x) = -\varphi(x)$.

   Now φ gives a continuous map $\bar{\varphi}\colon \mathbb{P}_n \to \mathbb{P}_{n-1}$ not trivializing the covering $S_{n-1} \to \mathbb{P}_{n-1}$, so that the following ring morphism fulfills $\bar{\varphi}^*(x) = x$; absurd:

$$\bar{\varphi}^*\colon \mathbb{F}_2[x]/(x^n) = H^\bullet(\mathbb{P}_{n-1}, \mathbb{F}_2) \longrightarrow H^\bullet(\mathbb{P}_n, \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1}).$$

12.7 Duality Theorem

Let X be a σ-compact space such that $H^p_c(X, \mathcal{F}) = 0$ for all $p > n$ and all sheaves $\mathcal{F}$ (as a topological manifold of dimension n, p. 346), and let $\mathcal{C}^\bullet$ be the Godement resolution of the constant sheaf defined by a principal ideal domain A, truncated at the n-th step:

$$0 \longrightarrow A \longrightarrow C^0A \longrightarrow C^1A \longrightarrow \ldots \longrightarrow C^{n-1}A \longrightarrow \mathcal{C}^n \longrightarrow 0.$$

Since this resolution $0 \to A \to \mathcal{C}^\bullet$ splits on stalks, for any sheaf of A-modules $\mathcal{M}$ we have a resolution $0 \to \mathcal{M} \to \mathcal{M} \otimes_A \mathcal{C}^\bullet$, which is Γ_c-acyclic since the sheaves $\mathcal{M} \otimes_A C^iA$ are $C^0\mathbb{Z}$-modules, and $H^p_c(X, \mathcal{M} \otimes_A \mathcal{C}^n) = H^{p+n}_c(X, \mathcal{M}) = 0$ (p. 350).

Let us fix an injective resolution $0 \to A \to I^0 \to I^1 \to 0$ of the ring A.

The contravariant functor $F(\mathcal{M}) = \mathrm{Hom}_A(\Gamma_c(X, \mathcal{M} \otimes_A \mathcal{C}^p, I^q)$ is exact (sheaves $\mathcal{C}^p$ are flat, sheaves $\mathcal{M} \otimes_A \mathcal{C}^p$ are Γ_c-acyclic and A-modules I^q are injective) and F transforms inductive limits into projective limits since Γ_c preserves inductive limits (p. 346). By Grothendieck's representability theorem[3] there is a sheaf of injective A-modules $\mathcal{D}^{-p,q}$ such that

$$\begin{aligned} \mathrm{Hom}_A(\mathcal{M}, \mathcal{D}^{-p,q}) &= \mathrm{Hom}_A(\Gamma_c(X, \mathcal{M} \otimes_A \mathcal{C}^p), I^q), \\ \mathrm{Hom}^\bullet_A(\mathcal{M}, \mathcal{D}) &= \mathrm{Hom}^\bullet_A(\Gamma_c(X, \mathcal{M} \otimes_A \mathcal{C}), I). \end{aligned}$$

where $\mathrm{Hom}^{p,q}_A(K, L) = \mathrm{Hom}_A(K^{-p}, L^q)$, with the differentials induced by those of $K^\bullet$ and $L^\bullet$.

Definition: The **dualizing complex** of X with coefficients in a principal ideal domain A is the simple complex $\mathcal{D}^A_X$ defined by the injective bicomplex $\mathcal{D}^{\bullet\bullet}$ and, by the following lemma, we have exact sequences

$$0 \longrightarrow \mathrm{Ext}^1_A(H^{p+1}_c(X, \mathcal{M}), A) \longrightarrow \mathbf{Ext}^{-p}(\mathcal{M}, \mathcal{D}^A_X) \longrightarrow \mathrm{Hom}_A(H^p_c(X, \mathcal{M}), A) \longrightarrow 0.$$

Lemma: *If $K^\bullet$ is a complex of A-modules, we have exact sequences*

$$0 \longrightarrow \mathrm{Ext}^1_A(H^{p+1}(K^\bullet), A) \longrightarrow H^{-p}\big[\mathrm{Hom}^\bullet_A(K, I)\big] \longrightarrow \mathrm{Hom}_A(H^p(K^\bullet), A) \longrightarrow 0.$$

Proof: $\mathrm{Hom}_A(H^p(K^\bullet), A)$ and $\mathrm{Ext}^1_A(H^p(K^\bullet), A)$ are the kernel and cokernel of

$$\varphi_p\colon \mathrm{Hom}_A(H^p(K^\bullet), I^0) \longrightarrow \mathrm{Hom}_A(H^p(K^\bullet), I^1).$$

On the other hand, we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}^\bullet_A(K, I^1[-1]) \longrightarrow \mathrm{Hom}^\bullet_A(K, I) \longrightarrow \mathrm{Hom}^\bullet_A(K, I^0) \longrightarrow 0,$$

and the connecting δ_{-p} of the induced cohomology exact sequence

$$\ldots \xrightarrow{\delta_{-(p+1)}} H^{-p}\big[\mathrm{Hom}^\bullet_A(K, I^1[-1])\big] \longrightarrow H^{-p}\big[\mathrm{Hom}^\bullet_A(K, I)\big] \longrightarrow H^{-p}\big[\mathrm{Hom}^\bullet_A(K, I^0)\big] \xrightarrow{\delta_{-p}} \ldots$$

[3] Any $\mathcal{O}$-module $\mathcal{M}$ admits an epimorphism $\bigoplus_I \mathcal{P} \to \mathcal{M}$, where $\mathcal{P} = \bigoplus_U \mathcal{O}_U$, because $\mathrm{Hom}_\mathcal{O}(\mathcal{O}_U, \mathcal{M}) = \mathcal{M}(U)$. Hence (p. 138) any minimal pair Q_ξ of F is fully determined by the subset $\mathrm{Hom}_\mathcal{O}(\mathcal{P}, \mathcal{M}) \subseteq F(\mathcal{P})$.

coincides with the morphism φ_p; i.e. the following square is commutative,

$$\begin{array}{ccc} H^{-p}\big[\mathrm{Hom}^\bullet_A(K,I^0)\big] & \xrightarrow{\delta_{-p}} & H^{-p+1}\big[\mathrm{Hom}^\bullet_A(K,I^1[-1])\big] \\ \| & & \| \\ \mathrm{Hom}_A(H^p(K^\bullet),I^0) & \xrightarrow{\varphi_p} & \mathrm{Hom}_A(H^p(K^\bullet),I^1) \end{array}$$

Duality Theorem: *If X is a topological manifold of dimension n, all the cohomology sheaves $\mathcal{H}^{-p}(\mathcal{D}^A_X)$ are null, except $\mathbb{T}^A_X = \mathcal{H}^{-n}(\mathcal{D}^A_X)$, which is a locally constant sheaf of stalk A, and for any sheaf of A-modules $\mathcal{M}$ we have exact sequences:*

$$0 \longrightarrow \mathrm{Ext}^1_A(H^{p+1}_c(X,\mathcal{M}),A) \longrightarrow \mathrm{Ext}^{n-p}_A(\mathcal{M},\mathbb{T}^A_X) \longrightarrow \mathrm{Hom}_A(H^p_c(X,\mathcal{M}),A) \longrightarrow 0.$$

Proof: If U is an open set in X, when $\mathcal{M} = A_U$ we have exact sequences

$$0 \longrightarrow \mathrm{Ext}^1_A(H^{p+1}_c(U,A),A) \longrightarrow H^{-p}[\mathcal{D}(U)] \longrightarrow \mathrm{Hom}_A(H^p_c(U,A),A) \longrightarrow 0.$$

If $U \simeq \mathbb{R}^n$, then $H^n_c(U,A) \simeq A$ and $H^p_c(X,A) = 0$, $p \neq n$; hence $H^{-n}[\mathcal{D}(U)] \simeq A$ and $H^{-p}[\mathcal{D}(U)] = 0$, $p \neq n$. Therefore $\mathcal{H}^{-p}(\mathcal{D}) = 0$, $p \neq n$, and the sheaf $\mathcal{H}^{-n}\mathcal{D}$ is locally constant because, when $X = \mathbb{R}^n$ and U is an open ball, we have $H^n_c(X-U,A) = 0$, and the natural morphism $H^n_c(U,A) \to H^n_c(X,A)$ is an isomorphism.

Now, since $\mathcal{D}$ has no term of degree $< -n$, we have an exact sequence

$$0 \longrightarrow \mathbb{T}^A_X \longrightarrow \mathcal{D}^{-n} \longrightarrow \mathcal{D}^{-n+1} \longrightarrow \ldots \longrightarrow \mathcal{D}^1 \longrightarrow 0,$$

which is an injective resolution of $\mathbb{T}^A_X$; hence $\mathbf{Ext}^{-p}_A(\mathcal{M},\mathcal{D}) = \mathrm{Ext}^{n-p}_A(\mathcal{M},\mathbb{T}^A_X)$.

Corollary: $\mathbb{T}^A_X(U) = \mathrm{Hom}_A(H^n_c(U,A),A)$, *and therefore* $\mathbb{T}^A_U = \mathbb{T}^A_X|_U$.

Corollary: $0 \longrightarrow \mathrm{Ext}^1_A(H^{p+1}_c(X,A),A) \longrightarrow H^{n-p}(X,\mathbb{T}^A_X) \longrightarrow \mathrm{Hom}_A(H^p_c(X,A),A) \longrightarrow 0.$

Corollary: $H^{n-p}(X,\mathbb{F}_2) = H^p_c(X,\mathbb{F}_2)^*$.

Proof: When $A = \mathbb{F}_2$, any locally constant sheaf is trivial, so that $\mathbb{T}^{\mathbb{F}_2}_X = \mathbb{F}_2$.

Definitions: The **orientation sheaf** of a topological manifold X is $\mathbb{T}_X = \mathbb{T}^{\mathbb{Z}}_X$, and X is **orientable** if $\mathbb{Z} \simeq \mathbb{T}_X$. The **orientations** of X at a point x are the two generators of the stalk $\mathbb{T}_{X,x}$, and the orientations of X are the isomorphisms $\mathbb{Z} \simeq \mathbb{T}_X$.

If X is connected of dimension n, the condition of being orientable is equivalent to

$$0 \neq \Gamma(X,\mathbb{T}_X) = \mathrm{Hom}_{\mathbb{Z}}(H^n_c(X,\mathbb{Z}),\mathbb{Z}),$$

and on any orientable manifold X of dimension n we have exact sequences

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H^{p+1}_c(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^{n-p}(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H^p_c(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow 0.$$

Proposition: $\mathbb{T}^A_X = \mathbb{T}_X \otimes_{\mathbb{Z}} A$; *hence,* $\mathbb{T}^A_X \simeq A$ *when* X *is orientable.*

Proof: The natural morphism $H^n_c(U,\mathbb{Z}) \otimes_{\mathbb{Z}} A \to H^n_c(U,A)$ is an isomorphism by the universal coefficients theorem. So we have a morphism

$$\mathbb{T}_X(U) \otimes_{\mathbb{Z}} A = \mathrm{Hom}_{\mathbb{Z}}(H^n_c(U,\mathbb{Z}),\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow \mathrm{Hom}_A(H^n_c(U,A),A) = \mathbb{T}^A_X(U),$$

which is an isomorphism when $U \simeq \mathbb{R}^n$, and it induces an isomorphism $\mathbb{T}_X \otimes_{\mathbb{Z}} A = \mathbb{T}^A_X$.

Corollary: $H^{n-p}(X,k) = H^p_c(X,k)^*$, *when* X *is an orientable manifold of dimension* n *and* k *is a field.*

Corollary: *If* $\operatorname{char} k \neq 2$, *a manifold* X *is orientable if and only if* $k \simeq \mathbb{T}^k_X$.

Proof: The sheaf Δ_X of orientations of X defines an **orientation covering** $\Delta^{\text{et}}_X \to X$ of degree 2, and it is trivial if and only if X is orientable.

When $\operatorname{char} k \neq 2$, the natural morphism $\Delta^{\text{et}}_X \hookrightarrow \mathbb{T}^{\text{et}}_X \to (\mathbb{T}^k_X)^{\text{et}}$ is injective, and if $(\mathbb{T}^k_X)^{\text{et}} \to X$ is a trivial covering, so is Δ^{et}_X, and X is orientable.

Corollary: *On any smooth manifold* X, *the topological and smooth orientations coincide.*

Proof: Let us consider the sheaf $\Delta^{\text{dif}}_X(U) = \{\text{smooth orientations of } U\}$.

Any smooth orientation of U defines an integration of n-forms with compact support and, by Stokes theorem, for any $(n-1)$-form with compact support ω_{n-1}

$$\int_U \mathrm{d}\omega_{n-1} = \int_{\partial U} \omega_{n-1} = \int_{\emptyset} \omega_{n-1} = 0.$$

So, any smooth orientation of U defines a non null linear form

$$\int_U : H^n_c(U,\mathbb{R}) \longrightarrow \mathbb{R},$$

and we obtain an injective morphism of sheaves $\Delta^{\text{dif}}_X \hookrightarrow \mathbb{T}^{\mathbb{R}}_X$.

Now, if X admits a smooth orientation, $\mathbb{T}^{\mathbb{R}}_X$ has a global section not vanishing at any point; hence $\mathbb{R} \simeq \mathbb{T}^{\mathbb{R}}_X$, and X is topologically orientable. Conversely, if $(\mathbb{T}^{\mathbb{R}}_X)^{\text{et}} \to X$ is a trivial covering, so is $(\Delta^{\text{dif}}_X)^{\text{et}} \to X$, and X admits a smooth orientation. q.e.d.

1. *If* X *is a compact orientable manifold of odd dimension* n, *then* $\chi(X) = 0$.

   $\chi(X) = \sum_p (-1)^p \dim H^p(X,\mathbb{Q}) = -\sum_p (-1)^{n-p} \dim H^{n-p}(X,\mathbb{Q}) = -\chi(X)$.

2. If X is a connected manifold of dimension n and $H^n_c(X,\mathbb{Q}) \neq 0$, then $\mathbb{T}^{\mathbb{Q}}_X$ has a non zero section and X is orientable; hence $H^n_c(X,\mathbb{Q}) \simeq \mathbb{Q}$.

3. *Any closed submanifold* Y *in* $\mathbb{R}^n$ *of codimension 1 is orientable. Hence, the real projective plane is not a closed submanifold of* $\mathbb{R}^3$.

   We may assume that Y is connected, so that (p. 365) $U = \mathbb{R}^n - Y$ has two connected components (obviously orientable) and the closed subspace exact sequence let us conclude

   $$0 \longrightarrow H^{n-1}_c(Y,\mathbb{Q}) \longrightarrow H^n_c(U,\mathbb{Q}) = \mathbb{Q}^2 \longrightarrow H^n_c(\mathbb{R}^n,\mathbb{Q}) = \mathbb{Q} \longrightarrow 0.$$

12.7.1 Degree Theory

Theorem: $H^n_c(X,\mathbb{T}_X) = \mathbb{Z}$, *when* X *is a connected manifold of dimension* n.

Proof: If k is field, by duality and universal coefficients,

$$k = \operatorname{Hom}_k(\mathbb{T}^k_X, \mathbb{T}^k_X) = H^n_c(X, \mathbb{T}_X \otimes_{\mathbb{Z}} k)^* = (H^n_c(X,\mathbb{T}_X) \otimes_{\mathbb{Z}} k)^*,$$

hence $H^n_c(X,\mathbb{T}_X) \otimes_{\mathbb{Z}} k = k$ and, if $H^n_c(X,\mathbb{T}_X)$ is finitely generated, $\mathbb{Z} = H^n_c(X,\mathbb{T}_X)$.

On the other hand, if a connected open set is a finite union $U = V_1 \cup \ldots \cup V_r$ of open sets $V_i \simeq \mathbb{R}^n$, the Mayer-Vietoris exact sequence

$$H^n_c(V_1,\mathbb{T}_X) \oplus H^n_c(V_2 \cup \ldots \cup V_r, \mathbb{T}_X) \to H^n_c(U,\mathbb{T}_X) \to H^{n+1}_c(V_1 \cap (V_2 \cup \ldots \cup V_r), \mathbb{T}_X) = 0$$

shows, by induction on r, that $H^n_c(U,\mathbb{T}_X)$ is finitely generated; hence $\mathbb{Z} = H^n_c(U,\mathbb{T}_X)$.

Such open sets U cover X, and we have (p. 346)

$$H^n_c(X,\mathbb{T}_X) = H^n_c(X, \varinjlim(\mathbb{T}_X)_U) = \varinjlim H^n_c(U,\mathbb{T}_X) = \mathbb{Z},$$

because any morphism $\mathbb{Z} = H^n_c(U,\mathbb{T}_X) \to H^n_c(U',\mathbb{T}_X) = \mathbb{Z}$ is an isomorphism, since so is the dual morphism (U and U' are connected):

$$\Gamma(U',\mathbb{Z}) = \mathrm{Hom}(H^n_c(U',\mathbb{T}_X),\mathbb{Z}) \longrightarrow \mathrm{Hom}(H^n_c(U,\mathbb{T}_X),\mathbb{Z}) = \Gamma(U,\mathbb{Z}).$$

Lemma: *If $\pi\colon Y \to X$ is a covering, then $\mathbb{T}_Y = \pi^*\mathbb{T}_X$.*

Proof: If $U \subset Y$ is an open set such that $\pi\colon U \to \pi(U)$ is a homeomorphism, then we have a natural isomorphism $\varphi_U\colon (\pi^*\mathbb{T}_X)|_U \to (\mathbb{T}_Y)|_U$, and $\varphi_V = (\varphi_U)|_V$ when V is an open set in U; hence it defines an isomorphism $\pi^*\mathbb{T}_X \to \mathbb{T}_Y$.

Theorem: *If X is a connected manifold of dimension n,*

$$H^n_c(X,\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } X \text{ is orientable} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } X \text{ is not orientable} \end{cases}$$

Proof: If X is orientable, $H^n_c(X,\mathbb{Z}) \simeq H^n_c(X,\mathbb{T}_X) = \mathbb{Z}$.

If X is not orientable and $\pi\colon Y \to X$ is the orientation covering, $\mathbb{T}_Y = \pi^*\mathbb{T}_X$ has a canonical non null section and Y is orientable.

We have an epimorphism $\mathrm{tr}\colon \pi_*\mathbb{Z} \to \mathbb{Z}$, $\mathrm{tr}(f)(x) = f(x_1) + f(x_2)$, where $\pi^{-1}(x) = \{x_1, x_2\}$, and the cohomology exact sequence

$$H^n_c(Y,\mathbb{Z}) = H^n_c(X,\pi_*\mathbb{Z}) \longrightarrow H^n_c(X,\mathbb{Z}) \longrightarrow H^{n+1}_c(X,\mathrm{Ker}\,\mathrm{tr}) = 0$$

shows that $H^n_c(X,\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$, where $m > 0$ since X is not orientable.

The morphism $\mathbb{Z} \xrightarrow{\pi^*} \pi_*\mathbb{Z} \xrightarrow{\mathrm{tr}} \mathbb{Z}$ is multiplication by 2; hence so is the composition

$$\mathbb{Z}/m\mathbb{Z} = H^n_c(X,\mathbb{Z}) \xrightarrow{\ \pi^*\ } H^n_c(X,\pi_*\mathbb{Z}) = \mathbb{Z} \xrightarrow{\ \mathrm{tr}\ } H^n_c(X,\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z},$$

and we see that $m = 1$ or 2. The case $m = 1$ is impossible since the universal coefficients formula gives $H^n_c(X,\mathbb{F}_2) = 0$, while by duality $H^n_c(X,\mathbb{F}_2)^* = H^0(X,\mathbb{F}_2) = \mathbb{F}_2$.

Corollary: *If X is a connected oriented smooth manifold of dimension n, the integration of n-forms induces an isomorphism*

$$\int_X \colon H^n_c(X,\mathbb{R}) \xrightarrow{\ \sim\ } \mathbb{R}.$$

Proof: The linear map $\int_X\colon \mathbb{R} = H^n_c(X,\mathbb{R}) \to \mathbb{R}$ is non null (just integrate forms with compact support contained in a coordinate neighborhood).

Definition: If X is a connected orientable manifold of dimension n, any orientation defines an isomorphism $\mathbb{Z} = H^n_c(X,\mathbb{Z})$, hence a generator ε_X of this group. If $\pi\colon Y \to X$ is a proper morphism between connected oriented manifolds of dimension n, it induces a morphism

$$\pi^*\colon \mathbb{Z}\varepsilon_X = H^n_c(X,\mathbb{Z}) \longrightarrow H^n_c(Y,\mathbb{Z}) = \mathbb{Z}\varepsilon_Y,$$

and the **degree** of π is the integer number $\deg\pi$ such that $\pi^*(\varepsilon_X) = (\deg\pi)\varepsilon_Y$.

Theorem: *If X is a connected orientable manifold of dimension n and $p \in X$, then the natural morphism $H^n_p(X,\mathbb{Z}) \longrightarrow H^n_c(X,\mathbb{Z})$ is an isomorphism, so that any orientation $\varepsilon_X \in H^n_c(X,\mathbb{Z})$ of X defines a normal orientation $\varepsilon_p \in H^n_p(X,\mathbb{Z})$ at any point p.*

Proof: If U is a connected open neighborhood of p, the morphism $H^n_c(U,\mathbb{Z}) \to H^n_c(X,\mathbb{Z})$ is an isomorphism, since so is the dual morphism, and $H^n_p(X,\mathbb{Z}) = H^n_p(U,\mathbb{Z})$.

Hence it is enough to check the theorem on one orientable manifold of dimension n.

When $X = S_n$, the theorem follows from the local cohomology exact sequence,

$$H^n_p(S_n,\mathbb{Z}) \longrightarrow H^n(S_n,\mathbb{Z}) \longrightarrow H^n(\mathbb{R}^n,\mathbb{Z}) = 0.$$

Definition: Let $\pi\colon Y \to X$ be a continuous map between manifolds of dimension n. If $p = \pi(q)$, we have a morphism

$$\pi^*\colon \mathbb{Z}\varepsilon_p = H^n_p(X,\mathbb{Z}) \longrightarrow H^n_q(Y,\mathbb{Z}) = \mathbb{Z}\varepsilon_q,$$

and the **degree** of π at $q \in Y$ is the integer number $\deg_q\pi$ such that $\pi^*(\varepsilon_p) = (\deg_q\pi)\varepsilon_q$.

Theorem: *Let $\pi\colon Y \to X$ be a proper morphism between connected oriented manifolds of dimension n. If the fibre of $p \in X$ is finite, $\pi^{-1}(p) = \{q_1,\dots,q_r\}$, then*

$$\deg\pi = \deg_{q_1}\pi + \ldots + \deg_{q_r}\pi.$$

Proof: It follows directly from the commutative square,

$$\begin{array}{ccc} H^n_p(X,\mathbb{Z}) & \xrightarrow{\pi^*} & H^n_{\pi^{-1}p}(Y,\mathbb{Z}) = \mathbb{Z}\varepsilon_{q_1} \oplus \ldots \oplus \mathbb{Z}\varepsilon_{q_r} \\ \downarrow & & \downarrow \\ H^n_c(X,\mathbb{Z}) & \xrightarrow{\pi^*} & H^n_c(Y,\mathbb{Z}) \end{array}$$

since $\pi^*\varepsilon_X = (\deg\pi)\varepsilon_Y$, $\pi^*\varepsilon_p = ((\deg_{q_1}\pi)\varepsilon_{q_1},\dots,(\deg_{q_r}\pi)\varepsilon_{q_r})$ and the vertical morphisms transform ε_p into ε_X, and ε_{q_i} into ε_Y, according to the above theorem.

Corollary: *If the degree of π is not zero, then π is surjective.*

Theorem: *Let $\pi\colon Y \to X$ a continuous map between manifolds of dimension n. If π is a local homeomorphism at $q \in Y$, then* $\deg_q\pi = \pm 1$.

Proof: The definition of $\deg_q\pi$ is local on X and Y, by excision; hence we may assume that π is a homeomorphism, an obvious case.

Corollary: *The degree of $z^n\colon \mathbb{C} \to \mathbb{C}$ at the origin is n.*

Proof: It has degree 1 at any point $p \neq 0$, because it preserves the orientation. q.e.d.

1. Any non constant proper morphism $X \to Y$ between Riemann surfaces has positive degree; hence it is surjective. If the degree is 1, then it is injective, so that it is an isomorphism. In particular any non constant polynomial $P\colon \mathbb{C} \to \mathbb{C}$ is surjective: it has a complex root.

2. Let D be a vector field on $\mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is a compact manifold with boundary and D does not vanish on the boundary, we may consider the degree of the proper morphism $\phi\colon \partial\Omega \to S_{n-1}$, $\phi(p) = D_p/|D_p|$. This degree is zero when D does not vanish on Ω, since ϕ may be extended to a neighborhood of Ω and

$$\int_{\partial\Omega}\phi^*\omega_{n-1} = \int_\Omega \mathrm{d}(\phi^*\omega_{n-1}) = \int_\Omega \phi^*(\mathrm{d}\omega_{n-1}) = \int_\Omega \phi^* 0 = 0.$$

Therefore, when Ω is a small ball centred at an isolated singularity, $D_p = 0$, the degree does not depend on the ball and we name it **index** of the vector field D at p.

In general, if all singularities of the vector field are isolated, considering small open balls B_i around each singularity $p_1, \dots, p_r$ contained in Ω, we have that D does not vanish on the manifold with boundary $\Omega - (B_1 \cup \ldots \cup B_r)$, so that the degree of ϕ coincides with the sum of the indices of the vector field at all singularities contained in Ω.

3. *If $\pi\colon S_n \to \mathbb{P}_n(\mathbb{R})$, $n \geq 2$, is a continuous map, there is no non empty open set $U \subset \mathbb{P}_n(\mathbb{R})$ such that π defines a homeomorphism $\pi^{-1}(U) \xrightarrow{\sim} U$.*

   In fact, otherwise we have an isomorphism

$$\pi^*\colon H^n_p(\mathbb{P}_n, \mathbb{F}_2) = H^n_p(U, \mathbb{F}_2) \longrightarrow H^n_p(\pi^{-1}(U), \mathbb{F}_2) = H^n_p(S_n, \mathbb{F}_2),$$

   and so is also $\pi^*\colon H^n(\mathbb{P}_n, \mathbb{F}_2) \to H^n(S_n, \mathbb{F}_2)$ since both groups are generated by the cohomology class of a point. Absurd: $\pi^*(x^n) = (\pi^* x)^n = 0$, since $\pi^* x \in H^1(S_n, \mathbb{F}_2) = 0$.

12.7.2 Lefschetz's Theorem

The duality isomorphisms $\mathrm{Hom}_A(\mathcal{M}, \mathcal{D}^{-p,q}) = \mathrm{Hom}_A(\Gamma_c(X, \mathcal{M} \otimes_A \mathcal{C}^p), I^q)$ are given by morphisms $\Gamma_c(X, \mathcal{D}^{-p,q} \otimes_A \mathcal{C}^p) \to I^q$, defining a morphism $\xi\colon \Gamma_c(X, \mathcal{D} \otimes_A \mathcal{C}) \to I$.

In the case of a manifold of constant dimension n, the morphism

$$\xi\colon H^n_c(X, \mathbb{T}^A_X) = \mathbf{H}^0_c(X, \mathcal{D}) \longrightarrow A$$

is surjective since otherwise there is no surjective morphisms $H^n_c(X, \mathcal{M}) \to A$, and there are such morphisms when $\mathcal{M} = \mathbb{T}^A_X$. Hence it is an isomorphism.

The isomorphism $\mathrm{Hom}^\bullet_A(\mathcal{M}, \mathcal{D}) = \mathrm{Hom}^\bullet_A(\Gamma_c(X, \mathcal{M} \otimes_A \mathcal{C}), I)$ is induced by the following pairing (where λ is the obvious morphism),

$$\Gamma(\underline{\mathrm{Hom}}^\bullet(\mathcal{M}, \mathcal{D})) \otimes \Gamma_c(\mathcal{M} \otimes \mathcal{C}^\bullet) \xrightarrow{\otimes} \Gamma_c(\underline{\mathrm{Hom}}^\bullet(\mathcal{M}, \mathcal{D}) \otimes (\mathcal{M} \otimes \mathcal{C}^\bullet)) \xrightarrow{\lambda} \Gamma_c(\mathcal{D} \otimes \mathcal{C}^\bullet) \xrightarrow{\xi} I.$$

When $\mathcal{M} = A$, we have that $\underline{\mathrm{Hom}}^\bullet(\mathcal{M}, \mathcal{D}) = \mathcal{D}$ is an injective resolution of $\mathbb{T}^A_X$, $\mathcal{M} \otimes \mathcal{C}^\bullet = \mathcal{C}^\bullet$ is a Γ_c-acyclic resolution of A, $\underline{\mathrm{Hom}}^\bullet(\mathcal{M}, \mathcal{D}) \otimes (\mathcal{M} \otimes \mathcal{C}^\bullet)$ is a Γ_c-acyclic resolution of $\mathbb{T}^A_X \otimes A$, and the epimorphism $H^{n-p}(X, \mathbb{T}^A_X) \to \mathrm{Hom}_A(H^p_c(X, A), A)$ is induced by the pairing

$$H^{n-p}(X, \mathbb{T}^A_X) \otimes_A H^p_c(X, A) \xrightarrow{\cup} H^n_c(X, \mathbb{T}^A_X \otimes_A A) \xrightarrow{\lambda} H^n_c(X, \mathbb{T}^A_X) \xrightarrow{\xi} A.$$

When X is an oriented compact manifold and $A = k$ is a field, the cup product defines a metric identifying $H^\bullet(X) = H^\bullet(X, k)$ with the dual $H^\bullet(X)^*$,

$$\langle c', c\rangle := \xi(c' \cup c).$$

By Künneth's theorem $H^\bullet(X \times X) = H^\bullet(X) \otimes H^\bullet(X)$, so that any orientation ε_X of X defines an orientation $\varepsilon_X \otimes \varepsilon_X$ of $X \times X$, and we fix the normal orientation of the diagonal $\Delta\colon X \to X \times X$ so that Δ_* preserves orientations, and by the projection formula

$$\langle \Delta^* a, b\rangle = \langle a, \Delta_* b\rangle.$$

Theorem: *The cohomology class of the diagonal $[\Delta]$ defines the metric of the cup product,*

$$\langle a \otimes b, [\Delta]\rangle = \langle a, b\rangle.$$

Proof: $\langle a\otimes b, [\Delta]\rangle = \langle a\otimes b, \Delta_*(1)\rangle = \langle \Delta^*(a\otimes b), 1\rangle = \langle a\cup b, 1\rangle = \langle a, b\rangle$.

Corollary: *If we fix a base (a_i) of $H^\bullet(X)$, and (b_i) is the dual base, $\langle a_i, b_j\rangle = \delta_{ij}$, then*

$$[\Delta] = \textstyle\sum_i (-1)^{\deg a_i} a_i \otimes b_i.$$

Proof: $\langle b_j\otimes a_i, \sum_i (-1)^{\deg a_i} a_i\otimes b_i\rangle = (-1)^{\deg a_i}\delta_{ij}\langle b_i\otimes a_i, a_i\otimes b_i\rangle = \delta_{ij}\langle b_i, a_i\rangle\langle a_i, b_i\rangle$
$= (-1)^{(\deg a_i)(\deg b_i)}\delta_{ij} = \langle b_j, a_i\rangle = \langle b_j\otimes a_i, [\Delta]\rangle$.

Definition: If X is an oriented compact manifold, the **Lefschetz number** Λ_f of a continuous map $f\colon X\to X$ is the global intersection number of the graph $\Gamma_f = f\times 1\colon X\to X\times X$ with the diagonal,

$$\Lambda_f := \langle [\Delta], [\Gamma_f]\rangle.$$

Lefschetz's Formula: $\Lambda_f = \sum\limits_{p=0}^{n} (-1)^p \operatorname{tr} f_p^*,\ \ f_p^*\colon H^p(X)\to H^p(X)$.

Proof: $\Lambda_f = \langle [\Delta], (f\times 1)_* 1\rangle = \langle (1\times f)^*[\Delta], 1\rangle = \langle (1\times f)^* \sum_i (-1)^{\deg a_i} a_i\otimes b_i, 1\rangle$
$= \sum_i (-1)^{\deg a_i}\langle f^* a_i, b_i\rangle = \sum_p (-1)^p \operatorname{tr} f_p^*$.

Corollary: *The self-intersection of the diagonal is the Euler-Poincaré characteristic,*

$$\Delta^*\big([\Delta]\big) = \chi(X)\cdot\varepsilon_X.$$

1. *If X is a compact orientable manifold of dimension $n = 4d+2$, then $\chi(X)$ is even.*

   Since $(-1)^p \dim H^p(X,\mathbb{Q}) = (-1)^{n-p}\dim H^{n-p}(X,\mathbb{Q})$, we only have to show that the dimension of $H^{2d+1}(X,\mathbb{Q})$ is even. Now, the cup product defines a non singular alternate metric $H^{2d+1}(X,\mathbb{Q})\times H^{2d+1}(X,\mathbb{Q}) \xrightarrow{\cup} \mathbb{Q}$.

2. *If a compact orientable smooth manifold X admits a continuous vector field without zeros, then $\chi(X) = 0$.*

   If the tangent bundle $\pi\colon TX\to X$ admits a continuous section $s\colon X\to TX$ without zeros, then $0 = s^*\big([s_0X]\big) = s_0^*\big([s_0X]\big)$. Now, since X admits a riemannian metric (p. 299) and TX is the normal bundle of the diagonal embedding $\Delta\colon X\to X\times X$, by the tubular neighborhood lemma

$$0 = s_0^*\big([s_0X]\big) = \Delta^*\big([\Delta]\big) = \chi(X)\cdot\varepsilon_X.$$

3. If τ is a homography of the complex projective line, the degree is 1 since it preserves orientations; hence $\Lambda_\tau = 2$. When it is parabolic, the topological intersection multiplicity of the diagonal and the graph at the unique fixed point is 2.

4. Any continuous map $f\colon S_{2n}\to S_{2n}$ of degree $\neq -1$ has Lefschetz number $\Lambda_f\neq 0$, and f has some fixed point. Analogously, any continuous map $S_{2n+1}\to S_{2n+1}$ of degree $\neq 1$ has some fixed point.

5. If $X = \mathbb{R}^2/\mathbb{Z}^2$, any matrix $A\in M_{2\times 2}(\mathbb{Z})$ induces a continuous map $f\colon X\to X$. The action of f^* on $H^1(X,\mathbb{R}) = \mathbb{R}^2$ is defined by A, and on $H^2(X,\mathbb{R}) = \mathbb{R}$ by the determinant. Hence $\Lambda_f = 1 - \operatorname{tr}A + \det A$.

6. If f is an analytic endomorphism of a complex torus $\mathbb{C}/(\mathbb{Z}\alpha + \mathbb{Z}\beta)$, the lifting of f to the universal covering is an endomorphism $\mathbb{C}\to\mathbb{C}$ fixing the origin; hence it is the product by a number complex $a+bi$, so that the degree of f is $d = a^2+b^2$, and $\Lambda_f = 1 - 2a + a^2 + b^2$. So we see that $|\Lambda_f - d - 1| \le 2\sqrt{d}$.

12.8 Characteristic Classes

Definition: If $E \to X$ is a real or complex ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) vector bundle, $\mathbb{P}(E)$ is the set of all vector subspaces of dimension 1 of the fibres E_x, with the quotient topology, and we have a natural projection $\pi\colon \mathbb{P}(E) \to X$. The vector bundle $\pi^*E = E \times_X \mathbb{P}(E)$ has a **tautological** line sub-bundle $\xi_E \hookrightarrow \pi^*E$, the fibre over a point $p \in \mathbb{P}(E)$ being the corresponding line of $E_{\pi(p)}$.

The tautological line bundle of the projective space $\mathbb{P}_d$ is denoted by ξ_d.

Lemma: *If L is a line bundle over a separated compact space X, then there exists a continuous map $f\colon X \to \mathbb{P}_d$ such that $L = f^*\xi_d$.*

Proof: If we see that the dual line bundle L^* is generated by a finite number of global sections $\{s_0, s_1, \dots, s_d\}$, then we have an epimorphism

$$X \times \mathbb{K}^{d+1} \longrightarrow L^*, \ (x, \lambda_0, \dots, \lambda_d) \mapsto \lambda_0 s_0(x) + \dots + \lambda_d s_d(x),$$

and the injection $L \to X \times \mathbb{K}^{d+1}$ defines a continuous map $f\colon X \to \mathbb{P}_d$ such that $L = f^*\xi_d$.

Now, since X is completely regular, for any point of X there is a global section generating the fibre on a neighborhood and, X being compact, a finite number of global sections generate the fibre at any point.

Corollary: *The obstruction class of the tautological line bundle generates the group $H^2(\mathbb{P}_d, \mathbb{Z})$ in the complex case, and the group $H^1(\mathbb{P}_d, \mathbb{F}_2)$ in the real case.*

Proof: Let $i\colon \mathbb{P}_{d-1} \to \mathbb{P}_d$ be a hyperplane. In the complex case, the closed subspace exact sequence shows that $i^*\colon H^2(\mathbb{P}_d, \mathbb{Z}) \to H^2(\mathbb{P}_{d-1}, \mathbb{Z})$ is an isomorphism. Since

$$i^*(\delta(\xi_d)) = \delta(i^*\xi_d) = \delta(\xi_{d-1}),$$

the index m of the subgroup generated by $\delta(\xi_d)$ in $H^2(\mathbb{P}_d, \mathbb{Z})$ does not depend on d.

The above lemma shows that any line bundle on $\mathbb{P}_1$ is a m-th power of a line bundle.

Absurd when $m \neq 1$, since $H^2(\mathbb{P}_1, \mathbb{Z}) \simeq \mathbb{Z}$.

The argument also holds in the real case. q.e.d.

From now on, in the complex case we shall always consider cohomology groups with coefficients in $\mathbb{Z}$, and in the real case with coefficients in $\mathbb{F}_2$, and we fix the orientations so that the obstruction class $\delta(\xi_d)$ is just the cohomology class of a hyperplane.

Hirsch-Leray Theorem: *Let E be a vector bundle of rank r over a σ-compact space X and let $x_E = \delta(\xi_E)$ be the obstruction class of the tautological bundle of $\mathbb{P}(E)$. Then $H^\bullet(\mathbb{P}(E))$ is a free $H^\bullet(X)$-module of base $\{1, x_E, x_E^2, \dots, x_E^{r-1}\}$.*

Proof: Let $\mathbb{Z} \to \mathcal{C}^\bullet$ be the Godement resolution of the constant sheaf $\mathbb{Z}$ on $\mathbb{P}(E)$.

Any cohomology class x_E^j, $0 \leq j \leq r-1$, is represented by a global section of $\mathcal{C}^{2j}$, and these sections define a morphism of complexes

$$\oplus_j \mathbb{Z}[-2j] \longrightarrow \pi_*\mathcal{C}^\bullet$$

which is a quasi-isomorphism (cohomology of the fibre, p. 352, and projective spaces, p. 366).

Now, since $\pi_*\mathcal{C}^\bullet$ is a complex of flasque sheaves, we have a group isomorphism

$$\oplus_j H^\bullet(X, \mathbb{Z})[-2j] = H^\bullet\big[\Gamma(X, \pi_*\mathcal{C}^\bullet)\big] = H^\bullet\big[\Gamma(\mathbb{P}(E), \mathcal{C}^\bullet)\big] = H^\bullet(\mathbb{P}(E), \mathbb{Z})$$

transforming $(a_0, \ldots, a_{r-1})$ into $\pi^*(a_0) + \pi^*(a_1)x_E + \ldots + \pi^*(a_{r-1})x_E^{r-1}$.

In fact $\Gamma(X, \pi_*\mathcal{C}^\bullet) \xrightarrow{\pi^*} \Gamma(\mathbb{P}(E), \pi^*\pi_*\mathcal{C}^\bullet) \to \Gamma(\mathbb{P}(E), \mathcal{C}^\bullet)$ is just the identity.

In the case of real vector bundles we replace $\mathbb{Z}$ by $\mathbb{F}_2$ and $2j$ by j.

Definitions: Let X be a σ-compact space. The **Chern classes** of a complex vector bundle $E \to X$ of rank r are the coefficients $c_i(E) \in H^{2i}(X, \mathbb{Z})$ of the characteristic polynomial of the endomorphism of the free $H^\bullet(X)$-module $H^\bullet(\mathbb{P}(E))$ defined by $x_E = \delta(\xi_E)$,

$$x_E^r + c_1(E)x_E^{r-1} + \ldots + c_r(E) = 0.$$

We agree that $c_0(E) = 1$ and $c_i(E) = 0$, $i > r$. The **total Chern class** is $c(E) = \sum_i c_i(E)$.

Analogously, when $E \to X$ is a real vector bundle, we have **Stieffel-Whitney classes** $w_i(E) \in H^i(X, \mathbb{F}_2)$, and we agree that $w_0(E) = 1$ and $w_i(E) = 0$, $i > r$. The **total Stieffel-Whitney class** is $w(E) = \sum_i w_i(E)$, and the Stieffel-Whitney classes $w_i(X)$ of a smooth manifold X are those of the tangent bundle, $w_i(X) = w_i(TX)$.

From now on we shall give statements and proofs only for the Chern classes, but they also hold for the Stieffel-Whitney classes.

Functoriality: $c_i(f^*E) = f^*(c_i(E))$, *for any continuous map* $f\colon T \to X$.

Proof: Let us consider the continuous map $1 \times f\colon \mathbb{P}(f^*E) = \mathbb{P}(E) \times_X T \longrightarrow \mathbb{P}(E) \times_X X = \mathbb{P}(E)$.

We have $\xi_{f^*E} = (1 \times f)^*\xi_E$, where $x_{f^*E} = (1 \times f)^*x_E$, and in $H^\bullet(\mathbb{P}(f^*E))$,

$$0 = (1 \times f)^*(x_E^r + a_1 x_E^{r-1} + \ldots + a_r) = x_{f^*E}^r + (f^*a_1)x_{f^*E}^{r-1} + \ldots + f^*a_r.$$

Theorem: $c_1(L) = -\delta(L)$, *for any line bundle* L. *Hence* $c_1(L \otimes L') = c_1(L) + c_1(L')$.

Proof: When L is a line bundle, $\mathbb{P}(L) = X$, $\xi_L = L$, $x_L = \delta(L)$, and $x_L + c_1(L) = 0$.

Theorem: *The class* $c_r(E)$, $r = \operatorname{rk} E$, *are the zeros of any continuous section* $s\colon X \to E$,

$$c_r(E) = s^*\big([s_0(X)]\big).$$

Proof: Let $\bar{x}$ be the obstruction class of the tautological bundle over $\bar{E} = \mathbb{P}(E \oplus 1)$.

By the Hirsch-Leray theorem, the cohomology class of the null section is

$$\big([s_0(X)]\big) = a_0\bar{x}^r + a_1\bar{x}^{r-1} + \ldots + a_r,\ a_i \in H^{2i}(X).$$

Its restriction to any fibre is the cohomology class of a point (vector bundles are locally trivial), which is just the restriction of $\bar{x}^r$.

Since the restriction of a_i, $i > 0$, is null, we see that $a_0 = 1$.

Now, the restriction of this class to the infinity $j\colon \mathbb{P}(E) \to \mathbb{P}(E \oplus 1)$ is null since the zero section s_0 does not intersect the infinity,

$$0 = j^*(\bar{x}^r + a_1\bar{x}^{r-1} + \ldots + a_r) = x_E^r + a_1 x_E^{r-1} + \ldots + a_r,$$

hence $a_i = c_i(E)$. Since the tautological bundle is trivial on the affine part E,

$$s^*\big([s_0(X)]\big) = s^*\big([s_0(X)]\big) = s^*(\bar{x}^r + a_1\bar{x}^{r-1} + \ldots + a_r) = a_r = c_r(E).$$

Splitting Principle: *Let* $E \to X$ *be a vector bundle of rank* r. *There exists a base change* $\pi\colon Y \to X$ *such that* π^*E *admits a filtration* $0 = E_r \subset \ldots \subset E_1 \subset E_0 = E$ *whose quotients* E_{i-1}/E_i *are line bundles and* $\pi^*\colon H^\bullet(X) \to H^\bullet(Y)$ *is injective.*

Proof: On $\mathbb{P}(E)$ we have that $\xi_E \hookrightarrow \pi^*E$ is a line bundle and π^*E/ξ_E is a vector bundle of rank $r-1$. Since $\pi^*\colon H^\bullet(X) \to H^\bullet(\mathbb{P}(E))$ is injective by the Hirsch-Leray theorem, we conclude by induction on the rank.

Additivity: *If* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of vector bundles, then*

$$c(E) = c(E') \cdot c(E'').$$

Proof: By the splitting principle there is a base change $p\colon Y \to X$ such that $p^*\colon H^\bullet(X) \to H^\bullet(Y)$ is injective and p^*E', p^*E'' admit filtrations with line quotients. Since p^* is injective and Chern classes are functorial, we are reduced to show that for any filtration $0 = E_r \subset \ldots \subset E_1 \subset E_0 = E$ with line quotients E_{i-1}/E_i we have $c(E) = (1+\alpha_1)\ldots(1+\alpha_r)$, where we put $\alpha_i = c_1(E_{i-1}/E_i)$.

We proceed by induction on rm and it is an identity when $r = 1$.

Let us consider the projection $\pi\colon \mathbb{P}(E) \to X$ and the inclusion $j\colon \mathbb{P}(E_1) \to \mathbb{P}(E)$.

The natural morphism $\xi_E \to \pi^*(E/E_1)$ defines a global section of $\pi^*(E/E_1) \otimes \xi_E^*$ vanishing on $\mathbb{P}(E_1)$, and transversally intersecting the zero section (check it locally). Hence

$$j_*(1) = c_1(\pi^*(E/E_1) \otimes \xi_E^*) = x_E + \alpha_1.$$

By induction $j^*\big[(x_E+\alpha_2)\ldots(x_E+\alpha_r)\big] = 0$, and applying j_*, the projection formula shows that $c_i(E)$ is just the i-th elementary symmetric function of $\alpha_1,\ldots,\alpha_r$,

$$0 = j_*j^*\big[(x_E+\alpha_2)\ldots(x_E+\alpha_r)\big] = (x_E+\alpha_2)\ldots(x_E+\alpha_r)j_*(1) = (x_E+\alpha_1)\ldots(x_E+\alpha_r).$$

Roots of a Vector Bundle: If $0 \to E_1 \to E \to E_2 \to 0$ is an exact sequence of vector bundles and we fix a hermitian metric (scalar product in the real case) on E, then $E = E_1 \oplus E_1^\perp$, and $E_1^\perp \simeq E_2$; hence $E \simeq E_1 \oplus E_2$.

Now, by the splitting principle, there is a base change $\pi\colon Y \to X$ such that $\pi^*\colon H^\bullet(X) \to H^\bullet(Y)$ is injective and π^*E is a direct sum of line bundles, $\pi^*E = L_1 \oplus \ldots \oplus L_r$, and we say that $\alpha_1 = c_1(L_1),\ldots,\alpha_r = c_1(L_r)$ are the "roots" of E, since $c_i(E)$ is just the i-th elementary symmetric function of $\alpha_1,\ldots,\alpha_r$. We also put $L_{\alpha_i} = L_i$.

1. $c_i(E^*) = (-1)^i c_i(E)$.

   If $E = L_{\alpha_1} \oplus \ldots \oplus L_{\alpha_r}$, then $E^* = L_{-\alpha_1} \oplus \ldots \oplus L_{-\alpha_r}$.

2. $c_1(E) = c_1(\Lambda^r E)$.

   If $E = L_{\alpha_1} \oplus \ldots \oplus L_{\alpha_r}$, then $\Lambda^r E = L_{\alpha_1} \otimes \ldots \otimes L_{\alpha_r} = L_{\alpha_1+\ldots+\alpha_r}$.

3. *If* $\mathfrak{p}_Y$ *is the sheaf of ideals of a hypersurface* Y *of a smooth manifold* X*, then* $w_1(\mathfrak{p}_Y) = [Y]$*; hence* Y *admits a global equation* $f = 0$ *with non null differential at any point of* Y *if and only if its cohomology class* $[Y] \in H^1(X, \mathbb{F}_2)$ *is null.*

   The inclusion $\mathfrak{p}_Y \to \mathcal{C}_X^\infty$ induces a section $s\colon \mathcal{C}_X^\infty \to \mathfrak{p}_Y^*$ transversally intersecting the zero section along Y; hence $[Y] = w_1(\mathfrak{p}_Y^*) = w_1(\mathfrak{p}_Y)$.

4. *If* N *is the normal bundle of a compact smooth submanifold* $i\colon Y \to X$ *of codimension* d*,* then $w_d(N) = i^*[Y]$.

   Since X admits a riemannian metric (p. 299) there is a neighborhood U of Y in X and a diffeomorphism $U \simeq N$ transforming i into the zero section (p. 422).

5. *Let* X *be a compact smooth manifold of dimension* n*. If* X *is a smooth submanifold of* $\mathbb{R}^{n+d}$*, then* $w(X)^{-1}$ *has degree* $< d$*.*

   We have an exact sequence $0 \to TX \to (T\mathbb{R}^{n+d})|_X \to N \to 0$ and he tangent bundle $T\mathbb{R}^{n+d}$ is trivial; hence $w(X)^{-1} = w(N) = 1 + w_1(N) + \ldots + w_d(N)$, and $w_d(N) = 0$ since the cohomology class of X in $\mathbb{R}^{n+d}$ is null.

6. $w(\mathbb{P}_{n,\mathbb{R}}) = (1+x)^{n+1}$.

   If $E = \mathbb{R}^{n+1}$, the projection $E - \{0\} \to \mathbb{P}_n$ induces, at any vector $e \neq 0$, an identification of $E/\mathbb{R}e$ with the tangent space to $\mathbb{P}_n$ at the point $\langle e \rangle$, so defining a canonical isomorphism $T\mathbb{P}_n = \mathrm{Hom}(\xi_n, E/\xi_n)$, and we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\xi_n, \xi_n) \longrightarrow \mathrm{Hom}(\xi_n, E) \longrightarrow T\mathbb{P}_n \longrightarrow 0,$$
$$w(\mathbb{P}_n) = w(\xi_n^*)^{n+1} = (1 + \delta(\xi_n))^{n+1} = (1+x)^{n+1}.$$

7. *$\mathbb{P}_4$ is not a smooth submanifold of $\mathbb{R}^7$.*

   $w(\mathbb{P}_4)^{-1} = (1 + x + x^4)^{-1} = 1 + x + x^2 + x^3$.

8. *If $\mathbb{P}_n$ is parallelizable, then $n+1$ is a power of 2.*

   If $n+1 = 2^r m$, with m odd, then the tangent bundle $T\mathbb{P}_n$ is not trivial because $w_{2^r}(\mathbb{P}_n) \neq 0$,

$$(1+x)^{n+1} = \left(1 + x^{2^r}\right)^m = 1 + mx^{2^r} + \dots$$

9. *If there is a bilinear product on $\mathbb{R}^n$ without zero divisors, then n is a power of 2.*

   If $a_1, \dots, a_n$ is a base of $\mathbb{R}^n$ and we put $v_i(x) = xa_1^{-1}a_i$, then $v_1(x) = x, v_2(x), \dots, v_n(x)$ are linearly independent for any $x \neq 0$. Hence $v_2, \dots, v_n$ define linearly independent sections of $T\mathbb{P}_{n-1} = \mathrm{Hom}(\xi, \mathbb{R}^n/\xi)$, and $\mathbb{P}_{n-1}$ is parallelizable.

12.9 Spectral Sequences

We work in the category of modules over a ring (or a sheaf of rings). Any exact triangle

$$\begin{array}{ccc} C_1 & \xrightarrow{i_1} & C_1 \\ {\scriptstyle \delta_1}\nwarrow & & \swarrow{\scriptstyle j_1} \\ & E_1 & \end{array}$$

defines a differential $\mathrm{d}_1 = j_1\delta_1 \colon E_1 \longrightarrow E_1$, $\mathrm{d}_1^2 = j_1\delta_1 j_1\delta_1 = 0$, and we put:

$C_2 = \mathrm{Im}\, i_1$, $E_2 = H(E_1) = \mathrm{Ker}\,\mathrm{d}_1/\mathrm{Im}\,\mathrm{d}_1$.

$i_2 \colon C_2 \to C_2$, the restriction of i_1 a C_2.

$\delta_2 \colon E_2 \to C_2$, the morphism induced by $\delta_1 \colon \mathrm{Ker}\,\mathrm{d}_1 \to C_2$ on the quotient.

$j_2 \colon C_2 \to E_2$, the morphism defined by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \delta_1(E_1) & \longrightarrow & C_1 & \xrightarrow{i_1} & C_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \\ 0 & \longrightarrow & j_1\delta_1(E_1) & \longrightarrow & \mathrm{Ker}\,\mathrm{d}_1 & \longrightarrow & E_2 & \longrightarrow & 0 \end{array}$$

Theorem: *So we obtain a derived exact triangle*

$$\begin{array}{ccc} C_2 & \xrightarrow{i_2} & C_2 \\ {\scriptstyle \delta_2}\nwarrow & & \swarrow{\scriptstyle j_2} \\ & E_2 & \end{array}$$

Proof: The equality $i_2(C_2) = i_1(\mathrm{Ker}\, j_1) = \mathrm{Ker}\, j_2$ follows from the snake's lemma, applied to the above diagram. The other two are immediate. q.e.d.

Iterating we obtain derived exact triangles and differentials

$$\begin{array}{ccc} C_r & \xrightarrow{i_r} & C_r \\ & {\scriptstyle \delta_r}\nwarrow \quad \swarrow{\scriptstyle j_r} & \\ & E_r & \end{array}, \quad \mathrm{d}_r = j_r\delta_r \colon E_r \longrightarrow E_r$$

Now let (M, d) be a filtered differential module: $M = \bigcup_p F^p$, $F^{p+1} \subseteq F^p$, $\mathrm{d}F^p \subseteq F^p$. The exact sequence

$$0 \longrightarrow \oplus_p F^{p+1} \xrightarrow{i} \oplus_p F^p \longrightarrow \oplus_p F^p/F^{p+1} \longrightarrow 0$$

induces (p. 318) an exact triangle
$$\begin{array}{ccc} C_1 & \xrightarrow{i_1} & C_1 = \oplus_p H(F^p) \\ & {\scriptstyle \delta_1}\nwarrow & \downarrow{\scriptstyle j_1} \\ & & E_1 = \oplus_p H(F^p/F^{p+1}) \end{array}$$

Let $i_k^p \colon H(F^p) \to H(F^{p-k})$ be the natural morphism. In any derived triangle we have

$$C_r = \operatorname{Im} i_{r-1} = \bigoplus_p \operatorname{Im} i_{r-1}^{p+r-1} \subseteq \bigoplus_p H(F^p).$$

Moreover $E_r = \oplus_p E_r^p$, and the derived triangles decompose as exact sequences

$$\begin{array}{ccc} C_r & \xrightarrow{i_r} & C_r = \operatorname{Im} i_{r-1} \\ & {\scriptstyle \delta_r}\nwarrow \quad \swarrow{\scriptstyle j_r} & \\ & E_r & \end{array}$$

$$\ldots \longrightarrow E_r^p \xrightarrow{\delta_r} \operatorname{Im} i_{r-1}^{p+r} \xrightarrow{i_r} \operatorname{Im} i_{r-1}^{p+r-1} \xrightarrow{j_r} E_r^{p+r-1} \xrightarrow{\delta_r} \ldots$$

and the differential d_r is given by morphisms

$$\mathrm{d}_r = j_r\delta_r \colon E_r^p \longrightarrow E_r^{p+r}.$$

Now we assume that M is a complex, $M = \oplus_n M^n$, $\mathrm{d}M^n \subseteq M^{n+1}$, with a compatible filtration, $F^p = \oplus_n (F^p \cap M^n) = \oplus_n M^{p,n-p}$.

Let $i_k^{p,n-p} \colon H^n(F^p) \to H^n(F^{p-k})$ be the natural morphism. Then

$$C_r = \operatorname{Im} i_{r-1} = \oplus_{p,n} \operatorname{Im} i_{r-1}^{p+r-1,n+1-p-r} \subseteq \oplus_{p,n} H^n(F^p).$$

Moreover $E_r = \oplus_{p,n} E_r^{p,n-p}$, the triangles decompose as exact sequences

$$\ldots \longrightarrow E_r^{p,q} \xrightarrow{\delta_r} \operatorname{Im} i_{r-1}^{p+r,q+1-r} \xrightarrow{i_r} \operatorname{Im} i_{r-1}^{p+r-1,q+2-r} \xrightarrow{j_r} E_r^{p+r-1,q+2-r} \longrightarrow \ldots$$

and the differential d_r is given by morphisms

$$\mathrm{d}_r = j_r\delta_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

On the other hand, the images of natural morphisms $i_\infty^{p,n-p} \colon H^n(F^p) \to H^n(M)$ define a filtration of $H^n(M)$, and we shall study whether the spectral sequence converges to

$$GH^n(M) = \bigoplus_p E_\infty^{p,n-p}, \quad E_\infty^{p,n-p} = \operatorname{Im} i_\infty^{p,n-p} / \operatorname{Im} i_\infty^{p+1,n-p-1}.$$

Definition: The filtration is **regular** if for all n we have $H^n(F^p) = 0$ when $p \gg 0$.

Theorem: *If the filtration is regular, then the spectral sequence* ***converges****,*

$$E_\infty^{p,q} = \varinjlim E_r^{p,q}$$

and in such a case we put $E_2^{p,q} \Rightarrow H^{p+q}(M)$.

Proof: Once we fix p and n, the morphism $\delta_r \colon E_r^{p,n-p} \to \operatorname{Im} i_{r-1}^{p+r,n+1-p-r}$ is null when $r \gg 0$, because $i_{r-1}^{p+r,n+1-p-r} \colon 0 = H^{n+1}(F^{p+r}) \to H^{n+1}(F^{p+1})$,

$$\operatorname{Im} i_{r-1}^{p+1,n-p-1} \longrightarrow \operatorname{Im} i_{r-1}^{p,n-p} \longrightarrow E_r^{p,n-p} \longrightarrow 0.$$

Moreover $\mathrm{d}_r = j_r \delta_r$ vanishes on $E_r^{p,n-p}$, and we have epimorphisms $E_r^{p,n-p} \to E_{r+1}^{p,n-p}$. Taking inductive limit on r, we conclude,

$$\operatorname{Im} i_\infty^{p,n-p} \longrightarrow \operatorname{Im} i_\infty^{p+1,n-p-1} \longrightarrow \varinjlim E_r^{p,n-p} \longrightarrow 0.$$

Theorem: *Let $\phi \colon M \to \bar{M}$ be a morphism of filtered complexes, $\phi(F^p) \subseteq \bar{F}^p$. If both filtrations are regular and ϕ induces isomorphisms $E_r^{p,q} \xrightarrow{\sim} \bar{E}_r^{p,q}$ for some index r, then ϕ is a quasi-isomorphism, $\phi \colon H^n(M) \xrightarrow{\sim} H^n(\bar{M})$.*

Proof: Since the spectral sequences converge, ϕ induces an isomorphism $GH^n(M) \xrightarrow{\sim} GH^n(\bar{M})$; hence also on completions (p. 203), but $H^n(M)$ is complete when the filtration is regular because the morphisms $i_\infty^{p,n-p} \colon H^n(F^p) \to H^n(M)$, $p \gg 0$, are null.

Bicomplex Spectral Sequence: A bicomplex $K^{\bullet\bullet}$ admits the filtration $F^p = \bigoplus_{i \geq p} \bigoplus_j K^{i,j}$,

$$E_1^{p,q} = H_{\mathrm{d}_2}^q(K^{p\bullet}),$$
$$E_2^{p,q} = H_{\mathrm{d}_1}^p(H_{\mathrm{d}_2}^q(K^{\bullet\bullet})).$$

If the filtration is regular (for example, if $K^{\bullet\bullet}$ has bounded below diagonals) we have a spectral sequence converging to the cohomology of the bicomplex

$$E_2^{p,q} = H_{\mathrm{d}_1}^p(H_{\mathrm{d}_2}^q(K^{\bullet\bullet})) \;\Rightarrow\; H^{p+q}(K^{\bullet\bullet}).$$

Analogously, if the bicomplex $K^{\bullet\bullet}$ has bounded above diagonals, we obtain a convergent spectral sequence when considering the filtration $F^p = \bigoplus_i \bigoplus_{j \geq p} K^{i,j}$,

$$E_2^{p,q} = H_{\mathrm{d}_2}^p(H_{\mathrm{d}_1}^q(K^{\bullet\bullet})) \;\Rightarrow\; H^{p+q}(K^{\bullet\bullet}).$$

Hypercohomology Spectral Sequence: Let $K^\bullet$ be a bounded below complex.

If we fix injective resolutions $I_B^{p,\bullet}$ and $I_H^{p,\bullet}$ of the boundaries B^p and the cohomology H^p of $K^\bullet$, then we have exact sequences

$$0 \longrightarrow I_B^{p,\bullet} \longrightarrow I_Z^{p,\bullet} \longrightarrow I_H^{p,\bullet} \longrightarrow 0,$$

where $I_Z^{p,\bullet}$ is an injective resolution of the cycles Z^p, and exact sequences

$$0 \longrightarrow I_Z^{p,\bullet} \longrightarrow I^{p,\bullet} \longrightarrow I_B^{p+1,\bullet} \longrightarrow 0,$$

where $I^{p,\bullet}$ is an injective resolution of K^p. By the bicomplex theorem, $K^\bullet \to I^{\bullet\bullet}$ is a quasi-isomorphism, and moreover cycles, boundaries and cohomology of $I^{\bullet\bullet}$ respect to the differential d_1 are injective resolutions of cycles, boundaries and cohomology of $K^\bullet$.

Now, if F is a left exact additive functor, then $H^n\big[F(I^{\bullet\bullet})\big] = \mathbf{R}^n F(K^\bullet)$, and

$$H^p_{\mathrm{d}_1}\big[F(I^{\bullet\bullet})\big] = F\big[H^p_{\mathrm{d}_1}(I^{\bullet\bullet})\big] = F(I^{p\bullet}_H),$$

since the cycles, boundaries and cohomology of the complex $(I^{\bullet q}, \mathrm{d}_1)$ are injective, so that any additive functor preserves them. Hence $H^p_{\mathrm{d}_2}(H^q_{\mathrm{d}_1}(I^{\bullet\bullet})) = R^pF(H^q(K^\bullet))$, and the second spectral sequence of the bicomplex $F(I^{\bullet\bullet})$ is

$$E_2^{p,q} = R^pF(H^q(K^\bullet)) \;\Rightarrow\; \mathbf{R}^{p+q}F(K^\bullet).$$

Grothendieck's Spectral Sequence: *Let $F\colon \mathbf{A} \rightsquigarrow \mathbf{B}$ and $G\colon \mathbf{B} \rightsquigarrow \mathbf{C}$ be left exact covariant functors. If F transforms injective objects into G-acyclic objects, then for any bounded below complex $K^\bullet$ of $\mathbf{A}$ we have*

$$\mathbf{R}(GF)(K^\bullet) \xrightarrow{\sim} \mathbf{R}G(\mathbf{R}F(K^\bullet)),$$
$$E_2^{p,q} = R^pG(R^qF(M)) \;\Rightarrow\; R^{p+q}(GF)(M).$$

Proof: Let $K^\bullet \xrightarrow{\sim} I^\bullet$ be an injective resolution. Since $F(I^\bullet)$ is G-acyclic,

$$\mathbf{R}(GF)(K^\bullet) = GF(I^\bullet) \xrightarrow{\sim} \mathbf{R}G(F(I^\bullet)) = \mathbf{R}G(\mathbf{R}F(K^\bullet)),$$

and we have the hypercohomology spectral sequence

$$E_2^{p,q} = R^pG(\mathbf{R}^qF(K^\bullet)) \;\Rightarrow\; \mathbf{R}^{p+q}G(F(I^\bullet)) = \mathbf{R}^{p+q}(GF)(K^\bullet).$$

Leray's Spectral Sequence: *If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then we have spectral sequences*

$$E_2^{p,q} = R^pg_*(R^qf_*\mathcal{F}) \;\Rightarrow\; R^{p+q}(gf)_*\mathcal{F}.$$

Proof: The functor f_* transforms flasque sheaves into flasque sheaves; hence g_*-acyclic. q.e.d.

1. $E_2^{p,q} = H^p(Y, R^qf_*\mathcal{F}) \;\Rightarrow\; H^{p+q}(X, \mathcal{F})$, (*when $f\colon X \to Y$ is a continuous map*).

2. $E_2^{p,q} = H^p_c(Y, R^qf_!\mathcal{F}) \;\Rightarrow\; H^{p+q}_c(X, \mathcal{F})$, (*if moreover X, Y are σ-compact*).

   $\Gamma_c(X,-) = \Gamma_c(Y,-)\circ f_!$, and $f_!(C^p\mathcal{F})$ is a $C^0\mathbb{Z}$-module; hence it is $\Gamma_c(Y,-)$-acyclic.

3. $E_2^{p,q} = H^p(X, \mathcal{H}^q_Y\mathcal{F}) \;\Rightarrow\; H^{p+q}_Y(X, \mathcal{F})$, (*when Y is a closed set in X*).

   $\Gamma_Y = \Gamma \circ \underline{\Gamma}_Y$ and, if $\mathcal{C}$ is a Godement sheaf, then $\underline{\Gamma}_Y\mathcal{C}$ is flasque (p. 364).

4. $E_2^{p,q} = H^p(X, \underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{M}, \mathcal{N})) \;\Rightarrow\; \mathrm{Ext}^{p+q}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$.

   $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M},-) = \Gamma \circ \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},-)$ and, if $\mathcal{I}$ is an injective $\mathcal{O}$-module, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{I})$ is a flasque sheaf since all the restriction morphisms are surjective,

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{I}) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}_U, \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{M}|_U, \mathcal{I}|_U).$$

Chapter 13

Analysis IV

13.1 Dirichlet Problem

Lemma: *Let $U \subseteq \mathbb{R}^d$ be an open set. If V is a connected open set with compact closure $\bar{V} \subset U$, then there is a constant c such that for any harmonic function $u > 0$ on U we have*

$$u(x) \leq cu(y) \ , \quad \forall x, y \in V.$$

Proof: Fix $0 < r < \frac{1}{2}\mathrm{d}(V, \partial U)$. If $x, y \in V$ and $|y - x| < r$, then $B(x, r) \subset B(y, 2r) \Subset U$, so that

$$u(x) = \frac{1}{\mathrm{Vol}\, B_r} \int_{B(x,r)} u\mathrm{d}m \leq \frac{\mathrm{Vol}\, B_{2r}}{\mathrm{Vol}\, B_r} \frac{1}{\mathrm{Vol}\, B_{2r}} \int_{B(y,2r)} u\mathrm{d}m = 2^d u(y).$$

Since $\bar{V}$ is compact, it admits a cover by a finite number m of balls of radius $r/2$.

Since V is connected, given $x, y \in V$ there is a sequence $x = x_0, \ldots, x_k = y$ with $|x_i - x_{i-1}| < r$ and $k \leq m + 1$. Hence

$$u(x) \leq 2^d u(x_1) \leq 2^{2d} u(x_2) \leq \ldots \leq 2^{kd} u(y) \leq 2^{(m+1)d} u(y).$$

Harnack's Theorem: *Let (u_n) be an increasing sequence of harmonic functions on a connected open set $U \subseteq \mathbb{R}^d$. If there is a point $p \in U$ such that the sequence $u_n(p)$ is bounded, then (u_n) uniformly converges on any compact set to an harmonic function u on U.*

Proof: Given $\varepsilon > 0$, there is an index k such that $u_n(p) - u_m(p) < \varepsilon$, for any $n \geq m \geq k$.

If V is a connected bounded neighborhood of p with $\bar{V} \subset U$, by the lemma

$$u_n(x) - u_m(x) \leq c\big(u_n(p) - u_m(p)\big) < c\varepsilon,$$

so that (u_n) uniformly converges on V, and the limit function is harmonic (p. 273).

Proposition: *Let $\mathcal{P}_k$ be the vector space of all homogeneous polynomials of degree k on $\mathbb{R}^d$, and $\mathcal{H}_k \subset \mathcal{P}_k$ the vector subspace of harmonic polynomials. If we put $r^2 = x_1^2 + \ldots + x_d^2$, then*

$$\mathcal{P}_k = \mathcal{H}_k \oplus r^2 \mathcal{H}_{k-2} \oplus r^4 \mathcal{H}_{k-4} \oplus \ldots$$

Proof: Any homogeneous polynomial $P = \sum \lambda_\alpha x^\alpha$ defines a differential operator $D_P = \sum \lambda_\alpha \partial_\alpha$, and $P \cdot Q := D_P(Q)$ defines a scalar product on $\mathcal{P}_k$ since $x^\alpha \cdot x^\beta = (\alpha!)\delta_{\alpha\beta}$. Moreover,

$$(\Delta P_k) \cdot Q_{k-2} = P_k \cdot (r^2 Q_{k-2}),$$

so that $\mathcal{H}_k = (r^2 \mathcal{P}_{k-2})^\perp$. Hence $\mathcal{P}_k = \mathcal{H}_k \perp (r^2 \mathcal{P}_{k-2})$, and we conclude by induction on k.

Dirichlet Problem (on Balls)**:** *Let $B \subset \mathbb{R}^d$ be an open ball. If $f \in \mathcal{C}(\partial B)$, then there is a unique extension $u \in \mathcal{C}(\bar{B})$ harmonic on B.*

Proof: By the maximum principle, the extension is unique, and to prove the existence we may assume that $B = B(0, r_0)$.

If f is the restriction of a homogeneous polynomial P_k, then $P_k = H_k + r^2 H_{k-2} + r^4 H_{k-4} + \dots$ for some harmonic polynomials H_i, and $u_k = H_k + r_0^2 H_{k-2} + r_0^4 H_{k-4} + \dots$ is the extension.

When f is the restriction of a polynomial $P = \sum_k P_k$, just put $u = \sum_k u_k$.

Finally, by the Stone-Weierstrass theorem, f is a uniform limit $f = \lim p_n$ of polynomial functions p_n on ∂B. If u_n is the harmonic extension of p_n, then (u_n) converges because by the maximum principle $\|u_n - u_m\|_{\bar{B}} \leq \|p_n - p_m\|_{\partial B}$, and $u = \lim u_n$ is the required extension.

Corollary: *Let p be a point of an open set $U \subseteq \mathbb{R}^d$. If an harmonic function u on $U - p$ is bounded in a neighborhood of p, it is a removable singularity. Hence, if u may be continuously extended to p, the extension is harmonic.*

Proof: We may assume that $p = 0$ and $\bar{B} \subset U$, where $B = B(0, 1)$.

By the above result, there exists an extension $v \in \mathcal{C}(\bar{B})$ of $u|_{\partial B}$, harmonic on B, and we conclude if we show that $v = u$ on $B - 0$.

Given $\varepsilon > 0$, the function $u - v + \varepsilon\big(1 - |x|^{2-d}\big)$, resp. $u - v + \varepsilon \ln |x|$ when $d = 2$, is continuous on $\bar{B} - 0$, harmonic on $B - 0$, null on ∂B and it goes to $-\infty$ at the origin. By the maximum principle, it is negative on $B - 0$. When $\varepsilon \to 0$, we see that $u - v \leq 0$ on $B - 0$.

The same argument, with $\varepsilon < 0$, shows that $u - v \geq 0$ on $B - 0$.

Definition: Let $U \subseteq \mathbb{R}^d$ be an open set, $d \geq 2$. A function $u \in \mathcal{C}(U)$ is **subharmonic** if any point $p \in U$ has a neighborhood V such that for any closed ball $\bar{B} \subset V$ with centre at p we have

$$u(p) \leq \frac{1}{\operatorname{Vol} B} \int_B u \,\mathrm{d}m.$$

The sum and the maximum of two subharmonic functions also are subharmonic functions.

Maximum Principle: *If a subharmonic function u on a connected open set $U \subseteq \mathbb{R}^d$ attains a maximum, then it is constant.*

Proof: The points $p \in U$ where u attains the maximum M form an open set because if u is not the constant M on a ball $B \subset U$ with centre at p, then $\frac{1}{\operatorname{Vol} B} \int_B u \,\mathrm{d}m < M = u(p)$.

Theorem: *If $u \in \mathcal{C}(U)$, then the following conditions are equivalent*:

1. *The function u is subharmonic on U.*

2. *If V is a connected open set with compact closure $\bar{V} \subset U$ and $v \in \mathcal{C}(\bar{V})$ is harmonic on V and $u \leq v$ on ∂V, then $u \leq v$ on V.*

3. *For any closed ball $\bar{B} = \bar{B}(p, r) \subset U$ we have* $u(p) \leq \dfrac{1}{\operatorname{Vol} S} \displaystyle\int_S u \,\mathrm{d}S$, where $S = \partial \bar{B}$.

4. *For any closed ball $\bar{B} = \bar{B}(p, r) \subset U$ we have* $u(p) \leq \dfrac{1}{\operatorname{Vol} \bar{B}} \displaystyle\int_{\bar{B}} u \,\mathrm{d}m$.

Proof: ($1 \Rightarrow 2$) The function $u-v$ is subharmonic on V. Since $u-v \le 0$ on ∂V, by the maximum principle $u-v \le 0$ on V.

($2 \Rightarrow 3$) Let us consider an extension $v \in \mathcal{C}(\bar{B})$ of $u|_S$, harmonic on B. By hypothesis $u \le v$ on B, so that

$$u(p) \le v(p) = \frac{1}{\operatorname{Vol} S}\int_S v \,\mathrm{d}S = \frac{1}{\operatorname{Vol} S}\int_S u \,\mathrm{d}S.$$

($3 \Rightarrow 4$) Since $u(p) - u$ has non-negative integral on any sphere $S(p,\rho)$, $0 < \rho \le r$, it also has (p. 272) non-negative integral on $\bar{B}(p,r)$. Finally, ($4 \Rightarrow 1$) is obvious.

Lemma: *Let $\bar{B} \subset U$ be a closed ball. If $u \in \mathcal{C}(U)$ is subharmonic, then the function $u_B \in \mathcal{C}(U)$, harmonic on B and such that $u_B = u$ on $U - B$, also is subharmonic.*

Proof: We have $u \le u_B$ by the point 2 of the characterization.

Hence on any small sphere S with centre at $p \in U - B$ we have

$$u_B(p) = u(p) \le \frac{1}{\operatorname{Vol} S}\int_S u \,\mathrm{d}S \le \frac{1}{\operatorname{Vol} S}\int_S u_B \,\mathrm{d}S.$$

Perron Method: *Let $U \subset \mathbb{R}^d$ be a connected open set with compact closure. If $f \in \mathcal{C}(\partial U)$, then $\mathcal{F} = \{u \in \mathcal{C}(\bar{U})\colon u|_{\partial U} \le f$ and u subharmonic on $U\}$ defines an harmonic function on U:*

$$h(x) = \sup_{u\in\mathcal{F}}\{u(x)\}.$$

Proof: By the maximum principle $u(x) \le \max_{\partial U} u \le \max_{\partial U} f < \infty$; hence h is well-defined.

Given $p \in U$, we consider a sequence $u_n \in \mathcal{F}$ such that $h(p) = \lim u_n(p)$. Replacing u_n by $\max\{u_1, \dots, u_n\}$, we may assume that $u_n \le u_{n+1}$. Fix a closed ball $\bar{B} \subset U$ with centre at p. Replacing u_n by $(u_n)_B$ we may also assume that u_n is harmonic on B (if $u \le u'$, then $u_B \le u'_B$ by the maximum principle).

Now, by Harnak's theorem, $u = \lim u_n$ is harmonic on B, and $u(p) = h(p)$.

To conclude, it is enough to show that $u = h$ on B:

If $u(q) < h(q)$ at some $q \in B$, then there is $g \in \mathcal{F}$ such that $u(q) < g(q) \le h(q)$. Let $\bar{u}_n = (\max\{u_n, g\})_B$, so that $\bar{u} = \lim \bar{u}_n$ is an harmonic function on B such that $u \le \bar{u}$ and $u(p) = \bar{u}(p) = h(p)$. By the minimum principle $u = \bar{u}$ on B. Absurd, $u(q) < \bar{u}(q)$. q.e.d.

The problem is to determine whether h defines a continuous extension of f.

Definition: A **barrier** for U at $y_0 \in \partial U$ is a function $b \in \mathcal{C}(\bar{\Omega})$, harmonic on U, such that $b > 0$ except $b(y_0) = 0$. A boundary point y_0 is **regular** if there is a barrier for U at y_0.

Examples: If there is a closed ball $\bar{B} = \bar{B}(p,r)$ such that $\bar{B} \cap \bar{U} = y_0$, then y_0 is a regular point. A barrier is $b(x) = r^{2-d} - |x-p|^{2-d}$, resp. $b(x) = \ln|x-p| - \ln r$ when $d = 2$.

If U is convex (contains the segment joining any two points of U), the boundary is regular.

If the boundary of U is a smooth manifold, the boundary is regular.

Lemma: *Let h be the harmonic function provided by the Perron method. If $y_0 \in \partial U$ is a regular point, then $\lim_{x\to y_0} h(x) = f(y_0)$, $(x \in U)$.*

Proof: Given $\varepsilon > 0$, since f and b are continuous on ∂U, there is $\lambda \gg 0$ such that

$$\begin{aligned} |f(y) - f(y_0)| &< \varepsilon + \lambda b(y) && , \quad \forall y \in \partial U, \\ f(y_0) - \lambda b(y) - \varepsilon < f(y) &< f(y_0) + \lambda b(y) + \varepsilon && , \quad \forall y \in \partial U. \end{aligned}$$

Hence, all the harmonic functions $f(y_0) - \lambda b(x) - \varepsilon$ are in $\mathcal{F}$, while $f(y_0) + \lambda b(x) + \varepsilon$ bounds above any function in $\mathcal{F}$. Therefore

$$f(y_0) - \lambda b(x) - \varepsilon \leq h(x) \leq f(y_0) + \lambda b(x) + \varepsilon \ , \quad \forall x \in U.$$

When $x \to y_0$, we conclude that $f(y_0) - \varepsilon \leq \lim_{x \to y_0} h(x) \leq f(y_0) + \varepsilon$.

Theorem: *Let $U \subset \mathbb{R}^d$ be a connected open set with compact closure and regular boundary. Any function $f \in \mathcal{C}(\partial U)$ admits a unique extension $h \in \mathcal{C}(\bar{U})$ harmonic on U.*

13.1.1 Uniformization Theorem

Definitions: Let X be a Riemann surface. The **laplacian** of $u \in \mathcal{C}^2(X)$ is the continuous 2-form $\Delta u := \mathrm{d}(*\mathrm{d}u)$, where $*$ is the transpose of J, and u is **harmonic** when $\Delta u = 0$.

In a local analytic coordinate $z = x + yi$ we have $*\mathrm{d}x = -\mathrm{d}y$, $*\mathrm{d}y = \mathrm{d}x$, so that

$$\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \mathrm{d}x \wedge \mathrm{d}y.$$

A function $u \in \mathcal{C}(X)$ is **subharmonic** when so it is in a coordinate neighborhood of each point of X. In fact, this condition does not depend on the fixed coordinates, by the point 2 of the characterization of subharmonic functions.

Examples: If $f = u + vi$ is an analytic function, then u and v are harmonic, by the Cauchy-Riemann equations. Conversely, if u is an harmonic function and $H^1(X, \mathbb{R}) = 0$, then there is a (unique up to the addition of a constant) function $v \in \mathcal{C}^2(X)$ such that $*\mathrm{d}u = \mathrm{d}v$, and $f = u + vi$ is analytic because locally it satisfies the Cauchy-Riemann equations.

If (U, z) is coordinate neighborhood of a point $p \in X$ and $z(p) = 0$, then $\ln |z|$ is harmonic on $U - p$. Locally, it is the real part of the analytic function $\ln z = \ln |z| + (\arg z)i$.

If f is an analytic function and $f(p) \neq 0$, then $\ln |f|$ is harmonic in a neighborhood of p, because it is the real part of $\ln f$.

Definition: Let $U \subset X$ be a relatively compact open set. A **Green's function** on U with pole at $p \in U$ is a function $g \in \mathcal{C}(\bar{U} - p)$ such that

1. $g = 0$ on the boundary ∂U, and g is harmonic on $U - p$.
2. $g - \ln |z|$ has a removable singularity at p, where z is a local coordinate such that $z(p) = 0$. (This condition does not depend on the fixed local coordinate).

If it exists, such function g is unique, since the difference of two Green's functions would be harmonic on U and null on ∂U.

If f is an analytic function, then $\ln |f|$ is the Green's function when $|f| = 1$ on ∂U and f has no zero on U, except a simple zero at z.

Imagine that the electrostatical force between punctual charges in $\mathbb{C}$ is proportional to the charges and inversely proportional to the distance (and fix the charge unit so that Coulomb's constant is 1). Then the electric potential of a charge e is just $e \ln r$. If $\mathbb{C} - U$ is filled with a conductor material and we put a positive unit charge at p, so that the negative charges in $\mathbb{C} - U$ move and attain an equilibrium position, the electric potential is just the Green's function of U with pole at p.

Theorem: *If the boundary ∂U is regular, then the Green's function with pole at p exists.*

Proof: We may assume that, in the local coordinate z, the unit disk $\bar{\mathbb{D}} \subset U$. To construct $-g$ with Perron method we consider the family $\mathcal{F}$ of functions $u \in \mathcal{C}(\bar{U} - p)$ such that

1. $u = 0$ on ∂U, and u is subharmonic on $U - p$.
2. $u + \ln|z|$ is subharmonic in a neighborhood of p.

This family $\mathcal{F}$ is not empty, because it has the function

$$u = \begin{cases} -\ln|z| & \text{when } |z| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let us see that the functions $u \in \mathcal{F}$ are uniformly bounded on $U - \mathbb{D}_\varepsilon$, $\varepsilon < 1$:

Let h be the continuous function on $\bar{U} - \mathbb{D}_\varepsilon$, harmonic on $U - \bar{\mathbb{D}}_\varepsilon$, such that $h = 1$ on $\partial\mathbb{D}_\varepsilon$ and $h = 0$ on ∂U, and let u_ε be the maximum of u on $\partial\mathbb{D}_\varepsilon$. Applying the maximum principle to the subharmonic function $u + \ln|z|$ on the unit disk $\mathbb{D}$, we obtain

$$u_\varepsilon + \ln\varepsilon \leq u_1 + \ln 1 = u_1.$$

Moreover, $u_\varepsilon h$ is harmonic on $U - \bar{\mathbb{D}}_\varepsilon$ and $u \leq u_\varepsilon h$ on the boundary; hence $u \leq u_\varepsilon h$ on $U - \mathbb{D}_\varepsilon$. In particular $u_1 \leq u_\varepsilon h_1$. Hence $u_\varepsilon + \ln\varepsilon \leq u_\varepsilon + h_1$, and we conclude that

$$\max_{x \in U - \mathbb{D}_\varepsilon} \{u(x)\} = u_\varepsilon \leq \frac{-\ln\varepsilon}{1 - h_1}.$$

Now the proof of the Perron method shows that the function $-g(x) := \sup_{u\in\mathcal{F}}\{u(x)\}$ is harmonic on $U - p$, and may be continuously extended by 0 on ∂U.

Finally, let us see that p is a removable singularity of the harmonic function $-g + \ln|z|$:

If $u \in \mathcal{F}$, then on $\mathbb{D}_\varepsilon$ we have

$$u + \ln|z| \leq u_\varepsilon + \ln\varepsilon \leq \frac{-\ln\varepsilon}{1 - h_1} + \ln\varepsilon = \frac{-h_1 \ln\varepsilon}{1 - h_1}$$

so that $-g + \ln|z|$ is bounded above on $\mathbb{D}_\varepsilon$. On the other hand, the function

$$u = \begin{cases} \ln\varepsilon - \ln|z| & \text{when } |z| \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

is in the family $\mathcal{F}$; hence $u \leq -g$, so that $\ln\varepsilon \leq -g + \ln|z|$ on $\mathbb{D}_\varepsilon$, and $-g + \ln|z|$ also is bounded below on $\mathbb{D}_\varepsilon$. We conclude that p is a removable singularity of $-g + \ln|z|$.

Lemma: *Let $U \subset X$ be a connected open set with regular boundary, and g the Green's function on U with pole at $p \in U$. If $H^1(U, \mathbb{R}) = 0$, then there exists $f \in \mathcal{O}(U)$ such that $|f| = e^g$.*

Proof: Given an harmonic function u, let us denote $\tilde{u}$ an analytic function (it exists locally) such that $u = \text{Re}(\tilde{u})$. The equation $|f| = e^g$ admits the local solution $f = e^{\tilde{g}}$ at any point, except at p, where locally we have $g = u + \ln|z|$ for some harmonic function u, and a local solution is $f = ze^{\tilde{u}}$. Moreover, any two local solutions have equal modulus, so that both differ in a of modulus 1. That is to say, the etalé space of the sheaf of solutions of $|f| = e^g$ is a principal covering of U of group $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and we conclude if we show that it is a trivial covering (i.e. it admits global sections).

Now, the principal coverings of U of group $\mathbb{T}$ are classified (p. 359) by $H^1(U, \mathbb{T})$, and the cohomology exact sequence of the following exact sequence of sheaves shows that $H^1(U, \mathbb{T}) = 0$, because $H^1(U, \mathbb{R}) = 0$ and $H^2(U, \mathbb{Z}) = 0$ (p. 363),

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi} \mathbb{R} \xrightarrow{e^{ti}} \mathbb{T} \longrightarrow 0.$$

Proposition: *Let $U \subset X$ be a connected open with regular boundary. If $H^1(U, \mathbb{R}) = 0$, then U is isomorphic to the unit disk $\mathbb{D}$.*

Proof: Let g be the Green's function with pole at $p \in U$, and fix $f \in \mathcal{O}(U)$ such that $|f| = e^g$.

Since $e^g = 1$ on ∂U, by the maximum principle we have a commutative square

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & \mathbb{D} \\ \downarrow & & \downarrow{\scriptstyle |-|} \\ \bar{U} & \xrightarrow{\ e^g\ } & [0,1] \end{array}$$

which shows that $f^{-1}(\mathbb{D}_r)$, $r < 1$, is compact; hence f is a proper map, of degree 1 because f has a unique simple zero at p. It is an isomorphism (p. 371).

Lemma: *Let X be a non-compact Riemann surface such that $H^1(X, \mathbb{R}) = 0$. There is a countable cover $X = \bigcup_n U_n$ by connected open sets U_n such that*

1. *$\bar{U}_n$ is a compact manifold with boundary and $\bar{U}_n \subset U_{n+1}$.*
2. *$H^1(U_n, \mathbb{R}) = 0$.*

Proof: Fix a point $p \in X$ and a proper map $f\colon X \to \mathbb{R}$ (p. 260). By Sard's theorem there is an increasing sequence of regular values $a_n \to \infty$, and we may assume that $f(p) < a_0$.

Let Y_n be the connected component of $\{x \in X : f(x) \leq a_n\}$ containing p, and let $V_1, \ldots, V_m$ be the relatively compact connected components of $X - Y_n$. We put (p. 100)

$$K_n = Y_n \cup \bar{V}_1 \cup \ldots \cup \bar{V}_m \quad , \quad U_n = \overset{\circ}{K}_n \ .$$

The open sets U_n fulfill the first condition. Let us see the second one:

The connected components of $X - K_n$ have non-compact closure; i.e. the connected components of $X - U_n$ are non-compact, so that $H^0_c(X - U_n, \mathbb{R}) = 0$.

The closed subspace exact sequence (by duality $H^1_c(X, \mathbb{R})^* = H^1(X, \mathbb{R}) = 0$)

$$0 = H^0_c(X - U_n, \mathbb{R}) \xrightarrow{\ \delta\ } H^1_c(U_n, \mathbb{R}) \longrightarrow H^1_c(X, \mathbb{R}) = 0$$

shows that $H^1_c(U_n, \mathbb{R}) = 0$, and again by duality $H^1(U_n, \mathbb{R}) = H^1_c(U_n, \mathbb{R})^* = 0$.

Uniformization Theorem: *Let X be a Riemann surface. If $H^1(X, \mathbb{R}) = 0$, then X is isomorphic to the complex plane $\mathbb{C}$, the complex projective line $\mathbb{P}_1$ or the unit disk $\mathbb{D}$.*

In particular, X is simply connected.

Proof: If X is not compact, fix the cover $X = \bigcup_n U_n$ of the lemma and a local coordinate at a point $p \in U_0$. By the above proposition, we have an isomorphism $f_n\colon U_n \to \mathbb{D}_{r_n}$, unique with the conditions $f(p) = 0$, $f'(p) = 1$, which determine the radius r_n.

Moreover $r_m < r_n$ when $m < n$. Just apply Schwarz's lemma to the composition

$$\varphi\colon \mathbb{D}_{r_m} \xrightarrow{\ f_m^{-1}\ } U_m \xrightarrow{\ f_n\ } \mathbb{D}_{r_n} \ , \quad \varphi(0) = 0, \ \varphi'(0) = 1.$$

By Koebe's theorem, the injective analytic functions $f_n\colon U_0 \to \mathbb{D}_R$, where $R = \lim r_n$ (and $\mathbb{D}_R := \mathbb{C}$ when $R = \infty$), admit a subsequence converging to an injective analytic function $f\colon U_0 \to \mathbb{D}_R$. Applying again Koebe's theorem to such subsequence on U_1, we may extend f

to U_1, and so on. Finally we obtain an injective analytic function $f\colon X \to \mathbb{D}_R$, so that X is isomorphic to an open subset of $\mathbb{D}_R \subseteq \mathbb{C}$.

By the Riemann mapping theorem, we conclude that X is isomorphic to $\mathbb{C}$ or $\mathbb{D}$.

If X is compact, the exact sequence of the closed subspace shows that $H^1_c(X - p, \mathbb{R}) = 0$ and, by duality, $H^1(X - p, \mathbb{R}) = 0$. Hence $X - p \simeq \mathbb{C}$ or $X - p \simeq \mathbb{D}$ by the non-compact case.

In the first case, the isomorphism extends to a homeomorphism $X \to \mathbb{P}_1$, which is an isomorphism by the removable singularity theorem.

In the second case, the isomorphism $X - p \simeq \mathbb{D}$ is a bounded analytic function; hence it extends to a (non constant) analytic function $X \to \mathbb{C}$. Absurd, X is compact.

Corollary: *The fundamental group of any Riemann surface X is torsion free.*

Proof: The universal covering $\tilde{X} \to X$ is a Galois covering of group $\pi_1(X)$.

If $H \subseteq \pi_1(X)$ is a finite subgroup, then $\pi_1(\tilde{X}/H) = H$, and by Hurewicz's theorem

$$H^1(\tilde{X}/H, \mathbb{R}) = \mathrm{Hom}_{\mathrm{gr}}(\pi_1(\tilde{X}/H), \mathbb{R}) = \mathrm{Hom}_{\mathrm{gr}}(H, \mathbb{R}) = 0.$$

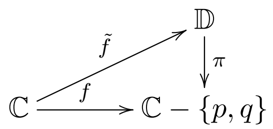
We conclude that $\tilde{X}/H$ is simply connected, so that $H = 0$.

Note: If $\tilde{X} \to X$ is the universal covering of a Riemann surface, then $X = \tilde{X}/G$, where G is a group of automorphisms of $\tilde{X}$ without fixed points. If $\tilde{X} = \mathbb{P}_1$, then $X = \mathbb{P}_1$, because any homography has a fixed point. If $\tilde{X} = \mathbb{C}$ then G is a discrete group of translations; hence (p. 311) X is $\mathbb{C}$, a cylinder $\mathbb{C}/\mathbb{Z}e$ or a torus $\mathbb{C}/(\mathbb{Z}e_1 + \mathbb{Z}e_2)$. In particular $\pi_1(X)$ is abelian.

The universal covering of any other Riemann surface is $\mathbb{D}$.

Picard Little Theorem: *Any non constant analytic function $f\colon \mathbb{C} \to \mathbb{C}$ omits at most one value.*

Proof: The universal covering of $\mathbb{C} - \{p, q\}$ is $\mathbb{D}$, because $\mathbb{C} - \{p, q\}$ is not compact and has non abelian fundamental group. Since $\mathbb{C}$ is simply connected, any analytic function $f\colon \mathbb{C} \to \mathbb{C} - \{p, q\}$ may be lifted to the universal covering



and $\tilde{f}\colon \mathbb{C} \to \mathbb{D}$ is a bounded analytic function. Hence it is constant and so is f.

13.2 Fréchet Spaces

Let us consider the category of topological $\mathbb{R}$-vector spaces and continuous $\mathbb{R}$-linear maps.

1. *If the origin of E is closed, then E is a separated space.*

   The diagonal is the inverse image of 0 by the continuous map $E \times E \xrightarrow{y-x} E$.

2. *If a linear map f is continuous at 0, then it is continuous.*

   Translations are continuous, and $f(x) = f(x - e) + f(e)$.

3. *The closure of a vector subspace also is a vector subspace.*

4. *Any neighborhood U of 0 is **absorbing** (if $e \in E$, then $\lambda e \in U$ for any λ sufficiently small) and there is a neighborhood V of 0 such that $\bar{V} \subseteq V + V \subseteq U$.*

   The continuity of the product $\mathbb{K} \times E \to E$ and the addition $E \times E \to E$ at 0.

5. *There is a base of **balanced** ($\lambda U \subseteq U$ for any scalar $|\lambda| \leq 1$) open neighborhoods of 0.*

   If U is a neighborhood of 0, by the continuity of the product, there is an open neighborhood V of 0 and $\varepsilon > 0$ such that $U' := \bigcup_{|\lambda|<\varepsilon} \lambda V \subseteq U$, and U' is a balanced open set.

6. A family $\mathcal{F}$ of continuous linear maps $E \to F$ is equicontinuous when for every neighborhood V of 0 in F there is a neighborhood U of 0 in E such that $f(U) \subseteq V$, $\forall f \in \mathcal{F}$. In particular, a family of linear forms on E is equicontinuous when there is a neighborhood U of 0 in E such that every $\omega \in \mathcal{F}$ is absolutely bounded by 1 on U.

7. Let B be the unit ball of a seminormed space E. A linear form $\omega\colon E \to \mathbb{R}$ is continuous if and only if $\|\omega\| = \sup_B |\omega(e)| < \infty$, and so we obtain a norm on the topological dual E'.

Proposition: *If V is a vector subspace of a topological vector space E, then $\bar{E} = E/V$, with the quotient topology, is a topological vector space and the canonical map $\pi\colon E \to \bar{E}$ is an open map such that any continuous linear map vanishing on V uniquely factors through π. The space $\bar{E}$ is separated if and only if V is closed. If the topology of E is defined by a family of seminorms $\{q_i\}$, then the topology of $\bar{E}$ is defined by the seminorms $\bar{q}_i(\bar{e}) = \inf_{v\in V} q_i(e+v)$.*

Proof: The map π is open because $\pi^{-1}(\pi U) = U + V = \bigcup_{v\in V} v + U$ is open when so is U.

Hence $E \times E \to \bar{E} \times \bar{E}$ and $\mathbb{R} \times E \to \mathbb{R} \times \bar{E}$ are open surjective continuous maps, so that the addition $\bar{E} \times \bar{E} \to \bar{E}$ and product $\mathbb{R} \times \bar{E} \to \bar{E}$ are continuous because so are the induced maps $E \times E \to \bar{E}$ and $\mathbb{R} \times E \to \bar{E}$. The other statements are easy to check.

Definition: A topological vector space E is **Fréchet** when it is complete (hence separated) and the topology may be defined by a countable family $\{q_n\}$ of seminorms (hence also by the translation-invariant metric $\mathrm{d}(x,y) = \sum_n 2^{-n} \min\{1, q_n(y-x)\}$).

Replacing q_n by $\max\{q_0, \ldots, q_n\}$, we may always assume that $q_{n-1} \leq q_n$.

1. *If V is a closed vector subspace of a Fréchet space E, then $\bar{E} = E/V$ also is Fréchet.* In fact, the translation-invariant metric $\bar{\mathrm{d}}(\bar{x},\bar{y}) = \inf_{u,v\in V} \mathrm{d}(x+u, y+v)$ defines the uniform structure of $\bar{E}$. Now, if $(\bar{x}_n)$ is a Cauchy sequence in $\bar{E}$, we may assume that $\bar{d}(\bar{x}_{n-1}, \bar{x}_n) < \frac{1}{2^{n+1}}$, and fix $x_n \in E$ such that $\bar{x}_n = \pi(x_n)$, $\mathrm{d}(x_{n-1}, x_n) < \frac{1}{2^n}$. Hence $\mathrm{d}(x_n, x_{n+m}) < \frac{1}{2^n}$, $\forall m \geq 0$, so that $x_n \to e \in E$, and $\bar{x}_n \to \pi(e)$.

2. $\mathcal{C}(X)$ when X is a σ-compact space (p. 120), $\mathcal{C}^m(X)$ when X is a smooth manifold (p. 261), $\Gamma(X, E)$ when E is a smooth vector bundle over a smooth manifold X (p. 362), and $\mathcal{O}(X)$ when X is a Riemann surface (p. 120) are Fréchet spaces.

3. Let E be a Fréchet space, K a compact space. The vector space of continuous maps $\mathcal{C}(K, E)$ is Fréchet with the topology of uniform convergence, defined by the seminorms $q(f) = \max_{x\in K} q\big(f(x)\big)$, where q runs over the continuous seminorms on E.

Definition: An exact sequence $\ldots \to E_n \xrightarrow{f_n} E_{n+1} \xrightarrow{f_{n+1}} E_{n+2} \to \ldots$ of continuous linear maps between topological vector spaces is **topologically exact** if the induced continuous linear bijections $E_n/\mathrm{Ker}\, f_n = \mathrm{Ker}\, f_{n+1}$ are isomorphisms (i.e. open maps).

Lemma: *Let $\phi\colon E \to F$ be a continuous linear map. If E is Fréchet and, for every neighborhood U of 0 in E, the closure $\overline{\phi(U)}$ is a neighborhood of 0 in F, then ϕ is open.*

Proof: As U varies, $\overline{U+U}$ runs over a base of neighborhoods of 0 in E; hence it is enough to show that any point $y \in \overline{\phi(U)}$ is contained in $\phi\big(\overline{U+U}\big)$.

Let $\{U_n\}$ be a base of neighborhoods of 0 in E such that $U_0 = U$ and $U_n + U_n \subseteq U_{n-1}$,

$$U_n + \ldots + U_{n+k} \subseteq U_{n-1}$$
$$U_0 + U_1 + \ldots + U_k \subseteq U + U$$

Since $y - \overline{\phi(U_1)}$ is a neighborhood of y, it intersects $\phi(U_0) = \phi(U)$ at some point y_0, so that $y - y_0 \in \overline{\phi(U_1)}$. Recursively, we construct a sequence y_n in F such that $y_n = \phi(x_n) \in \phi(U_n)$ and $y - (y_0 + \ldots + y_n) \in \overline{\phi(U_{n+1})}$. Now the series $x = \sum_n x_n$ converges, because E is complete and $x_n + \ldots + x_{n+k} \in U_{n-1}$, and we have $x \in \overline{U+U}$. Finally, $y = \sum_n y_n$ because $\overline{\phi(U_n)}$ is a base of neighborhoods of 0 in F, and $y = \sum_n \phi(x_n) = \phi(x)$.

Open Mapping Theorem: *Any exact sequence of continuous linear maps between Fréchet spaces is topologically exact.*

Proof: Since $E_n/\mathrm{Ker}\, f_n$ and $\mathrm{Ker}\, f_{n+1}$ are Fréchet, it is enough to prove that any continuous linear bijection $\phi\colon E \to F$ between Fréchet spaces is open.

If U is a neighborhood of 0 in E, then

$$F = \phi(E) = \phi\Big(\bigcup_n nU\Big) = \bigcup_n n\phi(U).$$

Baire theorem states that some $\overline{n\phi(U)} = n\overline{\phi(U)}$ contains an open set nV, so that the closure of $\phi(U-U) = \phi(U) - \phi(U)$ contains the neighborhood $V - V$ of 0.

As U varies, $U - U$ runs over a base of neighborhoods of 0. We conclude by the lemma.

Closed Graph Theorem: *Let $f\colon E \to F$ be a linear map between Fréchet spaces. If the graph $\Gamma_f = \{\big(e, f(e)\big); e \in E\}$ is closed in $E \times F$, then f is continuous.*

Proof: The first projection $p_1\colon \Gamma_f \to E$ is a continuous linear bijection; hence an isomorphism, and $f = p_2 \circ p_1^{-1}$ is continuous.

Theorem: *If K is a compact topological space, the functor* $\mathrm{Hom}_{\mathrm{Top}}(K, -)$ *preserves exact sequences of Fréchet spaces.*

Proof: If $E \to \bar{E}$ is a separated quotient of a Fréchet space E, we have to show that the induced map $\pi\colon \mathcal{C}(K, E) \to \mathcal{C}(K, \bar{E})$ is surjective. Let B the open unit ball of a continuous seminorm q on E, and $\bar{B}$ the open ball of the induced seminorm $\bar{q}$ on $\bar{E}$.

Let $f \in \mathcal{C}(K, \bar{B})$. Given $\varepsilon > 0$ and a continuous seminorm q' on E, since f is uniformly continuous there is a finite open cover $K = \bigcup_i U_i$ and points $t_i \in U_i$ such that $\bar{q}'\big(f(t) - f(t_i)\big) < \varepsilon$, $\forall t \in U_i$. Moreover, since $\bar{q}\big(f(t_i)\big) < 1$, we have $f(t_i) = \bar{e}_i$, where $q(e_i) < 1$.

Let $\{\phi_i\}$ be a partition of unity subordinated to $\{U_i\}$. We have $q\big(\sum_i \phi_i e_i\big) < 1$ and

$$\bar{q}'\big(f(t) - \textstyle\sum_i \phi_i(t)\bar{e}_i\big) = \bar{q}'\big(\textstyle\sum_i \phi_i(t)(f(t) - f(t_i)\big) < \varepsilon,$$

since we can restrict the sum to those i such that $t \in U_i$. Hence $\mathcal{C}(K, \bar{B}) \subseteq \overline{\pi(\mathcal{C}(K, B))}$.

By the lemma π is open, and therefore surjective.

Banach-Steinhaus Theorem: *A family $\mathcal{F}$ of continuous linear maps from a Fréchet space E to a topological vector space F is equicontinuous if and only if for every $x \in E$ the set*

$\mathcal{F}(x) = \{f(x); f \in \mathcal{F}\}$ *is* ***bounded*** *in* F. (A set $B \subset F$ is bounded if for every neighborhood V of 0 in F there is a scalar λ such that $B \subseteq \lambda V$).

Proof: If $\mathcal{F}$ is equicontinuous, there is a neighborhood U of 0 in E such that $f(U) \subseteq V$, $\forall f \in \mathcal{F}$. Now, given $x \in E$, we have $\lambda x \in U$ for some $\lambda \neq 0$; hence $\lambda\mathcal{F}(x) = \mathcal{F}(\lambda x) \subseteq V$.

Conversely, let V be a closed neighborhood of 0 in F. Then $U := \mathcal{F}^{-1}(V) = \bigcap_{f\in\mathcal{F}} f^{-1}(V)$ is a closed set, and it is absorbing when $\mathcal{F}(x)$ is bounded, $\forall x \in E$. By Baire theorem, U has non-empty interior; so that $\mathcal{F}^{-1}(V - V)$ contains the neighborhood $U - U$ of 0 in E.

As V varies, $\mathcal{F}^{-1}(V - V)$ runs over a base of neighborhoods of 0 in F, and we conclude.

Theorem: *Let E be a topological vector space. If the kernel of a linear form $\omega\colon E \to \mathbb{R}$ is closed, then ω is continuous.*

Proof: Replacing E by $E/\mathrm{Ker}\,\omega$, we may assume that E is separated of dimension 1 and ω is the inverse of a linear bijection $\mathbb{R} \to E$, $\lambda \mapsto \lambda e$.

Given $\varepsilon > 0$, there is a balanced neighborhood U of 0 such that $\varepsilon e \notin U$. Hence $U \subseteq (-\varepsilon, \varepsilon)e$, so that $\omega(U) \subseteq (-\varepsilon, \varepsilon)$ and we conclude that ω is continuous at 0.

Corollary: *Any separated topological vector space of finite dimension n is isomorphic to $\mathbb{R}^n$.*

Proof: By induction on n, and it is obvious when $n = 0$. If $n > 0$, the kernel N of any linear form $0 \neq \omega\colon E \to \mathbb{R}$ is separated, of dimension $n - 1$; hence $N \simeq \mathbb{R}^{n-1}$ is complete, so that it is closed in E, and ω is continuous. We conclude that any linear bijection $E \to \mathbb{R}^n$ is continuous, and the inverse also is continuous, because so is any linear map $\mathbb{R}^n \to E$.

Corollary: *Any finite dimensional subspace of a separated topological vector space is closed.*

Corollary: *A separated topological vector space E is finite dimensional if and only if some neighborhood P of 0 is precompact.*

Proof: If a vector subspace $H \subset \mathbb{R}^n$ does not contain a bounded set P, then $H + \frac{1}{2}P$ does not contain P (in $\mathbb{R}^n/H$ it is clear that $\frac{1}{2}P$ does not contain P).

Hence, if E were infinite dimensional, we could construct a sequence x_n in P such that $x_n \notin \langle x_1, \ldots, x_{n-1}\rangle + \frac{1}{2}P$, so that $x_n - x_m \notin \frac{1}{2}P$ whenever $n \neq m$. Absurd since (x_n) has an adherent point in the completion $\widehat{P}$ and $\frac{1}{2}P$ is a neighborhood of 0.

Lemma: *Let C be a convex set in a topological vector space E. If x is an interior point of C and $y \in \bar{C}$, then the segment $[x, y)$ is interior to C.*

Proof: Put $U = \overset{\circ}{C}$, and let us show that $\lambda x + (1-\lambda)y \in U$, $0 < \lambda < 1$. Now, $y + \frac{\lambda}{1-\lambda}(x - U)$ is open and contains y (since $x \in U$); hence it contains $z \in C$ (since $y \in \bar{C}$). Since C is convex, $\lambda U + (1-\lambda)z \subseteq U$, and $\lambda x + (1-\lambda)y \in \lambda U + (1-\lambda)z$ because $z \in y + \frac{\lambda}{1-\lambda}(x - U)$.

Hahn-Banach Theorem: *Let E be a topological vector space, U a convex open set, V a linear subvariety not meeting U. There is a closed hyperplane containing V and not meeting U.*

Proof: We may assume that V is a vector subspace. Let W be a maximal vector subspace containing V and not meeting U (it exists by Zorn's lemma). W is closed, because $\bar{W}$ is a vector subspace not meeting U, and we must show that $\dim E/W > 1$ is absurd.

Replacing E by E/W we may assume that $V = W = 0$ and that $E \simeq \mathbb{R}^2$.

Replacing U by $\bigcup_{\lambda>0} \lambda U$, we may assume that U is a **cone** ($\lambda U \subseteq U$ for every $\lambda > 0$).

Pick $0 \neq y \in \partial U$ (it exists, $\mathbb{R}^2 - \{0\}$ is connected). Since U is an open cone, $\lambda y \in \partial U$ when $\lambda > 0$; hence $\lambda y \notin U$.

Moreover $\lambda y \notin U$ when $\lambda < 0$, since otherwise by the lemma $0 \in [\lambda y, y) \subseteq U$.

We conclude that $\mathbb{R}y \cap U = \emptyset$, against the maximal character of $W = 0$.

Hahn-Banach Theorem: *Let V be a vector subspace of a seminormed space E. Any linear form ω on V of seminorm ≤ 1 is the restriction of a linear form on E of seminorm ≤ 1.*

Proof: The linear subvariety $L = \{v \in V : \omega(v) = 1\}$ does not meet the unit ball B of E, a convex open set. Hence there is a hyperplane $\omega'(x) = 1$ containing L (so that ω is the restriction of ω') and not meeting B (so that $\|\omega'\| \leq 1$).

Corollary: *If $0 \to V \to E \to W \to 0$ is a topologically exact sequence of locally convex spaces, then the induced sequence $0 \to W' \to E' \to V' \to 0$ is exact, and every equicontinuous set in V' is the image of an equicontinuous set in E'.*

Proof: A set $\mathcal{F}$ of linear forms on a locally convex space F is equicontinuous if and only if there is a continuous seminorm q on F such that every $\omega \in \mathcal{F}$ has seminorm ≤ 1.

Corollary: *Let E be a separated locally convex space. Any vector subspace $V \subseteq E$ of finite dimension admits a* ***topological supplement*** (a vector subspace W such that the natural map $V \times W \to E$ is an isomorphism).

Proof: The inclusion $V \hookrightarrow E$ admits a continuous retraction $E \to V$.

Theorem: *Let E be a separated locally convex space, $e \in E$. If $\omega(e) = 0$, $\forall \omega \in E'$, then $e = 0$.*

Corollary: *Let V be a vector subspace of a locally convex space E, and $e \in E$. If $\omega(e) = 0$ for any continuous linear form $\omega \in E'$ vanishing on V, then $e \in \bar{V}$.*

13.2.1 Duality

When we endow a real vector space E with the **weak** topology of a certain vector subspace $E' \subseteq E^*$ (the initial topology of the maps $\omega : E \to \mathbb{R}$, $\omega \in E'$, defined by the seminorms $q_\omega(e) = |\omega(e)|$, $\omega \in E'$) then any continuous linear form ω on E_{w} is bounded on some vector subspace $V = \langle \omega_1, \ldots, \omega_n \rangle^{\mathrm{o}}$, where $\omega_i \in V$. Hence $\omega(V) = 0$ and $\omega \in \langle \omega_1, \ldots, \omega_n \rangle \subseteq E'$, so that E' is the topological dual of E_{w}.

From now on E, E' will be a pair of real vector spaces with a non-singular bilinear pairing $E \times E' \to \mathbb{R}$, identifying each factor with a space of linear forms on the other one. Then we endow both factors with the weak topology of the pointwise convergence on the other one, so obtaining separated locally convex spaces[1] E_{w}, E'_{w} in a fully symmetric situation: any statement has an equivalent dual statement, interchanging E and E'. Moreover, we have on E a finer locally convex topology such that the topological dual is E', so that weakly closed sets are closed, compact sets are weakly compact, continuous linear maps are weakly continuous, etc.

Definition: A **disk** $D \subseteq E$ is a convex balanced set, and the **polar** of D is the disk

$$D^{\mathrm{o}} = \{\omega \in E' : |\omega(D)| \leq 1\} = \{\omega \in E' : \omega(D) \leq 1\} \subseteq E'.$$

When $V \subseteq E$ is a vector subspace, $V^{\mathrm{o}} = \{\omega \in E' : \omega(V) = 0\}$ is just the incident.

[1] Our basic objects of study $E_{\mathrm{w}}, E'_{\mathrm{w}}$ are not complete nor metrizable spaces even in the most simple cases.

Any absorbing disk $D \subseteq E$ defines a seminorm $q_D(x) = \inf\{|\lambda| \colon x \in \lambda D\}$ on E, and the unit ball B of a seminorm q on E is an absorbing disk such that $q = q_B$.

Lemma: *If U, V are disjoint open convex sets, some closed hyperplane separates U and V.*

Proof: By the Hahn-Banach theorem, some closed hyperplane $\omega(x) = 0$ does not meet $U - V$.
Hence $\omega(U)$ and $\omega(V)$ are disjoint intervals in $\mathbb{R}$, and we conclude.

Theorem: *Any closed convex subset C of E is an intersection of closed **half-spaces*** (subsets $\omega(x) \geq a$, where $\omega \in E'$, $a \in \mathbb{R}$). In particular C is weakly closed.

Proof: If $p \notin C$, there is a convex neighborhood U of 0 such that $p + U$ and $C + U$ do not meet.
By the lemma there is a closed half-space H such that $C \subseteq H$, $p \notin H$.

Corollary: *Any closed disk in E is weakly closed.*

Bipolar Theorem: *If D is a weakly closed disk in E, then $D = (D^{\mathrm{o}})^{\mathrm{o}}$. Hence, the polarity defines a lattice anti-isomorphism*

$$\begin{bmatrix}\text{(Weakly) closed}\\ \text{disks in } E\end{bmatrix} \xrightarrow{\ \sim\ } \begin{bmatrix}\text{Weakly closed}\\ \text{disks in } E'.\end{bmatrix}$$

where closed vector subspaces of E correspond to weakly closed vector subspaces of E', and disked closed neighborhoods of 0 in E correspond to equicontinuous weakly closed disks in E'.

Proof: If $p \notin D$, some closed hyperplane $\omega(x) \leq 1$ contains D and not p; hence $p \notin (D^{\mathrm{o}})^{\mathrm{o}}$.
Finally, a family $\mathcal{F} \subseteq E'$ is equicontinuous if and only if there is a disked neighborhood D of 0 such that $\mathcal{F} \subseteq D^{\mathrm{o}}$.

Corollary: *The topology of E is the topology of uniform convergence on the equicontinuous subsets of E' (or the equicontinuous weakly closed disks in E').*

Proof: If $D = (D^{\mathrm{o}})^{\mathrm{o}}$ is a disked closed neighborhood of 0, then $q_D(x) = \sup_{\omega \in D^{\mathrm{o}}} |\omega(x)|$.

Theorem: *Any equicontinuous family in E' has compact closure in E'_{w}. Hence, the topology of E is a locally convex topology intermediate between the weak topology and the weakly compact-convergence topology. Moreover, the dual of E for any such topology is E'.*

Proof: If D is a disked closed neighborhood of 0 in E, we have to show that

$$\mathcal{F} = \{\omega \in E' \colon |\omega(D)| \leq 1\} = \{\omega \in E^* \colon |\omega(D)| \leq 1\}$$

has compact closure in E'_{w}. Now, $\mathcal{F}(x) = \{\omega(x);\ \omega \in \mathcal{F}\} \subset \mathbb{R}$ is bounded (D is absorbing) and $\mathcal{F} = \{\omega \in E^* \colon |\omega(D)| \leq 1\}$ is closed in the compact space $\prod_{x \in E} \overline{\mathcal{F}(x)} \subset \mathcal{C}(E_{\mathrm{dis}}, \mathbb{R})$ (with the pointwise convergence topology). Hence $\mathcal{F}$ is compact.
Finally, if $\omega \in E^*$ is continuous for the weakly compact-convergence topology, there is a weakly compact disk $K \subset E'_{\mathrm{w}} \subseteq E^*_{\mathrm{w}}$ such that $|\omega(K^{\mathrm{o}})| \leq 1$.
By the bipolar theorem, applied to E^*_{w}, we have $\omega \in (K^{\mathrm{o}})^{\mathrm{o}} = K \subset E'$.

Example: Let E be a Fréchet space. By the Banach-Steinhaus theorem, any weakly bounded (hence any weakly compact) set in E' is equicontinuous, so that the topology of E is the weakly compact-convergence topology.

13.2.2 The Transpose Linear Mapping

Let E, F be separated locally convex spaces. Any weakly continuous linear map $f\colon E \to F$ induces a weakly continuous linear map $f^*\colon F' \to E'$, $f^*(\omega) = \omega \circ f$, and $f^{**} = f\colon E \to F$.

Definitions: A continuous linear map $f\colon E \to F$ is a **homomorphism** when the induced continuous linear bijection $E/\mathrm{Ker}\, f \to \mathrm{Im}\, f$ is an isomorphism (when $E/\mathrm{Ker}\, f$ is endowed with the quotient topology and $\mathrm{Im}\, f$ with the topology induced by F).

Proposition: *The functors $E \rightsquigarrow E_{\mathrm{w}}$ and $E \rightsquigarrow E'_{\mathrm{w}}$ preserve topologically exact sequences of separated locally convex spaces: If $V \subseteq E$ is a closed vector subspace, we have topologically exact sequences*

$$0 \longrightarrow V_{\mathrm{w}} \xrightarrow{\ i\ } E_{\mathrm{w}} \xrightarrow{\ \pi\ } (E/V)_{\mathrm{w}} \longrightarrow 0$$

$$0 \longrightarrow (E/V)'_{\mathrm{w}} \xrightarrow{\ \pi^*\ } E'_{\mathrm{w}} \xrightarrow{\ i^*\ } V'_{\mathrm{w}} \longrightarrow 0$$

Proof: The topology of V_{w} is induced by the topology of E_{w} because $E' \to V'$ is surjective.

If $\omega \in E'$, then the seminorm $q_\omega(e) = |\omega(e)|$ induces on E/V the null seminorm when $\omega(V) \neq 0$, and the seminorm $q_{\bar\omega}$ when $\omega(V) = 0$, where $\bar\omega(\bar e) = \omega(e)$. Hence, the weak topology of E/V is just the quotient topology of E_{w}.

The topology of $(E/V)'_{\mathrm{w}}$ is induced by the topology of E'_{w} because $E \to E/V$ is surjective.

If $e \in E$, then the seminorm $q_e(\omega) = |\omega(e)|$ induces on V' the null seminorm when $e \notin V$ (because $\inf_{\eta \in V^{\mathrm{o}}} |(\omega + \eta)(e)| = 0$, since $\eta(e) \neq 0$ for some $\eta \in V^{\mathrm{o}}$), and the seminorm q_e when $e \in V$. Hence, the weak topology of V' is just the quotient topology of E'_{w}.

Proposition: *Let $f\colon E_{\mathrm{w}} \to F_{\mathrm{w}}$ be a continuous linear map. The closure of* $\mathrm{Im}\, f$ *in F_{w} is the incident of* $\mathrm{Ker}\, f^*$*, and* $\mathrm{Ker}\, f$ *is the incident of* $\mathrm{Im}\, f^*$*:*

$$\overline{\mathrm{Im}\, f}^{\mathrm{w}} = (\mathrm{Ker}\, f^*)^{\mathrm{o}} \quad , \quad \mathrm{Ker}\, f = (\mathrm{Im}\, f^*)^{\mathrm{o}} = (\overline{\mathrm{Im}\, f^*}^{\mathrm{w}})^{\mathrm{o}}.$$

In particular, f is injective if and only if $\mathrm{Im}\, f^*$ *is dense in E'_{w}.*

Proof: It is clear that $\mathrm{Ker}\, f = (\mathrm{Im}\, f^*)^{\mathrm{o}}$; hence $\mathrm{Ker}\, f^* = (\mathrm{Im}\, f)^{\mathrm{o}} = (\overline{\mathrm{Im}\, f}^{\mathrm{w}})^{\mathrm{o}}$.

By the bipolar theorem we conclude that $\overline{\mathrm{Im}\, f}^{\mathrm{w}} = (\mathrm{Ker}\, f^*)^{\mathrm{o}}$.

Proposition: *A continuous linear map $f\colon E_{\mathrm{w}} \to F_{\mathrm{w}}$ is a homomorphism if and only if* $\mathrm{Im}\, f^*$ *is closed in E'_{w}. Hence, if* $\mathrm{Im}\, f^* = E'$*, then f is an injective weak homomorphism.*

Proof: Replacing E_{w} by $(E/\mathrm{Ker}\, f)_{\mathrm{w}} = E_{\mathrm{w}}/(\mathrm{Ker}\, f)_{\mathrm{w}}$, and F_{w} by $(\mathrm{Im}\, f)_{\mathrm{w}}$, we may assume that f is bijective, an obvious case (if $\mathrm{Im}\, f^*$ is closed, then $\mathrm{Im}\, f^* = 0^{\mathrm{o}} = E'$).

Proposition: *A continuous linear map $f\colon E \to F$ is a homomorphism if and only if it is a weak homomorphism* ($\mathrm{Im}\, f^*$ *is weakly closed in E'*) *and every equicontinuous family in* $\mathrm{Im}\, f^*$ *is the image by f^* of an equicontinuous family in F'.*

Proof: Replacing E_{w} by $(E/\mathrm{Ker}\, f)_{\mathrm{w}} = E_{\mathrm{w}}/(\mathrm{Ker}\, f)_{\mathrm{w}}$, and F_{w} by $(\mathrm{Im}\, f)_{\mathrm{w}}$, we may assume that f is bijective. Now, two (separated) locally convex topologies on a vector space are identical if and only if they have the same dual and equicontinuous families in the dual.

Definition: A linear map $f\colon E \to F$ is **compact** if there is a neighborhood U of 0 in E such that $f(U)$ has compact closure in F; a fortiori it is continuous, since for every neighborhood V of 0 in F there is an integer n such that $f(U) \subseteq nV$, i.e. $f\big(\frac{1}{n}U\big) \subseteq V$.

Lemma: *If $i\colon E \to F$ is an isomorphism onto a closed subspace of F and $f\colon E \to F$ is compact, then $h = i + f$ is a homomorphism onto a closed subspace, with finite dimensional kernel.*

Proof: Put $N = \operatorname{Ker} h$ and let U be a disked neighborhood of 0 such that $f(U)$ has compact closure. Then $V = N \cap U$ is a neighborhood of 0 in N such that

$$i(V) = -f(V) \subseteq -f(U) = f(U)$$

has compact closure, and therefore is precompact. Since i is an isomorphism, V is precompact, and N admits a precompact neighborhood of 0: it is finite dimensional.

To prove that f is a homomorphism onto a closed subspace of F, we may restrict i and f to a topological supplement of N (it exists because N is finite dimensional), so that we may assume that h is injective, and we have to prove that h is an isomorphism onto a closed subspace.

This means that any **ultrafilter**[2] $\mathfrak{m}$ in E converges whenever $h(\mathfrak{m})$ converges in F: the convergent ultrafilters determine the neighborhoods of 0 because (p. 228) any filter is an intersection of ultrafilters, and if $p \in \bar{E}$, any ultrafilter containing the intersections of E with the neighborhoods of p converges to p.

Let q be the seminorm of U. Then $\lim_{\mathfrak{m}} q(x) = a \in [0, \infty]$ exists because $[0, \infty]$ is a compact separated space. If $a < \infty$, there exists $A \in \mathfrak{m}$ such that $A \subseteq (a+1)U$, then $f(A)$ has compact closure, so that $\lim_{\mathfrak{m}} f(x)$ exists, and so does $\lim_{\mathfrak{m}} i(x) = \lim_{\mathfrak{m}} h(x) - \lim_{\mathfrak{m}} f(x) = y \in F$ because $\lim_{\mathfrak{m}} h(x)$ exists by hypothesis. But $i(E)$ being closed, we have $y = i(e)$, $e \in E$, and $\mathfrak{m}$ converges to e because $i\colon E \to i(E)$ is an isomorphism.

But $a = \infty$ is impossible:
Otherwise $\mathfrak{m}$ comes from an ultrafilter $\mathfrak{p}$ in $\{q(x) > 0\}$ and $\lim_{\mathfrak{p}} h\bigl(\frac{x}{q(x)}\bigr) = \lim_{\mathfrak{p}} \frac{h(x)}{q(x)} = 0$.
Moreover, $\lim_{\mathfrak{p}} q\bigl(\frac{x}{q(x)}\bigr) = 1$, and by the preceding argument, $\lim_{\mathfrak{p}} \frac{x}{q(x)} = e \in E$ exists.
Now $h(e) = \lim_{\mathfrak{p}} h\bigl(\frac{x}{q(x)}\bigr) = 0$, $q(e) = \lim_{\mathfrak{p}} q\bigl(\frac{x}{q(x)}\bigr) = 1$; absurd, h is injective.

Proposition: *If $f\colon E \to F$ is compact, so is $f^*\colon F'_c \to E'_c$ when we endow the duals with the compact convergence topology.*

Proof: Let D be a closed disked neighborhood of 0 in E such that the disk $K = f(D)$ has compact closure, then $f^*(K^{\text{o}}) \subseteq D^{\text{o}}$. Now, K^{o} is a neighborhood of 0 in F'_c, and D^{o} is weakly compact in E'. Since the family D^{o} is equicontinuous, D^{o} also is compact for compact convergence: D^{o} is compact in E'_c.

Schwartz Theorem: *Let $p\colon E \to F$ be a surjective linear map between Fréchet spaces. If $f\colon E \to F$ is compact, then $h = p + f$ is a homomorphism onto a closed subspace of finite codimension in F.*

Proof: Let E'_c, F'_c be the duals with the compact convergence topology (their duals are then E and F, because compact sets are weakly compact), so that f^* is compact.

$\operatorname{Im} p^*$ is closed because p is a homomorphism; hence a weakly homomorphism.

Any compact disk in F is the image by $p^{**} = p$ of a compact disk in E (because any compact set is the image of a compact set, p. 389). Since the equicontinuous sets in E are just the subsets of the compact disks, we see that the injective continuous linear map p^* is an isomorphism onto

[2] An ultrafilter in a set X is a maximal (or prime, since $\mathcal{P}(X)$ is a Boolean algebra) ideal $\mathfrak{m}$ of the algebra $\mathcal{P}(X)$ of subsets of X. Given a map $f\colon X \to Y$, then $f(\mathfrak{m}) := \{B \in \mathcal{P}(Y)\colon f^{-1}(B) \in \mathfrak{m}\}$, is an ultrafilter in Y and (when Y is a separated topological space) $\lim_{\mathfrak{m}} f(x)$ denotes the limit, if it exists, of $f(\mathfrak{m})$; i.e. $\lim_{\mathfrak{m}} f(x) = y \in Y$ means that for every neighborhood V of y we have $f(A) \subseteq V$ for some $A \in \mathfrak{m}$.

a closed subspace of E'_c and, by the lemma, $h^* = p^* + f^*$ is a homomorphism onto a closed subspace, with finite dimensional kernel.

The image of h is closed in F because h^* is a (weak) homomorphism, and of finite codimension because $\operatorname{Im} h = \overline{\operatorname{Im} h}^{\mathrm{w}} = (\operatorname{Ker} h^*)^{\mathrm{o}}$.

Finally, h is a homomorphism by the open mapping theorem.

13.3 Compact Riemann Surfaces

Let $\varphi\colon X \to Y$ be an analytic map between Riemann surfaces, $x \in X$, and $y = \varphi(x) \in Y$.

When φ is not constant, there are (p. 277) coordinate neighborhoods (U, u) and (V, v) centered at x and y respectively, such that φ has the normal form $v = u^n$, where n is the ramification index of φ at x. Hence, if $\mathcal{O}_{X,x}$ denotes the local ring of germs at x of analytic functions, then $\operatorname{ind}_x\varphi = \dim(\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x})$, where $\mathfrak{m}_y$ is the maximal ideal of the local ring $\mathcal{O}_{Y,y}$, $y = \varphi(x)$.

When $f\colon X \to \mathbb{P}_1$ is a non null meromorphic function, $\operatorname{ind}_x f$ is the number of zeros of f at x, when $f(x) = 0$, and the number of poles when $f(x) = \infty$.

Theorem: *Let $\varphi\colon X \to Y$ be a non constant analytic map. If X is compact, then Y is compact, φ is surjective, with finite fibres, and the number*

$$d(y) = \sum_{\varphi(x)=y} \operatorname{ind}_x\varphi$$

does not depend on $y \in Y$, and it is named **degree** of the morphism φ.

Proof: Since $\varphi(X)$ is a compact open set in Y, we have that $Y = \varphi(X)$ is compact.

The fibres are discrete; hence finite because X is compact.

Put $\varphi^{-1}(y) = \{x_1, \ldots, x_n\}$, $n_i = \operatorname{ind}_{x_i}\varphi$, and fix coordinate neighborhoods $U_1, \ldots, U_n, V$ centered at $x_1, \ldots, x_n, y$ where $\varphi\colon U_i \to V$ has normal form $v = u_i^{n_i}$ (to obtain the normal form, only a change of coordinate in U_i is required, not in V), so that the fibre of $\varphi\colon U_i \to V$ over any point $y' \in V$, $y' \neq y$, is formed by n_i points with ramification index 1.

We may assume that $U_1, \ldots, U_n$ are disjoint and, replacing V by $Y - \varphi(X - \bigcup_i U_i)$, that $\varphi^{-1}(V) = \amalg_i U_i$. Now $d(y') = \sum_i n_i = d(y)$, so that d is locally constant.

The Main Example: If X is a $\mathbb{C}$-scheme, then X_{an} will be the set of rational points, so that any $f \in \mathcal{O}_X(U)$ defines a map $f_{\mathrm{an}}\colon U_{\mathrm{an}} \to \mathbb{C}$, and we endow X_{an} with the initial topology of the maps f_{an} when U runs over the open subsets and $f \in \mathcal{O}_X(U)$; hence, any morphism of $\mathbb{C}$-schemes $\varphi\colon X \to Y$ induces a continuous map $\varphi_{\mathrm{an}}\colon X_{\mathrm{an}} \to Y_{\mathrm{an}}$.

Let C be the Riemann variety of a finite extension Σ of $\mathbb{C}(t)$, and $\mathbb{C}[x_1, \ldots, x_n]/I$ the ring of some affine open neighborhood U of a closed point $p \in C$. Since $\mathcal{O}_{C,p}$ is a regular ring, we may assume that the ideal I is generated by $n-1$ polynomials with linearly independent differentials at any point of the closed set $U_{\mathrm{an}} \subset \mathbb{C}^n$. Hence U_{an} is a closed smooth surface and $T_p^*U_{\mathrm{an}}$ is a complex vector subspace of $T_p^*\mathbb{C}^n$, isomorphic to $\mathfrak{m}_p/\mathfrak{m}_p^2$, the algebraic cotangent space. So we see that the topological space C_{an} has a natural smooth structure with a complex structure on the cotangent spaces.

Any generator z of $\mathfrak{m}_p$ has non-null differential in a neighborhood $V \subset C_{\mathrm{an}}$, and $\mathrm{d}_x z$ is $\mathbb{C}$-linear at any point $x \in V$. Hence, z is a local isomorphism (preserving the complex structure of the tangent spaces) so that C_{an} is a Riemann surface and z is a local coordinate at p.

Finally, any morphism $\mathbb{C}(t) \to \Sigma$ defines a morphism of $\mathbb{C}$-schemes $C \to \mathbb{P}_1$; hence an analytic map $\varphi\colon C_{\mathrm{an}} \to (\mathbb{P}_1)_{\mathrm{an}}$ such that the number of points of the fibres $\varphi^{-1}(t)$ (counted with the ramification index) is constant (p. 327). Hence, any point $t \in (\mathbb{P}_1)_{\mathrm{an}}$ has a compact neighborhood K such that $\varphi^{-1}(K)$ is compact, and we conclude that C_{an} is a compact Riemann surface. We shall see that there are no other compact Riemann surfaces, but first let us extend some algebraic constructions used in the course Algebraic Geometry I:

Definitions: A **divisor** on a Riemann surface X is a function $D\colon X \to \mathbb{Z}$ with support on a discrete closed set, hence finite if X is compact. The divisors form an abelian group $\mathrm{Div}(X)$, and we put $D \leq D'$ when $D(x) \leq D'(x)$, $\forall x \in X$. The **degree** of a divisor $D = \sum_{x\in X} n_x \cdot x$ is $\deg D = \sum_x n_x$, whenever the sum only has a finite number of non null terms.

Any meromorphic function $0 \neq f$ has a discrete set of zeros and poles and the divisor of f is defined to be

$$D(f) = \sum_{x\in X} v_x(f)\cdot x := \sum_{f(x)=0} (\mathrm{ind}_x f)\cdot x - \sum_{f(x)=\infty} (\mathrm{ind}_x f)\cdot x$$

and two divisors D, D' are **linearly equivalent**, and we put $D' \sim D$, when $D' = D + D(f)$ for some meromorphic function $f \neq 0$.

We have $D(fh) = D(f) + D(h)$, $D(1/f) = -D(f)$ and $D(f+h) \geq \min\{D(f), D(h)\}$.

Moreover, $D(f) \geq 0$ if and only if $f \in \mathcal{O}_X(X)$.

Hence, *when X is compact, any meromorphic function f is fully determined, up to a constant factor, by the divisor $D(f)$, and moreover,* $\deg D(f) = 0$ *by the above theorem.*

Definition: Any divisor D defines a subsheaf $\mathcal{L}_D$ of the sheaf $\mathcal{M}_X$ of meromorphic functions:

$$\mathcal{L}_D(U) = \{f \in \mathcal{M}_X(U)\colon D|_U + D(f) \geq 0\}\ , \quad \text{by convention } 0 \in \mathcal{L}_D(U),$$

and it is a line $\mathcal{O}_X$-module. In fact, if $D = \sum_x n_x \cdot x$, and (U, z) is a coordinate neighborhood centered at x such that $n_y = 0$ at any point $x \neq y \in U$, then $\mathcal{L}_D|_U = z^{-n_x}\mathcal{O}_X|_U$.

Hence, $\mathcal{L}_{D(f)} = \frac{1}{f}\mathcal{O}_X$, and $\mathcal{L}_{D'} \subseteq \mathcal{L}_D$ if and only if $D' \leq D$.

Moreover, it is easy to check that $\mathcal{L}_D \otimes_{\mathcal{O}_X} \mathcal{L}_{D'} = \mathcal{L}_{D+D'}$ and $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}_D, \mathcal{L}_{D'}) = \mathcal{L}_{D'-D}$.

Proposition: *The line $\mathcal{O}_X$-modules $\mathcal{L}_D$ and $\mathcal{L}_{D'}$ are isomorphic if and only if D and D' are linearly equivalent.*

Proof: We may assume that $D' = 0$. Now, if $D = D(f)$, then $\mathcal{L}_{D(f)} = \frac{1}{f}\mathcal{O}_X \simeq \mathcal{O}_X$.

Conversely, if $\varphi\colon \mathcal{O}_X \to \mathcal{L}_D$ is an isomorphism, put $f = \varphi(1)$. Then $\mathcal{L}_D = f\mathcal{O}_X = \mathcal{L}_{-D(f)}$ as subsheaves of $\mathcal{M}_X$, so that $D = -D(f)$.

Definitions: At any point $x \in X$ the complex structure of T_xX defines a $\mathbb{R}$-linear automorphism $J\colon T_pX \to T_pX$ such that $J^2 = -1$, inducing $\mathbb{C}$-linear automorphisms of $T_xX \otimes_{\mathbb{R}} \mathbb{C}$ and $T_x^*X \otimes_{\mathbb{R}} \mathbb{C}$. A smooth 1-form ω is of **type** (1,0) when $J(\omega) = i\omega$, and of type $(0,1)$ when $J(\omega) = -i\omega$. The sheaves of smooth 1-forms of type (1,0) and (0,1) are denoted $\Omega_X^{1,0}$ and $\Omega_X^{0,1}$ respectively. Both are modules over the sheaf $\mathcal{C}_X^\infty$ of complex smooth functions; hence acyclic sheaves (p. 341).

If $f \in \mathcal{C}^\infty(X)$, the decomposition $\mathrm{d}f = \partial f + \bar\partial f$ of $\mathrm{d}f$ as a sum of a 1-form ∂f of type (1,0) and a 1-form $\bar\partial f$ of type (0,1) defines derivations

$$\partial\colon \mathcal{C}_X^\infty \longrightarrow \Omega_X^{1,0} \qquad , \qquad \bar\partial\colon \mathcal{C}_X^\infty \longrightarrow \Omega_X^{0,1}.$$

In a coordinate open set (U, z), 1-forms of type (1,0) (resp. (0,1)) are just $f\mathrm{d}z$ (resp. $f\mathrm{d}\bar z$), $f \in \mathcal{C}^\infty(U)$. Hence,

$$\partial f = \frac{\partial f}{\partial z}\,\mathrm{d}z \qquad , \qquad \bar\partial f = \frac{\partial f}{\partial \bar z}\,\mathrm{d}\bar z$$

and the Cauchy-Riemann equations state that we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}^\infty_X \xrightarrow{\ \bar\partial\ } \Omega^{0,1}_X .$$

Note: Let $\mathcal{C}^\infty_c(\mathbb{C})$ (resp. $\mathcal{C}^\infty_K(\mathbb{C})$) be the complex vector space of all complex smooth functions $\rho\colon \mathbb{C} \to \mathbb{C}$ with compact support (resp. with support in a given compact set $K \subseteq \mathbb{C}$). Then $\mathcal{C}^\infty_K(\mathbb{C})$ is a closed subspace of $\mathcal{C}^\infty(\mathbb{C})$, hence a Fréchet space, and we equip $\mathcal{C}^\infty_c(\mathbb{C}) = \bigcup_K \mathcal{C}^\infty_K(\mathbb{C})$ with the inductive limit topology. The space $\mathcal{D}$ of $\mathbb{C}$-linear continuous maps $\mathcal{C}^\infty_c(\mathbb{C}) \to \mathbb{C}$, is the space of complex **distributions** on $\mathbb{C}$. Any point $p \in \mathbb{C}$ defines a distribution, the **Dirac delta** $\delta_p(\rho) = \rho(p)$, and any smooth function f also, $T_f(\rho) = \int_{\mathbb{C}} f\rho \mathrm{d}x\mathrm{d}y$. Now $\partial_{\bar z}$ acts on $\mathcal{D}$ by the transpose map, $(\partial_{\bar z}T)(\rho) = -T(\partial_{\bar z}\rho)$, up to a sign introduced so that $\partial_{\bar z}T_f = T_{\partial_{\bar z}f}$.

The following lemmas state that a solution of the equation $\partial_{\bar z} f = \delta_0$ is just $f = \frac{1}{\pi z}$, and that the convolution product $f = \frac{1}{\pi z} * \rho$ is a solution of $\partial_{\bar z} f = \delta_0 * \rho = \rho$.

Lemma: $\displaystyle\int_{\mathbb{C}} \frac{1}{z-u}\frac{\partial\rho(u)}{\partial\bar u}\,\mathrm{d}\bar u\wedge\mathrm{d}u = 2\pi i\rho(z)\ , \quad \rho\in\mathcal{C}^\infty_c(\mathbb{C}).$

Proof: Let C_ε be the boundary of the disk D_ε of radius ε with center at z. Then

$$\begin{aligned}
\int_{\mathbb{C}} \frac{1}{z-u}\frac{\partial\rho(u)}{\partial\bar u}\,\mathrm{d}\bar u\wedge\mathrm{d}u &= \int_{\mathbb{C}} \frac{\partial}{\partial\bar u}\left(\frac{\rho(u)}{z-u}\right)\mathrm{d}\bar u\wedge\mathrm{d}u \qquad \left(\text{because } \tfrac{1}{z-u} \text{ is analytic}\right)\\
&= \int_{\mathbb{C}} \mathrm{d}\left(\frac{\rho(u)}{z-u}\,\mathrm{d}u\right) = \lim_{\varepsilon\to0}\int_{\mathbb{C}-D_\varepsilon}\mathrm{d}\left(\frac{\rho(u)}{z-u}\,\mathrm{d}u\right) \overset{\text{Stokes}}{=\!=} -\lim_{\varepsilon\to0}\int_{C_\varepsilon}\frac{\rho(u)}{z-u}\,\mathrm{d}u\\
&\overset{u=z+\varepsilon e^{i\theta}}{=\!=} \lim_{\varepsilon\to0}\int_0^{2\pi}\frac{\rho(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\,\mathrm{d}(\varepsilon e^{i\theta}) = \lim_{\varepsilon\to0}\int_0^{2\pi} i\rho(z+\varepsilon e^{i\theta})\,\mathrm{d}\theta = 2\pi i\rho(z).
\end{aligned}$$

Lemma: *If $\rho\in\mathcal{C}^\infty_c(\mathbb{C})$, then a smooth solution of the equation $\partial_{\bar z} f = \rho$ is*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\rho(u)}{z-u}\,\mathrm{d}\bar u\wedge\mathrm{d}u.$$

Proof: Use polar coordinates $u = z + re^{i\theta}$ centered at z. Then $\mathrm{d}\bar u\wedge\mathrm{d}u = 2ir\mathrm{d}r\wedge\mathrm{d}\theta$, and

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\rho(u)}{z-u}\,\mathrm{d}\bar u\wedge\mathrm{d}u = \frac{1}{\pi}\int_{\mathbb{C}}\rho(z+re^{i\theta})e^{-i\theta}\mathrm{d}r\wedge\mathrm{d}\theta.$$

Since ρ has compact support, we may differentiate under the integral sign; hence (using that $\partial_{\bar z} = \partial_{\bar u}$ and the above lemma)

$$\frac{\partial f}{\partial\bar z}(z) = \frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\rho(z+re^{i\theta})}{\partial\bar z}e^{-i\theta}\mathrm{d}r\wedge\mathrm{d}\theta = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{\partial\rho(u)}{\partial\bar u}\frac{\mathrm{d}\bar u\wedge\mathrm{d}u}{z-u} = \rho(z).$$

Dolbeault Exact Sequence: *The sheaf $\mathcal{O}_X$ of analytic functions on a Riemann surface X admits the acyclic resolution*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}^\infty_X \xrightarrow{\ \bar\partial\ } \Omega^{0,1}_X \longrightarrow 0.$$

Proof: Any germ of $\Omega^{0,1}_X$ is the germ of a section with compact support.

Theorem: *If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then $H^p(X,\mathcal{E}) = 0$, $p \geq 2$.*

Proof: Applying $\otimes_{\mathcal{O}_X}\mathcal{E}$ to the above exact sequence, we obtain a resolution of $\mathcal{E}$ by $\mathcal{C}^\infty_X$-modules; hence an acyclic resolution,

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{C}^\infty_X\otimes_{\mathcal{O}_X}\mathcal{E} \longrightarrow \Omega^{0,1}_X\otimes_{\mathcal{O}_X}\mathcal{E} \longrightarrow 0.$$

Lemma: *Let $K \subset \mathbb{C}$ be a compact set, and $\omega\colon \mathcal{C}(K) \to \mathbb{C}$ a continuous linear form. Then the function $h(a) = \omega\big(\frac{1}{z-a}\big)$, $a \in \mathbb{C} - K$, is analytic on $\mathbb{C} - K$.*

Proof: If $a \in B(z_0, r) \subset \mathbb{C} - K$, then $\frac{1}{z-a} = \frac{1}{(z-z_0)-(a-z_0)} = \sum_n \frac{(a-z_0)^n}{(z-z_0)^{n+1}}$.

The series uniformly converges on $z \in K$, because $|a - z_0| < r \le |z - z_0|$. Hence

$$(*) \qquad h(a) = \omega\left(\frac{1}{z-a}\right) = \omega\left(\sum_{n=0}^{\infty} \frac{(a-z_0)^n}{(z-z_0)^{n+1}}\right) = \sum_{n=0}^{\infty} \omega\left(\frac{1}{(z-z_0)^{n+1}}\right)(a-z_0)^n.$$

Lemma: *Let K be a compact subset of an open set $U \subseteq \mathbb{C}$. If no connected component of $U - K$ has compact closure in U, then the restriction morphism $\mathcal{R}(U) \to \mathcal{R}(K)$ has dense image* (where $\mathcal{R}(Z) \subseteq \mathcal{C}(Z)$ denotes the subalgebra of rational functions without poles on Z).

Proof: Since the image is a subalgebra, so is the closure, and it is enough to show (p. 391) that $\omega\big(\frac{1}{z-a}\big) = 0$, $a \in \mathbb{C} - K$, for any continuous linear form $\omega\colon \mathcal{C}(K) \to \mathbb{C}$ vanishing on $\mathcal{R}(U)$.

Consider the function $h(a) = \omega\big(\frac{1}{z-a}\big)$ on $\mathbb{C} - K$, and use that it is analytic:

1. *$h(a) = 0$ when a is in the unbounded connected component of $\mathbb{C} - K$.*

   If $|a| > \max_K |z|$, then $\frac{1}{z-a} = -\sum_n \frac{z^n}{a^{n+1}}$, and the series uniformly converges on K. Since $\omega(z^n/a^{n+1}) = 0$ and ω is continuous, we see that $0 = \omega\big(\frac{1}{z-a}\big) = h(a)$.

2. *$h = 0$ on any connected component W of $\mathbb{C} - K$ intersecting $\mathbb{C} - U$.*

   If $z_0 \in W$, $z_0 \notin U$, then $\omega\big(\frac{1}{(z-z_0)^{n+1}}\big) = 0$, and $(*)$ shows that $h = 0$ on a disk $D \subseteq W$.

3. *$h = 0$ on $\mathbb{C} - K$.* If W is a bounded connected component of $\mathbb{C} - K$, then $\bar{W}$ is compact, so that $\bar{W}$ intersects $\mathbb{C} - U$ (if $\bar{W} \subset U$, any connected component of $U - K$ contained in W would be relatively compact in U), and W intersects $\mathbb{C} - U$ because $\partial W \subseteq K$.

Theorem: *If $U \subseteq \mathbb{C}$ is an open set, then $\mathcal{R}(U)$ is dense in $\mathcal{O}(U)$.*

Proof: Any compact subset of U is contained in a compact set K such that $U - K$ has no connected component with compact closure in U (p. 100) and it is enough to show that $\omega(f|_K) = 0$, $\forall f \in \mathcal{O}(U)$, for any continuous linear form $\omega\colon \mathcal{C}(K) \to \mathbb{C}$ vanishing on $\mathcal{R}(U)$.

By the lemma, $\omega\big(\frac{1}{z-a}\big) = 0$, $\forall a \in \mathbb{C} - K$.

Pick a smooth function ρ with compact support $\subset U$ and such that $\rho = 1$ on a neighborhood of K, so that $\partial_{\bar{z}}\rho$ has a compact support L not intersecting K.

Since $\partial_{\bar{u}} f = 0$ and $\mathrm{supp}\,(\partial_{\bar{u}}\rho) = L$, when $z \in K$ we have

$$f(z) = f(z)\rho(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{z-u}\frac{\partial(f\rho)}{\partial\bar{u}}(u)\,\mathrm{d}\bar{u}\wedge\mathrm{d}u = \frac{1}{2\pi i}\int_L \frac{f(u)}{z-u}\frac{\partial\rho(u)}{\partial\bar{u}}\,\mathrm{d}\bar{u}\wedge\mathrm{d}u.$$

Since $K \times L$ and K is compact, the integrand $R(u, z) := \frac{h(u)}{z-u}$ is uniformly continuous; hence, given $\varepsilon > 0$, each point $a \in L$ has a neighborhood C where $|R(z,u) - R(z,a)| < \varepsilon$, $\forall z \in K, u \in C$, so that we may find a finite partition $L = \amalg L_n$ by measurable sets, and points $a_n \in L_n$, such that the Riemann sums of $R(u, z)$ uniformly approximate the integral; i.e. for every $z \in K$,

$$(\mathrm{Vol}\,L)\varepsilon > \Big|f(z) - \sum_n R(a_n, z)(\mathrm{Vol}\,L_n)\Big| = \Big|f(z) - \sum_n \frac{c_n}{z - a_n}\Big|, \quad c_n = h(a_n)(\mathrm{Vol}\,L_n),$$

so that the functions $\sum_n \frac{c_n}{z-a_n}$ uniformly converge to $f(z)$ in $\mathcal{C}(K)$.

Since $\omega\big(\sum_n \frac{c_n}{z-a_n}\big) = \sum_n \omega\big(\frac{c_n}{z-a_n}\big) = 0$, because $a_n \notin K$, we have $\omega\big(f|_K\big) = 0$.

Corollary: *Let K be a compact subset of an open set $U \subseteq \mathbb{C}$. If no connected component of $U - K$ has compact closure in U, then the restriction morphism $\mathcal{R}(U) \to \mathcal{O}(K)$ has dense image* (where $\mathcal{O}(K) = \{f \in \mathcal{C}(K)\colon f$ may be extended to an analytic function on a neighborhood of $K\}$).

Proof: If V is an open neighborhood of K, in $\mathcal{C}(K)$ we have

$$\mathcal{O}(V) \subseteq \overline{\mathcal{O}(V)} \overset{\text{th.}}{=\!=} \overline{\mathcal{R}(V)} \subseteq \overline{\mathcal{R}(K)} \overset{\text{lem.}}{=\!=} \overline{\mathcal{R}(U)}.$$

Dolbeault-Grothendieck Lemma: *If $U \subseteq \mathbb{C}$ is an open set, we have an exact sequence*

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{C}^\infty(U) \xrightarrow{\bar\partial} \Omega^{0,1}(U) \longrightarrow 0.$$

Proof: Put $U = \bigcup_n K_n$ with K_n compact, $K_{n-1} \subseteq U_n := \overset{\circ}{K}_n$, and no connected component of $U - K_n$ with compact closure in U (p. 100).

We have to show that $\partial_{\bar z}\colon \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$ is surjective. Pick $f \in \mathcal{C}^\infty(U)$.

1. *There exist $u_n \in \mathcal{C}^\infty(U)$ such that $\partial_{\bar z} u_n = f$ on U_n and $|u_{n+1} - u_n| < 2^{-n}$ on K_{n-1}.*

   If $u_1, \ldots, u_n$ are constructed, pick $\tilde u_{n+1} \in \mathcal{C}^\infty(U)$ such that $\partial_{\bar z}\tilde u_{n+1} = f$ on U_{n+1} (it exists because f coincides on U_{n+1} with some $\rho \in \mathcal{C}^\infty(U)$ with compact support). Now $\tilde u_{n+1} - u_n$ is analytic on U_n, because $\partial_{\bar z}(\tilde u_{n+1} - u_n) = f - f = 0$ on U_n. Since no connected component of $U - K_{n-1}$ has compact closure in U, there is an analytic function $R(z)$ on U such that $|\tilde u_{n+1} - u_n - R| < 2^{-n}$ on K_{n-1}. Put $u_{n+1} := \tilde u_{n+1} - R$.

2. *The function $u(z) = \lim u_n(z) \in \mathcal{C}^\infty(U)$ is a solution of $\partial_{\bar z} u = f$.*

   $$u = u_n + \sum_{m=n}^{\infty} (u_{m+1} - u_m).$$

   On U_n, the right term is an absolutely convergent series of analytic functions; hence it converges to an analytic function h, so that $u = u_n + h$ is smooth on U_n and

   $$\partial_{\bar z} u = \partial_{\bar z} u_n - \partial_{\bar z} h = f + 0 = f.$$

Theorem: *If $U \subseteq \mathbb{C}$ is an open set, then $H^1(U, \mathcal{O}_U) = 0$.*

Corollary: $H^1(U, \mathcal{O}^*) = 0$; *hence any line $\mathcal{O}_U$-module is trivial* (p. 361).

Proof: $H^1(U, \mathcal{O}_U) = 0$ and $H^2(U, \mathbb{Z}) = 0$ (p.363). We conclude by the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_U \xrightarrow{\exp} \mathcal{O}_U^* \longrightarrow 0.$$

Weierstrass Theorem: *Any divisor in $U \subseteq \mathbb{C}$ is the divisor of some meromorphic function.*

Corollary: *Any meromorphic function f on $U \subseteq \mathbb{C}$ is the quotient of two analytic functions.*

Proof: The divisor of poles of f is the divisor of some $h \in \mathcal{O}(U)$. Then $fh \in \mathcal{O}(U)$.

Definition: The **principal parts** at a point x of a Riemann surface X are the elements of the quotient group $\mathcal{M}_{X,x}/\mathcal{O}_{X,x} = \{\frac{a_n}{z^n} + \ldots + \frac{a_1}{z}\}$, where z is a local coordinate centered at x.

Mittag-Leffler Theorem: *Given a principal part at any point of a discrete closed set in an open set $U \subseteq \mathbb{C}$, there is a meromorphic function on U with the prescribed principal parts and no other pole.*

Proof: We have an exact sequence $0 \to \mathcal{M} \to \mathcal{O} \to \mathcal{P} \to 0$, where $\mathcal{P}$ is the sheaf of principal parts. Now take sections on U and use that $H^1(U, \mathcal{O}) = 0$.

13.3.1 Riemann-Roch Theorem

Let L be a free A-module, eventually of infinite rank, let $\{\mathrm{d}x_i\}_{i\in I}$ denote a base, and fix a good order on the set of indices I. Then $\{\mathrm{d}x_{i_0}\wedge\ldots\wedge\mathrm{d}x_{i_p}\}_{i_0<\ldots<i_p}$ is a base of $\Lambda^{p+1}L$, and we have an exact sequence of A-modules

$$\ldots\Lambda^3 L\xrightarrow{i_D}\Lambda^2 L\xrightarrow{i_D} L\xrightarrow{i_D} A\longrightarrow 0\ ,\quad D=\textstyle\sum_i\partial_i.$$

In fact, $\operatorname{Im} i_D\subseteq\operatorname{Ker} i_D$ because$i_D\circ i_D=0$, and $\operatorname{Ker} i_D\subseteq\operatorname{Im} i_D$ because

$$i_D(\mathrm{d}x_1\wedge\omega)=\omega-\mathrm{d}x_1\wedge(i_D\omega).$$

Let $\mathfrak{U}=\{U_i\}_{i\in I}$ an open cover of X. The stalks of $\mathcal{L}=\bigoplus_i\mathbb{Z}_{U_i}$ are free $\mathbb{Z}$/modules; hence, if $\mathrm{d}x_i$ denotes a generator of the stalk of $\mathbb{Z}_{U_i}$, we have an exact sequence

$$\ldots\Lambda^3\mathcal{L}\xrightarrow{i_D}\Lambda^2\mathcal{L}\xrightarrow{i_D}\mathcal{L}\xrightarrow{i_D}\mathbb{Z}\longrightarrow 0\ ,\quad D=\textstyle\sum_i\partial_i.$$

The **Cech complex** of a sheaf $\mathcal{F}$ on X is $\check{C}^\bullet(\mathfrak{U},\mathcal{F})=\operatorname{Hom}(\Lambda^\bullet\mathcal{L},\mathcal{F})$:

$$\check{C}^p(\mathfrak{U},\mathcal{F})=\operatorname{Hom}\Big(\bigoplus_{i_0<\ldots<i_p}\mathbb{Z}_{U_{i_0}\cap\ldots\cap U_{i_p}},\mathcal{F}\Big)=\prod_{i_0<\ldots<i_p}\mathcal{F}(U_{i_0}\cap\ldots\cap U_{i_p})$$
$$\mathrm{d}\colon\check{C}^p(\mathfrak{U},\mathcal{F})\longrightarrow\check{C}^{p+1}(\mathfrak{U},\mathcal{F})\ ,\quad(\mathrm{d}s)_{i_0<\ldots<i_{p+1}}=\sum_{k=0}^{p+1}(-1)^k s_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}|_{U_{i_0}\cap\ldots\cap U_{i_{p+1}}}.$$

and the **Cech cohomology** groups are $\check{H}^p(\mathfrak{U},\mathcal{F})=H^p\big[\check{C}^\bullet(\mathfrak{U},\mathcal{F})\big]$.

Applying $\operatorname{Hom}(-,\mathcal{F})$ to the exact sequence $\Lambda^{\mathcal{L}}\to\mathcal{L}\to\mathbb{Z}\to 0$ we obtain an exact sequence $0\to\mathcal{F}(X)\to\check{C}^0(\mathfrak{U},\mathcal{F})\xrightarrow{\mathrm{d}}\check{C}^1(\mathfrak{U},\mathcal{F})$ so that

$$\check{H}^0(\mathfrak{U},\mathcal{F})=\mathcal{F}(X).$$

Leray's Theorem: *If $\mathcal{F}$ is acyclic on any intersection $U_{i_0}\cap\ldots\cap U_{i_p}$, then*

$$\check{H}^p(\mathfrak{U},\mathcal{F})=H^p(X,\mathcal{F})\ ,\quad p\geq 0.$$

Proof: Consider an injective resolution $0\to\mathcal{F}\to\mathcal{I}^\bullet$ and the bicomplexes with bounded diagonals

$$\operatorname{Hom}(\mathbb{Z},\mathcal{I}^\bullet)\longrightarrow\operatorname{Hom}(\Lambda^\bullet\mathcal{L},\mathcal{I}^\bullet)\longleftarrow\operatorname{Hom}(\Lambda^\bullet\mathcal{L},\mathcal{F})$$

Now $\operatorname{Hom}(\mathbb{Z},\mathcal{I}^p)\longrightarrow\operatorname{Hom}(\Lambda^\bullet\mathcal{L},\mathcal{I}^p)$ is a quasi-isomorphism because $\Lambda^\bullet\mathcal{L}\to\mathbb{Z}\to 0$ is an exact sequence and $\operatorname{Hom}(-,\mathcal{I}^p)$ is an exact functor.

By hypothesis $\mathcal{F}(U)=\operatorname{Hom}(\mathbb{Z}_U,\mathcal{F})\to\mathcal{I}^\bullet(U)=\operatorname{Hom}(\mathbb{Z}_U,\mathcal{I}^\bullet)$ is a quasi-isomorphism when $U=U_{i_0}\cap\ldots\cap U_{i_p}$; hence $\operatorname{Hom}(\Lambda^p\mathcal{L},\mathcal{F})\to\operatorname{Hom}(\Lambda^p\mathcal{L},\mathcal{I}^\bullet)$ is a quasi-isomorphism.

By the bicomplex theorem,

$$H^p(X,\mathcal{F})=H^p\big[\operatorname{Hom}(\mathbb{Z},\mathcal{I}^\bullet)\big]=H^p\big[\operatorname{Hom}(\Lambda^\bullet\mathcal{L},\mathcal{I}^\bullet)\big]=H^p\big[\operatorname{Hom}(\Lambda^\bullet\mathcal{L},\mathcal{F})\big]=\check{H}^p(\mathfrak{U},\mathcal{F}).$$

Lemma: *Let $V\subseteq U$ be open sets in a Riemann surface X. If V has compact closure $\bar{V}\subseteq U$, then the restriction morphism $\mathcal{O}_X(U)\to\mathcal{O}_X(V)$ is compact.*

Proof: Let $B\subset\mathcal{O}_X(U)$ be the unit ball of the seminorm defined by the compact set $K=\bar{V}$.

For any sequence (f_n) in B we have $|f|\leq 1$ on V; hence, by Montel's theorem, some subsequence uniformly converges on the compact subsets of V.

We conclude that the image of B in $\mathcal{O}_X(V)$ has compact closure.

Finiteness Theorem: *Let X be a compact Riemann surface, and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank. All the cohomology groups $H^p(X,\mathcal{E})$ are complex vector spaces of finite dimension.*

Proof: Let $\mathfrak{U} = \{U_i\}$ be a finite open cover of X by open sets U_i isomorphic to disks D_{r_i} such that the disks $D_{r_i/2}$ also define an open cover $\mathfrak{V} = \{V_i\}$ of X, and such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_X^n$. By Leray's theorem, $\check{H}^p(\mathfrak{U},\mathcal{E}) = H^p(X,\mathcal{E}) = \check{H}^p(\mathfrak{V},\mathcal{F})$.

$\mathcal{E}(U_{i_0} \cap \ldots \cap U_{i_p}) \simeq \mathcal{O}_X(U_{i_0} \cap \ldots \cap U_{i_p})^n$ inherit a natural Fréchet topology; hence so do the vector spaces $\check{C}^p(\mathfrak{U}) := \check{C}^p(\mathfrak{U},\mathcal{E})$, and the linear maps $\mathrm{d}\colon \check{C}^p(\mathfrak{U}) \to \check{C}^{p+1}(\mathfrak{U})$ are continuous because so are the restriction maps, so that $Z^p(\mathfrak{U}) := \operatorname{Ker} \mathrm{d}$ is a Fréchet space. Moreover, the linear maps $\check{C}^p(\mathfrak{U}) \to \check{C}^p(\mathfrak{V})$ are compact by the lemma; hence so are $Z^p(\mathfrak{U}) \to Z^p(\mathfrak{V})$.

(*Case* $p = 0$) The compact map $Z^0(\mathfrak{U}) \to Z^0(\mathfrak{V})$ is bijective; hence the Fréchet space $Z^0(\mathfrak{V}) = H^0(X,\mathcal{E})$ admits a compact neighborhood of 0: it is finite-dimensional (p. 390).

(*Case* $p = 1$) The restriction morphism $f\colon Z^1(\mathfrak{U}) \to Z^1(\mathfrak{V})$ is compact, and the cokernel of $\mathrm{d}\colon \check{C}^0(\mathfrak{V}) \to Z^1(\mathfrak{V})$ is just $\check{H}^1(\mathfrak{V},\mathcal{E}) = H^1(X,\mathcal{E})$. Now $h = \mathrm{d} + f\colon \check{C}^0(\mathfrak{V}) \oplus Z^1(\mathfrak{U}) \to Z^1(\mathfrak{V})$ is surjective, because $\check{H}^1(\mathfrak{U},\mathcal{E}) = H^1(X,\mathcal{E}) = \check{H}^1(\mathfrak{V},\mathcal{E})$, and Schwartz theorem let us conclude.

Definition: We put $h^p(\mathcal{E}) := \dim H^p(X,\mathcal{E})$, and $g = h^1(\mathcal{O}_X)$ is the **genus** of X.

Corollary: *Meromorphic functions separate points on any compact Riemann surface X.*

Proof: If $x \in X$, then $\dim \mathcal{O}_x/\mathfrak{m}_x^n = n$. When $n > g$, the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L}_{nx} \longrightarrow \mathcal{O}_x/\mathfrak{m}_x^n \longrightarrow 0$$

shows that $h^0(\mathcal{L}_{nx}) > 1$; hence there is a meromorphic function with a unique pole at x.

Corollary: *The topology of X is the initial topology of the meromorphic functions $X \to \mathbb{P}_1$.*

Proof: The continuous bijection $X \to X_{\text{in}}$ is a homeomorphism because X_{in} is compact (so is X) and separated, because the meromorphic functions separate points.

Corollary: *If $x \in X$, there is a meromorphic function on X with a simple pole at x.*

Proof: Since $\mathcal{O}_x/\mathfrak{m}_x$ is flasque, we have $h^1(\mathcal{O}_x/\mathfrak{m}_x) = 0$, and the exact sequence

$$0 \longrightarrow \mathcal{L}_{nx} \longrightarrow \mathcal{L}_{(n+1)x} \longrightarrow \mathcal{O}_x/\mathfrak{m}_x \longrightarrow 0$$

shows that $h^1(\mathcal{L}_{(n+1)x}) \leq h^1(\mathcal{L}_{nx})$. Now the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L}_{nx} \longrightarrow \mathcal{O}_x/\mathfrak{m}_x^n \longrightarrow 0$$

shows that $h^0(\mathcal{L}_{nx}) = n - \text{const}$, $n \gg 0$. Hence $h^0(\mathcal{L}_{(n-1)x}) < h^0(\mathcal{L}_{nx}) < h^0(\mathcal{L}_{(n+1)x})$ when $n \gg 0$, so that there are meromorphic functions $f, h \in \mathcal{M}(X)$ with a poles at x of order n and $n+1$ respectively, and f/h has a simple pole.

Corollary: *Any line sheaf $\mathcal{L}$ is isomorphic to the line sheaf of some divisor.*

Proof: If we have an injective morphism $\mathcal{L} \to \mathcal{O}_X$, then $\mathcal{O}_X/\mathcal{L}$ is supported on a finite number of points, so that it defines a positive divisor D and $\mathcal{L} \simeq \mathcal{L}_{-D}$.

In the general case, we consider a point $x \in X$ and the exact sequence

$$0 \longrightarrow \mathcal{L}_{-nx} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_x/\mathfrak{m}_x^n \longrightarrow 0.$$

After applying $\otimes\mathcal{L}_{nx}\otimes\mathcal{L}$, the cohomology exact sequence

$$0 \longrightarrow H^0(X,\mathcal{L}) \longrightarrow H^0(X,\mathcal{L}_{nx}\otimes\mathcal{L}) \longrightarrow \mathcal{O}_x/\mathfrak{m}_x^n \longrightarrow H^1(X,\mathcal{L})$$

shows that $h^0(\mathcal{L}_{nx}\otimes\mathcal{L}) \neq 0$ when $n > h^1(\mathcal{L})$.

Hence there is a non null morphism $\mathcal{O}_X \to \mathcal{L}_{nx}\otimes\mathcal{L}$, and dualizing we obtain an injective morphism $\mathcal{L}_{-nx}\otimes\mathcal{L}^* \to \mathcal{O}_X$. We conclude that $\mathcal{L}_{-nx}\otimes\mathcal{L}^* \simeq \mathcal{L}_{-D}$ and $\mathcal{L} \simeq \mathcal{L}_{D-nx}$.

Lemma: *Let $\mathcal{L}$ be a line sheaf such that $h^1(\mathcal{L}) \neq 0$. For any line sheaf $\mathcal{L}'$ we have a natural injective morphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}',\mathcal{L}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}\big(H^1(X,\mathcal{L}'),H^1(X,\mathcal{L})\big).$$

Proof: If $\varphi\colon \mathcal{L}' \to \mathcal{L}$ is non-null, then it is injective, and the exact sequence

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L} \longrightarrow \mathcal{T} \longrightarrow 0$$

(where $\mathcal{T}$ is flasque, because it has finite support) induces an exact sequence

$$H^1(X,\mathcal{L}') \xrightarrow{\ \varphi\ } H^1(X,\mathcal{L}) \longrightarrow H^1(X,\mathcal{T}) = 0$$

and $\varphi\colon H^1(X,\mathcal{L}') \to H^1(X,\mathcal{L})$ is non-null, because $H^1(X,\mathcal{L}) \neq 0$.

Lemma: *If D_n is a strictly increasing sequence of positive divisors, then $h^1(\mathcal{L}_{D_n}) = 0$, $n \gg 0$.*

Proof: The cohomology exact sequence of the exact sequence of sheaves

$$0 \longrightarrow \mathcal{L}_{D_n} \longrightarrow \mathcal{L}_{D_{n+1}} \longrightarrow \mathcal{T} \longrightarrow 0$$

where $\mathcal{T}$ is flasque (because of finite support) shows that $h^1(\mathcal{L}_{D_{n+1}}) \leq h^1(\mathcal{L}_{D_n})$, so that $h^1(\mathcal{L}_{D_n})$ stabilizes when $n \gg 0$. Moreover, the exact sequences

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L}_{D_n} \longrightarrow \mathcal{O}_{D_n} \longrightarrow 0$$

show that $h^0(\mathcal{L}_{D_n}) \to \infty$ when $n \to \infty$. The above lemma shows that $h^1(\mathcal{L}_{D_n}) = 0$, $n \gg 0$.

Theorem: *If X is a compact Riemann surface, $H^p(X,\mathcal{M}_X) = 0$, $p \geq 1$.*

Proof: We have $\mathcal{M}_X = \varinjlim \mathcal{L}_D$, where D runs over the positive divisors, and we conclude by the above lemma, because the cohomology commutes with inductive limits (p. 346)

$$H^p(X,\mathcal{M}_X) = \varinjlim H^p(X,\mathcal{L}_D) = 0 \ , \quad p \geq 1.$$

Theorem: *Let X be a compact Riemann surface. The field $\mathcal{M}(X)$ is a finite extension of $\mathbb{C}(t)$, and the points of X correspond to the discrete valuations of $\mathcal{M}(X)$ trivial over $\mathbb{C}$.*

Proof: Let $t\colon X \to \mathbb{P}_1$ be a non constant meromorphic function, of degree d, and let us see that $\mathbb{C}(t) \to \mathcal{M}(X)$ is a finite extension. It is enough to show that any meromorphic function $f \in \mathcal{M}(X)$ fulfills a relation of degree d

$$f^d - s_1(t)f^{d-1} + \ldots \pm s_d(t) \ , \quad s_i(t) \in \mathbb{C}(t).$$

Out of the ramification points, the definition of $s_i(t)$ is obvious: it is the i-th elementary symmetric function of the values of f on the points of the fibre of $t \in \mathbb{P}_1$. Let us see that the functions $s_i(t)$ may be extended to meromorphic functions on $\mathbb{P}_1$:

If $t_0 \in \mathbb{P}_1$ is a ramification point, and f has no pole in the fibre of t_0, then f is bounded on a neighborhood of the fibre, so that s_i is bounded on a neighborhood of t_0 and by the removable singularity theorem s_i may be extended to an analytic function. If f has poles in the fibre of t_0, we pick $h \in \mathbb{C}(t)$ with many so zeros at t_0, that $\bar{f} = h(t)f$ has no poles in the fibre. Now $\bar{s}_i = h^i s_i$ is analytic on a neighborhood of t_0, and we conclude that s_i is meromorphic.

Now, any point $x \in X$ defines a discrete valuation $v_x \colon \mathcal{M}(X) - \{0\} \to \mathbb{Z}$, trivial over $\mathbb{C}$:

$$v_x(f) = \text{number of zeros or poles of } f \text{ at } x$$

(the poles counted with negative sign) and $v_x \neq v_y$ when $x \neq y$ because meromorphic functions separate points. Finally, if some discrete valuation v were not defined by a point of X, the theory of algebraic curves (p. 329) shows the existence of a non constant $f \in \mathcal{M}(X)$ with a unique pole at v, so that f has no pole on X. Absurd. q.e.d.

If X is a compact Riemann surface, then X_{alg} will be the Riemann variety of the field $\mathcal{M}(X)$. Any non constant analytic morphism $\varphi \colon X \to Y$ induces a morphism of $\mathbb{C}$-algebras $\varphi^* \colon \mathcal{M}(Y) \to \mathcal{M}(X)$, $\varphi^*(f) = f \circ \varphi$; hence a morphism of $\mathbb{C}$-schemes $\varphi_{\text{alg}} \colon X_{\text{alg}} \to Y_{\text{alg}}$.

The $\mathbb{C}$-scheme X_{alg} is a complete non singular curve, and conversely, recall that any complete non singular curve C over $\mathbb{C}$ defines a compact Riemann surface C_{an}.

Theorem: *The functors $X \rightsquigarrow X_{\text{alg}}$ and $C \rightsquigarrow C_{\text{an}}$ define an equivalence of categories*

$$\begin{bmatrix}\text{Compact Riemann}\\ \text{surfaces}\end{bmatrix} \leftrightsquigarrow \begin{bmatrix}\text{Complete and non-singular}\\ \text{algebraic curves over } \mathbb{C}\end{bmatrix}$$

Proof: If X is a Riemann surface, the natural map $X \to (X_{\text{alg}})_{\text{an}}$ is an homeomorphism by the above results, and it is analytic; hence an isomorphism.

If C is the Riemann variety of a finite extension Σ of $\mathbb{C}(t)$, then the elements of Σ define meromorphic functions on C_{an}, because locally they are quotients of analytic functions. If the extension $\Sigma \to \mathcal{M}(C_{\text{an}})$ were of degree > 1, then the meromorphic functions $f \in \Sigma$ do not separate the discrete valuations of $\mathcal{M}(C_{\text{an}})$; i.e. do not separate points of C_{an}. Absurd. Hence $\Sigma = \mathcal{M}(C_{\text{an}})$ and $C = (C_{\text{an}})_{\text{alg}}$.

Corollary: *The non constant analytic morphisms $\varphi \colon X \to Y$ correspond to the morphisms of $\mathbb{C}$-algebras $\varphi^* \colon \mathcal{M}(Y) \to \mathcal{M}(X)$, and $\deg \varphi = [\mathcal{M}(X) : \mathcal{M}(Y)]$.*

Corollary: *$H^p(X, \mathcal{L}_D) = H^p(X_{\text{alg}}, L_D)$, for any divisor D.*

Proof: Let us consider the following exact sequence of sheaves on X_{alg}

$$0 \longrightarrow L_D \longrightarrow \mathcal{M}(X) \longrightarrow \mathcal{P}_D^{\text{alg}} \longrightarrow 0$$

where $\mathcal{M}(X)$ is the constant sheaf (hence flasque, because all Zariski open sets are connected), and the exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{L}_D \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{P}_D \longrightarrow 0$$

The cohomology exact sequences define a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathcal{M}(X) & \longrightarrow & H^0(X_{\text{alg}}, \mathcal{P}_D^{\text{alg}}) & \longrightarrow & H^1(X_{\text{alg}}, L_D) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
H^0(X, \mathcal{M}_X) & \longrightarrow & H^0(X, \mathcal{P}_D) & \longrightarrow & H^1(X, \mathcal{L}_D) & \longrightarrow & 0
\end{array}$$

We conclude because $\mathcal{M}(X) = H^0(X, \mathcal{M}_X)$, and also $H^0(X_{\mathrm{alg}}, \mathcal{P}_D^{\mathrm{alg}}) = H^0(X, \mathcal{P}_D)$ since both sheaves $\mathcal{P}_D^{\mathrm{alg}}$ and $\mathcal{P}_D$ have the same stalk at any point $x \in X$ and any global section is zero everywhere, up to a finite number of points. q.e.d.

Now all the results on the cohomology of line sheaves on complete non-singular algebraic curves (pp. 327-338) also hold for line sheaves on compact Riemann surfaces. In particular the **Riemann-Roch theorem:** *On any compact Riemann surface X there is a canonical line sheaf $\mathcal{L}_K$ such that $H^1(X, \mathcal{L})^* = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}_K)$, and for any divisor D,*

$$h^0(\mathcal{L}_D) = 1 - g + \deg D + h^0(\mathcal{L}_{K-D}).$$

But the calculation (p. 330) of the dualizing sheaf $\omega_X = L_K$ may be greatly simplified:

Recall that, since ω_X represents the functor $H^1(X, -)^*$, it comes equipped with a linear map $\xi \in H^1(X, \omega_X)^*$ such that for any linear form $\eta \in H^1(X, \mathcal{L})^*$ there is a unique morphism $\varphi\colon \mathcal{L} \to \omega_X$ such that $\varphi^*(\xi) = \eta$. Let Ω_X be the line sheaf of analytic 1-forms.

The exact sequence $0 \to \mathcal{O}_X \to \mathcal{M}_X \to \mathcal{P}_X \to 0$ induces an exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{M}_X \otimes_{\mathcal{O}_X} \Omega_X \longrightarrow \mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega_X \longrightarrow 0$$

where the sheaf $\mathcal{M}_X \otimes \Omega_X = (\varinjlim \mathcal{L}_D) \otimes \Omega_X = \varinjlim \mathcal{L}_{D_n}$ is acyclic. Now the exact sequence

$$(\mathcal{M}_X \otimes_{\mathcal{O}_X} \Omega_X)(X) \longrightarrow (\mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega_X)(X) \longrightarrow H^1(X, \Omega_X) \longrightarrow 0$$

shows that $H^1(X, \Omega_X)$ is just the group $(\mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega_X)(X) = \bigoplus_x \{a_n \frac{\mathrm{d}z}{z^n} + \ldots + a_1 \frac{\mathrm{d}z}{z}\}$ of principal parts of meromorphic forms (where z is a local coordinate centered at $x \in X$), modulo the subgroup defined by the global meromorphic forms $(\mathcal{M}_X \otimes_{\mathcal{O}_X} \Omega_X)(X)$.

Now, we have a linear map $\mathrm{Res} = \sum_x \mathrm{Res}_x\colon (\mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega_X)(X) \to \mathbb{C}$, and the residue theorem states that it factors through a (non null, since $\mathrm{Res} \frac{\mathrm{d}z}{z} = 1$) linear map

$$\mathrm{Res}\colon H^1(X, \Omega_X) \longrightarrow \mathbb{C}.$$

Theorem: *The dualizing sheaf is* (Ω_X, Res).

Proof: There is a (non null, hence injective) morphism $\varphi\colon \Omega_X \to \omega_X$ such that $\varphi^*(\xi) = \mathrm{Res}$; hence a commutative diagram with exact rows

$$\begin{array}{ccccc}
(\mathcal{M}_X \otimes_{\mathcal{O}_X} \Omega_X)(X) & \longrightarrow & (\mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega_X)(X) & \xrightarrow{\mathrm{Res}} & \mathbb{C} \\
\downarrow \varphi & & \downarrow \varphi & & \downarrow \mathrm{Id} \\
(\mathcal{M}_X \otimes_{\mathcal{O}_X} \omega_X)(X) & \longrightarrow & (\mathcal{P}_X \otimes_{\mathcal{O}_X} \omega_X)(X) & \xrightarrow{\xi} & \mathbb{C}
\end{array}$$

If $\varphi\colon \Omega_{X,x} \to \omega_{X,x}$ were not surjective at some point $x \in X$, then $\frac{\mathrm{d}z}{z} \in \mathcal{D}_x$, so that $\frac{\mathrm{d}z}{z}$ has null image by φ in $(\mathcal{P}_X \otimes_{\mathcal{O}_X} \omega_X)(X)$. Absurd, $0 = \xi\big(\varphi(\frac{\mathrm{d}z}{z})\big) = \mathrm{Res}\big(\frac{\mathrm{d}z}{z}\big) = 1$.

Corollary: *A finite family $(\omega_x) \in \bigoplus_x \{a_n \frac{\mathrm{d}z}{z^n} + \ldots + a_1 \frac{\mathrm{d}z}{z}\}$ of principal parts of meromorphic forms is defined by a global meromorphic form if and only if $\sum_x \mathrm{Res}_x(\omega_x) = 0$.*

Corollary: *The genus $g = h^1(\mathcal{O}_X)$ is the maximal number of linearly independent analytic forms on X, and it coincides with the topological genus, $2g = \dim_{\mathbb{Q}} H^1(X, \mathbb{C})$.*

Proof: By duality we have

$$g = h^1(\mathcal{O}_X) = h^0\big(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \Omega_X)\big) = h^0(\Omega_X),$$
$$1 = h^0(\mathcal{O}_X) = h^0\big(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega_X, \Omega_X)\big) = h^1(\Omega_X).$$

Moreover, since X is a compact oriented surface, $H^0(X, \mathbb{C}) = H^2(X, \mathbb{C}) = \mathbb{C}$.
We conclude by the cohomology exact sequence of the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X \longrightarrow 0.$$

Chapter 14

Differential Geometry II

14.1 Valued Differential Calculus

Let $\mathcal{C}_X^\infty$ be the sheaf of smooth functions on a smooth manifold X of dimension n, and recall that the sheaf $\mathcal{D}$ of (smooth) vector fields is a locally free $\mathcal{C}_X^\infty$-module of rank n.

Definitions: Let us fix a locally free $\mathcal{C}_X^\infty$-module $\mathcal{E}$ of rank r.

An $\mathcal{E}$-valued **differential** p**-form** is an alternate $\mathcal{C}_X^\infty$-multilinear morphism of sheaves

$$\omega\colon \mathcal{D}\times \overset{p}{\ldots} \times\mathcal{D} \longrightarrow \mathcal{E}$$

and we agree that valued 0-forms are just global sections of $\mathcal{E}$.

The **interior contraction** of ω with a vector field D is the $\mathcal{E}$-valued $(p-1)$-form

$$(i_D\omega)(D_2,\ldots,D_p) = \omega(D,D_2,\ldots,D_p).$$

Given a $\mathcal{C}_X^\infty$-bilinear morphism of sheaves $\mathcal{E}\times\mathcal{E}' \xrightarrow{\cdot} \mathcal{E}''$, the **exterior product** of an $\mathcal{E}$-valued p-form ω with an $\mathcal{E}'$-valued q-form ω' is the $\mathcal{E}''$-valued $(p+q)$-form

$$(\omega\wedge\omega')(D_1,\ldots,D_{p+q}) = \tfrac{1}{p!q!}\sum_{\sigma\in S_{p+q}} \operatorname{sgn}(\sigma)\,\omega(D_{\sigma(1)},\ldots,D_{\sigma(p)})\cdot\omega'(D_{\sigma(p+1)},\ldots,D_{\sigma(p+q)}).$$

The proofs of p. 66 show that

$$i_D(\omega\wedge\omega') = (i_D\omega)\wedge\omega' + (-1)^p\omega\wedge(i_D\omega')$$
$$\omega\wedge\omega' = (-1)^{pq}\omega'\wedge\omega$$

when $\omega'\wedge\omega$ is considered respect to the product $e'\cdot e = e\cdot e'$.

In particular, if $\mathcal{E}'=\mathcal{E}$, we have $\omega\wedge\omega' = (-1)^{pq}\omega'\wedge\omega$ when the product is commutative.

Examples: Ordinary p-forms are just $\mathcal{C}_X^\infty$-valued p-forms.

The sheaf $\mathcal{T}_p^q$ of tensor fields of type (p,q) is a locally free $\mathcal{C}_X^\infty$-module; hence any (p,q)-tensor field may be viewed as a $\mathcal{T}_p^q$-valued 0-form.

An ordinary p-form ω_p and a global section e of $\mathcal{E}$, define an $\mathcal{E}$-valued p-form $\omega_p\otimes e$,

$$(\omega_p\otimes e)(D_1,\ldots,D_p) = \omega_p(D_1,\ldots,D_p)\,e.$$

The identity $I\colon\mathcal{D}\to\mathcal{D}$, $I(D)=D$, is a $\mathcal{D}$-valued 1-form.

The torsion tensor $\operatorname{Tor}(D_1,D_2) = D_1^\nabla D_2 - D_2^\nabla D_1 - [D_1,D_2]$ of a linear connection ∇ is a vector valued 2-form, and the curvature $R(D_1,D_2) = D_1^\nabla D_2^\nabla - D_2^\nabla D_1^\nabla - [D_1,D_2]^\nabla$ is an $\underline{\operatorname{End}}(\mathcal{D})$-valued 2-form.

Lemma: *If $(U; u_1, \ldots, u_n)$ is a coordinate open set in X where $\mathcal{E}$ is trivial, and $\{e_1, \ldots, e_r\}$ is a base of $\mathcal{E}(U)$, then $\mathcal{E}$-valued p-forms on U form a free $\mathcal{C}^\infty(U)$-module of base*

$$\{ (\mathrm{d}u_{i_1} \wedge \ldots \wedge \mathrm{d}u_{i_p}) \otimes e_j \}_{1 \le i_1 < \ldots < i_p \le n, 1 \le j \le r}.$$

Definition: A morphism of sheaves $\nabla\colon \mathcal{D} \times \mathcal{E} \longrightarrow \mathcal{E}$ is a **linear connection** in $\mathcal{E}$ if

1. $D^\nabla(e_1 + e_2) = D^\nabla e_1 + D^\nabla e_2$,
   $(fD)^\nabla e = fD^\nabla e$.
2. $(D_1 + D_2)^\nabla e = D_1^\nabla e + D_2^\nabla e$,
   $D^\nabla(fe) = (Df)e + fD^\nabla e$.

Examples: In $\mathcal{C}_X^\infty$ we have a natural connection $D^\nabla f = Df$, implicitly used in the differential calculus with ordinary forms.

Connections in $\mathcal{D}$ are just linear connections (p. 293) on the manifold X.

If we have connections in $\mathcal{E}$ and $\mathcal{E}'$, then we have natural connections in

$$\begin{array}{lcl} \mathcal{E} \oplus \mathcal{E}' & , & D^\nabla(e + e') = D^\nabla e + D^\nabla e' \\ \mathcal{E} \otimes_{\mathcal{C}_X^\infty} \mathcal{E}' & , & D^\nabla(e \otimes e') = (D^\nabla e) \otimes e' + e \otimes (D^\nabla e') \\ \mathcal{E}^* & , & (D^\nabla \omega)(e) = D(\omega(e)) - \omega(D^\nabla e) \\ \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{E}') & , & (D^\nabla T)(e) = D^\nabla(Te) - T(D^\nabla e) \end{array}$$

Definition: The **Lie derivative** of an $\mathcal{E}$-valued p-form ω with a vector field D is the $\mathcal{E}$-valued p-form

$$(D^L\omega)(D_1, \ldots, D_p) = D^\nabla(\omega(D_1, \ldots, D_p)) - \sum_{i=1}^{p} \omega(D_1, \ldots, [D, D_i], \ldots, D_p).$$

If we have a bilinear product $\mathcal{E} \times \mathcal{E}' \stackrel{\cdot}{\to} \mathcal{E}''$, and compatible connections in the sense that $D^\nabla(e \cdot e') = (D^\nabla e) \cdot e' + e \cdot (D^\nabla e')$, it is easy to check that

$$D^L(\omega \wedge \omega') = (D^L\omega) \wedge \omega' + \omega \wedge (D^L\omega').$$

Theorem: *Let $\Omega_X^p \bigotimes \mathcal{E}$ be the sheaf of $\mathcal{E}$-valued p-forms. There are unique $\mathbb{R}$-linear morphisms of sheaves $\mathrm{d}\colon \Omega_X^p \bigotimes \mathcal{E} \longrightarrow \Omega_X^{p+1} \bigotimes \mathcal{E}$ such that for any vector field D,*

$$D^L = \mathrm{d} \circ i_D + i_D \circ \mathrm{d}.$$

Proof: Let us show the uniqueness by induction on p.

If $p = 0$, then $D^\nabla e = D^L e = i_D(\mathrm{d}e)$; hence $(\mathrm{d}e)(D) = D^\nabla e$.

If $p \ge 1$, then $i_D(\mathrm{d}\omega) = D^L\omega - \mathrm{d}(i_D\omega)$ and, by induction, $\mathrm{d}(i_D\omega)$ is fully determined.

To prove the existence, we define recursively the exterior differential. If $p = 0$,

$$(\mathrm{d}e)(D) = D^\nabla e$$

and, once we have defined d on $(p-1)$-forms, for any p-form ω we put

$$(\mathrm{d}\omega)(D, D_1, \ldots, D_p) = (D^L\omega)(D_1, \ldots, D_p) - (\mathrm{d}i_D\omega)(D_1, \ldots, D_p).$$

We must show that $\mathrm{d}\omega$ is a $(p+1)$-form. It is $\mathbb{R}$-multilinear and alternate.

In fact, it is clear that $(\mathrm{d}\omega)(D,\ldots,D_i,\ldots,D_i,\ldots)=0$, and

$$(\mathrm{d}\omega)(D,D,D_2,\ldots,D_p)=(D^L\omega)(D,D_2,\ldots,D_p)-(\mathrm{d}i_D\omega)(D,D_2,\ldots,D_p)=$$
$$=D^\nabla(\omega(D,D_2,\ldots,D_p))-\sum_{i=2}^{p}\omega(D,\ldots,[D,D_i],\ldots)-(i_D\mathrm{d}i_D\omega)(D_2,\ldots,D_p)=0$$

since $(i_D\mathrm{d}i_D\omega)(D_2,\ldots,D_p)=D^L D(i_D\omega)(D_2,\ldots,D_p)-(\mathrm{d}i_D i_D\omega)(D_2,\ldots,D_p)$

$$=D^\nabla(\omega(D,D_2,\ldots,D_p))-\sum_{i=2}^{p}\omega(D,\ldots,[D,D_i],\ldots).$$

Finally, let us show that it is $\mathcal{C}^\infty_X$-multilinear. Linearity in the variables $D_1,\ldots,D_p$ is clear; hence also in the first variable, since it is alternate.

Cartan's Formula: $(\mathrm{d}\omega)(D_1,D_2)=D_1^\nabla(\omega(D_2))-D_2^\nabla(\omega(D_1))-\omega([D_1,D_2])$.

Proof: $(\mathrm{d}\omega)(D_1,D_2)=(D_1^L\omega)(D_2)-(\mathrm{d}i_{D_1}\omega)(D_2)$
$$=D_1^\nabla(\omega(D_2))-\omega([D_1,D_2])-D_2^\nabla(\omega(D_1)).$$

Corollary: *The exterior differential of the identity is the torsion 2-form,*

$$\mathrm{Tor}_\nabla=\mathrm{d}I.$$

Proof: $(\mathrm{d}I)(D_1,D_2)=D_1^\nabla(I(D_2))-D_2^\nabla(I(D_1))-I([D_1,D_2])=\mathrm{Tor}_\nabla(D_1,D_2)$.

Theorem: *If we have a bilinear product $\mathcal{E}\times\mathcal{E}'\dot\to\mathcal{E}''$, and compatible connections in the sense that $D^\nabla(e\cdot e')=(D^\nabla e)\cdot e'+e\cdot(D^\nabla e')$, then*

$$\mathrm{d}(\omega_p\wedge\omega'_q)=(\mathrm{d}\omega_p)\wedge\omega'_q+(-1)^p\omega_p\wedge(\mathrm{d}\omega'_q).$$

Proof: Since $i_D\mathrm{d}=D^L-\mathrm{d}i_D$, it follows directly from the equalities

$$i_D(\omega_p\wedge\omega'_q)=(i_D\omega_p)\wedge\omega'_q+(-1)^p\omega_p\wedge(i_D\omega'_q),$$
$$D^L(\omega_p\wedge\omega'_q)=(D^L\omega_p)\wedge\omega'_q+\omega_p\wedge(D^L\omega'_q).$$

Example: In a galilean space-time X, a punctual body of mass m is described by a trajectory $\sigma(t)=\big(t,x_1(t),x_2(t),x_3(t)\big)$ with a tangent vector field I, the impulse, such that $\omega(I)=m$,

$$I=mU=m(\partial_t+x_1'(t)\partial_1+x_2'(t)\partial_2+x_3'(t)\partial_3)=m\partial_t+\vec{p},$$

where $U=\partial_t+\vec{v}$ is the 4-velocity of the particle (an intrinsic concept) and $\vec{p}=m\vec{v}$ is the momentum (it depends on the inertial reference system). Hence, once an orientation $[\mathrm{d}X]$ of X is fixed, a continuous matter distribution is described by a vector-valued 3-form Π_3, the **impulse** 3-form, whose intuitive meaning is that $\Pi_3(D_1,D_2,D_3)$ is the sum, with the sign of $\mathrm{d}X(I,D_1,D_2,D_3)$, of the impulse of the particles crossing the infinitesimal parallelogram determined by D_1,D_2,D_3. As any other vector-valued 3-form, we have $\Pi_3=C_1^1(\mathrm{d}X\otimes T^2)$ for a unique 2-contravariant tensor T^2, named **matter** tensor, which does not depend on the orientation. In an inertial reference system, we have

$$T^2(\mathrm{d}t,\mathrm{d}t)=\mathrm{d}t\big(\Pi_3(\partial_1,\partial_2,\partial_3)\big)=\text{ mass density.}$$
$$T^2(\mathrm{d}t,\mathrm{d}x_j)=\mathrm{d}x_j\big(\Pi_3(\partial_1,\partial_2,\partial_3)\big)=\text{ j-momentum density.}$$
$$T^2(\mathrm{d}x_i,\mathrm{d}t)=\text{ mass flux acroos a unit area perpendicular to } \partial_i.$$
$$T^2(\mathrm{d}x_i,\mathrm{d}x_j)=\text{ j-momentum flux acroos a unit area perpendicular to } \partial_i.$$

When $T^2 = \rho U \otimes U$, where $\omega(U) = 1$, the matter is a dust of mass density ρ and mean velocity U. When $T^2 = \rho U \otimes U + ph$, the matter is a perfect fluid of mass density ρ, mean velocity U and pressure p. We always assume that the matter tensor T^2 is symmetric.

When we consider the exterior differential defined by the Cartan connection ∇, the equation $\mathrm{d}\Pi_3 = 0$ encodes the mass conservation law and the newtonian motion law (the equation of continuity and the Euler equations in presence of a gravitational force). In terms of the matter tensor, this equation $\mathrm{d}\Pi_3 = 0$ states that $\mathrm{div}_\nabla T^2 = 0$, where the **divergence** of a tensor T (with some contravariant index) is $\mathrm{div}_\nabla T := C_1^1(\nabla T)$.

14.1.1 Curvature

Definition: In general $\mathrm{d}^2 \neq 0$. For example, for any section e we have

$$\begin{aligned}(\mathrm{d}^2 e)(D_1, D_2) &= D_1^\nabla((\mathrm{d}e)(D_2)) - D_2^\nabla((\mathrm{d}e)(D_1)) - (\mathrm{d}e)([D_1, D_2]) \\ &= D_1^\nabla(D_2^\nabla e) - D_2^\nabla(D_1^\nabla e) - [D_1, D_2]^\nabla e = R(D_1, D_2)(e)\end{aligned}$$

where the **curvature** R is an $\underline{\mathrm{End}}(\mathcal{E})$-valued 2-form; i.e., $\mathrm{d}^2 e = R \wedge e$, the exterior product being considered with respect to the product $\underline{\mathrm{End}}(\mathcal{E}) \times \mathcal{E} \to \mathcal{E}$, $T \cdot e = T(e)$.

Theorem: $\mathrm{d}^2\omega = R \wedge \omega$.

Proof: If the locally free sheaf $\mathcal{E}$ is trivial on an open set U, and $\{e_1, \ldots, e_r\}$ is a base of $\mathcal{E}(U)$, then we have $\omega = \sum_i \omega_i \otimes e_i = \sum_i \omega_i \wedge e_i$, for some ordinary p-forms ω_i, and

$$\begin{aligned}\mathrm{d}\omega &= \textstyle\sum_i \big[(\mathrm{d}\omega_i) \wedge e_i + (-1)^p \omega_i \wedge (\mathrm{d}e_i)\big], \\ \mathrm{d}^2\omega &= \textstyle\sum_i \big[(\mathrm{d}^2\omega_i) \wedge e_i + (-1)^{p+1}(\mathrm{d}\omega_i) \wedge (\mathrm{d}e_i) + (-1)^p(\mathrm{d}\omega_i) \wedge (\mathrm{d}e_i) + \omega_i \wedge (\mathrm{d}^2 e_i)\big] \\ &= \textstyle\sum_i \omega_i \wedge (\mathrm{d}^2 e_i) = \sum_i \omega_i \wedge (R \wedge e_i) = \sum_i (-1)^{2p} R \wedge \omega_i \wedge e_i = R \wedge \omega.\end{aligned}$$

Bianchi's Differential Identity: $\mathrm{d}R = 0$.

Proof: $\mathrm{d}^3 e = \mathrm{d}^2(\mathrm{d}e) = R \wedge (\mathrm{d}e)$ for any section e of $\mathcal{E}$, and

$$\mathrm{d}^3 e = \mathrm{d}(\mathrm{d}^2 e) = \mathrm{d}(R \wedge e) = (\mathrm{d}R) \wedge e + R \wedge (\mathrm{d}e);$$

hence $(\mathrm{d}R) \wedge e = 0$. That is to say, $(\mathrm{d}R)(D_1, D_2, D_3)(e) = 0$, and $\mathrm{d}R = 0$.

Parallel Transport: The sheaf of smooth sections $\mathcal{E}$ of a real vector bundle $E \to X$ is a locally free $\mathcal{C}_X^\infty$-module, and $\mathcal{E}$-valued forms are just forms with values in the fibres of E.

Given a connection ∇ on $\mathcal{E}$, and a smooth map $\phi\colon Y \to X$, *there is a unique connection $\phi^*\nabla$ on the sheaf of smooth sections of the vector bundle $\phi^* E = E \times_X Y \to Y$ such that, for any local section $e \in \mathcal{E}(U)$,*

$$\phi^*(\mathrm{d}e) = \mathrm{d}(\phi^* e).$$

When $\mathcal{E}$ admits a base $e_1, \ldots, e_r$ we have $D^\nabla e_i = \sum_j \omega_{ij}(D) e_j$ for some ordinary 1-forms ω_{ij}, and the only possibility is just to consider the connection given by the 1-forms $\phi^*\omega_{ij}$,

$$\bar{D}^{\phi^*\nabla}(\phi^* e_i) = \textstyle\sum_j (\phi^*\omega_{ij})(\bar{D})(\phi^* e_j)$$

so that clearly $\phi^*(\mathrm{d}e_i) = \mathrm{d}(\phi^* e_i)$; hence also for any other local section e. By the uniqueness, these local connections coincide on intersections and they define a global connection.

For example, when $E = TX$ is the tangent bundle and $\sigma\colon I \to X$ is a smooth curve, fields with support on σ are just smooth sections of $\sigma^* TX$, and $\sigma^*\nabla$ is just the covariant derivative of fields with support introduced in p. 295.

In general, given a smooth curve $\sigma\colon I \to X$ and a vector $e_0 \in E_{x_0}$ in the fibre of $x_0 = \sigma(t_0)$, there exists a unique parallel section $e\colon I \to E$, $\mathrm{d}e = 0$, such that $e(t_0) = e_0$, and so we obtain a **parallel transport** (depending on σ, not just on the end points)

$$E_{\sigma(t_0)} \xrightarrow{\ \sim\ } E_{\sigma(t_1)}\ ;\ t_0, t_1 \in I.$$

Cartan Structure Equations: Let ∇ be a linear connection on X. If $\{D_1, \dots, D_n\}$ is a local base of vector fields, and $\{\theta_1, \dots, \theta_n\}$ is the dual base, we have

1. $\mathrm{d}D_j = \sum\limits_{i=1}^{n} \omega_{ij} \otimes D_i,\ \omega_{ij}(D) = \theta_i(D^\nabla D_j)$.
2. $\mathrm{d}^2 D_j = R \wedge D_j = \sum\limits_{i=1}^{n} \Omega_{ij} \otimes D_i,\ \Omega_{ij}(D, D') = \theta_i(R(D, D')D_j)$.
3. $\mathrm{Tor} = \mathrm{d}I = \sum\limits_{i=1}^{n} \Theta_i \otimes D_i,\ \Theta_i(D, D') = \theta_i(\mathrm{Tor}(D, D'))$.

Calculating the exterior differential of the identity $I = \sum_j \theta_j \otimes D_j = \sum_j \theta_j \wedge D_j$,

$$\begin{aligned}\mathrm{d}I &= \textstyle\sum_j \mathrm{d}\theta_j \wedge D_j - \sum_j \theta_j \wedge \mathrm{d}D_j = \sum_i \mathrm{d}\theta_i \otimes D_i - \sum_{i,j}(\theta_j \wedge \omega_{ij}) \otimes D_i \\ &= \textstyle\sum_i (\mathrm{d}\theta_i + \sum_j \omega_{ij} \wedge \theta_j) \otimes D_i\end{aligned}$$

and, comparing with 3, we obtain the **first structure equation**,

$$\mathrm{d}\theta_i + \textstyle\sum_j \omega_{ij} \wedge \theta_j = \Theta_i.$$

Differentiating 1 and comparing with 2, we obtain the **second structure equation**,

$$\mathrm{d}\omega_{ij} + \textstyle\sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}.$$

If we consider $\theta = (\theta_i)$, $\Theta = (\Theta_i)$ as $\mathbb{R}^n$-valued forms, and $\omega = (\omega_{ij})$, $\Omega = (\Omega_{ij})$ as forms with values in the $n \times n$ matrices, these equations are

$$\begin{aligned}\mathrm{d}\theta + \omega \wedge \theta &= \Theta \\ \mathrm{d}\omega + \omega \wedge \omega &= \Omega\end{aligned}$$

where the exterior products are considered with respect to the matrix product.

If we differentiate the first structure equation,

$$0 + (\mathrm{d}\omega) \wedge \theta - \omega \wedge (\mathrm{d}\theta) = \mathrm{d}\Theta$$

and we replace $\mathrm{d}\theta$ and $\mathrm{d}\omega$ by the values given by the structure equations, we obtain

Bianchi's Identity: $\Omega \wedge \theta - \omega \wedge \Theta = \mathrm{d}\Theta$.

If we differentiate the second structure equation,

$$0 + (\mathrm{d}\omega) \wedge \omega - \omega \wedge (\mathrm{d}\omega) = \mathrm{d}\omega$$

and we replace $\mathrm{d}\omega$ by the value given by the second structure equation, we obtain

Bianchi's Differential Identity: $\Omega \wedge \omega - \omega \wedge \Omega = \mathrm{d}\Omega$.

In the case of a symmetric connection (such as the Levi-Civita connection of a riemannian metric), we have $\Theta = 0$ and the structure equations and Bianchi's identities are

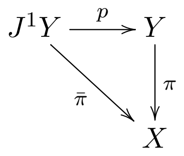
$$\left.\begin{aligned}\mathrm{d}\theta + \omega \wedge \theta &= 0 \\ \mathrm{d}\omega + \omega \wedge \omega &= \Omega\end{aligned}\right\}\quad,\quad \left.\begin{aligned}\Omega \wedge \theta &= 0 \\ \Omega \wedge \omega - \omega \wedge \Omega &= \mathrm{d}\Omega\end{aligned}\right\}$$

14.2 Calculus of Variations

Definition: Two smooth maps $\bar{s}, s\colon X \to Y$ defined on certain neighborhoods of a point $x \in X$ have equal k**-jet** at x when $s^*(f) \equiv \bar{s}^*(f)$ (mod. $\mathfrak{m}_x^{k+1}$) for all $f \in \mathcal{C}^\infty(Y)$. In particular, they have equal 1-jet when $\bar{s}(x) = s(x)$ and the tangent linear maps at x coincide.

Equivalence classes are k-jets at x of maps, and the k-jet at x of a map s is denoted by $j_x^k s$.

The set of 1-jets of local smooth sections of a regular projection $\pi\colon Y \to X$ is denoted by J^1Y, and it is endowed with natural projections



$$p(j_x^1 s) = s(x)$$

$$\bar{\pi}(j_x^1 s) = x$$

If $(x_1, \dots, x_n)$ are coordinates on an open set $U \subseteq X$, and $(x_1, \dots, x_n, y_1, \dots, y_m)$ on an open set $V \subseteq \pi^{-1}(U)$, then in $p^{-1}(V)$ the 1-jet $j_x^1 s$ of a section s is fully determined by the coordinates $(x_1, \dots, x_n, y_1, \dots, y_m)$ of $s(x)$ and the coefficients of the jacobian matrix

$$y_{j,i} = \frac{\partial y_j}{\partial x_i}(x).$$

On J^1Y we have a unique structure of smooth manifold such that $(x_i, y_j, y_{j,i})$ are local coordinate systems. Hence p and $\bar{\pi}$ are regular projections, and any section $s\colon X \to Y$ has a 1-jet extension $\bar{s}\colon X \to J^1Y$, $\bar{s}(x) = j_x^1 s$, which is smooth since the 1-jet extension $\bar{s}$ of a local section $y_j = y_j(x_1, \dots, x_n)$ is given by the equations

$$\begin{aligned} y_j &= y_j(x_1, \dots, x_n) \\ y_{j,i} &= \frac{\partial y_j(x_1, \dots, x_n)}{\partial x_i} \end{aligned}$$

Moreover, on J^1Y we have a structure 1-form θ with values in the inverse image p^*T^vY of the vertical bundle (the bundle of vectors tangent to Y with null projection on X),

$$\begin{aligned} \theta(D_{j_x^1 s}) &= p_*D - s_*\bar{\pi}_*D, \\ \theta &= \textstyle\sum_j (\mathrm{d}y_j - \sum_i y_{j,i}\mathrm{d}x_i) \otimes \partial_{y_j}. \end{aligned}$$

Definitions: The kernel of θ defines the structural Pfaff system P of J^1Y, and it is locally generated by the **structure** 1-**forms**

$$\theta_j = \mathrm{d}y_j - \textstyle\sum_i y_{j,i}\mathrm{d}x_i;\ j = 1, \dots, m,$$

and the 1-jet extension $\bar{s}$ of a section s is characterized by the condition of being tangent to the structural Pfaff system, $\bar{s}^*P = 0$.

The **contact ideal** I is the ideal generated by P in the exterior algebra $\bigoplus_p \Omega^p_{J^1Y}$.

A vector field $\tilde{D}$ on J^1Y is a **infinitesimal contact transformation** if it preserves the structural Pfaff system, $\tilde{D}^L P \subseteq P$, or equivalently the contact ideal, $\tilde{D}^L I \subseteq I$.

Proposition: *Any vector field D on Y is the projection of a unique contact infinitesimal transformation $\tilde{D}$. If $D = \sum_j h_j \partial_{y_j}$ is vertical,*

$$\tilde{D} = \sum_j h_j \frac{\partial}{\partial y_j} + \sum_{j,i} \left(\frac{\partial h_j}{\partial x_i} + \sum_k y_{k,i} \frac{\partial h_j}{\partial y_k} \right) \frac{\partial}{\partial y_{j,i}}.$$

Proof: If $f \in \mathcal{C}^\infty(Y)$, then $\mathrm{d}f \equiv \sum_i f_i \mathrm{d}x_i$ (mod P), for some functions f_i, since

$$\mathrm{d}y_j \equiv \textstyle\sum_i y_{j,i}\mathrm{d}x_i \quad (\text{mod. } P).$$

Now, if $D = \sum_i g_i \partial_{x_i} + \sum_j h_j \partial_{y_j}$, then $\tilde{D}x_i = g_i$, $\tilde{D}y_j = h_j$, and the functions $\tilde{D}y_{j,i}$ are fully determined by the following congruences (mod. P)

$$0 \equiv \tilde{D}^L\theta_i = \tilde{D}^L(\mathrm{d}y_j - \sum_i y_{j,i}\mathrm{d}x_i) = \mathrm{d}h_j - \sum_i (\tilde{D}y_{j,i})\mathrm{d}x_i - \sum_i y_{j,i}\mathrm{d}g_i,$$
$$\sum_i (\tilde{D}y_{j,i})\mathrm{d}x_i \equiv \mathrm{d}h_j - \sum_i y_{j,i}\mathrm{d}g_i \equiv \sum_i u_i \mathrm{d}x_i.$$

Definitions: Given a n-form $\mathbb{L}_n$ on J^1Y, the **critical** sections of the variational problem defined by $\mathbb{L}_n$ are the smooth sections $s\colon X \to Y$ such that

$$\int_{\bar{s}} \tilde{D}^L \mathbb{L}_n = 0$$

for any vertical vector field D on Y with compact support. If τ_t is the flow of D, this condition states that the derivative at $t=0$ of the integral of $\mathbb{L}_n$ on the 1-jet extension of $\tau_t(s)$ is 0.

Since $\int_{\bar{s}} \tilde{D}^L\omega_n = 0$ for any n-form $\omega_n \in I$, two n-forms $\mathbb{L}_n, \mathbb{L}'_n$ on J^1Y define the same variational problem when

$$\mathbb{L}'_n \equiv \mathbb{L}_n \quad (\text{mod. } I).$$

We always assume that locally $\mathbb{L}_n = \mathcal{L}\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ for some smooth function $\mathcal{L}$ on J^1Y, named **lagrangian**, and we put $\mathrm{d}X = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$.

Lemma: *If $(i_D\mathrm{d}\mathbb{L}_n)|_{\bar{s}} = 0$ for any vector field D on J^1Y, then s is a critical section.*
The converse holds when $\mathrm{d}\mathbb{L}_n \equiv 0$ (mod. I).

Proof: For any vector field D on J^1Y, with compact support on $\bar{s}$, we have

$$\int_{\bar{s}} D^L\mathbb{L}_n = \int_{\bar{s}} i_D\mathrm{d}\mathbb{L}_n + \int_{\bar{s}} \mathrm{d}i_D\mathbb{L}_n = \int_{\bar{s}} \mathrm{d}i_D\mathbb{L}_n \overset{\text{Stokes}}{=\!=} 0$$

since $i_D\mathbb{L}_n$ has compact support on $\bar{s}$, and s is a critical section.

Conversely, if $\mathrm{d}\mathbb{L}_n \in I$, locally $\mathrm{d}\mathbb{L}_n = \sum_j \theta_j \wedge \omega_j$ for some n-forms ω_j.

Now, if a section s is critical, for any vertical vector field D with compact support we have

$$0 = \int_{\bar{s}} \tilde{D}^L\mathbb{L}_n = \int_{\bar{s}} i_{\tilde{D}}\mathrm{d}\mathbb{L}_n + \int_{\bar{s}} \mathrm{d}i_{\tilde{D}}\mathbb{L}_n = \textstyle\sum_j \displaystyle\int_{\bar{s}} \theta_j(\tilde{D})\omega_j.$$

When $D = \rho\partial_{y_j}$, where $\rho \geq 0$ has compact support,

$$0 = \int_{\bar{s}} \theta_j(\tilde{D})\omega_j = \int_{\bar{s}} \rho\omega_j.$$

Since the support of ρ is arbitrarily small, $\omega_j|_{\bar{s}} = 0$.

Since also $\theta_j|_{\bar{s}} = 0$, then for any vector field D on J^1Y we have

$$(i_D\mathrm{d}\mathbb{L}_n)|_{\bar{s}} = \big(i_D\textstyle\sum_j \theta_j \wedge \omega_j\big)|_{\bar{s}} = \big(\sum_i \theta_j(D)\omega_j - \omega_j(D)\theta_j\big)|_{\bar{s}} = 0.$$

Fundamental Lemma: *There is a n-form Θ on J^1Y, defining the same variational problem as $\mathcal{L}\mathrm{d}X$, which is closed modulo the contact ideal,*

$$\Theta \equiv \mathcal{L}\mathrm{d}X \quad (\text{mod. } I),$$
$$\mathrm{d}\Theta \equiv 0 \qquad (\text{mod. } I).$$

Moreover Θ *is unique if we require that* $\Theta \equiv \mathcal{L}\mathrm{d}X$ (mod. $P \wedge \Omega_X^{n-1}$).

Proof: On J^1Y, a local base of $(n+1)$-forms which are multiple of $\mathrm{d}X$ is

$$\theta_j \wedge \mathrm{d}X = \mathrm{d}y_j \wedge \mathrm{d}X,\ \mathrm{d}\theta_j \wedge i_{\partial_{x_i}}\mathrm{d}X = -\mathrm{d}y_{j,i} \wedge \mathrm{d}X.$$

Therefore, locally there are functions f_{ji}, g_j on J^1Y such that

$$\begin{aligned} \mathrm{d}(\mathcal{L}\mathrm{d}X) &= \mathrm{d}\mathcal{L} \wedge \mathrm{d}X = \textstyle\sum_{j,i} f_{ji}\mathrm{d}\theta_j \wedge i_{\partial_{x_i}}\mathrm{d}X + \sum_j g_j\theta_j \wedge \mathrm{d}X, \\ \mathrm{d}(\mathcal{L}\mathrm{d}X) &\equiv \textstyle\sum_{j,i} f_{ji}\mathrm{d}\theta_i \wedge i_{\partial_{x_i}}\mathrm{d}X \quad (\text{mod. } I), \\ \Theta &= \mathcal{L}\mathrm{d}X - \textstyle\sum_{j,i} f_{ji}\theta_j \wedge i_{\partial_{x_i}}\mathrm{d}X \end{aligned}$$

and we have $\Theta \equiv \mathcal{L}\mathrm{d}X$, $\mathrm{d}\Theta \equiv 0$ (mod I).

The local uniqueness is clear since $\{\theta_j,\ \mathrm{d}y_{j,i},\ \mathrm{d}x_j\}$ is a local base of 1-forms, and

$$\mathrm{d}(f\theta_j \wedge i_{\partial_{x_i}}\mathrm{d}X) \equiv -f\mathrm{d}y_{j,i} \wedge \mathrm{d}X.$$

Finally, the local uniqueness let us conclude the global existence of the n-form Θ on J^1Y, named **Poincaré-Cartan form** of the variational problem.

Its local expression (where we put $\mathcal{L}_{y_{ji}} = \partial\mathcal{L}/\partial y_{j,i}$) is

$$\Theta = \mathcal{L}\mathrm{d}X - \textstyle\sum_{j,i}(-1)^i\mathcal{L}_{y_{ji}}\big(\mathrm{d}y_j - \sum_k y_{j,k}\mathrm{d}x_k\big) \wedge \mathrm{d}x_1 \wedge \ldots \widehat{\mathrm{d}x_i} \ldots \wedge \mathrm{d}x_n.$$

Theorem: *A smooth section* s *is critical if and only if for any vector field* D *on* J^1Y

$$(i_D\mathrm{d}\Theta)|_{\bar{s}} = 0.$$

Definitions: An **infinitesimal symmetry** of the variational problem defined by $\mathcal{L}\mathrm{d}X$ is a contact infinitesimal transformation D on J^1Y such that $D^L(\mathcal{L}\mathrm{d}X) \equiv 0$ (mod. I). The **Noether invariant** of a symmetry D is the $(n-1)$-form $-i_D\Theta$.

Since $\Theta \equiv \mathcal{L}\mathrm{d}X$ (mod. I), and $D^LI \subseteq I$, D is an infinitesimal symmetry if and only if

$$D^L\Theta \equiv 0 \quad (\text{mod. } I).$$

Noether's Theorem: *The Noether invariant of an infinitesimal symmetry* D *is a closed form on the 1-jet extension* $\bar{s}$ *of any critical section,*

$$(\mathrm{d}i_D\Theta)|_{\bar{s}} = 0.$$

Proof: Since $D^L\Theta \equiv 0$ (mod. I), on any section s we have $(D^L\Theta)|_{\bar{s}} = 0$.

If moreover s is critical, then $(i_D\mathrm{d}\Theta)|_{\bar{s}} = 0$, and

$$0 = (D^L\Theta)|_{\bar{s}} = (\mathrm{d}i_D\Theta + i_D\mathrm{d}\Theta)|_{\bar{s}} = (\mathrm{d}i_D\Theta)|_{\bar{s}}.$$

Note: When $X = \mathbb{R}$, Noether invariants are functions on J^1Y, constant on any critical section. In general, if $X = \mathbb{R} \times S$, where we identify the first factor with time, $\omega_{n-1} = (i_D\Theta)|_{\bar{s}}$ is a closed $(n-1)$-form on X and it gives all kind of conservation laws. For example, if the support of ω_{n-1} is compact, $\int_{t\times S}\omega_{n-1}$ does not depend on the instant t.

14.2.1 Problems in Dimension 1

Now we assume that $X=\mathbb{R}$, and we put $t=x_1$, $y'_j=y_{j,1}$.

J^1Y has odd dimension $2m+1$, so that the 2-form $\mathrm{d}\Theta$ has non null radical (p. 169) and condition $(i_D\mathrm{d}\Theta)|_{\bar{s}}=0$ states that $\bar{s}$ is tangent to the radical of

$$\mathrm{d}\Theta=\textstyle\sum_j(\mathrm{d}\mathcal{L}_{y'_j}-\mathcal{L}_{y_j}\mathrm{d}t)\wedge\theta_j$$

or equivalently that the 1-forms $\mathrm{d}\mathcal{L}_{y'_j}-\mathcal{L}_{y_j}\mathrm{d}t$ vanish on $\bar{s}$.

Euler-Lagrange Equations: *A smooth section s is critical if and only if on $\bar{s}$ we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{L}_{y'_j})-\mathcal{L}_{y_j}=0.$$

Definitions: A lagrangian $\mathcal{L}$ is **regular** if $\det(\mathcal{L}_{y'_iy'_j})\neq 0$ at any point of J^1Y, so that

$$\mathrm{d}t,\ \theta_j=\mathrm{d}y_j-y'_j\mathrm{d}t,\ \mathrm{d}\mathcal{L}_{y'_j}-\mathcal{L}_{y_j}\mathrm{d}t$$

define a local base of 1-forms, or equivalently the radical of $\mathrm{d}\Theta=\sum_j(\mathrm{d}\mathcal{L}_{y'_j}-\mathcal{L}_{y_j}\mathrm{d}t)\wedge\theta_j$ has dimension 1 and it is not vertical. In such a case, the radical $\langle Z\rangle$ of $\mathrm{d}\Theta$ is incident to $P=\langle\theta_j\rangle$, and the integral curves of the **lagrangian field** Z, normalized with condition $Zt=1$, are just the 1-jet extensions of critical sections.

Lemma: *Let D be a vector field on a manifold X. If there exists a function $t\in\mathcal{C}^\infty(X)$ such that $Dt>0$, then the integral curves of D are the fibres of a regular projection $X\to W$ onto a (eventually non separated) manifold.*

Proof: The integral curves of D are the fibres of a surjective map $\pi\colon X\to W$, and we consider on W the quotient topology ($U\subseteq W$ is open if and only if so is $\pi^{-1}U$) and the sheaf

$$\mathcal{O}(U)=\{f\in\mathcal{C}(U)\colon \pi^*f\in\mathcal{C}^\infty(\pi^{-1}U)\}.$$

Let us show that $(W,\mathcal{O})$ is a smooth manifold and π is a regular projection.

By hypothesis the hypersurfaces H_a of equation $t=a$ transversally intersect integral curves, and at a unique point since t is an increasing function on any integral curve.

Each point of X admits an open neighborhood V such that $V\to\pi(V)=H_a\cap V$ is a regular projection for some smooth structure on $\pi(V)$, coinciding with the structure induced by W.

In fact, if U is an open set in $H_a\cap V$, then

$$\pi^{-1}(U)=\bigcup_{V',t}\tau_t(V')$$

is an open set in X, where V' runs over all open subsets of U.

By the same reason, if f is smooth on U, then π^*f is smooth on $\pi^{-1}U$. q.e.d.

Since $Zt=1$, then the 1-jet extensions of critical sections are the fibres of a regular projection $J^1Y\to W$, and the 2-form $\mathrm{d}\Theta$ is projectable onto W since

$$i_Z\mathrm{d}\Theta=0,\ Z^L\mathrm{d}\Theta=\mathrm{d}i_Z\mathrm{d}\Theta+i_Z\mathrm{d}\mathrm{d}\Theta=0+0=0.$$

Definition: The projection ω_2 of $\mathrm{d}\Theta$ onto W is a non singular closed 2-form, and the (eventually non separated) symplectic manifold (W,ω_2) is the **variety of solutions** of the variational problem.

Proposition: *Any infinitesimal symmetry D of the variational problem, and the Noether invariant $f = -\Theta(D)$, are projectable onto the variety of solutions W, and $i_D\omega_2 = \mathrm{d}f$ on W.*

Proof: We have to show that D preserves the sheaf of first integrals of Z. If $Zf = 0$,

$$Z(Df) = [Z, D]f - D(Zf) = -(D^L Z)f.$$

Deriving with D^L the equality $0 = C_1^1(Z \otimes \mathrm{d}\Theta)$, and using that $D^L(\mathrm{d}\Theta) = 0$, we obtain that $C_1^1(D^L Z \otimes \mathrm{d}\Theta) = 0$. Hence $D^L Z$ is in the radical of $\mathrm{d}\Theta$ and it is proportional to Z, so that $(D^L Z)f = 0$ and we conclude that $Z(Df) = 0$.

Finally, $f = -\Theta(D)$ is constant on any solution, and it defines a smooth function on W.

Condition $D^L\Theta = 0$ shows that $i_D\mathrm{d}\Theta = \mathrm{d}f$; hence $i_D\omega_2 = \mathrm{d}f$ on W.

Hamilton's Equations: Assume that $Y = \mathbb{R} \times F$ and that $\mathcal{L}$ does not depend on t.

In this case ∂_t is an infinitesimal symmetry of the variational problem, and the Noether invariant $h = -\Theta(\partial_t)$ is the **energy** or **hamiltonian**. In coordinates $h = -\mathcal{L} - \sum_j y'_j \mathcal{L}_{y'_j}$.

Now $J^1Y = \mathbb{R} \times TF$, and the lagrangian field Z does not depend on t since ∂_t is an infinitesimal symmetry, $Z = \partial_t + \bar{Z}$ where $\bar{Z}$ is a vector field on the tangent bundle TF.

The 2-form $\mathrm{d}\Theta$ is invariant by ∂_t, and its restriction ω_2 to any fibre TF is non singular since fibres are transversal to $\langle Z \rangle = \mathrm{rad}\, \mathrm{d}\Theta$.

(TF, ω_2) is a symplectic manifold.

Since $i_Z\mathrm{d}\Theta = 0$ and $i_{\partial_t}\mathrm{d}\Theta = \partial_t^L\Theta - \mathrm{d}i_{\partial_t}\Theta = \mathrm{d}h$, we have

Hamilton's Equations: $i_{\bar{Z}}\omega_2 = \mathrm{d}h$.

The restriction of $\mathrm{d}\Theta$ to any fibre is $\omega_2 = \sum_j \mathrm{d}\mathcal{L}_{y'_j} \wedge \mathrm{d}y_j$; hence the functions

$$p_j = y_j, \ q_j = \mathcal{L}_{y'_j}$$

are **canonical coordinates** on TF, in the sense that $\omega_2 = \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$.

In canonical coordinates, Hamilton's equations are

$$\begin{cases} q'_j = h_{p_j} \\ p'_j = -h_{q_j} \end{cases}$$

14.3 Natural Bundles

Let us fix a smooth manifold X of constant dimension n.

Definitions: A **natural bundle** over a smooth manifold X is a regular projection $\pi\colon F \to X$, endowed with a lifting $\tau_*\colon F_U = \pi^{-1}(U) \to F_V = \pi^{-1}(U)$ of any diffeomorphism $\tau\colon U \to V$ between open sets in X, such that the following squares commute

$$\begin{array}{ccc} F_U & \xrightarrow{\tau_*} & F_V \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\tau} & V \end{array}$$

1. *Functoriality:* $\mathrm{Id}_* = \mathrm{Id}$, and $(\tau \circ \tau')_* = \tau_* \circ \tau'_*$.
2. *Local Character:* For any diffeomorphism $\tau\colon U \to V$ and any open set $U' \subset U$, the restriction of τ_* to $F_{U'}$ is the lifting of $\tau_{|U'}\colon U' \to \tau(U')$.

3. *Regularity:* If $\{\tau_t \colon U_t \to V_t\}_{t\in T}$ is a smooth family of diffeomorphisms, parameterized by a manifold T (i.e., $U = \bigcup_t U_t \times t$ and $V = \bigcup_t V_t \times t$ are open sets of $X \times T$, and $U \to V$, $(x,t) \mapsto (\tau_t x, t)$, is a diffeomorphism), then $\{\tau_{t*} \colon F_{U_t} \to F_{V_t}\}_{t\in T}$ is a smooth family of diffeomorphisms between open sets of F.

A **morphism** of natural bundles is a smooth map $\varphi \colon F \to F'$ commuting with the lifting of diffeomorphisms, $\varphi\tau_* = \tau_*\varphi$,

$$\begin{array}{ccc} F_U & \xrightarrow{\varphi} & F'_U \\ \downarrow{\scriptstyle \tau_*} & & \downarrow{\scriptstyle \tau_*} \\ F_V & \xrightarrow{\varphi} & F'_V \end{array}$$

A natural bundle $F \to X$ is of **order**[1] $\leq k$ if for any pair of diffeomorphisms $\tau', \tau \colon U \to V$, and any point $x \in U$ where $j^k_x\tau' = j^k_x\tau$, we have $\tau'_* = \tau_*$ on the fibre F_x of x.

Examples: The tangent bundle $TX \to X$, the cotangent bundle $T^*X \to X$ (where $\tau_* = (\tau^*)^{-1}$), and all tensor bundles $T^q_pX \to X$, are natural vector bundles of order 1.

If $F \to X$ is a natural bundle of order k, then $J^1F \to X$ is a natural bundle of order $k+1$.

If $F \to X$ is a natural bundle of order $\leq k$ and we fix a point $p \in X$, the Lie group $G_p = G^k_p$ of k-jets at p of diffeomorphisms $U \to V$ fixing p, acts on the fibre F_p,

$$g \cdot e = \tau_*(e);\ g = j^k_p\tau \in G^k_p,\ e \in F_p,$$

and this action is smooth (F_p is a G^k_p-manifold).

In fact, in a coordinate neighborhood of p it is clear the existence of a smooth family of diffeomorphisms $\tau_g \colon U_g \to V_g$, parameterized by $g \in G_p$, such that $g = j^k\tau_g$. By regularity, $(\tau_g)_*$ is a smooth family of diffeomorphisms of F_p, and $g \cdot e = (\tau_g)_*(e)$ is a smooth action.

Galois Theorem: *The fibre functor $F \rightsquigarrow F_p$ defines an equivalence of categories*

$$\begin{bmatrix} \text{Natural bundles} \\ \text{over } X \text{ of order} \leq k \end{bmatrix} \leftrightsquigarrow \left[G^k_p\text{-manifolds}\right].$$

Proof: The similarity with the Galois theory of coverings (p. 252) is clear; but paths joining two points p and x (defining an identification of the fibres) are replaced by diffeomorphisms $\sigma \colon U \to V$ such that $\sigma(x) = p$. So we shall obtain a universal natural bundle $P \to X$ of order k, and it is a Galois bundle, $P \times_X P = G_p \times P$, of group G_p. Moreover $P/G_p = X$, and P trivializes any natural bundle F of order $\leq k$, $F \times_X P = P \times F_p$, so that the action of G_p on F_p reconstructs the bundle, $F = (P \times F_p)/G_p$. Let us see the details:

The manifold P of k-jets $j^k_x\sigma$ of diffeomorphisms $\sigma \colon U \to V$, $\sigma(x) = p$, admits a regular projection $P \to X$, $j^k_x\sigma \mapsto x$, and the group G_p acts on P,

$$(j^k_p g) \cdot (j^k_x\sigma) = j^k_x(g\sigma).$$

This universal bundle P is a smooth **principal bundle** of group G_p (in the sense that locally we have diffeomorphisms $P_U = G_p \times U$ over X preserving the action of the Lie group G_p) and it is a natural bundle of order k.

The lifting of a diffeomorphism $\tau \colon U \to V$ is defined as follows

$$P_{|U} \xrightarrow{\tau_*} P_{|V},\ \tau_*(j^k_x\sigma) = j^k_{\tau(x)}(\sigma\tau^{-1}).$$

[1] In fact, any natural bundle has finite order; but in this notes we shall not prove this result.

Now, the associated bundle of a G_p-manifold F_p is defined to be

$$F = (P \times F_p)/G \longrightarrow X,\ [(j^k_x\sigma, e)] \mapsto x,$$

and it is a natural bundle of order $\leq k$, where the lifting of $\tau\colon U \to V$ is

$$F_{|U} = (P_U \times F_p)/G_p \xrightarrow{\ \tau_*\ } (P_V \times F_p)/G_p = F_{|V},\ \tau_*\left([j^k_x\sigma, e]\right) = [\tau_*(j^k_x\sigma), e].$$

This lifting is well defined since the actions of τ_* and G_p on P commute:

$$\tau_*(j_pg \cdot j_x\sigma) = \tau_*j_x(g\sigma) = j_{\tau(x)}(g\sigma\tau^{-1}) = j_pg \cdot j_{\tau(x)}(\sigma\tau^{-1}) = j_pg \cdot \tau_*j_x\sigma,$$

and we get a functor from the category of G_p-manifolds into the category of natural bundles, any G_p-morphism $f\colon F_p \to F'_p$ inducing the morphism of natural bundles

$$F = (P \times F_p)/G_p \xrightarrow{\mathrm{Id}\times f} (P \times F'_p)/G_p = F',\ [j^k_x\sigma, e] \mapsto [j^k_x\sigma, f(e)].$$

Let us see that this functor and the fibre functor define an equivalence of categories.

We have a G_p-diffeomorphism

$$F_p \ = \ [(P \times F_p)/G_p]_p\,,\ e \mapsto [(j^k_p\mathrm{Id}, e)],$$
$$j_pg \cdot [j_p\mathrm{Id}, e] = [g_*j_p\mathrm{Id}, e] = [j_pg^{-1}, e] = [j_pg \cdot (j_pg^{-1}, e)] = [j_p\mathrm{Id}, j_pg \cdot e].$$

On the other hand, if $F \to X$ is a natural bundle of order $\leq k$, we have an isomorphism of F onto the associated bundle of the fibre F_p,

$$(P \times F_p)/G_p = F,\ [j_x\sigma, e] \mapsto \sigma_*^{-1}e,$$

and it is well defined since, if we replace $(j_x\sigma, e)$ by the equivalent pair

$$j_pg \cdot (j_x\sigma, e) = (j_x(g\sigma), j_pg \cdot e) = (j_x(g\sigma), g_*e),$$

we have that $[j_x(g\sigma), g_*e] \ \mapsto\ (g\sigma)_*^{-1}g_*e = \sigma_*^{-1}e$.

Definition: Replacing p by the origin of $\mathbb{R}^n$ we shall see that a natural bundle over X defines a natural bundle over any manifold of dimension n.

References of order k at $x \in X$ are k-jets $j^k_x\sigma$ of diffeomorphisms σ of a neighborhood of $x \in X$ onto a neighborhood of the origin $0 \in \mathbb{R}^n$ such that $\sigma(x) = 0$.

References of order k at the origin of $\mathbb{R}^n$ form a Lie group G^k_n, acting on the bundle $R^k_X \to X$ of references of order k at points of X, which is a natural bundle of order k.

The **associated bundle** of a G^k_n-manifold F_0 is the natural bundle $(R^k_X \times F_0)/G^k_n \to X$, and it is of order $\leq k$.

Corollary: *The functor "associated bundle" defines an equivalence of categories*

$$\left[G^k_n\text{-manifolds}\right] \ \rightsquigarrow\ \begin{bmatrix}\text{Natural bundles} \\ \text{over } X \text{ of order} \leq k\end{bmatrix}.$$

Proof: Once we fix local coordinates at p, we have isomorphisms $G^k_n \simeq G^k_p$, $R^k_X \simeq P$, any G^k_n-manifold F_0 inherits a structure of G_p-manifold, and $(R^k_X \times F_0)/G^k_n \simeq (P \times F_0)/G_p$.

Now the result follows from the above theorem. q.e.d.

1. An inverse functor is the fibre functor $F \rightsquigarrow F_p$, where F_p is considered with the structure of G_n^k-manifold defined by a choice of coordinates. To obtain an intrinsic inverse functor, just take, as in the Galois theory of coverings (p. 250), the funtor
$$F \rightsquigarrow \mathrm{Hom}_{\mathrm{nat}}(R_X^k, F) = \mathrm{Hom}_{G_p^k}((R_X^k)_p, F_p) \simeq F_p.$$

2. Natural vector bundles of order ≤ 1 correspond to smooth linear representations of the group $G_n^1 = Gl(\mathbb{R}^n)$, the tangent bundle corresponding to the obvious action of $Gl(\mathbb{R}^n)$ on $\mathbb{R}^n$.

3. The sign of the determinant defines an action of $Gl(\mathbb{R}^n)$ on $\{\pm 1\}$, corresponding to the orientation covering $\Delta_X \to X$.

4. Natural coverings of order $\leq k$ correspond to actions of G_n^k on discrete manifolds. Such actions factor through the quotient by the connected component of the identity and, since G_n^k has two connected components, we see that natural coverings of finite order are disjoint unions of the trivial covering $X \to X$ and the orientation covering $\Delta_X \to X$.

14.4 Chern Classes and Curvature

Let ∇ be a linear connection on a complex vector bundle $E \to X$ of rank r.

Any homogeneous polynomial $P(x_{ij})$ of degree p on the matrices $M_{r\times r}$ corresponds to a symmetric linear map $\tilde{P}\colon \bigotimes^p M_{r\times r} \to \mathbb{C}$, $P(A) = \tilde{P}(A\otimes \overset{p}{\ldots} \otimes A)$.

When P is invariant (under the action of the linear group) $\tilde{P}$ is well defined on the endomorphisms of any complex vector space of dimension r.

Hence it induces a morphism $\tilde{P}\colon \bigotimes^p \mathrm{End}(E) \to \mathbb{C}_X$, and we obtain an ordinary $2p$-form
$$P(\Omega) = \tilde{P}(R\wedge \overset{p}{\ldots} \wedge R).$$

If we fix a local base of sections, this complex $2p$-form is $P(\Omega_{ij})$, where Ω_{ij} are the curvature 2-forms (p. 411) and the product of 2-forms is the exterior product.

As an example we have the invariant polynomials $c_p(A) = \mathrm{tr}(\Lambda^p A)$, and
$$|I + A| = 1 + c_1(A) + \ldots + c_r(A).$$

Theorem: *If a polynomial $P(x_{ij})$ is invariant under the linear group, then the differential form $P(\Omega)$ is closed, and its class in $H^\bullet(X,\mathbb{C})$ does not depend on the linear connection ∇.*

Proof: By the Bianchi's identity $\mathrm{d}R = 0$, we have
$$\mathrm{d}(R\wedge \overset{p}{\ldots} \wedge R) = (\mathrm{d}R)\wedge \overset{p}{\ldots} \wedge R + \ldots + R\wedge \overset{p}{\ldots} \wedge(\mathrm{d}R) = 0.$$

Since P is invariant, the morphism $\tilde{P}\colon \bigotimes^p \mathrm{End}(E) \to \mathbb{C}_X$ commutes with the parallel transport; hence with the covariant derivative of sections and the exterior differential,
$$\mathrm{d}P(\Omega) = \mathrm{d}(\tilde{P}(R\wedge \overset{p}{\ldots} \wedge R)) = \tilde{P}(\mathrm{d}(R\wedge \overset{p}{\ldots} \wedge R)) = 0.$$

Now, given two connections ∇_0, ∇_1 on E, we consider the projection $\pi\colon X \times \mathbb{R} \to X$ and the connection $\nabla = t\pi^*\nabla_1 + (1-t)\pi^*\nabla_0$ on π^*E, so that $\big[P(\Omega_i)\big] = s_i^*\big[P(\Omega)\big]$, $i = 0, 1$.

We conclude because $s_0^* = s_1^*$, since $\pi^*\colon H^\bullet(X,\mathbb{C}) \to H^\bullet(X\times\mathbb{R},\mathbb{C})$ is an isomorphism.

Lemma: *If Ω is the curvature 2-form of a linear connection on a complex line bundle L, then the obstruction class of L is $\delta(L) = \big[\frac{\Omega}{2\pi i}\big] \in H^2(X,\mathbb{C})$.*

Proof: Let $\mathcal{O}$ be the sheaf of complex smooth functions, and $\mathcal{Z}^p$ the sheaf of closed p-forms.

According to the following commutative diagram with exact rows, where $\mathrm{dl}(f) = f^{-1}\mathrm{d}f$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi i} & \mathcal{O} & \xrightarrow{e^f} & \mathcal{O}^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 2\pi i} & & \downarrow & & \downarrow{\scriptstyle \mathrm{dl}} & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{O} & \xrightarrow{\mathrm{d}} & \mathcal{Z}^1 & \longrightarrow & 0
\end{array}$$

we must prove that $\mathrm{dl}\colon H^2(X,\mathbb{Z}) = H^1(X,\mathcal{O}^*) \to H^1(X,\mathcal{Z}^1) = H^2(X,\mathbb{C})$ takes $\delta(L)$ into $[\Omega]$.

Let us consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^* & \longrightarrow & C^0\mathcal{O}^* & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{dl}} & & \downarrow{\scriptstyle \mathrm{dl}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Z}^1 & \longrightarrow & C^0\Omega^1_X & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{Z}^1 & \longrightarrow & \Omega^1_X & \xrightarrow{\mathrm{d}} & \mathcal{Z}^2 & \longrightarrow & 0
\end{array}$$

Let $\{U_i\}$ be an open cover of X where there are continuous sections $e_i\colon U_i \to L$ not vanishing at any point, and we put $e_i = g_{ij}e_j$ on $U_i \cap U_j$.

Once we fix generators of the fibres, e_i defines a section f_i of $C^0\mathcal{O}^*$ on U_i, and $f_i/f_j = g_{ij}$ in $\mathcal{O}^*(U_i \cap U_j)$, so that the sections f_i define a global section f of $\mathcal{F}$, representing $\delta(L)$.

We must prove that f and Ω define sections of $\mathcal{G}$ differing in a global section of $C^0\Omega^1_X$.

Now, we have the connection 1-forms θ_i, and $\Omega = \mathrm{d}\theta_i - \theta_i \wedge \theta_i = \mathrm{d}\theta_i$ by the Cartan's structure equations; hence we must show that the germ of $\theta_i - \mathrm{dl}(f_i)$ at a point does not depend on the index i; i.e., that we have $\theta_i - \theta_j = \mathrm{dl}(f_i/f_j) = \mathrm{dl}(g_{ij})$,

$$\begin{aligned}
D^\nabla e_i &= \theta_i(D)e_i = g_{ij}\theta_i(D)e_j, \\
D^\nabla e_i &= D^\nabla(g_{ij}e_j) = D(g_{ij})e_j + g_{ij}D^\nabla e_j = (\mathrm{d}g_{ij} + g_{ij}\theta_j)(D)e_j, \\
\theta_i &= \theta_j + g_{ij}^{-1}\mathrm{d}g_{ij}.
\end{aligned}$$

Theorem: *The Chern classes of any complex vector bundle E are*

$$c_p(E) = \left[c_p\left(\frac{-\Omega}{2\pi i}\right)\right].$$

Proof: Two connections ∇', ∇'' on certain vector bundles E', E'', with curvature 2-forms Ω', Ω'', induce a connection ∇ on $E' \oplus E''$ with curvature 2-form

$$\begin{gathered}
\Omega = \begin{pmatrix} \Omega' & 0 \\ 0 & \Omega'' \end{pmatrix} \\
\det\big(I + \tfrac{-1}{2\pi i}\Omega\big) = \det\big(I + \tfrac{-1}{2\pi i}\Omega'\big) \wedge \det\big(I + \tfrac{-1}{2\pi i}\Omega''\big) \\
c_n\big(\tfrac{-1}{2\pi i}\Omega\big) = \sum_{p+q=n} c_p\big(\tfrac{-1}{2\pi i}\Omega\big) \wedge c_q\big(\tfrac{-1}{2\pi i}\Omega\big)
\end{gathered}$$

and if the theorem holds for E' and E'', then it also holds for $E' \oplus E''$ by Whitney's formula, since the exterior product represents the cup product (p. 355).

After a base change $Y \to X$, injective at the cohomology level, we may assume that E decomposes as a direct sum of line bundles (p. 376) and we conclude by the former lemma.

Theorem: *If X is a connected orientable manifold of dimension n, then integer cohomology classes in $H_c^n(X,\mathbb{R})$ are just classes of integer integral,*

$$\int_X \varepsilon_X = 1.$$

Proof: If U is a connected open subset, the natural morphism $H_c^n(U,\mathbb{Z}) \to H_c^n(X,\mathbb{Z})$ is an isomorphism, since the dual is the restriction morphism $\mathbb{T}_X(X) \to \mathbb{T}_X(U)$.

Hence we are reduced to the case $X = \mathbb{R}^n$.

Now, by the Künneth and Fubini's theorems,

$$\int_{\mathbb{R}^n} \varepsilon_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \varepsilon_{\mathbb{R}} \wedge \ldots \wedge \varepsilon_{\mathbb{R}} = \Big(\int_{\mathbb{R}} \varepsilon_{\mathbb{R}}\Big)^n$$

and we are reduced to one of the cases $X = \mathbb{R}$, or $\mathbb{R}^2$, or the sphere S_2 (remark that $\int_X \varepsilon_X$ is always positive since we integrate with the orientation defined by ε_X).

Let us prove it on the complex projective line $\mathbb{P}_1$, that we identify with the sphere of radius 1 in $\mathbb{R}^3$ through the stereographic projection.

If ξ is the tautological line bundle on $\mathbb{P}_1$, the tangent bundle is $\xi^* \otimes \xi^*$ (p. 330) and

$$c_1(\xi^* \otimes \xi^*) = -2c_1(\xi) = 2\varepsilon_{\mathbb{P}_1}.$$

The Levi-Civita connection of the sphere is a connection on the complex vector bundle $\xi^*\otimes\xi^*$ since it preserves rotations.

The curvature 2-form is $\Omega = -i\omega_2$, where ω_2 is the area form; hence

$$c_1(\xi^* \otimes \xi^*) = \big[\frac{i}{2\pi}\Omega\big] = \big[\frac{1}{2\pi}\omega_2\big],$$
$$\int_{\mathbb{P}_1} 2\varepsilon_{\mathbb{P}_1} = \frac{1}{2\pi}\int_{S_2} \omega_2 = \frac{4\pi}{2\pi} = 2.$$

Corollary: *If $\pi\colon Y \to X$ is a proper morphism between connected oriented smooth manifolds of dimension n, and ω_n is a n-form with compact support in X,*

$$\int_Y \pi^*\omega_n = (\deg \pi)\int_X \omega_n.$$

Proof: The integration of n-forms defines isomorphisms $\int_X\colon H_c^n(X,\mathbb{R}) \simeq \mathbb{R}$, $\int_Y\colon H_c^n(Y,\mathbb{R}) \simeq \mathbb{R}$ (p. 370), and $\pi^*(\varepsilon_X) = (\deg\pi)\varepsilon_Y$ by the definition of $\deg\pi$ (p. 370).

14.4.1 Gauss-Bonnet Theorem

The tangent bundle of a compact oriented riemannian surface X is a complex line bundle with the structure defined by the automorphism J attached to the area form ω_2,

$$J(D)\cdot D' = \omega_2(D,D').$$

and the Levi-Civita connection ∇ preserves the scalar product and the area form; hence the complex structure, and it is a connection on this complex line bundle.

If K is the scalar curvature of the surface, the curvature 2-form of this connection is

$$\Omega = -iK\omega_2.$$

Gauss-Bonnet Theorem: $\displaystyle\frac{1}{2\pi}\int_X K\omega_2 = \chi(X).$

Proof: Since the tangent bundle TX is the normal bundle of the diagonal map $X \to X \times X$, by the following lemma we have a diffeomorphism of a neighborhood of the zero section s_0 in TX onto a neighborhood of the diagonal Δ in $X \times X$; hence

$$c_1(TX) = s_0^*(s_{0,*}(1)) = \Delta^*(\Delta_*(1)) = \chi(X)\varepsilon_X,$$
$$\chi(X) = \int_X c_1(TX) = \int_X \frac{i\Omega}{2\pi} = \frac{1}{2\pi}\int_X K\omega_2.$$

Tubular Neighborhood Lemma: *Let N be the normal bundle of a compact submanifold $i\colon Y \to X$ of a riemannian manifold X. There is an open neighborhood U of Y in X and a diffeomorphism $\varphi\colon N \to U$ transforming the zero section into the embedding, $i = \varphi \circ s_0$.*

Proof: Let us consider the neighborhoods $U_\varepsilon = \{D_y \in N\colon \|D_y\| < \varepsilon\}$ of the zero section $s_0(Y)$.

By the theorem of smooth dependence on the initial conditions of the solutions of a differential equation (p. 264) the **exponential map**

$$\exp\colon U_\varepsilon \longrightarrow X,\ \exp(D_y) = \sigma(1),$$

where $\sigma(t)$ is the geodesic curve tangent to D_y at $t = 0$, is well-defined and it is smooth if ε is sufficiently small.

Moreover, at any point $y \in Y$, the tangent linear map

$$\exp_*\colon T_y(N) = T_yY \oplus N_y \longrightarrow T_yX = T_yY \oplus N_y$$

is the identity. In fact, it is obvious on T_yY, and on N_y we have that the line $\gamma(t) = tD_y$ is tangent to D_y at $t = 0$, and $\exp\gamma(t) = \sigma(t)$ is tangent to D_y at $t = 0$.

Hence we may fix ε so that exp is a local diffeomorphism at any point, and even injective. Otherwise for any natural number n there are vectors D_n, D'_n at some points $y_n, y'_n \in Y$, of modulus $< 1/n$, such that $\exp(D_n) = \exp(D'_n)$.

Since Y is compact, we may assume that the sequences converge

$$(y, 0) = \lim(D_n)_{y_n},$$
$$(y', 0) = \lim(D'_n)_{y'_n}.$$

Since exp is continuous, we have $y = y'$, so that the equality $\exp(D_n) = \exp(D'_n)$, for large n, contradicts the statement that exp is a local diffeomorphism at y.

Finally, the diffeomorphism $U_\varepsilon \to N$, $D_y \mapsto \sqrt{1 - \|D_y\|^2/\varepsilon^2}D_y$ is the identity on $s_0(Y)$.

Part V

Fifth Year

Chapter 15

Algebraic Geometry II

15.1 Injective Modules

All rings are assumed to be noetherian.

Definitions: An injective morphism of A-modules $M \hookrightarrow M_e$ is an **essential extension** if any non null submodule of M_e has non null intersection with M, and we name it an **injective hull** of M if moreover M_e is an injective A-module.

Lemma: *If $M \hookrightarrow M_e$ is an essential extension and $M \hookrightarrow I$ is an injective morphism, then any extension $M_e \to I$ of j is injective* (since the kernel intersects M at 0, it is null).

Theorem: *Any A-module M admits a unique injective hull $E(M)$.*

Proof: We know that M is a submodule of an injective module I (p. 127) and by Zorn's lemma there is a maximal essential extension $M \hookrightarrow M_e \hookrightarrow I$. Again Zorn's lemma shows the existence of a maximal submodule $N \subset I$ such that $M_e \cap N = 0$.

Since $M_e \hookrightarrow I/N$ is essential, by the above lemma any extension $I/N \to I$ is injective; hence $M_e \hookrightarrow I/N \hookrightarrow I$, and $M_e \hookrightarrow I/N$ is an isomorphism.

Then M_e is a direct summand of I; hence M_e is injective, and it is an injective hull of M.

If $M \hookrightarrow M_{e'}$ is another injective hull, by the above lemma $M_e \hookrightarrow M_{e'}$.

If it is not surjective, M_e being injective, it has a non null supplement in $M_{e'}$; absurd since $M_{e'}$ is an essential extension of M. Hence $M_e \simeq M_{e'}$.

Theorem: *Any injective A-module is $I \simeq \oplus_j E(A/\mathfrak{p}_j)$, where $\mathfrak{p}_j$ are prime ideals.*

Proof: Let $\{I_j\}$ be a maximal family of submodules $I_j \simeq E(A/\mathfrak{p}_j)$ such that the sum $J = \sum_j I_j$ is direct (it exists by Zorn's lemma).

Direct sums of injective modules are injective by the ideal criterion (A is noetherian).

Hence $I = J \oplus I'$. If $I' \neq 0$, some element has prime annihilator $\mathfrak{p}$ (p. 200) and $E(A/\mathfrak{p}) \hookrightarrow I'$, against the maximal character of the family.

Lemma: *The annihilator of any non null element of $E(A/\mathfrak{p})$ is a $\mathfrak{p}$-primary ideal.*

Proof: If $A/I \hookrightarrow E(A/\mathfrak{p})$ and $\bar{\mathfrak{p}}$ is an associated prime of the ideal I, then $A/\bar{\mathfrak{p}} \hookrightarrow A/I$ (p. 200), and the annihilator of $0 \neq m \in (A/\mathfrak{p}) \cap (A/\bar{\mathfrak{p}})$ is $\bar{\mathfrak{p}} = \mathfrak{p}$. Hence I is $\mathfrak{p}$-primary.

Theorem: *The sheaf $\tilde{I}$ induced by an injective A-module I is flasque.*

Proof: Since Spec A is noetherian, direct sums of flasque sheaves are flasque (p. 336), and we may assume that $I = E(A/\mathfrak{p})$.

If a prime ideal $\mathfrak{p}'$ does not contain $\mathfrak{p}$, by the lemma $I_{\mathfrak{p}'} = 0$, and otherwise $I = I_{\mathfrak{p}'}$, since the kernel of the epimorphism (injective modules are divisible) $I \to I_{\mathfrak{p}'}$ intersects $A/\mathfrak{p}$ at 0.

Hence $\tilde{I}$ is the constant sheaf I concentred on $(\mathfrak{p})_0$, which has a dense point, so that it is a flasque sheaf. q.e.d.

1. This theorem shows again that the sheaves $\tilde{M}$ are acyclic (p. 322), at least when the ring A is noetherian: if $0 \to M \to I^\bullet$ is an injective resolution, the flasque resolution $0 \to \tilde{M} \to \tilde{I}^\bullet$ calculates $H^p(\operatorname{Spec} A, \tilde{M})$, and $\Gamma(\operatorname{Spec} A, \tilde{I}^\bullet) = I^\bullet$.

2. If Σ is the field of fractions of a principal ideal domain A, then $E(A) = \Sigma$, and the injective hull of $A/\mathfrak{m}$ is $A/\mathfrak{m} \simeq A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m} \hookrightarrow \Sigma/\mathfrak{m}A_\mathfrak{m} = E(A/\mathfrak{m})$.

   In particular, the injective hull of the $k[x]$-module $k[x]/(x)$ is just $k[x^{-1}]$.

3. Let $\mathfrak{m}$ be a maximal ideal of a k-algebra A such that $[A/\mathfrak{m} : k]$ is finite. Then $E(A/\mathfrak{m}) = \varinjlim (A/\mathfrak{m}^n)^*$. In particular, when $A = k[x,y]$ and $\mathfrak{m} = (x,y)$, we have that $E(A/\mathfrak{m}) = k[x^{-1}, y^{-1}]$.

15.2 Local Algebra

In this section $\mathcal{O}$ is a local noetherian ring, $\mathfrak{m}$ is the maximal ideal, $X = \operatorname{Spec} \mathcal{O}$, $x \in X$ is the closed point, $k = \mathcal{O}/\mathfrak{m}$ is the residue field, and M is a finitely generated $\mathcal{O}$-module.

If $a \in \mathfrak{m}$, the complex $K_1 = \mathcal{O}e \xrightarrow{\mathrm{d}} \mathcal{O} = K_0$, $\mathrm{d}(e) = a$, is denoted by $K(a)$.

If $M_\bullet$ is a complex, we put $M_\bullet(a) = M_\bullet \otimes_\mathcal{O} K(a)$, and we have exact sequences

$$0 \longrightarrow M_\bullet \longrightarrow M_\bullet(a) \xrightarrow{\ \pi\ } M_\bullet[1] \longrightarrow 0\ ,\ \pi(m + m' \otimes e) = m',$$
$$\ldots \longrightarrow H_{p+1}(M_\bullet(a)) \longrightarrow H_p(M_\bullet) \xrightarrow{\cdot a} H_p(M_\bullet) \longrightarrow H_p(M_\bullet(a)) \longrightarrow \ldots \qquad (15.1)$$

Definitions: The **Koszul complex** of M and a sequence $a_1, \ldots a_r \in \mathfrak{m}$ is

$$K_M(a_1, \ldots, a_r) = M \otimes_\mathcal{O} K(a_1) \otimes_\mathcal{O} \ldots \otimes_\mathcal{O} K(a_r),$$

and $H_0(K_M(a_1, \ldots, a_r)) = M/(a_1, \ldots, a_r)M$. If we put $L = \mathcal{O}\omega_1 \oplus \ldots \oplus \mathcal{O}\omega_r$, the Koszul complex is just the complex $\bigoplus_p \Lambda^p L \otimes_\mathcal{O} M$, with the differential

$$\mathrm{d}(\omega_{i_1} \wedge \ldots \wedge \omega_{i_p} \otimes m) = \textstyle\sum_j (-1)^{j-1} a_{i_j}\, \omega_{i_1} \wedge \ldots \wedge \widehat{\omega}_{i_j} \wedge \ldots \wedge \omega_{i_p} \otimes m.$$

A sequence $a_1, \ldots, a_r \in \mathfrak{m}$ is M-**regular** (or just regular when $M = \mathcal{O}$) if a_i does not divide 0 in $M/(a_1, \ldots, a_{i-1})M$; i.e., for any index i we have an exact sequence

$$0 \longrightarrow M/(a_1, \ldots, a_{i-1})M \xrightarrow{a_i} M/(a_1, \ldots, a_{i-1})M$$

Theorem: *The following conditions are equivalent,*

1. *The sequence $a_1, \ldots, a_r \in \mathfrak{m}$ is M-regular.*

2. *The complex $K_M(a_1, \ldots, a_r)$ is a free resolution of $M/(a_1, \ldots, a_r)M$.*

3. $H_1(K_M(a_1, \ldots, a_r)) = 0$.

Proof: $(1 \Rightarrow 2)$. When $p > 1$, we have the exact sequence (15.1)

$$0 = H_p(K_M(a_1, \dots, a_{r-1})) \longrightarrow H_p(K_M(a_1, \dots, a_r)) \longrightarrow H_{p-1}(K_M(a_1, \dots, a_{r-1})) = 0$$

and we see that $H_p(K_M(a_1, \dots, a_r)) = 0$ by induction on r. If $p = 1$, we conclude since a_r does not divide 0 in $M/(a_1, \dots, a_{r-1})M$ and we have an exact sequence

$$0 \longrightarrow H_1(K_M(a_1, \dots, a_r)) \longrightarrow M/(a_1, \dots, a_{r-1})M \xrightarrow{\cdot a_r} M/(a_1, \dots, a_{r-1})M \qquad (15.2)$$

$(3 \Rightarrow 1)$. By the Nakayama's lemma and the exact sequence (15.1)

$$H_1(K_M(a_1, \dots, a_{r-1})) \xrightarrow{\cdot a_r} H_1(K_M(a_1, \dots, a_{r-1})) \longrightarrow H_1(K_M(a_1, \dots, a_r)) = 0$$

we have $H_1(K_M(a_1, \dots, a_{r-1})) = 0$; hence $a_1, \dots, a_{r-1}$ is M-regular by induction on r.

Now the exact sequence (15.2) shows that $a_1, \dots, a_r$ is a M-regular sequence.

Example: If $\mathcal{O}$ is a regular ring and $\mathrm{d}f_1, \dots, \mathrm{d}f_r \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent, then the ring $\mathcal{O}/(f_1, \dots, f_i)$ is regular; hence integral, and the sequence $f_1, \dots, f_r$ is regular.

Corollary: *If an ideal is generated by a regular sequence, $I = (a_1, \dots, a_r)$, then I/I^2 is a free $\mathcal{O}/I$-module of rank r.*

Proof: Put $L = \bigoplus_{i=1}^r \mathcal{O}\omega_i$ and tensor the exact sequence $\Lambda^2 L \to L \to I \to 0$ with $\mathcal{O}/I$.

Since the differential $(\Lambda^2 L) \otimes_{\mathcal{O}} \mathcal{O}/I \to L \otimes_{\mathcal{O}} \mathcal{O}/I$ is null, we have $L \otimes_{\mathcal{O}} \mathcal{O}/I \simeq I/I^2$.

Theorem: *If an ideal I is generated by a regular sequence, then the graded ring $\mathcal{O}[I] = \oplus_n I^n$ is the symmetric algebra of I, and the graded ring $G_I\mathcal{O} = \oplus_n I^n/I^{n+1}$ is the symmetric algebra of the free $\mathcal{O}/I$-module I/I^2,*

$$(\mathcal{O}/I)[x_1, \dots, x_r] \simeq S^\bullet_{\mathcal{O}/I}(I/I^2) = G_I\mathcal{O}.$$

Proof: If I is generated by a regular sequence $a_1, \dots, a_r$, the Koszul complex shows that I is the quotient of $\mathcal{O}x_1 \oplus \dots \oplus \mathcal{O}x_r$ by the submodule generated by the elements $y_{ij} = a_i x_j - a_j x_i$; hence $S^\bullet I$ is the quotient of the polynomial ring $A[x_1, \dots, x_r]$ by the ideal $J = (y_{ij})$, and we must prove that J is just the kernel of the morphism $A[x_1, \dots, x_r] \longrightarrow \oplus_n I^n$, $x_i \mapsto a_i$.

Let $P_n(x_1, \dots, x_r)$ be a homogeneous polynomial of degree n such that $P_n(a_1, \dots, a_r) = 0$. To show that $P_n \in J$, we proceed by induction on r and n, and we put $\bar{\mathcal{O}} = \mathcal{O}/a_1\mathcal{O}$.

The reduction $\bar{P}_n(0, x_2, \dots, x_r) \in \bar{\mathcal{O}}[x_2, \dots, x_r]$ fulfills $\bar{P}_n(\bar{a}_2, \dots, \bar{a}_r) = 0$; hence it is in the ideal generated by $\bar{a}_i x_j - \bar{a}_j x_i$. Since $\bar{\mathcal{O}}[x_2, \dots, x_r] = \mathcal{O}[x_1, \dots, x_r]/(a_1, x_1)$,

$$P_n(x_1, \dots, x_r) \equiv a_1 S_n(x_1, \dots, x_r) + x_1 T_{n-1}(x_1, \dots, x_r) \pmod{J}.$$

Since $a_1 x_i \equiv a_i x_1 \pmod{J}$, we see that $P_n \equiv x_1 Q_{n-1} \pmod{J}$.

Now, a_1 is not a zero divisor and $0 = P_n(a_1, \dots, a_r) = a_1 Q_{n-1}(a_1, \dots, a_r)$.

Hence $Q_{n-1}(a_1, \dots, a_r) = 0$, and $Q_{n-1} \in J$ by induction on n, so that $P_n \in J$.

Finally, $G_I\mathcal{O} = \mathcal{O}[I] \otimes_{\mathcal{O}} (\mathcal{O}/I) = (S^\bullet_{\mathcal{O}} I) \otimes_{\mathcal{O}} (\mathcal{O}/I) = S^\bullet_{\mathcal{O}/I}(I/I^2)$.

15.2.1 Regular Rings

Lemma: *A finitely generated $\mathcal{O}$-module M is free if and only if* $\mathrm{Tor}_1^{\mathcal{O}}(M,k)=0$.

Proof: Let us consider an epimorphism $L\to M$, where L is free and $L\otimes_{\mathcal{O}} k \xrightarrow{\sim} M\otimes_{\mathcal{O}} k$.

The exact sequence $0\to N\to L\to M\to 0$ induces an exact sequence

$$0=\mathrm{Tor}_1(M,k)\longrightarrow N\otimes_{\mathcal{O}} k\longrightarrow L\otimes_{\mathcal{O}} k \xrightarrow{\ \sim\ } M\otimes_{\mathcal{O}} k\longrightarrow 0$$

so that $N\otimes_{\mathcal{O}} k=0$. By Nakayama's lemma $N=0$, and M is free.

Definitions: In general, if we have a resolution by free modules L_i of finite rank

$$0\longrightarrow N\longrightarrow L_{n-1}\longrightarrow \ldots \longrightarrow L_0\longrightarrow M\longrightarrow 0$$

and $\mathrm{Tor}_{n+1}(M,k)=0$, then (p. 350) $0=\mathrm{Tor}_1(N,k)$ and N is free. The **projective dimension** of M is the minimal length of a projective resolution of M. It is the least number n such that $\mathrm{Tor}_p(M,N)=0$, $p>n$, for any module N (or just $\mathrm{Tor}_{n+1}(M,k)=0$). The **global dimension** of $\mathcal{O}$ is the supremum of the projective dimensions of all finite $\mathcal{O}$-modules, and it is the first number n such that $\mathrm{Tor}_{n+1}(k,k)=0$ (or $\mathrm{Tor}_p(M,N)=0$, $p>n$, for any module N).

Lemma: *If $f\in\mathcal{O}$ does not divide 0 in $\mathcal{O}$ nor in M, then for any $\mathcal{O}/f\mathcal{O}$-module N,*

$$\mathrm{Tor}_p^{\mathcal{O}}(M,N)=\mathrm{Tor}_p^{\mathcal{O}/f\mathcal{O}}(M/fM,N),\ p\geq 0.$$

Proof: $\mathrm{Tor}_p^{\mathcal{O}}(\mathcal{O}/f\mathcal{O},M)=0$, $p\geq 1$ since by hypothesis we have exact sequences

$$0\longrightarrow \mathcal{O}\xrightarrow{\cdot f}\mathcal{O}\longrightarrow \mathcal{O}/f\mathcal{O}\longrightarrow 0$$
$$0\longrightarrow M\xrightarrow{\cdot f} M\longrightarrow M/fM\longrightarrow 0$$

Now, if $L_\bullet\to M\to 0$ is a free resolution, then $L_\bullet/fL_\bullet\to M/fM\to 0$ is a resolution by free $\mathcal{O}/f\mathcal{O}$-modules, and

$$\mathrm{Tor}_p^{\mathcal{O}}(M,N)=H_p(L_\bullet\otimes_{\mathcal{O}} N)=H_p\big[(L_\bullet/fL_\bullet)\otimes_{\mathcal{O}/f\mathcal{O}} N)\big]=\mathrm{Tor}_p^{\mathcal{O}/f\mathcal{O}}(M/fM,N).$$

Serre's Theorem: *The ring $\mathcal{O}$ is regular if and only if it has finite global dimension.*

Proof: If $\mathcal{O}$ is a regular ring of dimension n, and $\mathfrak{m}=(f_1,\ldots,f_n)$, then the Koszul complex $K(f_1,\ldots,f_n)$ is a free resolution of k, and the differential of $K(f_1,\ldots,f_n)\otimes_{\mathcal{O}} k$ is null.

Hence $\mathrm{Tor}_n(k,k)\simeq k$ and $\mathrm{Tor}_{n+1}(k,k)=0$, and the global dimension of $\mathcal{O}$ is n.

We prove the converse by induction on $n=\dim\mathcal{O}$.

There is no $0\neq f\in\mathfrak{m}$ such that $f\mathfrak{m}=0$; in fact, if $0\to L_d\to\ldots\to L_0\to k\to 0$ is a free resolution, we may assume that $L_d\subseteq \mathfrak{m}L_{d-1}$, and $0\neq fL_d\subseteq f\mathfrak{m}L_{d-1}=0$.

Hence $\mathfrak{m}$ is not one of the associated primes $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ of the ideal 0.

There exists $f\in\mathfrak{m}-(\mathfrak{p}_1\cup\ldots\cup\mathfrak{p}_r\cup\mathfrak{m}^2)$. In fact, if $f_1\in\mathfrak{m}-(\mathfrak{p}_2\cup\ldots\cup\mathfrak{p}_r\cup\mathfrak{m}^2)$ and $f_1\in\mathfrak{p}_1$, we take $f_2\in(\mathfrak{p}_2\cap\ldots\cap\mathfrak{p}_r\cap\mathfrak{m}^2)-\mathfrak{p}_1$, and $f=f_1+f_2$.

Now $f\in\mathfrak{m}-\mathfrak{m}^2$ and $\dim(\mathcal{O}/f\mathcal{O})=n-1$.

Let us see that $\mathcal{O}/f\mathcal{O}$ has finite global dimension (hence it is regular).

Since $0\to\mathfrak{m}/f\mathcal{O}\to\mathcal{O}/f\mathcal{O}\to k\to 0$ is exact, we are reduced to show that the projective dimension of $\mathfrak{m}/f\mathcal{O}$ is finite. Since the exact sequence

$$0\longrightarrow\langle\bar{f}\rangle\longrightarrow\mathfrak{m}/f\mathfrak{m}\longrightarrow\mathfrak{m}/f\mathcal{O}\longrightarrow 0$$

splits (a retract is the composition $\mathfrak{m}/f\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to \langle \bar{f} \rangle$ defined by any supplement of $\langle \bar{f} \rangle$ in $\mathfrak{m}/\mathfrak{m}^2$), we are reduced to see that $\mathfrak{m}/f\mathfrak{m}$ has finite projective dimension.

We conclude by the lemma, since f is not a zero divisor, $\mathrm{Tor}_p^{\mathcal{O}/f\mathcal{O}}(\mathfrak{m}/f\mathfrak{m}, k) = \mathrm{Tor}_p^{\mathcal{O}}(\mathfrak{m}, k)$.

Corollary: *If $\mathcal{O}$ is a regular local ring, then $\mathcal{O}_\mathfrak{p}$ is regular for any prime ideal $\mathfrak{p}$.*

Proof: The Tor functors localize, $\mathrm{Tor}_n^{\mathcal{O}_\mathfrak{p}}(\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}) = \mathrm{Tor}_n^{\mathcal{O}}(\mathcal{O}/\mathfrak{p}, \mathcal{O}/\mathfrak{p})_\mathfrak{p} = 0$.

Definition: A noetherian ring A is **regular** if so is any local ring $A_\mathfrak{p}$.

If the dimension of A is finite, by Serre's theorem it is equivalent to the existence of a number n such that $\mathrm{Tor}_p^A(M, N) = 0$, $p > n$, for any pair of finitely generated A-modules M, N.

Corollary: *Let A be a finite type k-algebra and $k \to L$ an extension. If A_L is regular, so is A.*

Proof: $\mathrm{Tor}_p^A(M, N)_L = \mathrm{Tor}_p^{A_L}(M_L, N_L)$.

Definition: The **height** of a prime ideal $\mathfrak{p}$ of a ring A is the dimension of the local ring $A_\mathfrak{p}$.

Lemma: *A noetherian domain is a UFD if and only if any height 1 prime ideal $\mathfrak{p}$ is principal.*

Proof: If A is a DFU and $p \in \mathfrak{p}$ is irreducible, then the ideal $pA \subseteq \mathfrak{p}$ is prime, and $pA = \mathfrak{p}$.

Conversely, by noetherianity any element is a product of irreducible elements, and uniqueness is obvious if we prove that any irreducible element p generates a prime ideal.

Any minimal prime $\mathfrak{p}$ containing pA has height 1 because $\dim A_\mathfrak{p}/pA_\mathfrak{p} = 0$.

Hence $\mathfrak{p} = aA$, and $p = ab$. Since p is irreducible, b is invertible, and $pA = \mathfrak{p}$ is prime.

Theorem: *Any regular local ring $\mathcal{O}$ is a unique factorization domain.*

Proof: We proceed by induction on $n = \dim \mathcal{O}$.

We take $f \in \mathfrak{m} - \mathfrak{m}^2$, so that $\mathcal{O}/f\mathcal{O}$ is regular, and $f\mathcal{O}$ is a prime ideal.

(1) $\mathcal{O}_f$ is a DFU: If $\mathfrak{p} \subset \mathcal{O}$ is a prime of height 1 and $f \notin \mathfrak{p}$, since $\mathcal{O}_f$ is a regular ring of dimension $< n$, by induction $\mathfrak{p}_f$ is a line $\mathcal{O}_f$-module.

Since $\mathfrak{p}$ admits a finite resolution by free $\mathcal{O}$-modules, $\mathfrak{p}_f$ admits a finite resolution by free $\mathcal{O}_f$-modules, and $\mathfrak{p}_f = \mathcal{O}_f$ in the K-group of locally free $\mathcal{O}_f$-modules. Since $L \mapsto \Lambda^{\mathrm{rk}\, L} L$ is an additive function with values in $\mathrm{Pic}(\mathcal{O}_f)$, we see that $\mathfrak{p}_f \simeq \mathcal{O}_f$ is principal.

(2) $\mathcal{O}$ is a DFU: Let $\mathfrak{p} \subset \mathcal{O}$ be a prime of height 1. If $f \in \mathfrak{p}$, then $f\mathcal{O} = \mathfrak{p}$. If $f \notin \mathfrak{p}$, then $\mathfrak{p}_f$ is principal, $\mathfrak{p}_f = p\mathcal{O}_f$, and by noetherianity we may assume that $p \in \mathfrak{p}$ is not multiple of f.

If $a \in \mathfrak{p}$ is not a multiple of p, we have $af^r = pb$, $r > 0$, $b \notin f\mathcal{O}$, contradicting that $f\mathcal{O}$ is prime. Hence $\mathfrak{p} = p\mathcal{O}$ is principal.

15.2.2 Depth

When A is a noetherian ring, the primary decomposition of ideals may be easily extended to finitely generated modules: a submodule $N \subset M$ is primary if any homothety $M/N \xrightarrow{a\cdot} M/N$ is injective or nilpotent, and in such a case $\mathrm{Ann}\,(M/N)$ is a $\mathfrak{p}$-primary ideal.

Irreducible submodules are primary, so that any submodule is an intersection of primary submodules, $N = \bigcap_i N_i$, the associated primes $\mathfrak{p}_i = \mathrm{rad}\big(\mathrm{Ann}\,(M/N_i)\big)$ being the prime ideals coinciding with the annihilator of some element of M/N, and $\bigcup_i \mathfrak{p}_i$ is the set of elements of A dividing 0 in M/N.

Lemma: *There is a M-regular element if and only if* $\mathrm{Hom}_{\mathcal{O}}(k,M)=0$.

Proof: Consider the associated primes $\mathfrak{p}_i$ of the submodule 0. The existence of a M-regular element states that $\mathfrak{m}$ strictly contains $\bigcup_i \mathfrak{p}_i$; i.e. $\mathfrak{m}$ is not an associated prime, no element of M has annihilator $\mathfrak{m}$.

Theorem: *There is a M-regular sequence $f_1,\ldots,f_r$ if and only if* $\mathrm{Ext}^p_{\mathcal{O}}(k,M)=0$, $0\leq p<r$.

Proof: By induction on r. If the sequence $f_1,\ldots,f_r$ is regular, we have exact sequences

$$0\longrightarrow M \xrightarrow{\cdot f_1} M\longrightarrow M/f_1M\longrightarrow 0$$
$$0=\mathrm{Ext}^{p-1}(k,M/f_1M)\longrightarrow \mathrm{Ext}^p(k,M)\xrightarrow{\cdot f_1}\mathrm{Ext}^p(k,M),\ p<r.$$

Since $\mathrm{Ext}^p(k,M)$ is annihilated by $\mathfrak{m}$ (consider an injective resolution of M), it is null.

Conversely, if $\mathrm{Hom}_{\mathcal{O}}(k,M)=0$, by the lemma there is a M-regular element f_1, and the exact sequence

$$0=\mathrm{Ext}^p(k,M)\longrightarrow \mathrm{Ext}^p(k,M/f_1M)\longrightarrow \mathrm{Ext}^{p+1}(k,M)=0$$

shows that $\mathrm{Ext}^p(k,M/f_1M)=0$, $p<r-1$.

Hence there exists a (M/f_1M)-regular sequence $f_2,\ldots,f_r$; and $f_1,f_2,\ldots,f_r$ is M-regular.

Definitions: The **depth** of M is the first integer p such that $\mathrm{Ext}^p(k,M)\neq 0$, and the above argument shows that any M-regular sequence may be extended so as to have length p.

M is **Cohen-Macaulay** if the depth coincides with the dimension of $\mathrm{supp}\,M$.

So, $\mathcal{O}$ is a Cohen-Macaulay ring when $\mathrm{Ext}^p(k,\mathcal{O})=0$, $p<\dim\mathcal{O}$.

For example, regular rings are Cohen-Macaulay.

If $f_1,\ldots,f_r$ is a regular sequence, $\mathcal{O}$ is Cohen-Macaulay if and only if so is $\mathcal{O}/(f_1,\ldots,f_r)$.

$\mathcal{O}$ is Cohen-Macaulay if and only if so is $\widehat{\mathcal{O}}$, because $\mathrm{Ext}^p_{\mathcal{O}}(k,\mathcal{O})\otimes_{\mathcal{O}}\widehat{\mathcal{O}}=\mathrm{Ext}^p_{\widehat{\mathcal{O}}}(k,\widehat{\mathcal{O}})$.

Ischebeck's Theorem: *If M is a finitely generated module of depth p and N is a finitely generated module with support of dimension d, then*

$$\mathrm{Ext}^n_{\mathcal{O}}(N,M)=0,\ n<p-d.$$

Proof: By induction on d. Taking a filtration of N with quotients $\simeq\mathcal{O}/\mathfrak{p}_i$ (p. 200) we are reduced to the case $N=\mathcal{O}/\mathfrak{p}$, and when $d=0$ it follows from the above theorem.

If $d=\dim\mathcal{O}/\mathfrak{p}>0$, and we take $f\in\mathfrak{m}-\mathfrak{p}$, then $\dim\mathcal{O}/(\mathfrak{p},f)<d$, and by induction

$$\mathrm{Ext}^n(\mathcal{O}/(\mathfrak{p},f),M)=0,\ n<p-d+1.$$

Now the exact sequence $0\to\mathcal{O}/\mathfrak{p}\xrightarrow{\cdot f}\mathcal{O}/\mathfrak{p}\to\mathcal{O}/(\mathfrak{p},f)\to 0$ induces an exact sequence

$$0=\mathrm{Ext}^n(\mathcal{O}/(\mathfrak{p},f),M)\longrightarrow\mathrm{Ext}^n(\mathcal{O}/\mathfrak{p},M)\xrightarrow{\cdot f}\mathrm{Ext}^n(\mathcal{O}/\mathfrak{p},M)\longrightarrow\mathrm{Ext}^{n+1}(\mathcal{O}/(\mathfrak{p},f),M)=0$$

where $n<p-d$. Now $\mathrm{Ext}^n(\mathcal{O}/\mathfrak{p},M)=0$ by Nakayama's lemma. q.e.d.

1. *If $\mathcal{O}$ is Cohen-Macaulay, and $\mathfrak{p}$ is an associated prime of 0, then* $\dim\mathcal{O}=\dim\mathcal{O}/\mathfrak{p}$, *and $\mathfrak{p}$ is a minimal prime.*

   $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p},\mathcal{O})\neq 0$; hence $0\geq\mathrm{depth}\,\mathcal{O}-\dim\mathcal{O}/\mathfrak{p}=\dim\mathcal{O}-\dim\mathcal{O}/\mathfrak{p}$.

2. *If $\mathcal{O}$ is Cohen-Macaulay, then* $\dim \mathcal{O} = \dim(\mathcal{O}/\mathfrak{p}) + \dim \mathcal{O}_\mathfrak{p}$ *for any prime ideal* $\mathfrak{p}$.

   If $\mathfrak{p}$ is a minimal prime, then $\dim \mathcal{O} = \dim \mathcal{O}/\mathfrak{p}$, and we conclude because $\dim \mathcal{O}_\mathfrak{p} = 0$. If $\mathfrak{p}$ is not minimal, there exists $f \in \mathfrak{p}$ not dividing 0. Hence $\bar{\mathcal{O}}$ is a Cohen-Macaulay local ring od dimension $\dim \mathcal{O} - 1$, and we have $\mathcal{O}/\mathfrak{p} = \bar{\mathcal{O}}/\bar{\mathfrak{p}}$, and $\dim \bar{\mathcal{O}}_{\bar{\mathfrak{p}}} = \dim \mathcal{O}_\mathfrak{p} - 1$. We conclude by induction on $\dim \mathcal{O}$.

3. *If $\mathcal{O}$ is Cohen-Macaulay, so is $\mathcal{O}_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$;* and we may define **Cohen-Macaulay** rings to be noetherian rings A such that A_x is Cohen-Macaulay $\forall x \in \operatorname{Spec} A$.

   If $\dim \mathcal{O}_\mathfrak{p} = 0$, then $\mathcal{O}_\mathfrak{p}$ is Cohen-Macaulay. If $\mathfrak{p}$ is not minimal, by (1) there exists $f_1 \in \mathfrak{p}$ not dividing 0. By induction $(\mathcal{O}/f_1\mathcal{O})_\mathfrak{p} = \mathcal{O}_\mathfrak{p}/f_1\mathcal{O}_\mathfrak{p}$ is Cohen-Macaulay; hence so is $\mathcal{O}_\mathfrak{p}$.

4. *If $\mathcal{O}$ is Cohen-Macaulay of dimension n, and* $\dim \mathcal{O}/(f_1, \ldots, f_n) = 0$, *then $f_1, \ldots, f_n$ is a regular sequence.*

   Since $\dim \mathcal{O}/(f_1, \ldots, f_i) = n - i$, then f_1 does not divide 0 and $\mathcal{O}/f_1\mathcal{O}$ is Cohen-Macaulay. We conclude by induction on n.

5. *Let $A \to B$ an injective finite morphism of integral rings. If A is regular and B is Cohen-Macaulay, then the morphism is flat.*

   We may assume that A is local of dimension n, and $\mathfrak{m} = (f_1, \ldots, f_n)$. Since the morphism is finite, $\dim B/(f_1, \ldots, f_n) = 0$, and $f_1, \ldots, f_n$ is a regular sequence at any closed point y of $\operatorname{Spec} B$ since B_y is Cohen-Macaulay of dimension n (p.209). We conclude that B is a free A-module: $\operatorname{Tor}_1^A(A/\mathfrak{m}, B) = H_1\big[K_B(f_1, \ldots, f_n)\big] = 0$.

15.2.3 Local Cohomology

Lemma: $H^p_x(X, N) = \varinjlim \operatorname{Ext}^p_\mathcal{O}(\mathcal{O}/\mathfrak{m}^n, N)$, *for any $\mathcal{O}$-module N.*

Proof: Since the sheaves $\tilde{I}^p$ are flasque (p. 425), the local cohomology groups may be calculated with an injective resolution $0 \to N \to I^\bullet$. Moreover $\Gamma_x(X, \tilde{N}) = \varinjlim \operatorname{Hom}_\mathcal{O}(\mathcal{O}/\mathfrak{m}^n, N)$,

$$H^p_x(X, \tilde{N}) = H^p\big[\Gamma_x(X, \tilde{I}^\bullet)\big] = \varinjlim H^p\big[\operatorname{Hom}_\mathcal{O}(\mathcal{O}/\mathfrak{m}^n, I^\bullet)\big] = \varinjlim \operatorname{Ext}^p_\mathcal{O}(\mathcal{O}/\mathfrak{m}^n, N).$$

Theorem: *The depth of M is the first integer p such that $H^p_x(X, M) \neq 0$, and in such a case we have that* $\operatorname{Ext}^p_\mathcal{O}(k, M) \hookrightarrow H^p_x(X, M)$.

Proof: If $\operatorname{Ext}^i(k, M) = 0$, then $\operatorname{Ext}^i(\mathfrak{m}^n/\mathfrak{m}^{n+1}, M) = 0$.

$$0 \longrightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathcal{O}/\mathfrak{m}^{n+1} \longrightarrow \mathcal{O}/\mathfrak{m}^n \to 0$$

and the Ext exact sequence shows that $\operatorname{Ext}^i(\mathcal{O}/\mathfrak{m}^n, M) = 0$, and $H^i_x(X, M) = 0$.

Moreover, $\operatorname{Ext}^{i+1}(\mathcal{O}/\mathfrak{m}^n, M) \hookrightarrow \operatorname{Ext}^{i+1}(\mathcal{O}/\mathfrak{m}^{n+1}, M)$, and $\operatorname{Ext}^{i+1}(\mathcal{O}/\mathfrak{m}, M) \hookrightarrow H^{i+1}_x(X, M)$.

Corollary: *$\mathcal{O}$ is Cohen-Macaulay if and only if $H^p_x(X, \mathcal{O}) = 0$, $p \neq \dim \mathcal{O}$.*

Proof: Since $H^p(X, M) = 0$, $p \geq 1$, and $H^p(X - x, M) = 0$, $p \geq \dim(X - x) = \dim \mathcal{O} - 1$, we always have $H^p_x(X, M) = 0$, $p > \dim \mathcal{O}$.

Lemma: *Let $\mathbf{C}_{\rm fl}$ be the category of finite length $\mathcal{O}$-modules. An $\mathcal{O}$-linear contravariant functor $\mathbf{C}_{\rm fl} \rightsquigarrow \mathbf{C}_{\rm fl}$, $M \rightsquigarrow M^*$, is representable by an injective hull I of the residue field k if and only if it is exact and $k^* \simeq k$. In such a case the natural map $M \to M^{**}$ is an isomorphism for any $\mathcal{O}$-module M of finite length, we have* $\operatorname{Hom}_\mathcal{O}(I, I) = \widehat{\mathcal{O}}$ *and*

$$I = \varinjlim (\mathcal{O}/\mathfrak{m}^r)^*.$$

Proof: If M^* is exact, by the representability theorem it is representable by an inductive limit I of modules of finite length, $I = \varinjlim \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}^r, I) = \varinjlim (\mathcal{O}/\mathfrak{m}^r)^*$.

Let us see, using the ideal criterion, that I is injective. If J is an ideal, any morphism $J \to I$ factors through $J/\mathfrak{m}^r J$; hence through $J/\mathfrak{m}^s \cap J$ by Artin-Rees. Since the functor is exact on finite length modules, the morphism may be extended to $\mathcal{O}/\mathfrak{m}^s$, hence to $\mathcal{O}$.

Now, I is a direct sum of injective hulls of k because $\operatorname{supp} I = x$.

The sum has a unique term when $k^* \simeq k$.

Conversely, if I is an injective hull of k, it is clear that $M^* = \operatorname{Hom}_{\mathcal{O}}(M, I)$ is an exact functor and $k^* \simeq k$. Hence $k = k^{**}$ and, by induction on the length, $M = M^{**}$ for any finite length module. Finally,

$$\operatorname{Hom}_{\mathcal{O}}(I, I) = \varprojlim \operatorname{Hom}_{\mathcal{O}}\big((\mathcal{O}/\mathfrak{m}^r, I)^*, I\big) = \varprojlim (\mathcal{O}/\mathfrak{m}^r)^{**} = \varprojlim \mathcal{O}/\mathfrak{m}^r = \widehat{\mathcal{O}}.$$

Theorem: *If $\mathcal{O}$ is a regular local ring of dimension n, then $H^n_x(X, \mathcal{O})$ is an injective hull of the residue field k.*

Proof: Using the Koszul complex, one may easily see that

$$\operatorname{Ext}^p_{\mathcal{O}}(k, \mathcal{O}) = \begin{cases} k & p = n \\ 0 & p \neq n \end{cases}$$

and, by induction on the length, we have $\operatorname{Ext}^p_{\mathcal{O}}(M, \mathcal{O}) = 0$, $p \neq n$, for any module of finite length M. Hence, the functor $F(-) = \operatorname{Ext}^n_{\mathcal{O}}(-, \mathcal{O})$ is exact on the category of finite length $\mathcal{O}$-modules and $F(k) \simeq k$, and it corresponds to an injective hull of k,

$$I = \varinjlim \operatorname{Ext}^n_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}^r, \mathcal{O}) = H^n_x(\mathcal{O}).$$

Example: If $\mathcal{O}$ is a local ring, there is no canonical injective hull of the residue field $\mathcal{O}/\mathfrak{m}$. But when $\mathcal{O}$ is a local k-algebra and the extension $k \to \mathcal{O}/\mathfrak{m}$ is finite, then $M^* = \operatorname{Hom}_k(M, k)$ is an exact functor on the category of finite length $\mathcal{O}$-modules, and we have an isomorphism $\mathcal{O}/\mathfrak{m} \simeq (\mathcal{O}/\mathfrak{m})^*$, because the $\mathcal{O}$-module $(\mathcal{O}/\mathfrak{m})^*$ is annihilated by $\mathfrak{m}$. Hence this exact functor corresponds to an injective hull $\varinjlim (\mathcal{O}/\mathfrak{m}^n)^*$ of $\mathcal{O}/\mathfrak{m}$.

So, if A is a local finite k-algebra, the injective hull of $A/\mathfrak{m}$ is just $A^* = \operatorname{Hom}_k(A, k)$.

Residues on Curves

Let C be a non singular complete curve over an algebraically closed field k, and $\mathcal{O}$ the local ring at a closed point $x \in C$. Since $\mathcal{O}$ is regular, an injective hull of the residue field $\mathcal{O}/\mathfrak{m} = k$ is $I := H^1_x(\Omega_{\mathcal{O}}) \simeq H^1_x(\mathcal{O}) \simeq \varinjlim (\mathcal{O}/\mathfrak{m}^n)^*$, and $I^* \simeq (\varinjlim (\mathcal{O}/\mathfrak{m}^n)^*)^* = \varprojlim \mathcal{O}/\mathfrak{m}^n = \widehat{\mathcal{O}}$.

In the surface $S = C \times_k C$ we have $\Omega^2_S = \Omega_C \otimes_k \Omega_C$; hence, considering the diagonal embedding $\Delta \colon C \to S$, we have $(\Omega^2_S \otimes \Delta^{-1}) \otimes \mathcal{O}_C = \Omega_C \otimes \Omega_C \otimes (\Omega_C)^* = \Omega_C$, so that, tensoring with $\Omega^2_S \otimes \Delta^{-1}$ the exact sequence $0 \to \Delta \to \mathcal{O}_S \to \mathcal{O}_C \to 0$, we obtain an exact sequence

$$0 \longrightarrow \Omega^2_{C \times C} \longrightarrow \Omega^2_{C \times C} \otimes \Delta^{-1} \longrightarrow \Omega_C \longrightarrow 0;$$

hence a commutative square, where δ_x is an injective morphism of $\mathcal{O} \otimes_k \mathcal{O}$-modules because $\mathcal{O} \otimes_k \mathcal{O}$ has depth 2 at (x, x), so that $H^1_{(x,x)}(\Omega^2_{C \times C} \otimes \Delta^{-1}) \simeq H^1_{(x,x)}(\mathcal{O} \otimes_k \mathcal{O}) = 0$,

$$\begin{array}{ccccc} H^1(\Omega_C) & \xrightarrow{\ \delta\ } & H^2(\Omega^2_S) & \xleftarrow{\ \sim\ } & H^1(\Omega_C) \otimes_k H^1(\Omega_C) \\ \uparrow & & \uparrow & & \uparrow \\ H^1_x(\Omega_C) & \xrightarrow{\ \delta_x\ } & H^2_{(x,x)}(\Omega^2_S) & \xleftarrow{\ \sim\ } & H^1_x(\Omega_C) \otimes_k H^1_x(\Omega_C) \end{array} \qquad (15.3)$$

Lemma: *If $\mu\colon I \hookrightarrow I \otimes_k I$ is an injective morphism of $\mathcal{O} \otimes_k \mathcal{O}$-modules, then $\mu^*\colon I^* \otimes_k I^* \to I^*$ defines a structure of k-algebra on the k-vector space I^*. In fact, $(\widehat{\mathcal{O}}, \cdot) \simeq (I^*, \mu^*)$.*

Proof: We have $\mu(k) \subseteq k \otimes_k k$ because $\mu(k)$ is annihilated by $\mathfrak{m} \otimes_k \mathcal{O} + \mathcal{O} \otimes_k \mathfrak{m}$.

Hence we have commutative squares

$$\begin{array}{ccc} I & \xrightarrow{\mu} & I \otimes_k I \\ \uparrow & & \uparrow \\ k & \xrightarrow[\sim]{\mu} & k \otimes_k k \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{O}} \otimes_k \widehat{\mathcal{O}} \simeq I^* \otimes_k I^* & \xrightarrow{\mu^*} & I^* \simeq \widehat{\mathcal{O}} \\ \downarrow & & \downarrow \\ k \otimes_k k & \xrightarrow[\sim]{\mu^*} & k \end{array}$$

and $\mu^*\colon \widehat{\mathcal{O}} \otimes_k \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$ is $\widehat{\mathcal{O}} \otimes_k \widehat{\mathcal{O}}$-linear; hence $\mu^*(a \otimes b) = abu$ for some $u \in \widehat{\mathcal{O}}$.

Now, u is invertible because $\mu^*(k \otimes_k k) = k$, so that $(\widehat{\mathcal{O}}, \mu^*) \xrightarrow{u\cdot} (\widehat{\mathcal{O}}, \cdot)$ is a k-linear isomorphism preserving the product. q.e.d.

Hence $\delta_x^*\colon H_x^1(\Omega_{\mathcal{O}})^* \otimes_k H_x^1(\Omega_{\mathcal{O}})^* \to H_x^1(\Omega_{\mathcal{O}})^*$ defines a canonical ring structure on $H_x^1(\Omega_{\mathcal{O}})^*$, with a unity $r\colon H_x^1(\Omega_{\mathcal{O}}) \to k$ providing a purely local definition of the residue. A tedious calculation of δ_x would show that it coincides with the local residue defined in p. 333, but at least it is easy to see that both coincide up to a non null factor:

Corollary: $\mathrm{Res}_x = \lambda r$*, where* $0 \neq \lambda \in k$.

Proof: The image of $\mathrm{Res} \in H^1(\Omega_C)^*$ by the natural map $H^1(\Omega_C)^* \to H_x^1(\Omega_C)^*$ is Res_x, by definition, and dualizing 15.3 we obtain a commutative square[1]

$$\begin{array}{ccc} H^1(\Omega_C)^* & \xleftarrow{\delta^*} & H^1(\Omega_C)^* \otimes_k H^1(\Omega_C)^* \\ \downarrow & & \downarrow \\ H_x^1(\Omega_{\mathcal{O}})^* & \xleftarrow{\delta_x^*} & H_x^1(\Omega_{\mathcal{O}})^* \otimes_k H_x^1(\Omega_{\mathcal{O}})^* \end{array}$$

so that $\mathrm{Res}_x^2 = \lambda \mathrm{Res}_x$, $\lambda \in k$, in the ring $I^* \simeq \widehat{\mathcal{O}}$. Since $\widehat{\mathcal{O}}$ is an integral local ring, and $\mathrm{Res}_x \neq 0$ (recall that $\mathrm{Res}_x\left(\frac{\mathrm{d}t}{t}\right) = 1$), we conclude that $\mathrm{Res}_x = \lambda$.

Theorem: *Let $\pi\colon \bar{C} \to C$ be a morphism between non singular complete curves over k. If π is unramified at a point $\bar{x} \in \bar{C}$, for any meromorphic differential form ω on C we have*

$$x = \pi(\bar{x}) \qquad \mathrm{Res}_x(\omega) = \mathrm{Res}_{\bar{x}}(\pi^*\omega).$$

Proof: Let $\bar{\mathcal{O}}$ be the local ring of $\bar{C}$ at $\bar{x}$.

Since π is unramified, the natural morphisms $\mathcal{O}/\mathfrak{m}^n \to \bar{\mathcal{O}}/\bar{\mathfrak{m}}^n$, $\Omega_{\mathcal{O}} \otimes_{\mathcal{O}} \bar{\mathcal{O}} \to \Omega_{\bar{\mathcal{O}}}$ are isomorphisms. Hence so is the morphism $H_x^1(\Omega_{\mathcal{O}}) \to H_{\bar{x}}^1(\Omega_{\bar{\mathcal{O}}})$, which is compatible with the connecting, so that $H_{\bar{x}}^1(\Omega_{\bar{\mathcal{O}}})^* \to H_x^1(\Omega_{\mathcal{O}})^*$ is an isomorphism preserving the product, hence the unity.

It follows that $\mathrm{Res}_{\bar{x}} = \alpha \mathrm{Res}_x$, $0 \neq \alpha \in k$.

Now, if $\mathfrak{m} = t\mathcal{O}$, then $\bar{\mathfrak{m}} = t\bar{\mathcal{O}}$, so that $\mathrm{Res}_{\bar{x}}\left(\frac{\mathrm{d}t}{t}\right) = 1 = \mathrm{Res}_x\left(\frac{\mathrm{d}t}{t}\right)$, and $\alpha = 1$.

Theorem: *If t is a local parameter at $x \in C$, then* $\mathrm{Res}_x\left(\frac{\mathrm{d}t}{t^n}\right) = 0$, $n \geq 2$.

Proof: The morphism $t\colon C \to \mathbb{P}_1$ is unramified at x, so that we may assume that x is the origin of $\mathbb{P}_1$. Now, $\frac{\mathrm{d}t}{t^n}$ has a unique pole at the origin; hence the residue is 0, because the sum of all residues of a meromorphic form is 0.

[1]where δ^* defines in fact an algebra structure on $H^1(\Omega_C)^*$, the global residue Res is the unity, and the natural morphism $H^1(\Omega_C)^* \to H_x^1(\Omega_C)^*$ is a morphism of k-algebras.

15.3 Quasi-Coherent Sheaves

Definition: If $j\colon U \to X$ is an open set and $\mathcal{F}$ is a sheaf on X, we put $\mathcal{F}^U = j_*j^*\mathcal{F}$; i.e. $\mathcal{F}^U(V) = \mathcal{F}(U \cap V)$, so that we have a natural map $\mathcal{F}^U \to \mathcal{F}^{U'}$ whenever $U' \subset U$.

Let $\mathfrak{U} = \{U_1, \ldots, U_n\}$ be a finite family of open sets in X. We put $\check{C}^0(\mathfrak{U}, \mathcal{F}) = \prod_i \mathcal{F}^{U_i}$, so that we have a natural morphism $\mathcal{F} \to \check{C}^0(\mathfrak{U}, \mathcal{F})$. In general we put

$$\check{C}^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_p} \mathcal{F}^{U_{i_0} \cap \ldots \cap U_{i_p}}$$

$$\mathrm{d}\colon \check{C}^p\mathcal{F} \longrightarrow \check{C}^{p+1}\mathcal{F},\ (\mathrm{d}s)_{i_0 \ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_0 \ldots \widehat{i_k} \ldots i_{p+1}}|_{U_{i_0} \cap \ldots \cap U_{i_{p+1}}}; \quad \mathrm{d}^2 = 0.$$

and we say that $\check{C}^\bullet\mathcal{F} = \check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is the **Cech complex** of $\mathcal{F}$ attached to $\mathfrak{U}$.

Theorem: *If $\mathfrak{U} = \{U_1, \ldots, U_n\}$ is a finite open cover of X, then the Cech complex defines a finite resolution of any sheaf of abelian groups $\mathcal{F}$; i.e. we have an exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \check{C}^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{\ \mathrm{d}\ } \ldots \xrightarrow{\ \mathrm{d}\ } \check{C}^{n-1}(\mathfrak{U}, \mathcal{F}) \longrightarrow 0.$$

Proof: When $X = U_i$ (and we may assume that $i = 1$), we put $\mathfrak{U}' = \{U_2, \ldots, U_n\}$, so that

$$\check{C}^p(\mathfrak{U}, \mathcal{F}) = \prod_{1 = i_0 < i_1 < \ldots < i_p} \mathcal{F}^{U_{i_1} \cap \ldots \cap U_{i_p}} \times \prod_{1 \neq i_0 < \ldots < i_p} \mathcal{F}^{U_{i_0} \cap \ldots \cap U_{i_p}} = \check{C}^{p-1}(\mathfrak{U}', \mathcal{F}) \times \check{C}^p(\mathfrak{U}', \mathcal{F}),$$

the differential (in germs) being $\mathrm{d}(a, b) = (b - \mathrm{d}'a, \mathrm{d}'b)$, where d' is the differential of $\check{C}^\bullet(\mathfrak{U}', \mathcal{F})$.

If $\mathrm{d}(a, b) = 0$, then $b = \mathrm{d}'a$ and $(a, b) = \mathrm{d}(0, a)$. The sequence is exact.

In the general case, since $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})|_V$ is just the Cech complex of the sheaf $\mathcal{F}|_V$ associated to the open cover $\{U_1 \cap V, \ldots, U_n \cap V\}$ of V, the sequence is exact on any open set U_i.

Definition: A morphism of schemes $X \to S$ is said to be **separated** when the diagonal morphism $\Delta\colon X \to X \times_S X$ is a closed embedding, and a scheme X is separated when so is the morphism $X \to \operatorname{Spec} \mathbb{Z}$. In such a case, any intersection $U \cap V$ of affine open sets is affine, since it is a closed subscheme of the affine scheme $U \times_{\mathbb{Z}} V$.

For example, $\operatorname{Spec} A$ and $\operatorname{Proj} R$ always are separated schemes.

Theorem: *Let $\mathcal{M}$ be a quasi-coherent sheaf on a noetherian separated scheme X. For any cover $X = U_1 \cup \ldots \cup U_n$ by affine open sets, we have that the Cech complex $\check{C}^\bullet\mathcal{M}$ is a finite quasi-coherent acyclic resolution of $\mathcal{M}$. Moreover, if*

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

is an exact sequence of quasi-coherent sheaves, then we also have an exact sequence

$$0 \longrightarrow \check{C}^\bullet\mathcal{M}' \longrightarrow \check{C}^\bullet\mathcal{M} \longrightarrow \check{C}^\bullet\mathcal{M}'' \longrightarrow 0.$$

Proof: When $i\colon U \to X$ is an affine open set, i_* preserves the cohomology of quasi-coherent sheaves (pp. 322, 323), so that $\mathcal{M}^U$ is acyclic, and the functor $\mathcal{M} \rightsquigarrow \mathcal{M}^U$ is exact.

Moreover $\mathcal{M}^U$ is quasi-coherent since the restriction to any affine open set V coincides with $j_*(\mathcal{M}|_{U \cap V})$, where $j\colon U \cap V \to V$ is the inclusion, and $U \cap V$ is affine.

Corollary: *If $f\colon X \to S$ is a morphism of noetherian separated schemes, and $\mathcal{M}$ is a quasi-coherent sheaf on X, the sheaves $R^i f_*\mathcal{M}$ are quasi-coherent.*

Proof: We may assume that $S = \operatorname{Spec} A$.

If $i\colon U \to X$ is an affine open set, then the sheaf $f_*\mathcal{M}^U = (fi)_*(\mathcal{M}|_U)$ is quasi-coherent (because fi is a morphism of affine schemes), and $\mathcal{M}^U$ is f_*-acyclic.

Hence $f_*(\check{C}^p\mathcal{M})$ is quasi-coherent, and so is $R^pf_*\mathcal{M} = \mathcal{H}^p\big[f_*(\check{C}^\bullet\mathcal{M})\big]$.

Definition: A morphism of schemes $\phi\colon T \to S$ is **flat** when so is any morphism $\mathcal{O}_{S,s} \to \mathcal{O}_{T,t}$, where $s = \phi(t)$, so that the functor $\phi^*\colon \mathcal{O}_S$**-mod** $\rightsquigarrow \mathcal{O}_T$**-mod** preserves exact sequences.

Theorem: *Given a fibred product of morphisms of noetherian separated schemes*

$$\begin{array}{ccc} X_T & \xrightarrow{\bar\phi} & X \\ \downarrow{\scriptstyle \bar f} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\phi} & S \end{array}$$

and a quasi-coherent sheaf $\mathcal{M}$ on X, if the base change ϕ is flat, then

$$\phi^*(R^if_*\mathcal{M}) = R^i\bar f_*(\bar\phi^*\mathcal{M}).$$

Proof: To show that the natural morphism $\phi^*(R^if_*\mathcal{M}) \to R^i\bar f_*(\bar\phi^*\mathcal{M})$ defined by the inverse image is an isomorphism, we may assume that $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$, and we must prove that $H^i(X,\mathcal{M}) \otimes_A B = H^i(X_B, \bar\phi^*\mathcal{M})$.

Let $\{U_i\}$ be a finite cover of X by affine open sets.

Then $\{\bar\phi^{-1}(U_i)\}$ is a cover of X_B by affine open sets, and

$$\Gamma(X, \check{C}^\bullet\mathcal{M}) \otimes_A B = \Gamma(X_B, \check{C}^\bullet\bar\phi^*\mathcal{M}).$$

Taking cohomology we conclude since B is a flat A-module.

Deligne's Formula: *Let $\mathfrak{p}$ be a coherent sheaf of ideals on a noetherian scheme X and let $U = X - (\mathfrak{p})_0$. If $\mathcal{N}$ is coherent and $\mathcal{M}$ is quasi-coherent,*

$$\varinjlim \operatorname{Hom}_X(\mathfrak{p}^n\mathcal{N}, \mathcal{M}) = \operatorname{Hom}_U(\mathcal{N}|_U, \mathcal{M}|_U).$$

Proof: First we assume that $X = \operatorname{Spec} A$, $\mathcal{M} = \widetilde{M}$, $\mathcal{N} = \widetilde{N}$, and $\mathfrak{p} = fA$.

If $N = A$, the annihilator ideal of f^n stabilizes when $n \geq r$, and any element $m/f^i = f^rm/f^{i+r}$ of M_f comes from the (well defined) morphism $\frac{m}{f^i}\colon f^{i+r}A \to M$.

Hence $\varinjlim \operatorname{Hom}_A(f^nA, M) = M_f$, and the formula holds when N is free.

If N is not free, $N = L/K$ where L is a free module of finite rank. By the Artin-Rees lemma, there is an exponent r such that $K \cap f^{n+r}L = f^n(K \cap f^rL)$, and replacing N by f^rN we may assume that there is a presentation $L' \to L \to N \to 0$ such that $f^nL' \to f^nL \to f^nN \to 0$ remains exact. Taking $\operatorname{Hom}_A(-, M)$ and $\varinjlim$, we conclude.

If $\mathfrak{p} = (f_1, \dots, f_r)$, we put $\mathfrak{p}_1 = (f_1)$, $\mathfrak{p}_2 = (f_2, \dots, f_r)$. We have the exact sequence

$$0 \longrightarrow \mathfrak{p}_1^nN \cap \mathfrak{p}_2^nN \longrightarrow \mathfrak{p}_1^nN \oplus \mathfrak{p}_2^nN \longrightarrow \mathfrak{p}_1^nN + \mathfrak{p}_2^nN \longrightarrow 0.$$

The filtration $\mathfrak{p}_1^nN + \mathfrak{p}_2^nN$ is equivalent to $\mathfrak{p}^nN$, and the filtration $\mathfrak{p}_1^nN \cap \mathfrak{p}_2^nN$ is equivalent to $(\mathfrak{p}_1\mathfrak{p}_2)^nN$, because $\mathfrak{p}_1^n\mathfrak{p}_2^nN \supseteq \mathfrak{p}_1^{n+r}N \cap \mathfrak{p}_2^nN \supseteq \mathfrak{p}_1^{n+r}N \cap \mathfrak{p}_2^{n+r}N$ by the Artin-Rees lemma.

Taking $\operatorname{Hom}_A(-, M)$ and inductive limit, by induction on r we obtain the following exact sequence, which let us conclude, where $U_i = X - (\mathfrak{p}_i)_0$,

$$0 \longrightarrow \varinjlim \mathrm{Hom}_A(\mathfrak{p}^n N, M) \longrightarrow \bigoplus_{i=1}^{2} \mathrm{Hom}_{U_i}(\mathcal{N}|_{U_i}, \mathcal{M}|_{U_i}) \longrightarrow \mathrm{Hom}_{U_1\cap U_2}(\mathcal{N}|_{U_1\cap U_2}, \mathcal{M}|_{U_1\cap U_2})$$

In general, when X is not affine, the morphism of sheaves

$$\varinjlim \underline{\mathrm{Hom}}_X(\mathfrak{p}^n\mathcal{N}, \mathcal{M}) \longrightarrow \underline{\mathrm{Hom}}_X(\mathcal{N}, \mathcal{M})^U$$

induces an isomorphism on sections over any affine open set.

Hence it is an isomorphism of sheaves, and we conclude taking global sections.

Note: Again we obtain (see p. 350) that any injective module I over a noetherian ring A defines a flasque sheaf, because the morphisms $I \to \varinjlim \mathrm{Hom}_A(\mathfrak{p}^n, I) = \widetilde{I}(U)$ are surjective.

If X is a scheme, any $\mathcal{O}_X$-module $\mathcal{M}$ admits an injective morphism $\mathcal{M} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module, and the functor $\mathrm{Hom}_X(-, \mathcal{I})$ is representable on the category of quasi-coherent sheaves: there exists a quasi-coherent sheaf $\mathcal{I}_{\mathrm{qc}}$ such that for any quasi-coherent sheaf $\mathcal{N}$

$$\mathrm{Hom}_X(\mathcal{N}, \mathcal{I}_{\mathrm{qc}}) = \mathrm{Hom}_X(\mathcal{N}, \mathcal{I}).$$

Therefore, $\mathcal{I}_{\mathrm{qc}}$ is injective in the category of quasi-coherent sheaves.

When $\mathcal{M}$ is quasi-coherent, the morphism $\mathcal{M} \to \mathcal{I}_{\mathrm{qc}}$ is injective, and we see that $\mathcal{M}$ admits a resolution by injective quasi-coherent sheaves.

Lemma: *If X is noetherian, any injective quasi-coherent sheaf I is flasque. Moreover, the sheaf $\underline{\mathrm{Hom}}_X(\mathcal{M}, I)$ is flasque for any quasi-coherent sheaf $\mathcal{M}$.*

Proof: Let U be an open set in X. If $\mathcal{M}$ is coherent, we have an epimorphism

$$\mathrm{Hom}_X(\mathcal{M}, I) \longrightarrow \varinjlim \mathrm{Hom}_X(\mathfrak{p}^n\mathcal{M}, I) = \mathrm{Hom}_U(\mathcal{M}|_U, I|_U).$$

In general, given a morphism $s \colon \mathcal{M}|_U \to I|_U$, let $\mathcal{N} \subseteq \mathcal{M}$ be a maximal submodule so that there exists a morphism $t \colon \mathcal{N} \to I$ such that $t|_U = s|_{\mathcal{N}|_U}$ (it exists by Zorn's lemma).

If $\mathcal{N} \neq \mathcal{M}$, we take $\mathcal{N} \subset \mathcal{N}'$ such that $\mathcal{N}'/\mathcal{N}$ is coherent, and an extension $t' \colon \mathcal{N}' \to I$ of t.

Since the morphism $s - t' \colon \mathcal{N}'|_U \to I|_U$ vanishes on $\mathcal{N}|_U$, and $\mathcal{N}'/\mathcal{N}$ is coherent, there is a morphism $\bar{t} \colon \mathcal{N}' \to I$ such that $\bar{t}|_U = s - t'$.

Now $t' + \bar{t} \colon \mathcal{N}' \to I$ coincides with s on U, against the maximal character of $\mathcal{N}$.

15.4 K-Theory

All schemes are assumed to be noetherian and separated, $K(X)$ is the K-group (p.182) of coherent $\mathcal{O}_X$-modules and $K'(X)$ is the K-group of locally free coherent $\mathcal{O}_X$-modules.

If Y is a closed subscheme of X, we put $Y = \mathcal{O}_Y = \mathcal{O}_X/\mathfrak{p}_Y \in K(X)$.

If $\mathcal{L}$ is locally free, then $\mathcal{L}' \mapsto \mathcal{L} \otimes \mathcal{L}' \in K'(X)$ is additive and defines a group morphism $h_{\mathcal{L}} \colon K'(X) \to K'(X)$, $h_{\mathcal{L}}(\mathcal{L}') = \mathcal{L} \otimes \mathcal{L}'$. Now $\mathcal{L} \mapsto h_{\mathcal{L}} \in \mathrm{End}(K'(X))$ is additive and defines a group morphism $K'(X) \to \mathrm{End}(K'(X))$; hence $K'(X)$ is a ring with the product

$$\mathcal{L} \cdot \mathcal{L}' = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'.$$

Analogously, $K(X)$ is a $K'(X)$-module with the product $\mathcal{L} \cdot \mathcal{M} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$.

If $f \colon X \to S$ is a morphism of schemes, then $\mathcal{L} \mapsto f^*\mathcal{L} \in K'(X)$ is additive and it defines a ring morphism

$$f^! \colon K'(S) \longrightarrow K'(X),\ \ f^!(\mathcal{L}) = f^*\mathcal{L}.$$

When f is flat, it also defines a group morphism $f^!\colon K(S) \to K(X)$, $f^!(\mathcal{M}) = f^*\mathcal{M}$.

If $f\colon X \to S$ is a **projective morphism** (it factors $X \xrightarrow{i} \mathbb{P}_n \times S \xrightarrow{\pi_2} S$, where i is a closed embedding) and $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, then the sheaves $R^p f_*\mathcal{M}$ are coherent, and nulls when $p \gg 0$ (p. 337). The function $\mathcal{M} \mapsto \sum_p (-1)^p R^p f_*\mathcal{M} \in K(S)$ is additive, and it defines Grothendieck's **admirable direct image**

$$f_!\colon K(X) \longrightarrow K(S),\ \ f_!(\mathcal{M}) = \textstyle\sum_p (-1)^p R^p f_*\mathcal{M}.$$

Theorem: $(f \circ g)_! = f_! \circ g_!$.

Proof: In K-theory, we have $\mathcal{M} = \sum_n \mathcal{M}_n/\mathcal{M}_{n+1}$ for any finite filtration $\mathcal{M} = \bigcup_n \mathcal{M}_n$, and $\sum_p (-1)^p \mathcal{M}^p = \sum_p (-1)^p \mathcal{H}^p(\mathcal{M}^\bullet)$ for any bounded complex $\mathcal{M}^\bullet$.

Now the result follows from Leray's spectral sequence $R^p f_*(R^q g_*\mathcal{M}) \Rightarrow R^{p+q}(fg)_*\mathcal{M}$,

$$f_!(g_!\mathcal{M}) = \sum_{p,q} (-1)^{p+q} R^p f_*(R^q g_*\mathcal{M}) = \sum_n (-1)^n R^n (fg)_*\mathcal{M} = (fg)_!\mathcal{M}.$$

Projection Formula: $f_!\big(f^!(s) \cdot x\big) = s \cdot f_!(x)$.

Proof: The natural morphism $\mathcal{L} \otimes_{\mathcal{O}_S} R^p f_*\mathcal{M} \to R^p f_*(f^*\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M})$ is an isomorphism when $\mathcal{L}$ is locally free; since it is a local question and the case $\mathcal{L} = \mathcal{O}_S$ is obvious.

Theorem: $\phi^! f_! = \bar{f}_! \bar{\phi}^!\colon K(X) \to K(T)$, *for any flat base change* $\phi\colon T \to S$.

Proof: We have $\phi^*(R^p f_*\mathcal{M}) = R^p \bar{f}_*(\bar{\phi}^*\mathcal{M})$, (see p. 435).

Theorem: *The natural morphism $K'(X) \to K(X)$ is an isomorphism when X is a (noetherian and separated) regular scheme of finite dimension.*

Proof: First we show that any coherent sheaf $\mathcal{M}$ is a quotient of a locally free sheaf.

In fact, if U is an affine open set, y is a generic point of $Y = X - U$ and $\mathcal{O} = \mathcal{O}_{X,y}$, then $\operatorname{Spec}\mathcal{O} - y = U \cap \S\mathcal{O} = \operatorname{Spec} A$ is affine (X is separated) and the exact sequence

$$0 = H^0_y(\operatorname{Spec}\mathcal{O}, \widetilde{\mathcal{O}}) \longrightarrow \mathcal{O} = H^0(\operatorname{Spec}\mathcal{O}, \widetilde{\mathcal{O}}) \longrightarrow A = H^0(\operatorname{Spec}\mathcal{O} - y, \widetilde{\mathcal{O}}) \longrightarrow H^1_y(\operatorname{Spec}\mathcal{O}, \widetilde{\mathcal{O}})$$

shows that $H^1_y(\operatorname{Spec}\mathcal{O}, \widetilde{\mathcal{O}}) \neq 0$, and $\mathcal{O}$ is a regular local ring of depth ≤ 1.

Hence $\dim\mathcal{O} \leq 1$, and the ideal $\mathfrak{p}$ of Y is a line sheaf (where non null).

There is a morphism $\oplus_i \mathfrak{p}^{n_i} \to \mathcal{M}$ surjective on U, since the stalk of $\mathcal{M}$ at any point of U is generated by $\mathcal{M}(U) = \varinjlim \operatorname{Hom}_X(\mathfrak{p}^n, \mathcal{M})$. Taking an affine open cover of X we conclude.

Now, by Serre's theorem, $\mathcal{M}$ admits a finite resolution $\mathcal{L}_\bullet \to \mathcal{M} \to 0$ by locally free sheaves, and the element $\sum_p (-1)^p \mathcal{L}_p$ of $K'(X)$ defined by $\mathcal{L}_\bullet$ does not depend on the resolution.

In fact, if $\mathcal{L}'_\bullet \to \mathcal{M} \to 0$ is a resolution and there is an epimorphism $\mathcal{L}'_\bullet \to \mathcal{L}_\bullet$ (inducing the identity on $\mathcal{M}$), the kernel $\mathcal{N}_\bullet$ is an exact sequence; hence $\sum_p (-1)^p \mathcal{N}_p = 0$ and

$$\textstyle\sum_p (-1)^p \mathcal{L}'_p = \sum_p (-1)^p \mathcal{L}_p + \sum_p (-1)^p \mathcal{N}_p = \sum_p (-1)^p \mathcal{L}_p.$$

In general we take a resolution $\mathcal{L}''_\bullet$ with an epimorphism onto $\mathcal{L}_\bullet$ and $\mathcal{L}'_\bullet$.

Once we construct the step $p-1$, the next one is (taking $Z''_p \to Z_p$ and $Z''_p \to Z'_p$ surjective)

$$\begin{array}{ccccccccc}
 & & \mathcal{L}_p & \longrightarrow & Z_p & \longrightarrow & \mathcal{L}_{p-1} & \xrightarrow{\ \mathrm{d}\ } & \mathcal{L}_{p-2} & \qquad Z_p = \operatorname{Ker}\mathrm{d} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\mathcal{L}''_p & \xrightarrow{\ \mathrm{epi}\ } & \mathcal{L}_p \times_{Z_p} Z''_p \times_{Z'_p} \mathcal{L}'_p & \longrightarrow & Z''_p & \longrightarrow & \mathcal{L}''_{p-1} & \xrightarrow{\ \mathrm{d}''\ } & \mathcal{L}''_{p-2} & \qquad Z''_p = \operatorname{Ker}\mathrm{d}'' \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & \mathcal{L}'_p & \longrightarrow & Z'_p & \longrightarrow & \mathcal{L}'_{p-1} & \xrightarrow{\ \mathrm{d}'\ } & \mathcal{L}'_{p-2} & \qquad Z'_p = \operatorname{Ker}\mathrm{d}'
\end{array}$$

Now, given an exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$, we take epimorphisms $\mathcal{L}_0' \to \mathcal{M}'$, $\mathcal{L}_0'' \to \mathcal{M}$, and we have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_0' & \longrightarrow & \mathcal{L}_0' \oplus \mathcal{L}_0'' & \longrightarrow & \mathcal{L}_0'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}'' & \longrightarrow & 0 \end{array}$$

Repeating the argument with the kernels of the vertical morphisms we obtain resolutions $\mathcal{L}_\bullet', \mathcal{L}_\bullet, \mathcal{L}_\bullet''$ such that $\mathcal{L}_p = \mathcal{L}_p' \oplus \mathcal{L}_p''$.

Hence $\mathcal{M} \mapsto \sum_p (-1)^p \mathcal{L}_p \in K'(X)$ is additive, and it defines a morphism $K(X) \to K'(X)$, inverse of the natural morphism $K'(X) \to K(X)$. q.e.d.

1. If $\mathcal{M} = \sum_p (-1)^p \mathcal{L}_p$ and $\mathcal{N} = \sum_q (-1)^q \mathcal{L}_q'$, then $\mathcal{M} \cdot \mathcal{N} = \sum_{p,q} (-1)^{p+q} \mathcal{L}_p \otimes \mathcal{L}_q'$.

   Since $\mathcal{L}_\bullet \otimes \mathcal{L}_\bullet'$ coincides with the alternate sum of the cohomology sheaves in K-theory, when X is regular of finite dimension, the product of coherent sheaves in $K(X)$ is

   $$\mathcal{M} \cdot \mathcal{N} = \textstyle\sum_p (-1)^p \, \underline{\mathrm{Tor}}_p^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}),$$

   where $\underline{\mathrm{Tor}}_p^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is the associated sheaf of the presheaf $U \rightsquigarrow \mathrm{Tor}_p^{\mathcal{O}_X(U)}(\mathcal{M}(U), \mathcal{N}(U))$. On any affine open set $U = \operatorname{Spec} A$ it is the sheaf defined by $\mathrm{Tor}_p^A(\mathcal{M}(U), \mathcal{N}(U))$.

2. Analogously $f^! \mathcal{M} = \sum_p (-1)^p \mathcal{L}_p \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ and, when X, Y are regular of finite dimension, the inverse image $f^! \colon K(X) \to K(Y)$ of coherent sheaves is (and this formula let us also define the inverse image $j^! \colon K(X) \to K(Y)$ for any regular closed embedding $j \colon Y \to X$)

   $$f^! \mathcal{M} = \textstyle\sum_p (-1)^p \, \underline{\mathrm{Tor}}_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}).$$

3. If k is a field, the dimension defines an isomorphism $K(\operatorname{Spec} k) = \mathbb{Z}$.

   When $\pi \colon X \to \operatorname{Spec} k$ is a projective variety, $\pi_! \colon K(X) \to K(\operatorname{Spec} k) = \mathbb{Z}$ coincides with the Euler-Poincaré characteristic, $\pi_!(\mathcal{M}) = \sum_p (-1)^p \dim H^p(X, \mathcal{M}) = \chi(X, \mathcal{M})$.

   When $\pi \colon X \to \operatorname{Spec} k$ is a regular projective variety, given closed subschemes Y, Z of complementary codimension, we have a global intersection number

   $$\langle Y, Z \rangle := \pi_!(Y \cdot Z) = \textstyle\sum_p (-1)^p \chi(X, \underline{\mathrm{Tor}}_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)),$$

   and when $Y \cap Z$ is 0-dimensional, it is just the number of common points, counting any point x with the degree $[\kappa(x) : k]$ and Serre's **intersection multiplicity**

   $$(Y \cap Z)_x := \textstyle\sum_p (-1)^p \, l(\mathrm{Tor}_p^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y,x}, \mathcal{O}_{Z,x})).$$

4. The ideal of a projective plane curve C_n of degree n is $\simeq \mathcal{O}_{\mathbb{P}_2}(-n)$. Hence $C_n = 1 - \mathcal{O}_{\mathbb{P}_2}(-n)$ in $K(\mathbb{P}_2)$, and we obtain **Bézout's Theorem** (p. 337):

   $$\begin{aligned} \langle C_n, C_m \rangle &= \chi(C_n \cdot C_m) = \chi(1 - \mathcal{O}(-n) - \mathcal{O}(-m) + \mathcal{O}(-n-m)) \\ &= 1 - \tbinom{n-1}{2} - \tbinom{m-1}{2} + \tbinom{n+m-1}{2} = nm. \end{aligned}$$

5. If C is a smooth projective curve of genus g, in $K(C \times C)$ we have $\mathcal{O}_\Delta = 1 - L_{-\Delta}$ and $\Omega_C = L_{-\Delta} - L_{-2\Delta}$, where Δ is the diagonal; hence

   $$\begin{aligned} \Delta \cdot \Delta &= (1 - L_{-\Delta})^2 = 1 - 2L_{-\Delta} + L_{-2\Delta} = \mathcal{O}_\Delta - \Omega_C, \\ \langle \Delta, \Delta \rangle &= \pi_!(\Delta \cdot \Delta) = \chi(\mathcal{O}_C) - \chi(\Omega_C) = 2 - 2g. \end{aligned}$$

In higher dimensions, using the Koszul complex, one sees that

$$\underline{\mathrm{Tor}}_p^{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)=\Lambda^p\underline{\mathrm{Tor}}_1^{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)=\Lambda^p(\mathfrak{p}_\Delta/\mathfrak{p}_\Delta^2)=\Omega_X^p,$$
$$\Delta\cdot\Delta=\textstyle\sum_p(-1)^p\Omega_X^p,$$
$$\langle\Delta,\Delta\rangle=\pi_!(\Delta\cdot\Delta)=\textstyle\sum_p(-1)^p\chi(\Omega_X^p)=\sum_{p,q}(-1)^{p+q}\mathrm{dim}_kH^p(X,\Omega_X^q).$$

Lemma: *The sheaves $\mathcal{O}_Y$, with Y an irreducible closed set, generate the group $K(X)$.*

Proof: If $\mathcal{M}$ is a coherent sheaf, we proceed by induction on the support and the lengths of stalks $\mathcal{M}_x$ at the generic points x of $Z=\operatorname{supp}\mathcal{M}$.

Let $\mathfrak{p}$ be the ideal of x. If Z has some other generic point, by induction the lemma holds for $\mathcal{M}/\mathfrak{p}\mathcal{M}$ and $\mathfrak{p}\mathcal{M}$ (since $\mathfrak{p}_x\mathcal{M}_x\neq\mathcal{M}_x$); hence for $\mathcal{M}$.

If $Z=\overline{x}$, then $\mathcal{M}$ is annihilated by some power of $\mathfrak{p}$, and we only have to prove the lemma for the quotients $\mathfrak{p}^i\mathcal{M}/\mathfrak{p}^{i+1}\mathcal{M}$. When $\mathfrak{p}\mathcal{M}=0$, we may assume that $X=\overline{x}$ is integral.

If $\mathcal{M}_x$ is the constant sheaf of stalk $\mathcal{M}_x$, the lemma holds for the kernel of $\mathcal{M}\to\mathcal{M}_x$ by induction, and we may assume that $\mathcal{M}$ is torsion free.

Now a rational section $0\neq s\in\mathcal{M}_x$ defines an exact sequence

$$0\longrightarrow I\longrightarrow\mathcal{M}\longrightarrow\mathcal{M}/I\mathcal{M}\longrightarrow 0\ ,\quad I(U)=\{f\in\mathcal{O}_X(U)\colon fs\in\mathcal{M}(U)\},$$

where the lemma holds for $\mathcal{M}/I\mathcal{M}$ and $\mathcal{O}_X/I$ by induction.

Since it holds for $\mathcal{O}_X$, we conclude that it also holds for I and $\mathcal{M}$.

Example: When X is a nonsingular curve, for any divisor D we have $D=L_D-1$ in $K(X)$. In fact this equality is obvious when $D=0$, and if it holds for D so it does for $D\pm x$, as the exact sequence $0\to L_D\to L_{D+x}\to\mathcal{O}_X/\mathfrak{m}_x\to 0$ shows.

Now, the morphism $\mathbb{Z}\oplus\mathrm{Pic}(X)\to K(X)$, $(n,D)\mapsto n+D$, is an isomorphism, the inverse being $K(X)\to\mathbb{Z}\oplus\mathrm{Pic}(X)$, $\mathcal{L}\mapsto(r,\Lambda^r\mathcal{L})$, where $r=\mathrm{rk}\,\mathcal{L}$. In fact, by the lemma (or p. 328) the line sheaves generate the group $K(X)$. Hence $K(k[t])=\mathbb{Z}$ and $K(\mathbb{P}_1)=\mathbb{Z}\oplus\mathbb{Z}$.

Theorem (Gysin): *If $j\colon Y\to X$ is a closed subscheme, and $i\colon U\to X$ is the complementary open subscheme, we have an exact sequence*

$$K(Y)\ \xrightarrow{j_!}\ K(X)\ \xrightarrow{i^!}\ K(U)\longrightarrow 0$$

Proof: It is obvious that $i^!j_!=0$, and $i^!$ is surjective since any coherent sheaf $\mathcal{M}$ on U is the restriction of a coherent sheaf $\widetilde{\mathcal{M}}$ on X. In fact, if $i_*\mathcal{M}=\bigcup_k\mathcal{M}_k$, with $\mathcal{M}_k$ coherent, then $\mathcal{M}=(i_*\mathcal{M})|_U=\bigcup_k\mathcal{M}_k|_U$, and $\mathcal{M}=\mathcal{M}_k|_U$ for some index k.

Let us show that $\operatorname{Ker} i^!\subseteq\operatorname{Im} j_!$.

Two coherent extensions $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}'}$ of $\mathcal{M}$ differ in $K(X)$ in sheaves concentred on Y.

In fact, the kernels of $\widetilde{\mathcal{M}}\to i_*\mathcal{M}$ and $\widetilde{\mathcal{M}'}\to i_*\mathcal{M}$ are concentred on Y, and the images differ from the sum in sheaves concentred on Y.

Now, if $F_Y(X)$ is the subgroup of $K(X)$ generated by all sheaves concentred on Y, the morphism $s\colon K(U)\to K(X)/F_Y(X)$, $\mathcal{M}\mapsto\widetilde{\mathcal{M}}$, is well defined, hence $\operatorname{Ker} i^!\subseteq F_Y(X)$.

Finally, $\operatorname{Im} j_!=F_Y(X)$ since any coherent sheaf $\mathcal{N}$ concentred on Y coincides, in K-theory, with the graded module of the finite filtration $\{\mathfrak{p}_Y^n\mathcal{N}\}$, which is an $\mathcal{O}_Y$-module.

Definition: The **vector** and **projective bundles** of a locally free $\mathcal{O}_X$-module E are

$$\pi\colon E=\operatorname{Spec}S^\bullet E^*\longrightarrow X$$
$$\pi\colon\mathbb{P}(E)=\operatorname{Proj}S^\bullet E^*\longrightarrow X$$

and the universal property (see p. 335) of the projective bundle $\mathbb{P}(E)$ is just

$$\mathrm{Hom}_X(T, \mathbb{P}(E)) = \begin{bmatrix} \text{Line quotients} \\ \text{of } E^* \otimes_{\mathcal{O}_X} \mathcal{O}_T \end{bmatrix} = \begin{bmatrix} \text{Line sub-bundles} \\ \text{of } E \otimes_{\mathcal{O}_X} \mathcal{O}_T \end{bmatrix}$$

An **affine bundle** of associated vector bundle E is a morphism of schemes $P \to X$ endowed with an action $E \times_X P \to P$ locally (on X) isomorphic to the additive action of E on E.

Theorem: *If $E \to X$ is a vector bundle, then $\pi^!\colon K(X) \to K(E)$ is an isomorphism.*

Proof: The morphism $\pi^!$ is injective since the zero section $s\colon X \to E$ induces a morphism $s^!\colon K(E) \to K(X)$, and $s^!\pi^! = \mathrm{Id}$ since (just take a free resolution of M)

$$\mathrm{Tor}_p^{A[t_1,\ldots,t_n]}(M \otimes_A A[t_1,\ldots,t_n], A) = 0, \ p \geq 1.$$

To show that $\pi^!$ is surjective, by noetherian induction and Gysin exact sequence, we may assume that $X = \mathrm{Spec}\, A$, $E = \mathbb{A}^n_X$, and $n = 1$.

Let Z be a irreducible closed set in $\mathbb{A}^1_X$, let $\mathfrak{q}$ be the corresponding prime ideal in $A[t]$, and put $\mathfrak{p} = \mathfrak{q} \cap A$. If $\mathfrak{q} = \mathfrak{p}A[t]$, we conclude: $A[t]/\mathfrak{q} = \pi^!(A/\mathfrak{p})$.

Otherwise $\mathfrak{q}$ defines a non null prime ideal in $(A/\mathfrak{p})[t]$, and it becomes principal after localizing by some function $0 \neq \bar{f} \in A/\mathfrak{p}$, so that $\mathfrak{q} = (\mathfrak{p}, Q(t))$ in $A_f[t]$.

Then $A_f[t]/\mathfrak{q} = 0$ in K-theory, as shows the following exact sequence (where $B = A_f/\mathfrak{p}A_f$)

$$0 \longrightarrow B[t] \xrightarrow{\cdot Q(t)} B[t] \longrightarrow A_f[t]/\mathfrak{q} \longrightarrow 0$$

and $Z = 0$ in $K(\mathbb{A}^1 \times U_f)$. By Gysin, Z comes from $K(\mathbb{A}^1_Y)$, where $Y = (f)_0$.

By noetherian induction $K(Y) \to K(\mathbb{A}^1_Y)$ is surjective, and we conclude.

Periodicity Theorem: *Let E be a locally free $\mathcal{O}_X$-module of rank $r+1$, and let us consider the class $x_E = 1 - \mathcal{O}_{\mathbb{P}(E)}(-1)$ in $K'(\mathbb{P}(E))$. We have an isomorphism,*

$$\textstyle\bigoplus_{r+1} K(X) \xrightarrow{\sim} K(\mathbb{P}(E)), \ (a_0, \ldots, a_r) \mapsto \sum_i \pi^!(a_i) x_E^i.$$

Proof: If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, then $R^p\pi_*(\pi^*\mathcal{M} \otimes \mathcal{O}_{\mathbb{P}(E)}(n)) = \mathcal{M} \otimes R^p\pi_*(\mathcal{O}_{\mathbb{P}(E)}(n))$.

According to the cohomology (p. 337) of the sheaves $\mathcal{O}_{\mathbb{P}_r}(n)$, if we put $\xi = \mathcal{O}(-1)$,

$$\pi_!(\pi^!(a)\xi^n) = \begin{cases} a & n = 0 \\ 0 & 1 \leq n \leq r \end{cases}$$

Let us see that $1, x_E, \ldots, x_E^r$ are "linearly independent":
Since $x_E = 1 - \xi$, it is enough to prove that so are $1, \xi, \ldots, \xi^r$.
If $\pi^!(a_0) + \pi^!(a_1)\xi + \ldots + \pi^!(a_r)\xi^r = 0$, applying $\pi_!$ we see that $a_0 = 0$.
Now, multiplying by ξ^{-1} and applying $\pi_!$, we see that $a_1 = 0$, and so on.

Now let us prove that $1, x_E, \ldots, x_E^r$ "generate":
By noetherian induction and Gysin we are reduced to an open set where E is trivial.
We consider a hyperplane $j\colon \mathbb{P}_{r-1} \times X \to \mathbb{P}_r \times X = \mathbb{P}(E)$ and we put $x_r = 1 - \mathcal{O}_{\mathbb{P}_r}(-1)$.
Now we proceed by induction on r, and we have a Gysin exact sequence

$$K(\mathbb{P}_{r-1} \times X) \xrightarrow{j_!} K(\mathbb{P}_r \times X) \xrightarrow{i^!} K(\mathbb{A}^r \times X) \longrightarrow 0$$

By the projection formula, $j_!(x_{r-1}^d) = j_!(j^!(x_r^d)) = j_!(1)x_r^d = x_r^{d+1}$, so that $x_r, x_r^2, \ldots, x_r^r$ "generate" the image of $j_!$, and we conclude because $K(\mathbb{A}^r \times X) = K(X)$.

Corollary: *If* $\pi\colon P \to X$ *is an affine bundle, then* $\pi^!\colon K(X) \to K(P)$ *is an isomorphism.*

Proof: If V is the associated vector bundle, the construction of the projective closure of an affine space (pg. 164, and note that the function 1 defines a linear 1-form on E vanishing on V) shows that we have an exact sequence of locally free $\mathcal{O}_X$-modules $0 \to V \to E \to \mathcal{O}_X \to 0$ such that P is just the complement of the closed embedding $i\colon \mathbb{P}(V) \to \mathbb{P}(E)$.

The natural morphism $\mathcal{O}_{\mathbb{P}(E)}(-1) \to E \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{P}(E)} \to \mathcal{O}_{\mathbb{P}(E)}$ identifies $\mathcal{O}_{\mathbb{P}(E)}(-1)$ with the ideal of $\mathbb{P}(V)$. Hence $i_!(1) = x_E$, and the projection formula shows that

$$i_!(1) = x_E,\ i_!(x_V) = x_E^2,\ \dots\ ,\ i_!(x_V^{r-1}) = x_E^r.$$

Now the Gysin exact sequence $K(\mathbb{P}(V)) \xrightarrow{j_!} K(\mathbb{P}(E)) \xrightarrow{i^!} K(P) \to 0$ and the Periodicity theorem turn obvious the statement.

Corollary: $K(\mathbb{P}_r) = \mathbb{Z}[x]/(x^{r+1})$, *where* $x^d = \mathbb{P}_{n-d}$.

Proof: By induction on r, since $j_!((j^!x)^d) = j_!(j^!(x^d)) = x^{d+1}$. q.e.d.

1. If C is a plane curve of degree d, then $C = 1-(1-x)^d = dx - \binom{d}{2}x^2$.

2. In $\mathbb{P}_n$ we have $\chi(x^r) = 1$, $0 \le r \le n$. Now, if C is a curve of degree d in $\mathbb{P}_3$, we have $\chi(C\cdot x) = d$, $\chi(C\cdot x^2) = \chi(C\cdot x^3) = 0$, and $C = dx^2 + ax^3$. Since $d + a = \chi(C) = 1-\pi$, where π is the arithmetic genus, we see that $C = dx^2 + (1-d-\pi)x^3$.

   If S is a surface of degree d in $\mathbb{P}_3$, then we have $\chi(S\cdot x^2) = d$ and $\chi(S\cdot x^3) = 0$; hence $S = dx + ax^2 + bx^3$. Since $1-\pi = \chi(S\cdot x) = d + a$, where π is the arithmetic genus of a hyperplane section, and $1-p_a = \chi(S) = d+a+b$, where p_a is the arithmetic genus of S, we see that $S = dx + (1-d-\pi)x^2 + (\pi - p_a)x^3$.

3. A rational 1-form ω on $\mathbb{P}_2$ defines a line sheaf $(\omega)(U) = \{f\omega \in \Omega_{\mathbb{P}_2}(U)\colon f \in k(x,y)\}$ and, if $(\omega) \simeq \mathcal{O}(-n)$, we have an exact sequence

   $$(*) \qquad 0 \longrightarrow (\omega) \longrightarrow \Omega^1_{\mathbb{P}_2} \xrightarrow{\ \wedge\omega\ } \Omega^2_{\mathbb{P}_2} \otimes \mathcal{O}(n) \longrightarrow \mathfrak{C} \longrightarrow 0$$

   If the local equation at a point p is $\omega = h(f\mathrm{d}x + g\mathrm{d}y)$, where $f,g \in \mathcal{O}_p$ have no common factor, then $\mathfrak{C}_p = \mathcal{O}_p/(f,g)$. In $K(\mathbb{P}_2)$,

   $$\mathfrak{C} = \delta x^2,\ \text{where } \delta = \textstyle\sum_p \dim_k \mathfrak{C}_p,$$
   $$\Omega^1_{\mathbb{P}_2} = 3t - 1 = 3(1-x) - 1 = 2 - 3x,$$
   $$\Omega^2_{\mathbb{P}_2} \otimes \mathcal{O}(n) = \mathcal{O}(n-3) = (1-x)^{3-n},$$

   since there is an exact sequence $0 \to \Omega^1_{\mathbb{P}_2} \to \mathcal{O}(-1)^3 \to \mathcal{O} \to 0$. Now $(*)$ gives

   $$2 - 3x + \delta x^2 = (1-x)^n + (1-x)^{3-n} 2 - 3x + (n^2 - 3n + 3)x^2,$$
   $$3 = \delta - n^2 + 3n.$$

15.4.1 Graded K-Theory

All schemes will be **algebraic varieties** over a field k (separated k-schemes of finite type).

Let $F^p(X)$ be the subgroup of $K(X)$ generated by coherent sheaves supported on points x of codimension $\dim \mathcal{O}_{X,x} \ge p$. So we get a filtration of $K(X)$ and we put

$$GK^\bullet(X) = \textstyle\bigoplus_p GK^p(X) = \bigoplus_p F^p(X)/F^{p+1}(X).$$

Any irreducible closed set Y of codimension p defines a class $[Y] = [\mathcal{O}_Y] \in GK^p(X)$, and the class in $GK^p(X)$ of a coherent sheaf $\mathcal{M}$ with support of codimension p is

$$[\mathcal{M}] = \sum_{\mathrm{cod}\, y=p} l_{\mathcal{O}_{X,y}}(\mathcal{M}_y)\,[Y] \in GK^p(X),$$

where Y is the closure of the point y. Hence $GK^p(X)$ is a quotient of the group $\sum_{\mathrm{cod}\, y=p} \mathbb{Z}y$ of **cycles** of codimension p, so that it is a group of equivalence classes of cycles.

If $f\colon X \to S$ is a projective morphism and X, S are irreducible, then $f_! F^p(X) \subseteq F^{p+d}(S)$, $d = \dim S - \dim X$, and it induces a morphism $f_*\colon GK^\bullet(X) \to GK^\bullet(S)$ of degree d.

Now we study the compatibility of this filtration with inverse images (obvious in the case of open embeddings, vector bundles and projective bundles).

Lemma: *If L is a line sheaf, then $1 - L \in F^1(X)$ and $(1-L)\cdot F^p(X) \subseteq F^{p+1}(X)$, so that it induces a homogeneous morphism $[1-L]\colon GK^\bullet(X) \to GK^\bullet(X)$ of degree 1.*

Proof: If $\mathrm{codim}\,(\mathrm{supp}\,\mathcal{M}) = p$, then $\mathcal{M}$ and $L \otimes_{\mathcal{O}_X} \mathcal{M}$ coincide in $GK^p(X)$ since both have equal length at any point of codimension p; hence $(1-L)\mathcal{M} \in F^{p+1}(X)$.

Example: Let X be a projective variety of dimension d over a field k, and $\pi\colon \bar{X} = \mathrm{Proj} \oplus I^n \to X$ the **blowup** of X along an ideal $I \subset \mathcal{O}_X$. If $E = \pi^{-1}\big((I)_0\big)$ is the exceptional fibre, defined by the ideal $I\mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{X}}(1)$, by the lemma we have $E^i = 0$, $i > d$, in $K(\bar{X})$.

Moreover, since $I^n\mathcal{O}_{\bar{X}} = \mathcal{O}_{\bar{X}}(n)$, when $n \gg 0$ we have

$$\pi_*(I^n\mathcal{O}_{\bar{X}}) = I^n$$
$$R^p\pi_*(I^n\mathcal{O}_{\bar{X}}) = 0,\ p \geq 1,$$

so that $\chi(X, I^n) = \chi(\bar{X}, I^n\mathcal{O}_{\bar{X}})$. Hence the **Samuel function** $S_I(n) = \chi(X, \mathcal{O}_X/I^n)$ and the **Hilbert function** $H_I(n) = \chi(X, I^n/I^{n+1})$ are polynomial functions when $n \gg 0$,

$$I^n\mathcal{O}_{\bar{X}} = (I\mathcal{O}_{\bar{X}})^n = (1-E)^n = 1 + \sum_{i=1}^{d}(-1)^i\binom{n}{i}E^i,$$
$$S_I(n) = \chi(X, \mathcal{O}_X/I^n) = \chi(X, \mathcal{O}_X) - \chi(X, I^n) = \chi(X, \mathcal{O}_X) - \chi(\bar{X}, I^n\mathcal{O}_{\bar{X}})$$
$$= \chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) - \sum_{i=1}^{d}(-1)^i\chi(\bar{X}, E^i)\binom{n}{i}.$$
$$H_I(n) = \Delta S_I(n) = S_I(n+1) - S_I(n) = \sum_{i=0}^{d-1}(-1)^i\chi(\bar{X}, E^{i+1})\binom{n}{i}.$$

and we obtain a geometric interpretation of the coefficients of the Samuel and Hilbert polynomials $S(n)$ and $H(n)$, the independent term of $S(n)$ being $\chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}})$.

Moreover, when $n \gg 0$ we have

$$\sum_{i=0}^{n-1} H_I(i) = S_I(n) = S(n) = \chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) + \sum_{i=0}^{n-1} P(i),$$
$$\chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) = \sum_{n\geq 0}\big(H_I(n) - P(n)\big).$$

and we see that $\chi(X, \mathcal{O}_X) = \chi(\bar{X}, \mathcal{O}_{\bar{X}})$ when the Hilbert function is polynomial for all $n \geq 0$.

For example, if we blow-up a rational point x of a regular surface,

$$S(n) = \dim_k(\mathcal{O}_{X,x}/\mathfrak{m}^n) = \tbinom{n+1}{2} = 0 + \tbinom{n}{1} + \tbinom{n}{2}$$

and we see that $\chi(X,\mathcal{O}_X)=\chi(\bar{X},\mathcal{O}_{\bar{X}})$, that $\chi(E,\mathcal{O}_E)=1$ and that the self-intersection of the exceptional fibre is $(E\cap E)=-1$.

If we blow-up a rational point x of multiplicity m of a hypersurface X of a projective space $\mathbb{P}_d$, the Samuel polynomial (p. 205) is $\binom{n+d-1}{d}-\binom{n+d-m-1}{d}$, and the coefficient of degree zero is $(-1)^d\binom{m}{d}$. In particular, $\chi(X,\mathcal{O}_X)=\chi(\bar{X},\mathcal{O}_{\bar{X}})$ when $m<d$.

Periodicity Theorem: *Let E be a locally free $\mathcal{O}_X$-module of rank $r+1$, and let us consider the morphism $x_E=[1-\mathcal{O}_{\mathbb{P}(E)}(-1)]\colon GK^\bullet(\mathbb{P}(E))\to GK^\bullet(\mathbb{P}(E))$. We have an isomorphism*

$$\bigoplus_{r+1} GK^\bullet(X) \xrightarrow{\sim} GK^\bullet(\mathbb{P}(E)),\ (a_0,\dots,a_r)\mapsto \textstyle\sum_i \pi^*(a_i)x_E^i.$$

Proof: We put $x=1-\mathcal{O}_{\mathbb{P}(E)}(-1)\in K'(\mathbb{P}(E))$, so that $\pi_!(x^i)=1$ when $0\le i\le r$.

If $a\in F^p(X)$, then $\pi_*(\pi^*(a)\cdot x_E^i)$ is the class of $\pi_!(\pi^!(a)x^i)=a\pi_!(x^i)=a$ mod. $F^{p+i-r+1}(X)$, so that it is null if $i<r$, and

$$\pi_*(\pi^*(a)\cdot x_E^i)=\begin{cases}0 & i<r\\ a & i=r\end{cases}$$

Now the argument of p. 440 shows that $1,x_E,\dots,x_E^r$ are "linearly independent".

Now we take $0\ne[\mathcal{M}]\in GK^p(\mathbb{P}(E))$, and we know that $\mathcal{M}=\pi^!(a_0)+\ldots+\pi^!(a_r)x^r$.

If $0\ne\bar{a}_i\in GK^{n_i}(X)$, and $m=\min\{n_i+i\}<p$, then the class of $\mathcal{M}$ in $GK^m(\mathbb{P}(E))$ is not null, absurd. Hence $m\ge p$, and $[\mathcal{M}]=\sum_i\pi^*(\bar{a}_i)\cdot x_E^i$, with $\bar{a}_i\in GK^{p-i}(X)$.

Lemma: *If $i\colon U\to X$ is an open subscheme, then $i^!F^p(X)=F^p(U)$; so that $i^!$ induces an epimorphism $i^*\colon GK^\bullet(X)\to GK^\bullet(U)$.*

Proof: In the argument of p. 439, if a coherent sheaf $\mathcal{M}$ on U has support of codimension $\ge p$, then the coherent extension $\widetilde{\mathcal{M}}\subset i_*\mathcal{M}$ has support $\operatorname{supp}(\widetilde{\mathcal{M}})\subseteq\operatorname{supp}(i_*\mathcal{M})\subseteq\overline{\operatorname{supp}\mathcal{M}}$ of codimension $\ge p$, and we conclude.

Corollary: *If $P\to X$ is an affine bundle, then $\pi^*\colon GK^\bullet(X)\longrightarrow GK^\bullet(P)$ is an isomorphism.*

Proof: Since $\pi^!\colon K(X)\to K(P)$ is an isomorphism compatible with filtrations, it is enough to show that $\pi^*\colon GK^\bullet(X)\to GK^\bullet(P)$ is surjective.

The projective closure $i\colon P\to\mathbb{P}(E)$ is an open embedding, so that

$$i^*\colon GK^\bullet(\mathbb{P}(E))=GK^\bullet(X)\oplus GK^\bullet(X)x_E\oplus\ldots\oplus GK^\bullet(X)x_E^r\longrightarrow GK^\bullet(P)$$

is surjective. Since $i^*(x_E)=\ldots=i^*(x_E^r)=0$, because $i^*\mathcal{O}_{\mathbb{P}(E)}(-1)$ is trivial, we conclude.

Corollary: *If $s\colon X\to E$ is the zero section of a vector bundle, then $s^!(F^p(E))=F^p(X)$, so that $s^!$ induces a morphism $s^*\colon GK^\bullet(E)\to GK^\bullet(X)$.*

Proof: By the above corollary, $\pi^!F^p(X)=F^p(E)$, and $s^!$ is the inverse of $\pi^!$.

Lemma: *If $i\colon H\to X$ is a hypersurface of locally principal ideal $\mathfrak{p}$, then $i^!F^p(X)\subseteq F^p(H)$, so that $i^!$ induces a morphism $i^*\colon GK^\bullet(X)\to GK^\bullet(H)$. Moreover, $i_*i^*(x)=[1-\mathfrak{p}]x$.*

Proof: Let Y be a irreducible closed set in X of codimension $\ge p$.

By definition $i^!(\mathcal{O}_Y)=\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{O}_H-\underline{\operatorname{Tor}}_1^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_H)$, and the locally free resolution

$$0\longrightarrow\mathfrak{p}\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_H\longrightarrow 0$$

shows that $i^!(\mathcal{O}_Y) = \mathcal{O}_{Y\cap H} \in F^p(H)$ when H does not contain Y.

Moreover, $i_!(i^!\mathcal{O}_Y) = \mathcal{O}_{Y\cap H} = \mathcal{O}_Y - \mathfrak{p}\otimes\mathcal{O}_Y = (1-\mathfrak{p})\cdot\mathcal{O}_Y$.

If H contains Y, then $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_H = \mathcal{O}_Y$ and

$$\underline{\mathrm{Tor}}_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_H) = \mathfrak{p}\otimes_{\mathcal{O}_X}\mathcal{O}_Y,$$

so that $i^!(\mathcal{O}_Y) = (1-\mathfrak{p}|_H)\cdot\mathcal{O}_Y \in F^p(H)$, and $i_!(i^!\mathcal{O}_Y) = \mathcal{O}_Y - \mathfrak{p}\otimes\mathcal{O}_Y = (1-\mathfrak{p})\cdot\mathcal{O}_Y$.

Deformation to the Normal Cone: Let $Y \to X$ be a closed embedding of ideal $\mathfrak{p}$. We put

$$\mathcal{O}_X[\mathfrak{p}] = \mathcal{O}_X \oplus \mathfrak{p} \oplus \ldots \oplus \mathfrak{p}^n \oplus \ldots$$
$$G_{\mathfrak{p}}\mathcal{O}_X = \mathcal{O}_X/\mathfrak{p} \oplus \mathfrak{p}/\mathfrak{p}^2 \oplus \ldots \oplus \mathfrak{p}^n/\mathfrak{p}^{n+1} \oplus \ldots$$

and we say that $C = \mathrm{Spec}\, G_{\mathfrak{p}}\mathcal{O}_X$ is the **normal cone** of Y in X.

The natural morphism $G_{\mathfrak{p}}\mathcal{O}_X \to \mathcal{O}_X/\mathfrak{p}$ defines the zero section $Y \to C$.

If the embedding $Y \to X$ is regular, then (p. 427) we have $G_{\mathfrak{p}}\mathcal{O}_X = S^\bullet(\mathfrak{p}/\mathfrak{p}^2)$, and C is the **normal bundle** $N_{Y/X} \to Y$ defined by the locally free $\mathcal{O}_Y$-module $(\mathfrak{p}/\mathfrak{p}^2)^*$.

The **blowup** $\bar{X} = \mathrm{Proj}\,\mathcal{O}_X[\mathfrak{p}] \to X$ of X along Y is an isomorphism over $X - Y$, and the fibre over Y is just $\mathrm{Proj}\,\mathcal{O}_X[\mathfrak{p}]/\mathfrak{p}\mathcal{O}_X[\mathfrak{p}] = \mathrm{Proj}\, G_{\mathfrak{p}}\mathcal{O}_X = \mathbb{P}(C)$.

The natural morphism $(G_{\mathfrak{p}}\mathcal{O}_X)[x_0] \to G_{\mathfrak{p}}\mathcal{O}_X$ defines a closed embedding

$$\mathbb{P}(C) \to \mathbb{P}(1\oplus C) = \mathrm{Proj}\,(G_{\mathfrak{p}}\mathcal{O}_X)[x_0],$$

with complement $U_{x_0} = C$, and we say that $\mathbb{P}(1\oplus C)$ is the **projective closure** of C.

Let $\bar{Z}$ be the blowup of $X \times_k \mathbb{A}_1$ along $Y \times 0$, and $\pi\colon \bar{Z} \to \mathbb{A}_1$ the natural morphism.

$$\begin{aligned} \pi^{-1}(\mathbb{A}_1 - 0) &= X \times_k (\mathbb{A}_1 - 0) \quad \text{since } \bar{Z} \to X\times_k \mathbb{A}_1 \text{ is an isomorphism out of } Y\times 0, \\ \pi^{-1}(0) &= \mathrm{Proj}\,\mathcal{O}_{X\times\mathbb{A}_1}[\mathfrak{p}+(t)]/t\mathcal{O}_{X\times\mathbb{A}_1}[\mathfrak{p}+(t)] = \mathrm{Proj}\,(\mathcal{O}_X[\mathfrak{p}]\otimes_{\mathcal{O}_X}[t])/(\mathfrak{p}t) \\ &= \mathrm{Proj}\,(\mathcal{O}_X[\mathfrak{p}]\otimes_{\mathcal{O}_X}[t])/(\mathfrak{p}) \cup \mathrm{Proj}\,(\mathcal{O}_X[\mathfrak{p}]\otimes_{\mathcal{O}_X}[t])/(t) \\ &= \mathrm{Proj}\,(G_{\mathfrak{p}}\mathcal{O}_X)\otimes_{\mathcal{O}_Y}\mathcal{O}_Y[t] \cup \mathrm{Proj}\,\mathcal{O}_X[\mathfrak{p}] = \mathbb{P}(1\oplus C)\cup\bar{X}. \end{aligned}$$

Moreover, $\mathbb{P}(1\oplus C)\cap\bar{X} = \mathbb{P}(C)$. If we remove $\bar{X}$, the fibre over 0 is the normal cone C.

Hence, if we put $Z = \bar{Z} - \bar{X}$, we have a commutative triangle (an algebraic variant of the Tubular Neighborhood Lemma) where π is flat and j is a closed embedding,

$$\begin{array}{ccc} Y\times\mathbb{A}_1 & \xrightarrow{\ j\ } & Z \\ & \searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathbb{A}_1 & \end{array}$$

1. $\pi^{-1}(\mathbb{A}_1 - 0) = X\times(\mathbb{A}_1 - 0)$, and $j\colon Y\times(\mathbb{A}_1-0) \to X\times(\mathbb{A}_1-0)$ is the obvious morphism.

2. $\pi^{-1}(0) = C$, and the embedding $j\colon Y\times 0 \to C$ is the zero section.

Lemma: *If $i\colon Y\to X$ is a regular closed embedding, then $i^!F^p(X)\subseteq F^p(Y)$, so that $i^!$ induces a morphism $i^*\colon GK^\bullet(X)\to GK^\bullet(Y)$.*

Proof: $j_1\colon Y\to Z$ is the composition of $i\colon Y\to X$, the section $X\times 1\to X\times(\mathbb{A}_1-0)$, and the open embedding $X\times(\mathbb{A}_1-0)\to Z$. Since the maps $F^p(Z)\to F^p(X\times(\mathbb{A}_1-0))\to F^p(X)$ are surjective, we are reduced to prove the lemma for $j_1^!\colon K(Z)\to K(Y)$.

But the morphisms $j_1^!, j_0^!\colon K(Z)\to K(Y\times\mathbb{A}_1)\rightrightarrows K(Y)$ coincide since both morphisms $K(Y\times\mathbb{A}_1)\rightrightarrows K(Y)$ are just the inverse of the isomorphism $\pi^!\colon K(Y)\xrightarrow{\sim} K(Y\times\mathbb{A}_1)$.

Hence we are reduced to prove the lemma for $j_0^!: K(Z) \to K(Y)$; but $j_0: Y \to Z$ is the composition of the zero section $Y \to N_{Y/X}$ with the closed embedding $N_{Y/X} \to Z$, defined by a locally principal ideal, and the lemma holds for both morphisms.

Definition: An algebraic variety X of dimension d over a field k is **smooth** if the diagonal $\Delta: X \to X \times_k X$ is a regular embedding of codimension d.

Theorem: *If X is a smooth variety, the product of $K(X)$ is compatible with the filtration, $F^p(X) \cdot F^q(X) \subseteq F^{p+q}(X)$, and it induces a ring structure on $GK^\bullet(X)$.*

If $f: Y \to X$ is a morphism of smooth varieties, then $f^!(F^p(X)) \subseteq F^p(Y)$, so that $f^!$ induces a ring morphism $f^*: GK^\bullet(X) \to GK^\bullet(Y)$.

Proof: If Y, Z are integral subvarieties of codimension p, q, we must show that $Y \cdot Z \in F^{p+q}(X)$.

If $\Delta: X \to X \times_k X$ is the diagonal morphism, then $Y \cdot Z = \Delta^!(Y \times_k Z)$ since the formula $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{L} = \Delta^*(\mathcal{L}' \otimes_k \mathcal{L})$ is immediate for locally free sheaves, which generate $K(X)$.

Since $Y \times Z$ is of codimension $p+q$ and Δ is a regular embedding, the lemma let us conclude.

Finally, $f: Y \to X$ is the composition of the graph $1 \times f: Y \to Y \times_k X$ with the projection $\pi: Y \times_k X \to X$, and both $(1 \times f)^!$ and $\pi^!$ are compatible with the filtrations.

The first since $1 \times f$ is a regular closed embedding, and the second since the codimension of $Y \times_k Z$ in $Y \times_k X$ coincides with the codimension of Z in X.

Definition: Let X be a smooth variety. The **Chern classes** $c_i(E) \in GK^i(X)$ of a locally free $\mathcal{O}_X$-module E of rank r are the coefficients of the unique relation

$$x_E^r + c_1(E) x_E^{r-1} + \ldots + c_r(E) = 0$$

in $GK^\bullet(\mathbb{P}(E))$, and the Chern classes of X are those of the tangent bundle $T_X = (\Omega_X^1)^*$.

The proofs given in the topological case (p. 375), but now $\xi_E = \mathcal{O}_{\mathbb{P}(E)}(-1)$, show that the Chern class of a line sheaf is $c_1(L) = -[1 - L] = [1 - L^*]$, that Chern classes are functorial, $c_i(f^*E) = f^*c_i(E)$, that the total class $c(E) = 1 + c_1(E) + \ldots + c_r(E)$ satisfies Whitney's formula, and that the last Chern class is $c_r(E) = s^*[s_{0*}(1)]$ for any global section $s: X \to E$.

1. If $\pi: X \to \operatorname{Spec} k$ is a projective smooth variety, we have a **degree** morphism
$$\deg = \pi_*: GK^d(X) = F^d(X) \longrightarrow \mathbb{Z},\ \deg[\mathcal{M}] = \chi(X, \mathcal{M}),$$
where $d = \dim X$; hence a pairing $GK^p(X) \otimes_{\mathbb{Z}} GK^{d-p}(X) \to \mathbb{Z}$, $\langle z_1, z_2 \rangle = \deg(z_1 z_2)$, defining global intersection numbers of cycles of complementary codimension,
$$\langle [\mathcal{M}], [\mathcal{N}] \rangle = \textstyle\sum_i (-1)^i \chi(X, \underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})).$$

2. We have a ring isomorphism $GK(\mathbb{P}_d) = \mathbb{Z}[x]/(x^{d+1})$, where $x = [1 - \mathcal{O}(-1)] \in GK^1(\mathbb{P}_d)$, and the exact sequence $0 \to \Omega_{\mathbb{P}_d}^1 \to \mathcal{O}_{\mathbb{P}_d}(-1)^{d+1} \to \mathcal{O}_{\mathbb{P}_d} \to 0$ shows that $c_i(\mathbb{P}_d) = \binom{d+1}{i} x^i$.

3. If E is a locally free $\mathcal{O}_X$-module of rank r, then $c_1(E) = c_1(\Lambda^r E)$ and $c_i(E^*) = (-1)^i c_i(E)$.

4. $\deg c_n(X) = \sum_{p,q} (-1)^{p+q} \dim H^p(X, \Omega_X^q)$ for any smooth projective variety X of dimension n, and $c_d(N_{Y/X}) = i^*(i_*(1))$ for any smooth closed subvariety $i: Y \to X$ of codimension d.

   In fact, the Koszul complex shows that $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \Lambda^i(\mathfrak{p}_Y/\mathfrak{p}_Y^2)$; hence
$$i^*(i_*(1)) = \textstyle\sum_i (-1)^i \underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \sum_i (-1)^i \Lambda^i(\mathfrak{p}_Y/\mathfrak{p}_Y^2) = c_d((\mathfrak{p}_Y/\mathfrak{p}_Y^2)^*).$$

15.4.2 Cohomology theories and Chern Classes

Jouanolou's Trick: *Let X be a* ***quasi-projective*** *k-variety (an open subscheme of a projective k-scheme). There exists an affine bundle $\pi\colon P \to X$ such that P is an affine scheme. Hence, for any line sheaf L on X, we have $\pi^*L \simeq f^*\mathcal{O}_{\mathbb{P}_n}(1)$ for some k-morphism $f\colon P \to \mathbb{P}_n$.*

Proof: The incidence divisor H in $\mathbb{P}(E)\times\mathbb{P}(E^*)$, of equation $\sum_i u_i x_i = 0$, is an hyperplane section of the natural embedding $\mathbb{P}(E)\times\mathbb{P}(E^*) \to \mathbb{P}(E\otimes E^*)$; hence the complement U is an affine scheme, and the natural projection $\pi\colon U \to \mathbb{P}(E^*)$ is an affine bundle of associated vector bundle $\mathcal{O}(-1)\otimes\mathcal{O}(-1)^{\mathrm{o}}$. (See p. 164.)

Alternative Proof: The rank 1 idempotent endomorphisms, $T^2 = T$ and $c_T(x) = x^n(x-1)$, define a closed (hence affine) subscheme P of $\mathrm{End}_k E$ such that the morphism $\pi\colon P \to \mathbb{P}(E)$, $\pi(T) = \mathrm{Im}\, T$, is an affine bundle of associated vector bundle $\underline{\mathrm{Hom}}(E/\mathcal{O}(-1), \mathcal{O}(-1))$.

If $X \to \mathbb{P}(E)$ is a closed subscheme, then $P_X = \pi^{-1}(X) \to X$ is the required affine bundle.

Now, if $U \to X$ is an open subscheme, blowing up the complement, we may assume that $Y = X - U$ is defined by a locally principal ideal; hence so is the ideal of P_Y in P_X, so that $P_U \to P_X$ is an affine morphism, and we see that P_U is an affine scheme.

Finally, π^*L is generated (as any coherent sheaf on an affine scheme) by a finite number of global sections, and we have $\pi^*L \simeq f^*\mathcal{O}_{\mathbb{P}_n}(1)$ by the universal property of $\mathbb{P}_n$.

Definition: A **cohomology theory** is a contravariant functor A from the category of smooth quasi-projective k-varieties into the category of commutative rings, endowed with a functorial morphism of $A(X)$-modules $f_*\colon A(Y) \to A(X)$ for projective morphisms $f\colon Y \to X$ ($\mathrm{Id}_* = \mathrm{Id}$, $(fg)_* = f_*g_*$ and the projection formula $f_*(f^*(x)y) = xf_*(y)$ holds); hence a Chern class $c_1^A(L) = s_0^*(s_{0*}(1)) \in A(X)$ of line bundles $L \to X$ (where $s_0\colon X \to L$ is the zero section), and a cohomology class $[Y]^A = i_*(1) \in A(X)$ of smooth closed subvarieties $i\colon Y \to X$, such that

1. $i_1^* + i_2^*\colon A(X_1 \oplus X_2) \to A(X_1)\oplus A(X_2)$ is an isomorphism; hence $A(\emptyset) = 0$.
2. $\pi^*\colon A(X) \to A(P)$ is an isomorphism for any affine bundle $\pi\colon P \to X$.

   Hence $c_1(L) = s^*(s_{0*}(1))$ for any section s of a line bundle L, because $s^* = s_0^*$, both being the inverse of $\pi^*\colon A(X) \to A(L)$.
3. For any smooth closed subvariety $i\colon Y \to X$, with complement $j\colon U \to X$, we have an exact sequence
$$A(Y) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(U)$$
4. If $E \to X$ is a vector bundle of rank $r+1$, and we put $y_E = c_1^A(\mathcal{O}_{\mathbb{P}(E)}(-1))$, then
$$A(\mathbb{P}(E)) = A(X) \oplus A(X)y_E \oplus \ldots \oplus A(X)y_E^r.$$
5. For any projective bundle $\pi\colon \mathbb{P}(E) \to X$ and any morphism $f\colon Y \to X$, we have a commutative square
$$\begin{array}{ccc} A(\mathbb{P}(E)) & \xrightarrow{f^*} & A(\mathbb{P}(f^*E)) \\ \downarrow{\scriptstyle\pi_*} & & \downarrow{\scriptstyle\pi_*} \\ A(X) & \xrightarrow{f^*} & A(Y) \end{array}$$
6. If $f\colon \bar{X} \to X$ is transversal to a smooth closed subvariety $i\colon Y \to X$, in the sense that $\bar{Y} = Y \times_X \bar{X}$ is smooth and the natural morphism $f^*N_{Y/X} \to N_{\bar{Y}/\bar{X}}$ is an isomorphism, we have a commutative square (in particular, $f^*[Y] = 0$ when $f^{-1}(Y) = \emptyset$)

$$\begin{array}{ccc} A(Y) & \xrightarrow{f^*} & A(\bar{Y}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ A(X) & \xrightarrow{f^*} & A(\bar{X}) \end{array}$$

and a **morphism** of cohomology theories $\mathrm{ch}\colon A \to \bar{A}$ is a natural transformation preserving direct images. That is to say, $\mathrm{ch}\colon A(X) \to \bar{A}(X)$ is a ring morphism preserving inverse images, $\mathrm{ch}(f^*(a)) = f^*(\mathrm{ch}(a))$, and direct images, $\mathrm{ch}(f_*(a)) = f_*(\mathrm{ch}(a))$. Remark that:

1. Both i_1, i_2 are transversal to i_1, i_2; hence $i_2^* i_{1*} = 0$, $i_1^* i_{2*} = 0$, $i_1^* i_{1*} = \mathrm{Id}$, $i_2^* i_{2*} = \mathrm{Id}$, so that the inverse of $i_1^* + i_2^*\colon A(X_1 \oplus X_2) \xrightarrow{\sim} A(X_1) \oplus A(X_2)$ is just $i_{1*} + i_{2*}$. Hence $f_* = f_{1*} + f_{2*}\colon A(X_1) \oplus A(X_2) \to A(S)$ when $f = f_1 \oplus f_2\colon X_1 \oplus X_2 \to S$ is projective.

2. The zero section $s_0\colon X \to L$ of a line bundle L is transversal to the natural morphism $f^*L \to L$, for any morphism of schemes $f\colon \bar{X} \to X$; hence $f^*c_1(L) = c_1(f^*L)$.

3. In the projective bundle $\pi\colon \mathbb{P}(E) \to X$ of a locally free $\mathcal{O}_X$-module of rank r we have that $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is a submodule of π^*E, and $\bar{E} = (\pi^*E)/\mathcal{O}(-1)$ is locally free of rank $r-1$. Since $\pi^*\colon A(X) \to A(\mathbb{P}(E))$ is injective by condition 4, by induction on r we obtain the **Splitting Principle:** *There is a base change $\pi\colon \bar{X} \to X$ such that $\pi^*\colon A(X) \to A(\bar{X})$ is injective and $\pi^*E = L_1 + \ldots + L_r$ is a sum of line sheaves in $K(\bar{X})$.*

4. Given a smooth closed hypersurface $Y \to X$, the line sheaf L_Y admits a section transversal to the zero section and vanishing at Y. Hence $c_1(\mathcal{O}_X) = 1$, and $c_1(L_Y) = [Y]$.

Examples: Let L_x be a line sheaf of Chern class x. We say that A follows the additive law $x+y$ when $c_1(L_x \otimes L_y) = x+y$, and the multiplicative[2] law $x+y-xy$ when $c_1(L_x \otimes L_y) = x+y-xy$.

1. The K-theory $K(X)$ is a cohomology theory. Condition 6 holds since Y is locally defined by a regular sequence which is a regular sequence in $\bar{X}$, so that for any locally free $\mathcal{O}_Y$-module $\mathcal{L}$ we have $\underline{\mathrm{Tor}}_n^{\mathcal{O}_X}(\mathcal{O}_{\bar{X}}, \mathcal{L}) = 0,\ n \geq 1$.

   The cohomology class of a smooth closed subvariety $i\colon Y \to X$ is $i_!(1) = \mathcal{O}_Y \in K(X)$.

   If L is a line $\mathcal{O}_X$-module, then $c_1^K(L) = s_0^!(s_{0!}(1)) = 1 - L^* \in K(X)$. The K-theory follows the law $x+y-xy$ since $1 - (L' \otimes L) = (1-L') + (1-L) - (1-L')(1-L)$.

2. The rational graded K-theory $GK^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a cohomology theory, but we shall not prove it, since the proofs of condition 3 that we know involve the theory of the Chow ring, that we deliberately avoid in these notes.

   The Chern class of a line bundle L is $c_1^{GK}(L) = [1 - L^*] \in GK^1(X)$, and the theory follows the additive law $x+y$ because $(1-L')(1-L) \in F^2(X)$.

3. In the complex case, $A(X) = H^{2\bullet}(X_{\mathrm{an}}, \mathbb{Q}) = \oplus_p H^{2p}(X_{\mathrm{an}}, \mathbb{Q})$ defines a cohomology theory following the additive law $x+y$ (where X_{an} denotes the topological space of all closed points of X, with the coarsest topology such that the maps $f\colon U_{\mathrm{an}} \to \mathbb{C}$, $f \in \mathcal{O}_X(U)$, are continuous). For an arbitrary projective morphism $f\colon Y \to X$, the projection formula states that the direct image f_* may defined to be the adjoint of the inverse image $f^*\colon H_c^{2\bullet}(X_{\mathrm{an}}, \mathbb{Q}) \to H_c^{2\bullet}(Y_{\mathrm{an}}, \mathbb{Q})$ by Poincaré's duality. Condition 6 holds since the morphism $f^*\colon f^*\mathbb{T}_{Y/X} \to \mathbb{T}_{\bar{Y}/\bar{X}}$ defined by the inverse image is an isomorphism (use the argument of p. 365, showing that the multiplicity is ± 1 in any transversal intersection).

[2]The group law of the multiplicative group, since we have $(1-x)(1-y) = 1 - (x+y-xy)$.

4. If A is a cohomology theory defined on the k-varieties, so is $X \rightsquigarrow A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and we denote it $A \otimes \mathbb{Q}$. Moreover, for any field extension $k_0 \to k$ we have that $X \rightsquigarrow A(X_k)$ is a cohomology theory on the k_0-varieties.

Definition: The **Chern classes** of a locally free $\mathcal{O}_X$-module E of rank r are the coefficients $c_i^A(E)$, or simply $c_i(E)$, of the characteristic polynomial $c_E(y)$ of the endomorphism of the free $A(X)$-module $A(\mathbb{P}(E))$ defined by the product with $y_E = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$,

$$c_E(t) = y^r - c_1^A(E)y^{r-1} + \ldots + (-1)^r c_r^A(E) \in A(X)[t],$$

up to a sign that we introduce so that the first Chern class of any line sheaf L coincides with the previous one. In fact, $\mathbb{P}(L) = X$ and $\mathcal{O}_{\mathbb{P}(L)}(-1) = L$.

Functoriality: *$c_i(f^*E) = f^*(c_i(E))$ for any morphism $f\colon \bar{X} \to X$.*

Proof: We have $f^*c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) = c_1(f^*\mathcal{O}_{\mathbb{P}(E)}(-1)) = c_1(\mathcal{O}_{\mathbb{P}(f^*E)}(-1))$.

Additivity: *For any exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of locally free $\mathcal{O}_X$-modules,*

$$c_n(E) = \sum_{i+j=n} c_i(E_1) \cdot c_j(E_2).$$

Proof: If $\operatorname{rk} E_1 = 1$, then $i\colon X = \mathbb{P}(E_1) \to \mathbb{P}(E)$ is a section of $\mathbb{P}(E) \to X$ and i_* is injective.

Let us consider the complement $j\colon U \to \mathbb{P}(E)$ of $\mathbb{P}(E_1)$. The map $U \to \mathbb{P}(E_2)$ is an affine bundle of the vector bundle $\underline{\operatorname{Hom}}(\mathcal{O}_{\mathbb{P}(E_2)}(-1), E_1)$, so that $j^*\colon A(\mathbb{P}(E)) \to A(U) = A(\mathbb{P}(E_2))$ is surjective (in fact, $j^*(y_E^n) = y_{E_2}^n$). Therefore we have a commutative diagram with exact rows (since $y_{E_1} = i^*y_E$, the first square commutes by the projection formula)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(\mathbb{P}(E_1)) & \xrightarrow{i_*} & A(\mathbb{P}(E)) & \xrightarrow{j^*} & A(\mathbb{P}(E_2)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle y_{E_1}} & & \downarrow{\scriptstyle y_E} & & \downarrow{\scriptstyle y_{E_2}} & & \\
0 & \longrightarrow & A(\mathbb{P}(E_1)) & \xrightarrow{i_*} & A(\mathbb{P}(E)) & \xrightarrow{j^*} & A(\mathbb{P}(E_2)) & \longrightarrow & 0
\end{array}$$

and the additivity of the characteristic polynomial shows that $c_E(y) = c_{E_1}(y)c_{E_2}(y)$.

Now we proceed by induction on $\operatorname{rk} E_1$, and by the splitting principle we may assume the existence of a line sheaf $L \subset E_1$ such that $\bar{E}_1 = E_1/L$ and $\bar{E} = E/L$ are locally free.

We have an exact sequence $0 \to \bar{E}_1 \to \bar{E} \to E_2 \to 0$ and we conclude,

$$c_E(y) = c_L(y)c_{\bar{E}}(y) = c_L(y)c_{\bar{E}_1}(y)c_{E_2}(y) = c_{E_1}(y)c_{E_2}(y).$$

Definition: Since $1 + c_1(E)t + \ldots + c_r(E)t^r$ is an additive function with values in the multiplicative group of invertible formal series with coefficients in $A(X)$, it induces a group morphism on $K(X)$, so defining **Chern classes** $c_i(x) \in A(X)$ for every $x \in K(X)$.

Corollary: *The cohomology ring of the projective space is $A(\mathbb{P}_d) = A(\operatorname{Spec} k)[x]/(x^{d+1})$, where x corresponds to the cohomology class $x_d = [\mathbb{P}_{d-1}]$ of any hyperplane.*

Proof: Put $y_d = c_1(\mathcal{O}_{\mathbb{P}_d}(-1))$. By additivity, trivial bundles have null Chern classes.

Hence in $A(\mathbb{P}_d) = A(\operatorname{Spec} k) \oplus A(\operatorname{Spec} k)y_d \oplus \ldots \oplus A(\operatorname{Spec} k)y_d^d$ we have $y_d^{d+1} = 0$.

In $A(\mathbb{P}_1)$, the exact sequence $0 \to \mathcal{O}_{\mathbb{P}_1}(-1) \to \mathcal{O}_{\mathbb{P}_1}^2 \to \mathcal{O}_{\mathbb{P}_1}(1) \to 0$ shows that $x_1 = -y_1$.

Considering a line in $\mathbb{P}_d$, we see that $x_d = -y_d + a_2y_d^2 + \ldots + a_dy_d^d$ in $A(\mathbb{P}_d)$.

We conclude that $x_d^{d+1} = 0$ in $A(\mathbb{P}_d)$.

Corollary: *Chern classes are always nilpotent.*

Proof: Let $L \to X$ be a line bundle. By Jouanolou's trick there is an affine bundle $\pi\colon P \to X$ such that $\pi^* L = f^*(\mathcal{O}(1))$ for some morphism $f\colon P \to \mathbb{P}_d$. Therefore $\pi^* c_1(L) = f^*(x_d)$ is nilpotent, since so is x_d, and $c_1(L)$ is nilpotent because π^* is an isomorphism.

We conclude since any Chern class, after a base change injective in cohomology, is a sum of products of Chern classes of line bundles.

Example: To compute the Chern classes in the K-theory of a locally free $\mathcal{O}_X$-module E of rank r, we may assume that it is a sum of line sheaves, $E = L_1 + \ldots + L_r \in K(X)$; hence,

$$\begin{aligned} c_1^K(E) &= \textstyle\sum_i c_1^K(L_i) = \sum_i (1 - L_i^*) = \operatorname{rk} E - E^*, \\ c_r^K(E) &= \textstyle\prod_i c_1^K(L_i) = \prod_i (1 - L_i^*) = 1 - E^* + \Lambda^2 E^* + \ldots + (-1)^r \Lambda^r E^*. \end{aligned} \tag{15.4}$$

Definition: Given a locally free $\mathcal{O}_X$-module E, there is a base change $\pi\colon \bar{X} \to X$ such that $\pi^*\colon A(X) \to A(\bar{X})$ is injective and $\pi^* E = L_{\alpha_1} + \ldots + L_{\alpha_r}$ is a sum of line sheaves in $K(\bar{X})$, so that $c_i(E)$ is the i-th elementary symmetric function of the "roots" $\alpha_1, \ldots, \alpha_r$. For any formal series $F(t) = \sum_n a_n t^n$ with coefficients in $A(\operatorname{Spec} k)$ we put

$$F_+(E) = F(\alpha_1) + \ldots + F(\alpha_r) \in A(X),$$

(where we view the sum as a power series in the elementary symmetric functions $c_i(E)$, and the sum is finite because Chern classes are nilpotent) and F_+ is an additive function, so defining a functorial group morphism $F_+\colon K(X) \to A(X)$ named **additive extension** of F. Analogously, when a_0 is invertible, we have a **multiplicative extension** $F_\times\colon K(X) \to A(X)^*$ such that

$$F_\times(E) = F(\alpha_1) \cdot \ldots \cdot F(\alpha_r) \in A(X)^*.$$

15.4.3 Grothendieck's Riemann-Roch Theorem

Universal Property: *If a cohomology theory A follows the group law $x+y-xy$ of the K-theory, then there is a unique morphism of cohomology theories* $\operatorname{ch}\colon K \to A$.

Proof: If E is a locally free $\mathcal{O}_X$-module, by 15.4 we have $E = \operatorname{rk} E - c_1^K(E^*)$ in $K(X)$.

Hence the unique possible morphism of cohomology theories $\operatorname{ch}\colon K(X) \to A(X)$ is

$$\operatorname{ch}(E) = \operatorname{rk} E - c_1^A(E^*). \tag{15.5}$$

This function ch is additive on the locally free $\mathcal{O}_X$-modules, since so are the rank and c_1^A; hence it defines a group morphism $\operatorname{ch}\colon K(X) \to A(X)$, and it commutes with inverse images because so do the rank and c_1^A.

It preserves products of line bundles because A follows the law $x + y - xy$:

$$\begin{aligned} \operatorname{ch}(L_1 \cdot L_2) &= 1 - c_1^A(L_1^* \otimes L_2^*) = 1 - c_1^A(L_1^*) - c_1^A(L_2^*) + c_1^A(L_1^*) c_1^A(L_2^*) \\ &= \big(1 - c_1^A(L_1^*)\big)\big(1 - c_1^A(L_2^*)\big) = \operatorname{ch}(L_1) \cdot \operatorname{ch}(L_2). \end{aligned}$$

Now, by the splitting principle, ch preserves arbitrary products.

We only have to prove that ch preserves direct images, and the theorem follows from the following lemma, since ch preserves Chern classes of line sheaves,

$$\operatorname{ch}\big(c_1^K(L)\big) = \operatorname{ch}(1 - L^*) = 1 - \operatorname{ch}(L^*) = 1 - (1 - c_1^A(L)) = c_1^A(L).$$

Panin's Lemma: *If a functorial ring morphism* $\mathrm{ch}\colon A \to \bar{A}$ *between cohomology theories preserves the cohomology class of hyperplanes,* $\mathrm{ch}[c_1^A(\mathcal{O}_{\mathbb{P}_d}(1))] = c_1^{\bar{A}}(\mathcal{O}_{\mathbb{P}_d}(1))$*, then it preserves direct images; i.e., for any projective morphism* $f\colon Y \to X$*,*

$$\mathrm{ch}(f_*(a)) = f_*(\mathrm{ch}(a)) \ , \quad a \in A(Y).$$

Proof: By Jouanolou's trick, the natural transformation ch preserves Chern classes of line sheaves; hence it also preserves cohomology classes of smooth hypersurfaces.

By definition, any projective morphism $f\colon Y \to X$ is the composition of a closed embedding $Y \to \mathbb{P}_n \times X$ with the natural projection $\pi_2\colon \mathbb{P}_n \times X \to X$.

If the lemma holds for two morphisms, so it holds for the composition; hence we must prove the lemma for a closed embedding $i\colon Y \to X$ and the projection $\pi_2\colon \mathbb{P}_n \times X \to X$.

1. *If the lemma holds for the zero section* $s\colon Y \to \bar{N} = \mathbb{P}(1 \oplus N_{Y/X})$ *of the projective closure of the normal bundle, then it holds for* $i\colon Y \to X$.

   The deformation $\tilde{Z}$ to the normal cone, without removing the blowup $\tilde{X}$ of X (p. 427), gives a (commutative by condition 6) diagram, where $U = \tilde{Z} - (Y \times \mathbb{A}_1)$,

$$\begin{array}{ccc} & & \bar{A}(U) \\ & & \uparrow{\scriptstyle j^*} \\ \bar{A}(\bar{N}) & \xleftarrow{\ i_0^*\ } & \bar{A}(\tilde{Z}) \\ {\scriptstyle s_*}\uparrow & & {\scriptstyle i_*}\uparrow \\ \bar{A}(Y) & \underset{\sim}{\xleftarrow{\ i_0^*\ }} & \bar{A}(Y \times \mathbb{A}_1) \end{array}$$

   So we see that $(\operatorname{Ker} i_0^*) \cap (\operatorname{Ker} j^*) = 0$, because the column is exact by condition 3 and s_* is injective (since $p_* s_* = \mathrm{Id}$, where $p\colon \bar{N} \to Y$ is the natural projection). Now we have a commutative diagram, where the lemma states the coincidence of the vertical pairs,

$$\begin{array}{ccccc} & & \bar{A}(U) & & \\ & & \uparrow{\scriptstyle j^*} & & \\ \bar{A}(\bar{N}) & \xleftarrow{\ i_0^*\ } & \bar{A}(\tilde{Z}) & \xrightarrow{\ i_1^*\ } & \bar{A}(X) \\ \uparrow\uparrow & & \uparrow\uparrow & & \uparrow\uparrow \\ A(Y) & = & A(Y \times \mathbb{A}_1) & = & A(Y) \end{array}$$

   The difference of the first par is null by hypothesis; hence so is the difference of the central pair as we have seen, and we conclude that the difference of the last pair is null.

2. *If* L *is a line sheaf over* Y*, the lemma holds for the zero section* $s\colon Y \to \bar{L} = \mathbb{P}(1 \oplus L)$*; hence it holds for every closed embedding* $Y \to X$ *of codimension 1.*

   We have $\mathrm{ch}(s_*(1)) = s_*(1)$, because Y is an hypersurface in $\bar{L}$. Now, if $a \in A(Y)$, we may put $a = s^*b$ because $s^*\colon A(\bar{L}) \to A(Y)$ is surjective, and

$$\mathrm{ch}(s_*a) = \mathrm{ch}(s_*s^*b) = \mathrm{ch}(bs_*(1)) = \mathrm{ch}(b)s_*(1) = s_*(s^*\mathrm{ch}(b)) = s_*(\mathrm{ch}(a)).$$

3. *If* E *is a vector bundle on* Y*, the lemma holds for the zero section* $s\colon Y \to \bar{E} = \mathbb{P}(1 \oplus E)$*; hence it holds for every closed embedding* $Y \to X$*.*

   If E admits a filtration $\{E_i\}$ with quotients E_i/E_{i-1} of rank 1, the lemma holds for the zero section $Y \to \bar{E}_1$ and for the morphisms $\bar{E}_1 \to \bar{E}_2 \to \ldots \to \bar{E}_r = \bar{E}$; hence it holds for the composition $s\colon Y \to \bar{E}$.

   In general, we have a morphism $\pi\colon Y' \to Y$ such that π^* is injective and $E' = \pi^*E$ admits such filtration. The lemma holds for the zero section $s'\colon Y' \to \bar{E}'$, and we conclude applying condition 6 to the morphisms $\pi\colon \bar{E}' \to \bar{E}$ and $s\colon Y \to \bar{E}$,

$$\pi^*s_*(\mathrm{ch}(a)) = s'_*\pi^*\mathrm{ch}(a) = s'_*\mathrm{ch}(\pi^*a) = \mathrm{ch}(s'_*\pi^*a) = \mathrm{ch}(\pi^*s_*a) = \pi^*\mathrm{ch}(s_*a).$$

4. *If the lemma holds for $\mathbb{P}_n \to \operatorname{Spec} k$, so it does for $\pi_2 \colon \mathbb{P}_n \times X \to X$,* by condition 5.

5. *The lemma holds for the projection $p\colon \mathbb{P}_n \to \operatorname{Spec} k$ onto a point, and we conclude.*

Proof: We consider the embedding $i\colon \mathbb{P}_{n-1} \to \mathbb{P}_n$, and we put

$$A = A(\operatorname{Spec} k),\ x_n = c_1(\mathcal{O}_{\mathbb{P}_n}(1)) = i_*(1) \in A(\mathbb{P}_n)$$
$$\bar{A} = \bar{A}(\operatorname{Spec} k),\ \bar{x}_n = \bar{c}_1(\mathcal{O}_{\mathbb{P}_n}(1)) = \bar{i}_*(1) \in \bar{A}(\mathbb{P}_n)$$

By hypothesis $\operatorname{ch}(x_n) = \bar{x}_n$; hence $\operatorname{ch}(x_n^r) = \bar{x}_n^r$, and the ring morphism $\operatorname{ch}\colon A(\mathbb{P}_n) \to \bar{A}(\mathbb{P}_n)$ induces an isomorphism of $\bar{A}$-algebras $A(\mathbb{P}_n) \otimes_A \bar{A} = \bar{A}(\mathbb{P}_n)$. We have to prove that the 1-form $\bar{p}_*\colon \bar{A}(\mathbb{P}_n) \to \bar{A}$ is just the base change of the 1-form $p_*\colon A(\mathbb{P}_n) \to A$.

Now, if we consider the cohomology class $\Delta_n = \Delta_*(1) \in A(\mathbb{P}_n \times \mathbb{P}_n) = A(\mathbb{P}_n) \otimes_A A(\mathbb{P}_n)$ of the diagonal embedding $\Delta\colon \mathbb{P}_n \to \mathbb{P}_n \times \mathbb{P}_n$, we have

$$(p_* \otimes 1)(\Delta_n) = \pi_*\Delta_*(1) = \operatorname{Id}_*(1) = 1,$$

where $\pi\colon \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$ stands for the second projection. That is to say, by means of the polarity $\omega \mapsto (\omega \otimes 1)(\Delta_n)$ defined by the diagonal, p_* corresponds to the unity.

According to the following proposition, this equality fully determines the 1-form p_*, and since the cohomology class of the diagonal is stable under the base change $A \to \bar{A}$ (because the lemma holds for the diagonal embedding), we conclude that p_* also is stable.

Proposition: *The polarity $A(\mathbb{P}_n)^* \to A(\mathbb{P}_n)$, $\omega \mapsto (\omega \otimes 1)(\Delta_n)$, is an isomorphism.*

Proof: By induction on n we shall prove that

$$\Delta_n = \sum_{r,s=0}^{n} a_{rs} x_n^r \otimes x_n^s = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \diagup & \bullet \\ 0 & \diagup & \iddots & \vdots \\ 1 & \bullet & \cdots & \bullet \end{pmatrix}$$

where $a_{rs} = 0$ when $r + s < n$, and $a_{rs} = 1$ when $r + s = n$. In fact,

$$i_*(x_{n-1}^r) = i_* i^*(x_n^r) = x_n^r \cdot i_*(1) = x_n^{r+1},$$

and by condition 6 we have that $(i^* \otimes 1)(\Delta_n)$ is the cohomology class in $\mathbb{P}_{n-1} \times \mathbb{P}_n$ of the diagonal of $\mathbb{P}_{n-1}$. Now, if we put $\Delta_{n-1} = \sum_{rs} a'_{rs} x_{n-1}^r \otimes x_{n-1}^s$, then

$$(i^* \otimes 1)(\Delta_n) = \textstyle\sum_{r,s} a_{rs} x_{n-1}^r \otimes x_n^s$$
$$(1 \otimes i_*)(\Delta_{n-1}) = \textstyle\sum_{r,s} a'_{rs} x_{n-1}^r \otimes x_n^{s+1}$$

By induction, we obtain the stated result for the coefficients a_{rs}, $r < n$.

By symmetry it also holds for a_{rs}, $s < n$, and we conclude. q.e.d.

Now, *given the functor A, the direct image may be modified:* Let $F_\times$ be the multiplicative extension of an invertible series $F = a_0 + a_1 t + \dots$. For any projective morphism $f\colon Y \to X$ we consider the virtual **relative tangent bundle** $T_f = T_Y - f^! T_X \in K(Y)$ and we put

$$f_*^{\text{new}}(a) = f_*\big(F_\times(-T_f)\, a\big) = F_\times(T_X) f_*\big(F_\times(T_Y)^{-1} a\big) \in A(X) \qquad (15.6)$$

so that the following square (with vertical isomorphisms) is commutative:

$$\begin{array}{ccc} A(Y) & \xrightarrow{f_*} & A(X) \\ {\scriptstyle \cdot F_\times(T_Y)}\downarrow\wr & & \wr\downarrow{\scriptstyle \cdot F_\times(T_X)} \\ A(Y) & \xrightarrow{f_*^{\rm new}} & A(X) \end{array}$$

Lemma: $(A, f_*^{\rm new})$ *is also a cohomology theory, and* $c_1^{\rm new}(L_x) = xF(x)$.

Proof: In the case of the zero section $s_0\colon X \to L_x$ of a line bundle, we have $-T_{s_0} = L_x$, and s_0^* is surjective; hence:

$$c_1^{\rm new}(L_x) = s_0^*(s_{0*}^{\rm new}(1)) = s_0^*[s_{0*}(F_\times(L_x))] = s_0^*[s_{0*}(F(x))] = F(x)\cdot s_{0*}(1) = F(x)\cdot x.$$

Moreover, all the conditions of a cohomology theory are easy to check, except condition 4:
Put $y = y_E \in A(\mathbb{P}(E))$ and $z = y_E^{\rm new} = yF(y) = a_0y + a_1y^2 + \ldots \in A(\mathbb{P}(E))$.
Then $z^n = a_0^n y^n + \ldots$ and we have $z^d = a_0^d y^d$ when $y^{d+1} = 0$.
Since the powers of y generate the $A(X)$-module $A(P(E))$, so do the powers of z.
Now, $A(P(E))$ is a free $A(X)$-module of rank $r+1$; hence $1, z, \ldots, z^r$ define a basis (consider the characteristic polynomial of the endomorphism of $A(\mathbb{P}(E))$ defined by z). q.e.d.

Now, if A follows the additive law $x+y$, we may modify the direct image of $A\otimes\mathbb{Q}$ with an exponential so that it follows the multiplicative law $x+y-xy$ of the K-theory.
Since $e^{ax} = 1-(1-e^{ax})$ and $1-e^{ax} = -ax+\ldots$, it is convenient to fix $a=-1$.
Hence we modify the direct image of $A\otimes\mathbb{Q}$ with the invertible series

$$F(t) = \frac{1-e^{-t}}{t} = \left(1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \frac{t^4}{5!} - \ldots\right)$$
$$c_1^{\rm new}(L_x) = xF(x) = 1-e^{-x}$$

and we obtain a new cohomology theory $(A\otimes\mathbb{Q})^{\rm new}$ following the $x+y-xy$ law. By the universal property of the K-theory we have a morphism of cohomology theories ${\rm ch}\colon K \to (A\otimes\mathbb{Q})^{\rm new}$ and, by 15.5 (pg. 449) it is just the additive extension of the series e^t, named **Chern character**,

$${\rm ch}(L_x) = 1 - c_1^{\rm new}(L_x^*) = 1 - c_1^{\rm new}(L_{-x}) = 1-(1-e^x) = e^x.$$

Grothendieck's Riemann-Roch Theorem: *If a cohomology theory A on the smooth quasi-projective varieties over a field follows the additive law $x+y$, then for any projective morphism $f\colon Y\to X$ we have the following commutative square, where the **Todd class*** Td *is the multiplicative extension of the formal series* $F(t)^{-1} = \frac{t}{1-e^{-t}} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \ldots$,

$$\begin{array}{ccc} K(Y) & \xrightarrow{f_!} & K(X) \\ {\scriptstyle {\rm Td}(T_Y)\cdot{\rm ch}}\downarrow & & \downarrow{\scriptstyle {\rm Td}(T_X)\cdot{\rm ch}} \\ A(Y)\otimes\mathbb{Q} & \xrightarrow{f_*} & A(X)\otimes\mathbb{Q} \end{array}$$

$${\rm Td}(T_X)\cdot{\rm ch}(f_!(y)) = f_*\big[{\rm Td}(T_Y)\cdot{\rm ch}(y)\big]$$

Proof: ${\rm ch}(f_!(y)) = f_*^{\rm new}({\rm ch}(y)) \overset{15.6}{=} F(T_X)f_*\big[F(T_Y)^{-1}{\rm ch}(y)\big]$.

Definition: A **graded cohomology theory** $A^\bullet$ is a cohomology theory with values in the category of graded commutative rings, $A^\bullet(X) = \bigoplus_{n\geq 0} A^n(X)$, such that for any projective morphism $f\colon Y \to X$ between connected varieties, the direct image $f_*\colon A^n(Y) \to A^{n+d}(X)$ changes the degree in the **codimension** $d = \dim X - \dim Y$.

Remark that elements of negative degree are assumed to be null, and that the cohomology class of any hypersurface has degree 1, so that $c_i(x) \in A^i(X)$ for any $x \in K(X)$.

For example, $GK^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H^{2\bullet}(X_{\text{an}}, \mathbb{Z})$ are graded cohomology theories.

Lemma: *Any graded cohomology theory $A^\bullet$ follows the additive law $c_1(L_x \otimes L_y) = x + y$.*

Proof: Since $A^\bullet(\mathbb{P}_m \times \mathbb{P}_n) = A^\bullet(\operatorname{Spec} k)[x_m, x_n]/(x_m^{m+1}, x_n^{n+1})$, there exist $a, b \in A^0(\operatorname{Spec} k)$ and $c \in A^1(\operatorname{Spec} k)$ such that $c_1\big(\pi_1^*\mathcal{O}_{\mathbb{P}_m}(1) \otimes_{\mathcal{O}_{\mathbb{P}_m \times \mathbb{P}_n}} \pi_2^*\mathcal{O}_{\mathbb{P}_n}(1)\big) = ac_1\big(\pi_1^*\mathcal{O}_{\mathbb{P}_m}(1)\big) + bc_1\big(\pi_2^*\mathcal{O}_{\mathbb{P}_n}(1)\big) + c$.

Restricting to $\mathbb{P}_m \times \text{pt}$, $\text{pt} \times \mathbb{P}_n$ and $\text{pt} \times \text{pt}$ we see that $a = b = 1$, $c = 0$.

By Jouanolou's trick there is an affine bundle $\pi\colon P \to X$ and a morphism $f\colon P \to \mathbb{P}_m \times \mathbb{P}_n$ such that $\pi^*L_x = f^*\pi_1^*\mathcal{O}_{\mathbb{P}_m}(1)$ and $\pi^*L_y = f^*\pi_2^*\mathcal{O}_{\mathbb{P}_n}(1)$.

Hence $c_1(\pi^*L_x \otimes \pi^*L_y) = c_1(\pi^*L_x) + c_1(\pi^*L_y)$, so that $\pi^*(c_1(L_x \otimes L_y)) = \pi^*x + \pi^*y$.

We conclude because $\pi^*\colon A^\bullet(X) \to A^\bullet(P)$ is an isomorphism.

Theorem: *If $A^\bullet$ is a graded cohomology theory on the smooth quasi-projective varieties over a perfect field, there is a natural homogeneous ring morphism $GK^\bullet(X) \to A^\bullet(X) \otimes \mathbb{Q}$ preserving inverse and direct images (hence Chern classes and cohomology classes).*

Proof: Let $i\colon Y \to X$ be a closed subvariety of codimension d.

If Y is smooth, $\operatorname{ch}(\mathcal{O}_Y) = [Y] + \ldots \in \bigoplus_{p \geq d} A^p(X) \otimes \mathbb{Q}$ by the Riemann-Roch theorem.

In general Y is smooth outside a closed set Y_{sing} of codimension $> d$, when the base field is perfect, and if we put $j\colon U = X - Y_{\text{sing}} \to X$, repeatedly applying condition 3 we see that we have injective morphisms (recall that $A^i(Z) = 0$ when $i < 0$)

$$j^*\colon A^p(X) \otimes \mathbb{Q} \longrightarrow A^p(U) \otimes \mathbb{Q}\ , \quad p \leq d.$$

Since $j^*(\operatorname{ch}(\mathcal{O}_Y)) = \operatorname{ch}(j^!\mathcal{O}_Y) = [Y \cap U] + \ldots$, we see that $\operatorname{ch}\colon K(X) \to A^\bullet(X) \otimes \mathbb{Q}$ preserves filtrations; hence it induces a ring morphism $\varphi\colon GK^\bullet(X) \to A^\bullet(X) \otimes \mathbb{Q}$, $\varphi([x]_d) = [\operatorname{ch}(x)]_d$ when $[x]_d \in GK^d(X)$, and it preserves inverse images since so does ch.

By the Riemann-Roch theorem, it preserves direct images, since for any projective morphism $f\colon Y \to X$ of codimension d and any element $y \in F^n(Y)$ we have

$$\begin{aligned}\varphi f_*([y]_n) &= \varphi[f_!(y)]_{n+d} = [\operatorname{ch}(f_!(y))]_{n+d} = [f_*(\operatorname{Td}(T_f)\operatorname{ch}(y))]_{n+d} \\ &= f_*[(1+\ldots)\operatorname{ch}(y)]_n = f_*[\operatorname{ch}(y)]_n = f_*\varphi([y]_n).\end{aligned}$$

1. The Chern character and Todd class of a locally free sheaf $E = L_{\alpha_1} + \ldots + L_{\alpha_r}$ are
$$\begin{aligned}\operatorname{ch}(E) &= \textstyle\sum_i e^{\alpha_i} = \operatorname{rk} E + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \ldots \\ \operatorname{Td}(E) &= \textstyle\prod_i (1 + \frac{1}{2}\alpha_i + \frac{1}{12}\alpha_i^2 - \frac{1}{720}\alpha_i^4 + \ldots) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \ldots\end{aligned}$$

2. If C is a smooth projective curve, then $K = c_1(\Omega_C)$ is the class of any canonical divisor, and the Riemann-Roch theorem for the projection $C \to \operatorname{Spec} k$ onto a point gives, for any locally free sheaf E of rank r,
$$\chi(C, E) = \deg\,[\operatorname{Td}(L_{-K})\operatorname{ch}(E)] = \deg\,[(1 - \tfrac{1}{2}K)(r + c_1(E))] = \deg\,(c_1 - \tfrac{r}{2}K).$$

3. If S is a smooth projective surface, and we put $K = c_1(\Omega_S^1)$ and $\chi_{\text{top}} = \deg c_2(S)$, then we have (when $E = \mathcal{O}_S$, we obtain **Noether's identity** $\chi(S, \mathcal{O}_S) = \frac{1}{12}(K^2 + \chi_{\text{top}})$):
$$\begin{aligned}\operatorname{Td}(T_S) &= 1 - \tfrac{1}{2}K + \tfrac{1}{12}(K^2 + c_2(S)), \\ \operatorname{ch}(E) &= \operatorname{rk} E + c_1(E) + \tfrac{1}{2}\big(c_1(E)^2 - 2c_2(E)\big), \\ \chi(S, E) &= \tfrac{\operatorname{rk} E}{12}(K^2 + \chi_{\text{top}}) - \tfrac{1}{2}K \cdot c_1 + \tfrac{1}{2}c_1^2 - \deg c_2.\end{aligned}$$

4. The Riemann-Roch theorem for a smooth closed hypersurface $i\colon Y \to X$ gives the **adjunction formula**, $i_*K_Y = Y(K_X + Y)$. Just compare terms of degree 2 in

$$i_*(\mathrm{Td}(T_Y)) = \mathrm{ch}(\mathcal{O}_Y)\cdot \mathrm{Td}(T_X) = \mathrm{ch}(1 - L_{-Y})\cdot \mathrm{Td}(T_X)$$
$$i_*(1) - \tfrac{1}{2}i_*K_Y + \ldots = (Y - \tfrac{1}{2}Y^2 + \ldots)(1 - \tfrac{1}{2}K_X + \ldots)$$

5. Let ω be a rational 1-form on a smooth projective surface S and let us consider the line sheaf $(\omega)(U) = \{f\omega \in \Omega_S(U)\colon f \in \Sigma_S\} = L_D(U)$, where D is the divisor of zeros and poles of ω. We have an exact sequence

$$(*) \qquad 0 \longrightarrow (\omega) \longrightarrow \Omega^1_S \xrightarrow{\wedge\omega} \Omega^2_S \otimes L_{-D} \longrightarrow \mathfrak{C} \longrightarrow 0$$

If x, y are parameters at a point p, and $\omega = h(f\mathrm{d}x + g\mathrm{d}y)$, where $f, g \in \mathcal{O}_p$ have no common factor, a local equation of D is $h = 0$, and $\mathfrak{C}_p \simeq \mathcal{O}_p/(f,g)$. Now, $\mathrm{ch}(\mathcal{O}_S/\mathfrak{m}_p) = p$ by the Riemann-Roch theorem for $p \hookrightarrow S$, so that $\mathrm{ch}(\mathfrak{C}) = \sum_{p\in S} l(\mathfrak{C}_p)\cdot p$. If we put $K = c_1(\Omega^2_S)$, the exact sequence $(*)$ gives the **Zeuthen-Segre invariant**:

$$\mathrm{ch}(L_D) + \mathrm{ch}(L_{K-D}) = \mathrm{ch}(\Omega^1_S) + \mathrm{ch}(\mathfrak{C}),$$
$$e^D + e^{K-D} = 2 + K + \tfrac{1}{2}K^2 - c_2(S) + \mathrm{ch}(\mathfrak{C}),$$
$$c_2(S) = D(K-D) + \textstyle\sum_p l(\mathfrak{C}_p)\, p.$$

6. Let X be a smooth projective variety. If $i\colon Y \to X$ is a closed smooth subvariety of codimension d, then $\mathrm{ch}(\mathcal{O}_Y) = i_*(\mathrm{Td}(N_{Y/X})) = Y + \ldots$ in $A^\bullet(X)\otimes\mathbb{Q}$. Since (see problem 27 in page 493) $[\mathrm{ch}(\mathcal{O}_Y)]_r = P(c_1(\mathcal{O}_Y), \ldots, c_{r-1}(\mathcal{O}_Y)) - (-1)^r r c_r(\mathcal{O}_Y)$, we see that $c_1(\mathcal{O}_Y) = 0, \ldots, c_{d-1}(\mathcal{O}_Y) = 0$, $c_d(\mathcal{O}_Y) = (-1)^d(d-1)!\,Y$ in $A^\bullet(X)\otimes\mathbb{Q}$.

7. The above theorem shows that any numerical invariant of smooth projective varieties defined by means of a cohomology theory (using inverse and direct images, products and Chern classes) coincides with the corresponding invariant defined using the graded K-theory. For example, topological and algebraic global intersection numbers coincide. In particular, the topological and algebraic genus of a smooth projective curve C coincide, $\frac{1}{2}\dim_{\mathbb{Q}} H^1(C_{\mathrm{an}}, \mathbb{Q}) = \dim_{\mathbb{C}} H^0(C, \Omega_C)$, since for any complex smooth projective variety X we have that $\langle\Delta,\Delta\rangle_{\mathrm{top}}$ is (p. 373) the Euler-Poincaré characteristic and $\langle\Delta,\Delta\rangle_{\mathrm{alg}}$ is (p. 439) just $\sum_{p,q}(-1)^{p+q}\dim_{\mathbb{C}} H^p(X, \Omega^q_X)$,

$$\textstyle\sum_i (-1)^i \dim_{\mathbb{Q}} H^i(X_{\mathrm{an}}, \mathbb{Q}) = \sum_{p,q}(-1)^{p+q}\dim_{\mathbb{C}} H^p(X, \Omega^q_X).$$

15.5 Duality Theory

Now all schemes will be noetherian and separated, and $f\colon X \to S$ is a projective morphism.

If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, then the sheaves $R^p f_*\mathcal{M}$ are coherent (p. 337) and we fix a finite affine cover of X, affine over S (with images contained in affine open sets of S).

On the category of quasi-coherent $\mathcal{O}_X$-modules, the functor $\mathcal{M} \rightsquigarrow f_*\check{C}^p\mathcal{M}$ is exact and it commutes with inductive limits since global sections commute with inductive limits (p. 336).

Now the representability theorem[3] directly proves (p. 367) the

Duality Theorem: *For any bounded below complex I of injective quasi-coherent $\mathcal{O}_S$-modules there exists a bounded below complex $f^!I$ of injective quasi-coherent $\mathcal{O}_X$-modules such that, for*

[3]In the case of a contravariant functor F on the category of quasi-coherent sheaves, minimal pairs Q_ξ form a set because, according to Deligne's formula, anyone is fully determined by the set of all elements $\eta \in F(\mathfrak{p}^n)$ admitting a morphism of pairs $(\mathfrak{p}^n)_\eta \to Q_\xi$.

any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we have a natural $\mathcal{O}_S(S)$-linear isomorphism

$$\mathrm{Hom}^\bullet_X(\mathcal{M}, f^! I) = \mathrm{Hom}^\bullet_S\big(f_*(\check{C}^\bullet \mathcal{M}), I\big).$$

Definition: When I is an injective resolution of $\mathcal{O}_S$, we say that $D_{X/S} = f^! I$ is the **dualizing complex** of f and, up to quasi-isomorphisms, it does not depend on the affine cover of X (any two affine covers admit a common affine refinement) or the injective resolution of $\mathcal{O}_S$ (the cone of a quasi-isomorphism $I \simeq J$ is a split exact sequence).

When $S = \mathrm{Spec}\, k$ is the spectrum of a field, we may take $I = k$, and the dualizing complex $D_X = f^! k$ is a bounded complex of injective quasi-coherent sheaves.

Theorem: *If U is an open set in S, then $D_{X/S}|_{f^{-1}U} \xrightarrow{\sim} D_{f^{-1}U/U}$.*

For any coherent $\mathcal{O}_X$-module $\mathcal{M}$ we have a quasi-isomorphism

$$f_* \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}, D_{X/S}) \;\simeq\; \underline{\mathrm{Hom}}^\bullet_S(f_* \check{C}^\bullet \mathcal{M}, I).$$

Proof: Let U be an affine open set in S, and $\mathfrak{p}$ the sheaf of ideals of $S - U$.

Since the fixed open cover of X is affine over S, we have $f_* \check{C}^\bullet(f^*\mathfrak{p}^n \otimes \mathcal{M}) = \mathfrak{p}^n \otimes f_* \check{C}^\bullet \mathcal{M}$, and by Deligne's formula (the quasi-isomorphism is due to the following lemma)

$$\begin{aligned}\Gamma(U, f_* \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}, D_{X/S})) &= \varinjlim \mathrm{Hom}^\bullet_X(\mathfrak{p}^n, f_* \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}, D_{X/S}))\\ &= \varinjlim \mathrm{Hom}^\bullet_X(f^*\mathfrak{p}^n \otimes \mathcal{M}, D_{X/S}) = \varinjlim \mathrm{Hom}^\bullet_S(f_* \check{C}^\bullet(f^*\mathfrak{p}^n \otimes \mathcal{M}), I)\\ &= \varinjlim \mathrm{Hom}^\bullet_S(\mathfrak{p}^n \otimes f_* \check{C}^\bullet \mathcal{M}, I) \xrightarrow{\;\sim\;} \Gamma(U, \underline{\mathrm{Hom}}^\bullet_S(f_* \check{C}^\bullet \mathcal{M}, I)).\end{aligned}$$

Hence $f_* \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}, D_{X/S}) \simeq \underline{\mathrm{Hom}}^\bullet_S(f_* \check{C}^\bullet \mathcal{M}, I)$. Moreover, if we put $V = f^{-1}U$, then

$$\mathrm{Hom}^\bullet_V(\mathcal{M}|_V, D_{X/S}|_V) \xrightarrow{\;\sim\;} \mathrm{Hom}^\bullet_S((f|_V)_* \check{C}^\bullet(\mathcal{M}|_V), I|_U) = \mathrm{Hom}^\bullet_V(\mathcal{M}|_V, D_{V/U})$$

since $I|_U$ is an injective resolution of $\mathcal{O}_U$. Now, if $\mathfrak{q}$ is the sheaf of ideals of a closed subscheme in V, we put $\mathcal{M}|_V = \mathfrak{q}^n$, and taking inductive limit, we see that $D_{X/S}|_V \xrightarrow{\sim} D_{V/U}$.

Lemma: *If K is a bounded quasi-coherent complex of coherent cohomology sheaves $\mathcal{H}^p(K)$, then the natural morphism $\varinjlim \mathrm{Hom}^\bullet\big(\mathfrak{p}^n, \underline{\mathrm{Hom}}^\bullet(K, I)\big) \to \mathrm{Hom}^\bullet(K|_U, I|_U)$ is a quasi-isomorphism.*

Proof: First we see the existence of a quasi-isomorphic coherent subcomplex $K' \hookrightarrow K$.

If K^i is coherent for all $i > p$, we take a coherent submodule $\mathcal{M} \subset K^p$ such that $\mathrm{d}_p(\mathcal{M}) = \mathrm{d}_p(K^p)$, and a coherent submodule $\mathcal{N} \subset \mathrm{Ker}\, \mathrm{d}_p$ such that $\mathcal{N} \to \mathcal{H}^p(K) \to 0$.

Replacing K^p and K^{p-1} by $\mathcal{M} + \mathcal{N}$ and $\mathrm{d}_{p-1}^{-1}(\mathcal{M} + \mathcal{N})$, the complex is coherent in degree p.

Now we put $R = \underline{\mathrm{Hom}}^\bullet(K, I)$, $R' = \underline{\mathrm{Hom}}^\bullet(K', I)$, and we have a commutative square

$$\begin{array}{ccc}\varinjlim \mathrm{Hom}^\bullet(\mathfrak{p}^n, R) & \longrightarrow & \Gamma(U, R)\\ \downarrow & & \wr\downarrow \text{qis}\\ \varinjlim \mathrm{Hom}^\bullet(\mathfrak{p}^n, R') & \xrightarrow{\text{iso}} & \Gamma(U, R')\end{array}$$

where the quasi-isomorphism is due to the fact that $R \to R'$ is a quasi-isomorphism of flasque complexes (hence the cone is a flasque acyclic complex).

To conclude, we must show that $\varinjlim \mathrm{Hom}^\bullet(\mathfrak{p}^n, R) \to \varinjlim \mathrm{Hom}^\bullet(\mathfrak{p}^n, R')$ is a quasi-isomorphism.

If S is affine, $R(S) \simeq R'(S)$ induces a quasi-isomorphism (U is affine)

$$\varinjlim \mathrm{Hom}^\bullet(\mathfrak{p}^n, R) = \varinjlim \mathrm{Hom}^\bullet\big(\mathfrak{p}^n, \widetilde{R(S)}\big) = \Gamma\big(U, \widetilde{R(S)}\big) \xrightarrow{\sim} \Gamma\big(U, \widetilde{R'(S)}\big) = \varinjlim \mathrm{Hom}^\bullet(\mathfrak{p}^n, R').$$

In general it is a quasi-isomorphism since so it is on sections over any affine open set in S. Since both are flasque complexes, we conclude taking global sections.

Theorem: *If $\phi\colon T \to S$ is flat, we have a quasi-isomorphism $\bar{\phi}^* D_{X/S} \simeq D_{X_T/T}$.*

Proof: If $\phi^* I \to J$ is an injective quasi-coherent resolution, then J is a resolution of $\mathcal{O}_T$ because ϕ is flat, and for any coherent $\mathcal{O}_X$-module $\mathcal{M}$

$$\begin{aligned}
\bar{f}_* \underline{\mathrm{Hom}}^\bullet_{X_T}(\bar{\phi}^*\mathcal{M}, \bar{\phi} D_{X/S}) &= \bar{f}_* \bar{\phi}^* \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}, D_{X/S}) && \text{since } \phi \text{ is flat} \\
&= \phi^* f_* \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}, D_{X/S}) && \text{(p. 435)} \\
&\simeq \phi^* \underline{\mathrm{Hom}}^\bullet_S(f_* \check{C}^\bullet \mathcal{M}, I) && \text{by the previous theorem} \\
&\simeq \underline{\mathrm{Hom}}^\bullet_S(\phi^* f_* \check{C}^\bullet \mathcal{M}, J) && \text{by the following lemma} \\
&= \underline{\mathrm{Hom}}^\bullet_S(\bar{f}_* \bar{\phi}^* \check{C}^\bullet \mathcal{M}, J) && \text{(p. 435)} \\
&= \underline{\mathrm{Hom}}^\bullet_S\big(\bar{f}_* \check{C}^\bullet(\bar{\phi}^* \mathcal{M}), J\big) && \\
&\simeq \bar{f}_* \underline{\mathrm{Hom}}^\bullet_{X_T}(\bar{\phi}^* \mathcal{M}, D_{X_T/T}) && \text{by the previous theorem}
\end{aligned}$$

Lemma: *If K is a bounded quasi-coherent complex of coherent cohomology sheaves $H^p(K)$, and ϕ is flat, then $\phi^* \underline{\mathrm{Hom}}^\bullet(K, I) \xrightarrow{\sim} \underline{\mathrm{Hom}}^\bullet(\phi^* K, J)$; i.e.,*

$$\phi^* \mathbf{R}\underline{\mathrm{Hom}}^\bullet(K, I) \simeq \mathbf{R}\underline{\mathrm{Hom}}^\bullet(\phi^* K, \phi^* I).$$

Proof: Let $K' \subset K$ be a quasi-isomorphic coherent subcomplex.

Since I is injective, $\underline{\mathrm{Hom}}^\bullet(K, I) \simeq \underline{\mathrm{Hom}}^\bullet(K', I)$, and since it is a local question, we may assume that we have a resolution $L \to K'$ by finite free modules.

Now, using that ϕ is flat and the bicomplex spectral sequence, we have quasi-isomorphisms

$$\begin{aligned}
\phi^* \underline{\mathrm{Hom}}^\bullet(K, I) \simeq \phi^* \underline{\mathrm{Hom}}^\bullet(K', I) \simeq \phi^* \underline{\mathrm{Hom}}^\bullet(L, I) &= \underline{\mathrm{Hom}}^\bullet(\phi^* L, \phi^* I) \\
\simeq \underline{\mathrm{Hom}}^\bullet(\phi^* L, J) \simeq \underline{\mathrm{Hom}}^\bullet(\phi^* K', J) &\simeq \underline{\mathrm{Hom}}^\bullet(\phi^* K, J).
\end{aligned}$$

15.5.1 Calculation of the Dualizing Complex

Theorem: *When $i\colon Y \to X$ is a closed embedding, $D_{Y/S} \simeq \underline{\mathrm{Hom}}^\bullet_X(\mathcal{O}_Y, D_{X/S})$.*

Proof: We consider in Y the restriction of the fixed affine cover of X.

Then we have $i_* \check{C}^\bullet(i^* \mathcal{M}) = \check{C}^\bullet(i_* i^* \mathcal{M})$, and we conclude,

$$\begin{aligned}
\mathrm{Hom}^\bullet_X(\mathcal{M}, i_* D_{Y/S}) &= \mathrm{Hom}^\bullet_Y(i^* \mathcal{M}, D_{Y/S}) = \mathrm{Hom}^\bullet_S(f_* \check{C}^\bullet(i_* i^* \mathcal{M}), I) \\
&= \mathrm{Hom}^\bullet_X(i_* i^* \mathcal{M}, D_{X/S}) = \mathrm{Hom}^\bullet_X(\mathcal{M}, \underline{\mathrm{Hom}}_X(i_* \mathcal{O}_Y, D_{X/S})).
\end{aligned}$$

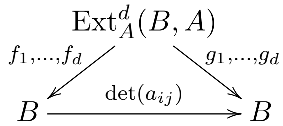
Lemma: *If $B = A/I$, where the ideal I is generated by a regular sequence, $I = (f_1, \ldots, f_d)$, then $\mathrm{Ext}^p_A(B, A) = 0$, $p \neq d$, and we have a canonical isomorphism*

$$\mathrm{Ext}^d_A(B, A) = \mathrm{Hom}_B(\Lambda^d(I/I^2), B).$$

If moreover $\mathrm{Tor}^A_p(B, M) = 0$, $p > 0$, then $\mathrm{Ext}^p_A(B, M) = \mathrm{Ext}^p_A(B, A) \otimes_A M$.

Proof: The Koszul complex K of $f_1, \ldots, f_d$ is a free resolution of B, and the dual complex only has a non null cohomology group, $H^d\big[\mathrm{Hom}_A(K, A)\big] \simeq B$.

If a regular sequence $g_i = \sum_j a_{ij} f_j$ generates I, we have an isomorphism $K(f_1, \ldots, f_d) \simeq K(g_1, \ldots, g_d)$, given in degree p by $\Lambda^p(a_{ij})$, and a commutative diagram



The same holds with the dual bases of $f_1 \wedge \ldots \wedge f_d$ and $g_1 \wedge \ldots \wedge g_d$ in $\Lambda^d(I/I^2)^*$, so that the isomorphism $\mathrm{Ext}^d_A(A/I, A) = \Lambda^d(I/I^2)^*$ is intrinsic.

Moreover $\mathrm{Hom}^\bullet_A(K, A) \otimes_A M = \mathrm{Hom}^\bullet_A(K, M)$ is a free resolution of B; hence

$$H^p\big[\mathrm{Hom}^\bullet_A(K, A)\big] \otimes_A M = H^p\big[\mathrm{Hom}^\bullet_A(K, M)\big].$$

Definition: A closed embedding $Y \to X$ is a **regular embedding** of codimension d if the sheaf of ideals $\mathfrak{p}_{Y/X}$ is locally generated by regular sequences of length d.

Theorem: *Let $\mathfrak{p}$ be the ideal of a regular embedding $i\colon Y \to X$ of codimension d. The cohomology sheaves of $D_{Y/X}$ are null, except $\omega_{Y/X} = \mathcal{H}^d(D_{Y/X}) = \Lambda^d(\mathfrak{p}/\mathfrak{p}^2)^*$. If moreover the sheaves $\mathcal{H}^p(D_{X/S})$ are i^*-acyclic, then*

$$\mathcal{H}^{p+d}(D_{Y/S}) = \omega_{Y/X} \otimes_{\mathcal{O}_Y} i^*\mathcal{H}^p(D_{X/S}).$$

Proof: The first statement follows directly from the lemma because

$$\mathcal{H}^p(D_{Y/X}) = \mathcal{H}^p\big[\mathbf{R}\underline{\mathrm{Hom}}_X(\mathcal{O}_Y, \mathcal{O}_X)\big] = \underline{\mathrm{Ext}}^p_X(\mathcal{O}_Y, \mathcal{O}_X).$$

Moreover, $D_{Y/S} = \underline{\mathrm{Hom}}_X(\mathcal{O}_Y, D_{X/S})$, and we have a spectral sequence (p. 380)

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p_X(\mathcal{O}_Y, \mathcal{H}^q(D_{X/S})) \;\Rightarrow\; \mathcal{H}^{p+q}(D_{Y/S}).$$

If the sheaves $\mathcal{H}^q(D_{X/S})$ are i^*-acyclic, by the lemma $E_2^{p,q} = 0$, $p \neq d$, and

$$\mathcal{H}^{d+q}(D_{Y/S}) = \underline{\mathrm{Ext}}^d_X(\mathcal{O}_Y, \mathcal{H}^q(D_{X/S})) = \underline{\mathrm{Ext}}^p_X(\mathcal{O}_Y, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{H}^q(D_{X/S}).$$

Definition: A flat morphism of finite type $X \to S$ is a **smooth** morphism of dimension d if the diagonal $\Delta\colon X \to X \times_S X$ is a regular embedding of codimension d.

Theorem: *If $X \to S$ is a projective smooth morphism of dimension d, then the dualizing complex $D_{X/S}$ has a unique non null cohomology sheaf, $\mathcal{H}^{-d}(D_{X/S}) = \Lambda^d\Omega_{X/S}$.*

Proof: Since the morphism $X \to S$ is flat,

$$\mathcal{H}^p(D_{X\times_S X/X}) = \mathcal{H}^p(\pi_1^*\mathcal{D}_{X/S}) = \pi_1^*\mathcal{H}^p(\mathcal{D}_{X/S}) = \mathcal{H}^p(D_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X\times X},$$

and these sheaves are Δ^*-acyclic because, if A is the local ring of $X \times_S X$ at a point of the diagonal, with the notations of the lemma, we have

$$\mathrm{Tor}^A_p(B, M \otimes_B A) = H^p[(M \otimes_B A) \otimes_A K] = H^p(M \otimes_B K) = \mathrm{Tor}^B_p(M, B),$$

K being a complex of flat B-modules. We conclude because the sheaves $\mathcal{H}^p(D_{X/X})$ are null, except $\mathcal{H}^0(D_{X/X}) = \mathcal{O}_X$, and according to the above theorem,

$$\mathcal{H}^{p+d}(D_{X/X}) = \omega_\Delta \otimes \Delta^*\mathcal{H}^p(D_{X\times_S X/X}) = (\Lambda^d\Omega_{X/S})^* \otimes \mathcal{H}^p(D_{X/S}).$$

Theorem: *If an open set $U \to X$ admits a closed embedding in $\mathbb{A}^n_S$,*

$$\mathcal{H}^p(D_{X/S})|_U \simeq \underline{\mathrm{Ext}}^{p+n}_{\mathbb{A}^n_S}(\mathcal{O}_U, \mathcal{O}_{\mathbb{A}^n_S}).$$

Proof: The ideal of the closed embedding $i\colon U \to \mathbb{A}^n_S \times_S X = \mathbb{A}^n_X$ is defined in the open set $\mathbb{A}^n_U$ by a regular sequence of length n, so that $\omega_{U/\mathbb{A}^n_X} \simeq \mathcal{O}_U$ is trivial.

As before (now A is the local ring of $\mathbb{A}^n_X$ at a point of U), the sheaves

$$\mathcal{H}^p(D_{\mathbb{A}^n_X/\mathbb{A}^n_S}) = \mathcal{H}^p(\pi_2^* D_{X/S}) = \pi_2^*\mathcal{H}^p(D_{X/S}) = \mathcal{H}^p(D_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathbb{A}^n_X}$$

are i^*-acyclic, and the lemma let us conclude,

$$i^*\pi_2^*\mathcal{H}^p(D_{X/S}) \simeq \omega_{U/\mathbb{A}^n_X} \otimes i^*\mathcal{H}^p(D_{\mathbb{A}^n_X/\mathbb{A}^n_S}) = \mathcal{H}^{p+n}(D_{U/\mathbb{A}^n_S}) = \underline{\mathrm{Ext}}^{p+n}_{\mathbb{A}^n_S}(\mathcal{O}_U, \mathcal{O}_{\mathbb{A}^n_S}). \qquad \text{q.e.d.}$$

1. If X is a projective scheme over a field, then we have $\mathrm{Hom}_X(\mathcal{M}, D_X) = \Gamma(X, \check{C}^\bullet\mathcal{M})^*$. Hence $\mathrm{Ext}^{-p}_X(\mathcal{M}, D_X) = H^p(X, \mathcal{M})^*$, and if the dualizing complex D_X has a unique non null cohomology sheaf $\omega_X = \mathcal{H}^{-d}(D_X)$, then $\mathrm{Ext}^{d-p}_X(\mathcal{M}, \omega_X) = H^p(X, \mathcal{M})^*$.

2. In the case of a finite k-algebra A, we have $\mathrm{Hom}_A(M, \omega_A) = M^*$ for any finitely generated A-module M. When $M = A$, we see that the dualizing module is $\omega_A = A^*$.

3. The dualizing sheaf of $\mathbb{P}_d$ is a line sheaf, $\omega_{\mathbb{P}_d} \simeq \mathcal{O}_{\mathbb{P}_d}(r)$, and $\omega_{\mathbb{P}_d} \simeq \mathcal{O}_{\mathbb{P}_d}(-d-1)$ since
$$H^d(\mathbb{P}_d, \mathcal{O}(n))^* = \mathrm{Hom}(\mathcal{O}(n), \omega_{\mathbb{P}_d}) = \Gamma(\mathbb{P}_d, \mathcal{O}(r-n)).$$

4. If Y is a hypersurface of a regular projective variety X, defined by a locally principal ideal $\mathfrak{p} = L_{-Y}$, then $\omega_Y = (\mathfrak{p}/\mathfrak{p}^2)^* \otimes \omega_X$, and the canonical divisor K_Y of Y is given by the **adjunction formula**, $K_Y = Y \cdot (K_X + Y)$.

5. If X is a smooth projective variety of dimension n, then $H^p(X, \Omega^q_X)^* = H^{n-p}(X, \Omega^{n-q}_X)$. In fact, $(\Omega^q_X)^* \otimes \Omega^n_X = \Omega^{n-q}_X$, and for any locally free $\mathcal{O}_X$-module $\mathcal{L}$ we have
$$H^p(X, \mathcal{L})^* = \mathrm{Ext}^{n-p}(\mathcal{L}, \Omega^n_X) = H^{n-p}(X, \mathcal{L}^* \otimes \Omega^n_X).$$

6. If K is the canonical divisor of an irreducible smooth projective surface S, by duality we have $h^2(D) = h^0(K-D)$ for any divisor D, and by the Riemann-Roch theorem (p. 453)
$$h^0(nD) + h^0(K-nD) \geq \chi(L_{nD}) = \tfrac{1}{2}(D \cdot D)n^2 - \tfrac{1}{2}(K \cdot D)n + \chi(\mathcal{O}_S).$$
When $D^2 > 0$, if $n \gg 0$ we see that $h^0(nD) \gg 0$ or $h^0(K-nD) \gg 0$, in which case $h^0(K+nD) = 0$ (so that $h^0(-nD) \gg 0$) because any non-null section of L_{K+nD} gives
$$h^0(K-nD) \leq h^0(K-nD+K+nD) = h^0(L_{2K}).$$
Hence $h^0(nD) > 1$ for some $n \in \mathbb{Z}$ and, S being irreducible, nD is linearly equivalent to an effective divisor, and we conclude that $D \cdot H > 0$ for any hyperplane section H. That is to say, the symmetric bilinear form $\mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$, $(D', D) \mapsto D' \cdot D$, has index 1 (**Hodge's index theorem**): $H \cdot H > 0$, and if $D \cdot H = 0$, then $D^2 \leq 0$ (and $D^2 = 0$ only if D is **numerically equivalent** to 0, in the sense that $D \cdot E = 0$ for any divisor E; that is to say, D is in the radical of the intersection metric).

7. When the surface is a direct product, $S = C' \times C$, of smooth projective curves with some rational point, the divisors $p' \times C$ and $C' \times p$ clearly form an hyperbolic pair, orthogonal to any divisor $\bar{D} = D - d(p' \times C) - d'(C' \times p)$, where d', d are the degrees of the divisor D over both factors. Hence $\bar{D}^2 \leq 0$, and we obtain **Castelnuovo's inequality**, $D \cdot D \leq 2d'd$, and the equality only holds if D is numerically equivalent to $d(p' \times C) + d'(C' \times p)$.

8. In the particular case of the graph Γ of a morphism $C \to C$ of degree d, the adjunction formula $\Gamma \cdot (K_S + \Gamma) = 2g - 2$, where g is the genus of C, states that $\Gamma^2 = (2-2g)d$. Now, Castelnuovo's inequality $(n\Gamma + m\Delta)^2 \leq 2(n+m)(dn+m)$ gives
$$q(n,m) = gdn^2 + (1 + d - \Gamma \cdot \Delta)nm + gm^2 \geq 0$$
because $\Delta^2 = 2 - 2g$. The discriminant of $q(n,m)$ is ≤ 0, and we obtain an estimate of the number $\Gamma \cdot \Delta$ of fixed points, counted with multiplicity, $|1 + d - \Gamma \cdot \Delta| \leq 2g\sqrt{d}$.

9. If moreover $k = \mathbb{F}_q$ is a finite field, we have a k-morphism $C \to C$, the identity on the underlying topological space and $\mathcal{O}_C(U) \to \mathcal{O}_C(U)$, $f \mapsto f^q$, on any open set U. By base change to an algebraic closure $k \to \bar{k}$ we obtain a **Frobënius morphism** $F\colon \bar{C} \to \bar{C}$, the fixed points being the rational points of C. This Frobënius morphism has degree q and it is purely inseparable (so that the graph transversally intersects the diagonal) and we obtain an estimate of the number N of rational points, $|1 + q^n - N| \leq 2g\sqrt{q}$, equivalent to the Riemann hypothesis for the zeta function of C (see [28]).

15.5.2 Local Duality

Let $\widehat{\mathcal{O}}$ be a complete noetherian local ring and let x be the closed point of $X = \operatorname{Spec}\widehat{\mathcal{O}}$. We put $U = X - x$, and for any $\widehat{\mathcal{O}}$-module M we put $M^* = \operatorname{Hom}_{\widehat{\mathcal{O}}}(M, I)$, where I is a fixed injective hull of the residue field $\widehat{\mathcal{O}}/\mathfrak{m}$. Now we try to determine the dual $H^p_x(X, M)^*$ of the local cohomology groups in terms of morphisms to a local dualizing complex $\widehat{D}$.

We have the local cohomology exact sequence

$$0 \longrightarrow H^0_x(X, M) \longrightarrow M \longrightarrow \widetilde{M}(U) \longrightarrow H^1_x(X, M) \longrightarrow 0$$

and $H^p_x(X, M) = H^{p-1}(U, \widetilde{M})$, $p \geq 2$. Hence, if we consider a finite open cover of U by basic open sets, the corresponding Cech complex $\check{C}^\bullet(\widetilde{M}|_U)$ is an acyclic resolution of $\widetilde{M}|_U$, and the cohomology groups of the following complex $K^\bullet(M)$:

$$K^0(M) = M \to K^1(M) = \Gamma(U, \check{C}^0\widetilde{M}|_U) \to \ldots \to K^p(M) = \Gamma(U, \check{C}^{p-1}\widetilde{M}|_U) \to \ldots$$

are just the local cohomology groups $H^p_x(X, M)$. Moreover, $K^p(M)$ is a linear exact functor and commutes with inductive limits (p. 336); hence, by the representability theorem, $K^p(M)^*$ is (linearly) representable by an injective $\widehat{\mathcal{O}}$-module $\widehat{D}^{-p}$.

Local Duality Theorem: *There is a bounded complex of injective $\widehat{\mathcal{O}}$-modules $\widehat{D}$, named **local dualizing complex** of $\widehat{\mathcal{O}}$, such that for any $\widehat{\mathcal{O}}$-module M, we have natural isomorphisms*

$$(K^\bullet M)^* = \operatorname{Hom}^\bullet_{\widehat{\mathcal{O}}}(M, \widehat{D}).$$

Corollary: $H^p_x(X, M)^* = \operatorname{Ext}^{-p}_{\widehat{\mathcal{O}}}(M, \widehat{D})$.

Theorem: *The local dualizing complex of a complete regular local ring of dimension n has a unique non zero cohomology module, $H^{-n}(\widehat{D}) \simeq \widehat{\mathcal{O}}$.*

Proof: We have $H^{-p}\widehat{D} = \operatorname{Ext}^{-p}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}, \widehat{D}) = H^p_x(\widehat{\mathcal{O}})^* = 0$ when $p \neq n$.

Moreover, $H^n_x(\widehat{\mathcal{O}})$ is an injective hull of $\mathcal{O}/\mathfrak{m}$, so that $H^n_x(\widehat{\mathcal{O}})^* \simeq (\widehat{\mathcal{O}})^{**} = \widehat{\mathcal{O}}$ (p. 431).

Lemma: *If $\widehat{\mathcal{O}}'$ is a quotient of $\widehat{\mathcal{O}}$, an injective hull of the residue field of $\widehat{\mathcal{O}}'$ is* $\operatorname{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', I)$.

Proof: On the category of $\widehat{\mathcal{O}}'$-modules of finite length, the functor $F(M) = \operatorname{Hom}_{\widehat{\mathcal{O}}}(M, I)$ is exact and $F(k) \simeq k$; hence it corresponds to an injective hull of the residue field,

$$\varinjlim \operatorname{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}'/\mathfrak{m}'^r, I) = \operatorname{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', I).$$

Theorem: *If $\widehat{\mathcal{O}}'$ is a quotient of $\widehat{\mathcal{O}}$, the local dualizing complex $\widehat{D}'$ of $\widehat{\mathcal{O}}'$ is*

$$\widehat{D}' \simeq \mathbf{R}\operatorname{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', \widehat{D}).$$

Proof: The basic open cover of $U = \operatorname{Spec}\widehat{\mathcal{O}} - x$ induces a basic open cover of $\operatorname{Spec}\widehat{\mathcal{O}}' - x$, and $\operatorname{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', I)$ is the injective hull of the residue field of $\widehat{\mathcal{O}}'$. Now, for any $\widehat{\mathcal{O}}'$-module M',

$$(K^\bullet M')^* = \operatorname{Hom}^\bullet_{\widehat{\mathcal{O}}'}\big(K^\bullet M', \operatorname{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', I)\big) = \operatorname{Hom}^\bullet_{\widehat{\mathcal{O}}}(K^\bullet M', I)$$
$$= \operatorname{Hom}^\bullet_{\widehat{\mathcal{O}}}(M', \widehat{D}) = \operatorname{Hom}^\bullet_{\widehat{\mathcal{O}}}\big(M, \operatorname{Hom}^\bullet_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', \widehat{D})\big).$$

Corollary: *Let D_X be the dualizing complex of a projective variety X over a field. At any point $x \in X$, the completion of the stalk $D_{X,x}$ is quasi-isomorphic to the local dualizing complex $\widehat{D}_x$ of the complete local ring $\widehat{\mathcal{O}}_{X,x}$.*

Proof: If X is a closed subscheme of a projective space $\mathbb{P}_d$, then $\mathcal{D}_X = \mathbf{R}\underline{\operatorname{Hom}}_{\mathcal{O}_{\mathbb{P}_d}}(\mathcal{O}_X, \Omega^d_{\mathbb{P}_d})$ and

$$\mathcal{D}_{X,x} = \mathbf{R}\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}_d,x}}(\mathcal{O}_{X,x}, \Omega^d_{\mathbb{P}_d,x}) \simeq \mathbf{R}\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}_d,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P}_d,x}),$$

since $\mathcal{D}_x$ is an injective resolution of $\Omega^d_{\mathbb{P}_d,x} \simeq \mathcal{O}_{\mathbb{P}_d,x}$ when $\mathcal{D}$ is an injective resolution of $\Omega^d_{\mathbb{P}_d}$.

Hence the completion of $D_{X,x}$ is just $\mathbf{R}\operatorname{Hom}_{\widehat{\mathcal{O}}_{\mathbb{P}_d,x}}(\widehat{\mathcal{O}}_{X,x}, \widehat{\mathcal{O}}_{\mathbb{P}_d,x}) = \widehat{D}_x$.

Corollary: *If $\widehat{\mathcal{O}}'$ is a quotient of a complete regular local ring $\widehat{\mathcal{O}}$, the local dualizing complex $\widehat{D}'$ is a bounded injective complex with finitely generated cohomology modules.*

Proof: The $\widehat{\mathcal{O}}'$-modules $\operatorname{Ext}^p_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', \widehat{D}) = \operatorname{Ext}^p_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}', \widehat{\mathcal{O}})$ are finitely generated.

15.5.3 Biduality

Definition: Let $\mathcal{O}$ be a local noetherian ring. A bounded complex D of injective $\mathcal{O}$-modules (or any quasi-isomorphic bounded complex of $\mathcal{O}$-modules) with finitely generated cohomology modules $H^p(D)$ is a **biduality complex** if the natural morphism $M^\bullet \to DD(M^\bullet)$ is a quasi-isomorphism for any bounded complex of $\mathcal{O}$-modules $M^\bullet$ with finitely generated cohomology modules $H^p(M^\bullet)$, where we put $D(M^\bullet) = \mathbf{R}\operatorname{Hom}^\bullet_{\mathcal{O}}(M^\bullet, D)$.

Theorem: *A bounded complex D of injective $\mathcal{O}$-modules, with finitely generated cohomology modules, is a biduality complex if and only if the natural morphism $\mathcal{O} \to DD(\mathcal{O})$ is a quasi-isomorphism, $\mathcal{O} \simeq \operatorname{Hom}^\bullet_{\mathcal{O}}(D, D)$.*

Proof: If D is a biduality complex, just put $M = \mathcal{O}$ in the definition.

Conversely, if M is a finitely generated $\mathcal{O}$-module and $L^\bullet \to M \to 0$ is a resolution by free modules of finite rank, by hypothesis the morphism $L^p \to \operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(L^p, D), D)$ is a quasi-isomorphism.

Hence, by the bicomplex theorem, $L^\bullet \xrightarrow{\sim} \operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(L^\bullet, D), D)$, and $M \simeq DD(M)$.

Finally, we proceed by induction on the number of non null componentes of the bounded complex $M^\bullet$ and, if M^p is the non null component of highest degree, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^p & \longrightarrow & M^\bullet & \longrightarrow & \bar{M}^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & DD(M^p) & \longrightarrow & DD(M^\bullet) & \longrightarrow & DD(\bar{M}^\bullet) & \longrightarrow & 0
\end{array}$$

Since $M^p \to DD(M^p)$, $\bar{M}^\bullet \to DD(\bar{M}^\bullet)$ are quasi-isomorphisms, so is $M^\bullet \to DD(M^\bullet)$.

Corollary: *If $\mathcal{O}$ is regular local ring, a biduality complex is $\mathcal{O}$.*

Proof: $\mathcal{O}$ has a finite injective resolution since $\mathrm{Ext}^n_{\mathcal{O}}(M,\mathcal{O}) = 0$, $n > \dim \mathcal{O}$, and the quasi-isomorphism $\mathcal{O} \simeq \mathbf{R}\mathrm{Hom}^\bullet(\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{O},\mathcal{O}),\mathcal{O})$ is obvious.

Corollary: *Any biduality complex of a field k is quasi-isomorphic to k, up to non canonical isomorphisms and re-grading.*

Proof: A complex D of vector spaces is quasi-isomorphic to $H(D) = \{H^p(D), \mathrm{d} = 0\}$, and

$$k \simeq \mathrm{Hom}^\bullet_k(D,D) \simeq \mathrm{Hom}^\bullet_k(H(D),H(D)).$$

Theorem: *If D is a biduality complex of $\mathcal{O}$, then $\mathbf{R}\mathrm{Hom}_{\mathcal{O}}(\bar{\mathcal{O}}, D)$ is a biduality complex of any quotient ring $\bar{\mathcal{O}}$.*

Proof: If Q is an injective $\mathcal{O}$-module, then $\mathrm{Hom}_{\mathcal{O}}(\bar{\mathcal{O}}, Q)$ is an injective $\bar{\mathcal{O}}$-module.

We conclude since $\mathrm{Hom}^\bullet_{\bar{\mathcal{O}}}(\bar{M}, \mathrm{Hom}^\bullet_{\mathcal{O}}(\bar{\mathcal{O}}, D)) = \mathrm{Hom}^\bullet_{\mathcal{O}}(\bar{M}, D)$ for any $\bar{\mathcal{O}}$-module $\bar{M}$.

Corollary: *Any quotient of a regular local ring posses a biduality complex.*

Corollary: *Let D_X be the dualizing complex of a projective variety X over a field. The stalk $D_{X,x}$ at any point is a biduality complex of $\mathcal{O}_{X,x}$-modules, and we have a quasi-isomorphism*

$$\mathcal{O}_X \simeq \mathbf{R}\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(D_X, D_X).$$

Proof: We have seen (p. 460) that $\mathcal{D}_{X,x} \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_d,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P}_d,x})$.

Hence it is a biduality complex of $\mathcal{O}_{X,x}$-modules.

Finally, D_X is quasi-isomorphic to a coherent subcomplex (p. 455), so that

$$\big(\mathbf{R}\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(D_X, D_X)\big)_x = \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{O}_{X,x}}(D_{X,x}, D_{X,x}) = \mathcal{O}_{X,x}.$$

Corollary: *The local dualizing complex $\widehat{D}$ of a complete local ring $\widehat{\mathcal{O}}$, quotient of a complete regular local ring, is a biduality complex of $\widehat{\mathcal{O}}$-modules.*

Note: We have shown (p. 460) that the completion of $D = D_{X,x}$ is quasi-isomorphic to the local dualizing complex $\widehat{D}$ of the completion $\widehat{\mathcal{O}}$ of the local ring $\mathcal{O} = \mathcal{O}_{X,x}$; but now, using that $\mathcal{O}_X = \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(D_X, D_X)$, we may exhibit a canonical quasi-isomorphism,

$$\begin{aligned}\widehat{D} &= \mathbf{R}\mathrm{Hom}_{\widehat{\mathcal{O}}}(\widehat{\mathcal{O}}, \widehat{D}) = (\mathbf{R}\Gamma_x \widehat{\mathcal{O}})^* = \varprojlim(\mathbf{R}\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}^n, \mathcal{O}))^* \\ &= \varprojlim(\mathbf{R}\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}^n, \mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{O}}(D,D)))^* = \varprojlim(\mathbf{R}\mathrm{Hom}^\bullet_{\mathcal{O}}(\mathcal{O}/\mathfrak{m}^n \otimes D, D))^* \\ &= \varprojlim(D/\mathfrak{m}^n D)^{**} = \varprojlim D/\mathfrak{m}^n D.\end{aligned}$$

Theorem: *A bounded complex D of injective $\mathcal{O}$-modules, with finitely generated cohomology modules, is a biduality complex if and only if the natural morphism $k \to DD(k)$ is a quasi-isomorphism, where $k = \mathcal{O}/\mathfrak{m}$ is the residue field.*

Proof: If D is a biduality complex, just put $M = k$ in the definition.

Conversely, by induction on the length, we see that $M \to DD(M)$ is a quasi-isomorphism for any finite length $\mathcal{O}$-module M; just consider an exact sequence $0 \to M' \to M \to M'' \to 0$, where $l(M'), l(M'') < l(M)$.

Now, by induction on the dimension of $\operatorname{supp} M$, we shall see that $M \to DD(M)$ is a quasi-isomorphism for any finitely generated $\mathcal{O}$-module M. If $M' \subset M$ is the submodule of elements with support $(\mathfrak{m})_0$, replacing M by M/M' we may assume that $\mathfrak{m}$ is not an associated prime of M, so that there is a M-regular element $t \in \mathfrak{m}$, and we have an exact sequence

$$0 \longrightarrow M \xrightarrow{t} M \longrightarrow M/tM \longrightarrow 0$$

When $p \neq 0$, the exact sequence $H^p(DD(M)) \xrightarrow{t} H^p(DD(M)) \to H^p(DD(M/tM)) = 0$ and Nakayama's lemma show that $H^p(DD(M)) = 0$. Now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{t} & M & \longrightarrow & M/tM & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\bar\alpha} & & \\ 0 & \longrightarrow & H^0(DD(M)) & \xrightarrow{t} & H^0(DD(M)) & \longrightarrow & H^0(DD(M/tM)) & \longrightarrow & 0 \end{array}$$

has exact rows and $\bar{\alpha}$ is an isomorphism; hence $\operatorname{Ker}\alpha \xrightarrow{t} \operatorname{Ker}\alpha$ and $\operatorname{Coker}\alpha \xrightarrow{t} \operatorname{Coker}\alpha$ are surjective by the Snake's lemma. Nakayama's lemma shows that α is an isomorphism.

Corollary: *The local dualizing complex $\widehat{D}$ of a complete local ring $\widehat{\mathcal{O}}$ is a biduality complex.*

Proof: We have $D(k) = \mathbf{R}\mathrm{Hom}_{\widehat{\mathcal{O}}}(k, \widehat{D}) \simeq (\mathbf{R}\Gamma_x(k))^* \simeq k^* = k$; hence $DD(k) \simeq D(k) \simeq k$.

Theorem: *Put $k = \mathcal{O}/\mathfrak{m}$. A bounded complex D of $\mathcal{O}$-modules, with finitely generated cohomology modules, is a biduality complex if and only if there is an integer n such that*

$$\operatorname{Ext}^p_{\mathcal{O}}(k, D) = \begin{cases} k & p = n \\ 0 & p \neq n \end{cases}$$

Proof: If D is a biduality complex, then $\mathbf{R}\mathrm{Hom}_{\mathcal{O}}(k, D)$ is a biduality complex of k; hence there is some integer number n such that $\operatorname{Ext}^n_{\mathcal{O}}(k, D) = k$, and $\operatorname{Ext}^p_{\mathcal{O}}(k, D) = 0$ when $p \neq n$.

For the converse, it is clear that $k \to DD(k)$ is a quasi-isomorphism, so we only have to show that D is quasi-isomorphic to a bounded complex of injective $\mathcal{O}$-modules.

First we prove that $\operatorname{Ext}^p_{\mathcal{O}}(M, D) = 0$, $p \notin [n-d, n]$, by induction on $d = \dim(\operatorname{supp} M)$.

We may assume that $\mathfrak{m}$ is not an associated prime of M, so that we have exact sequences

$$0 \longrightarrow M \xrightarrow{t} M \longrightarrow M/tM \longrightarrow 0$$
$$\operatorname{Ext}^p_{\mathcal{O}}(M, D) \xrightarrow{t} \operatorname{Ext}^p_{\mathcal{O}}(M, D) \longrightarrow \operatorname{Ext}^{p-1}_{\mathcal{O}}(M/tM, D) = 0$$

whenever $p \notin [n-d, n]$. Hence $\operatorname{Ext}^p_{\mathcal{O}}(M, D) = 0$ by Nakayama's lemma.

Now, if $D \simeq I^\bullet$ is an injective resolution, we conclude if we prove that the cycles $Z^r = B^r$ are an injective module when $r \gg 0$. Let us consider $K^\bullet\colon \ldots \to I^{r-1} \to I^r \to 0 \to \ldots$ and $\bar{K}^\bullet\colon \ldots \to I^{r-1} \to B^r \to 0 \to \ldots$, so that we have exact sequences

$$0 \longrightarrow \bar{K}^\bullet \longrightarrow K^\bullet \longrightarrow B^{r+1}[-r] \longrightarrow 0$$
$$\operatorname{Ext}^{r+1}_{\mathcal{O}}(M, K^\bullet) \longrightarrow \operatorname{Ext}^{r+1}_{\mathcal{O}}(M, B^{r+1}[-r]) \longrightarrow \operatorname{Ext}^{r+2}_{\mathcal{O}}(M, \bar{K}^\bullet)$$

where M is any $\mathcal{O}$-module. When $r \gg 0$, we have $\operatorname{Ext}^{r+2}_{\mathcal{O}}(M, \bar{K}^\bullet) = \operatorname{Ext}^{r+2}_{\mathcal{O}}(M, D) = 0$ and $\operatorname{Ext}^{r+1}_{\mathcal{O}}(M, K^\bullet) = 0$ because $K^\bullet$ is an injective complex without terms of degree $> r$.

Hence $0 = \operatorname{Ext}^{r+1}_{\mathcal{O}}(M, B^{r+1}[-r]) = \operatorname{Ext}^1_{\mathcal{O}}(M, B^{r+1})$ and B^{r+1} is an injective module.

Corollary: *A bounded complex D of $\mathcal{O}$-modules is a biduality complex if and only if $D \otimes_{\mathcal{O}} \widehat{\mathcal{O}}$ is a biduality complex of $\widehat{\mathcal{O}}$-modules.*

Proof: $\mathrm{Ext}^p_{\widehat{\mathcal{O}}}(k, D \otimes_{\mathcal{O}} \widehat{\mathcal{O}}) = \mathrm{Ext}^p_{\mathcal{O}}(k, D) \otimes_{\mathcal{O}} \widehat{\mathcal{O}} = \mathrm{Ext}^p_{\mathcal{O}}(k, D)$.

Theorem: *If it exists, the biduality complex D of a local noetherian ring $\mathcal{O}$ is unique up to (non-canonical) isomorphisms and regrading.*

Proof: Let D' be another biduality complex of $\mathcal{O}$-modules, and put $D'(M^\bullet) = \mathrm{Hom}^\bullet_{\mathcal{O}}(M^\bullet, D')$.

Let us consider the natural morphism (where the tensor product is derived: factors must be replaced by bounded above flat resolutions)

$$M^\bullet \underset{=}{\otimes} D'(D) = M^\bullet \underset{=}{\otimes} \mathrm{Hom}^\bullet(D, D') \longrightarrow \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(M^\bullet, D) D') = D'D(M^\bullet).$$

Since it is a quasi-isomorphism when $M^\bullet$ is a bounded above complex of free modules of finite rank, so it is when $M^\bullet$ is a bounded complex with finitely generated cohomology modules, because $M^\bullet$ is quasi-isomorphic to a finitely generated subcomplex (p. 455). Now,

$$D(D') \underset{=}{\otimes} D'(D) \simeq D'DD(D') = D'(D') \simeq \mathcal{O},$$

and the following lemma shows that $\mathrm{Hom}^\bullet(D, D') = D'(D) \simeq \mathcal{O}[n]$.

We conclude that $D' \simeq D'D(D) \simeq D \underset{=}{\otimes} D'(D) \simeq D \underset{=}{\otimes} \mathcal{O}[n] \simeq D[n]$.

Lemma: *Let K, L be two bounded above complexes with finitely generated cohomology modules. If $K \underset{=}{\otimes} L \simeq \mathcal{O}$, then $K \simeq \mathcal{O}[n]$ for some integer n.*

Proof: We may assume that $p = 0$ and $q = 0$ are the largest integers such that $H^p(K) \neq 0$ and $H^q(L) \neq 0$, so that $H^0(K \underset{=}{\otimes} L) = H^0(K) \otimes H^0(L) \neq 0$.

Therefore $H^0(K) \otimes H^0(L) = \mathcal{O}$ and we see that $H^0(K) = H^0(L) = \mathcal{O}$.

It follows that $K \simeq K_1 \oplus \mathcal{O}$ and $L \simeq L_1 \oplus \mathcal{O}$.

Since $\mathcal{O} \simeq K \underset{=}{\otimes} L \simeq \mathcal{O} \oplus K_1 \oplus L_1 \oplus (K_1 \underset{=}{\otimes} L_1)$, we have $K_1 \simeq 0$; hence $K \simeq \mathcal{O}$.

15.6 Formal Functions Theorem

In this section we shall omit many details and we don't distinguish between derived $R^n F$ and hyperderived $\mathbf{R}^n F$ functors. Let $f \colon X \to S = \operatorname{Spec} \mathcal{O}$ be a projective morphism, where $\mathcal{O}$ is a local noetherian ring, and we assume the existence of a biduality complex of $\mathcal{O}$-modules D_S (it exists when $\mathcal{O}$ is a quotient of a regular local ring).

Now $D_X = f^! D_S$ is a biduality complex at any closed point $x \in X$, because

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), D_{X,x}) = \mathbf{R}\mathrm{Hom}_{\mathcal{O}}(\kappa(x), D_S) = (\mathbf{R}\Gamma_s \kappa(x))^* \simeq \kappa(x),$$

where s is the closed point of S. Hence, for any bounded complex of $\mathcal{O}_X$-modules $\mathcal{M}^\bullet$, of coherent cohomology sheaves $\mathcal{H}^p(\mathcal{M}^\bullet)$, we have a natural quasi-isomorphism

$$\mathcal{M}^\bullet \longrightarrow \underline{\mathrm{Hom}}^\bullet_X(\underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}^\bullet, D_X), D_X).$$

Theorem: *If Y is the fibre of the closed point s of S, we have canonical isomorphisms (the dual * is considered with respect to the injective hull of the residue field of s)*

$$H^p_Y(X, \mathcal{M}^\bullet)^* = \mathrm{Ext}^{-p}_X(\mathcal{M}^\bullet, D_X)\widehat{\ }.$$

Proof: $H^p_Y(X,\mathcal{M}^\bullet)^* = H^p_s(S,Rf_*\mathcal{M}^\bullet)^* = \mathrm{Ext}^{-p}_{\mathcal{O}}(Rf_*\mathcal{M}^\bullet, D_S)\widehat{\ }$ by local duality,

$= \mathrm{Ext}^{-p}_X(\mathcal{M}^\bullet, D_X)\widehat{\ }$ by duality.

Corollary: $(R^p f_*\mathcal{M}^\bullet)\widehat{\ } = H^{-p}_Y(X, \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}^\bullet, D_X))^*$.

Proof: $(R^p f_*\mathcal{M}^\bullet)\widehat{\ } = \left[R^p f_* \underline{\mathrm{Hom}}^\bullet_X(\underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}^\bullet, D_X), D_X)\right]\widehat{\ } = H^{-p}_Y(X, \underline{\mathrm{Hom}}^\bullet_X(\mathcal{M}^\bullet, D_X))^*$.

Corollary: *If $\pi\colon Z \to X$ is a projective morphism and E is the fibre of a closed point $x \in X$,*

$$(R^{-p}\pi_* D_Z)\widehat{_x} = H^p_E(Z,\mathcal{O}_Z)^*,$$
$$(R^p\pi_*\mathcal{O}_Z)\widehat{_x} = H^{-p}_E(Z, D_Z)^*.$$

Proof: Let $Z_x = Z \times_X \mathrm{Spec}\,\mathcal{O}_{X,x} \to \mathrm{Spec}\,\mathcal{O}_{X,x}$ be the morphism induced by π.
Since $D_{X,x}$ is a dualizing complex of the local ring $\mathcal{O}_{X,x}$,

$$(R^p\pi_*\mathcal{O}_Z)\widehat{_x} = H^{-p}_E(Z_x, D_Z)^* = H^{-p}_E(Z, D_Z)^*,$$
$$(R^{-p}\pi_* D_Z)\widehat{_x} = H^p_E\big(Z_x, \underline{\mathrm{Hom}}^\bullet_Y(D_Z, D_Z)\big)^* = H^p_E(Z,\mathcal{O}_Z)^*.$$

Formal Functions Theorem: *If $\mathfrak{p}$ is the ideal of the fibre Y of the closed point, then*

$$(R^p f_*\mathcal{M}^\bullet)\widehat{\ } = \varprojlim H^p(X, \mathcal{M}^\bullet \underline{\underline{\otimes}} (\mathcal{O}_X/\mathfrak{p}^n)).$$

Proof: Take a quasi-isomorphism $P \to \mathcal{M}^\bullet$, where P is a bounded above complex of flat $\mathcal{O}_X$-modules (for example, direct sums of sheaves $\mathcal{O}_U$),

$$\begin{aligned}(R^p f_*\mathcal{M}^\bullet)\widehat{\ } &= H^{-p}_Y(X, \underline{\mathrm{Hom}}^\bullet(\mathcal{M}^\bullet, D_X))^* = \varprojlim \mathrm{Ext}^{-p}(\mathcal{O}_X/\mathfrak{p}^n, \underline{\mathrm{Hom}}^\bullet(P, D_X))^* \\ &= \varprojlim \mathrm{Ext}^{-p}(P\otimes(\mathcal{O}_X/\mathfrak{p}^n), D_X)^* = \varprojlim H^p(X, P\otimes(\mathcal{O}_X/\mathfrak{p}^n)).\end{aligned}$$

Note: Let us fix $p, n \geq 1$. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, then, by Artin-Rees, the morphisms $\underline{\mathrm{Tor}}_i(\mathcal{M}, \mathcal{O}_X/\mathfrak{p}^{n+r}) \longrightarrow \underline{\mathrm{Tor}}_i(\mathcal{M}, \mathcal{O}_X/\mathfrak{p}^n)$ are null when $r \gg 0$. Now, since the sheaves $\mathcal{M}/\mathfrak{p}^n\mathcal{M}$ are supported on Y, we have

$$\varprojlim H^p\big(X, \mathcal{M} \underline{\underline{\otimes}} (\mathcal{O}_X/\mathfrak{p}^n)\big) = \varprojlim H^p(X, \mathcal{M}/\mathfrak{p}^n\mathcal{M}) = \varprojlim H^p(Y, \mathcal{M}/\mathfrak{p}^n\mathcal{M}),$$

and we see that $(R^p f_*\mathcal{M})\widehat{\ } = \varprojlim H^p(Y, \mathcal{M}/\mathfrak{p}^n\mathcal{M})$.

1. $R^p f_*\mathcal{M} = 0$ when p is greater than the dimension of the fibre Y. In fact $R^p f_*\mathcal{M}$ is a finitely generated $\mathcal{O}$-module and $(R^p f_*\mathcal{M})\widehat{\ } = 0$, because $H^p(Y, \mathcal{M}/\mathfrak{p}^n\mathcal{M}) = 0$ when $p > \dim Y$.

2. If $f_*\mathcal{O}_X = \mathcal{O}$, then the fibre Y is connected, because otherwise the local ring $\widehat{\mathcal{O}}$ would decompose as a direct sum of two rings,

$$\widehat{\mathcal{O}} = (f_*\mathcal{O}_X)\widehat{\ } = \varprojlim H^0(Y_1 \oplus Y_2, \mathcal{O}/\mathfrak{p}^n) = \big(\varprojlim H^0(Y_1, \mathcal{O}/\mathfrak{p}^n)\big) \oplus \big(\varprojlim H^0(Y_2, \mathcal{O}/\mathfrak{p}^n)\big).$$

3. Assume now that both X and S are integral and that the field Σ_S is algebraically closed in the field Σ_X (for example if f is birrational). If $\mathcal{O}$ is normal, then the fibre Y is connected because, $f_*\mathcal{O}_X$ being a finite $\mathcal{O}$-algebra contained in Σ_X, we have $f_*\mathcal{O}_X = \mathcal{O}$.

4. If we put $X' = \mathrm{Spec}\, f_*\mathcal{O}_X$, we see that f factors into the composition of a projective morphism $Z \to X'$ with connected fibres and a finite morphism $X' \to S$, because $f_*\mathcal{O}_X$ is a finitely generated $\mathcal{O}$-module.

5. Now we assume that $X = \operatorname{Spec}\mathcal{O}$, where $\mathcal{O}$ is a Cohen-Macaulay local ring of dimension d with a dualizing module ω_X, so that $D_X \simeq \omega_X[-d]$.

   If $\pi\colon \bar{X} = \operatorname{Proj} \oplus_n I^n/I^{n+1} \to X = \operatorname{Spec}\mathcal{O}$ is the blow-up along an ideal $I \subset \mathcal{O}$, we have a cone $C = \operatorname{Spec} G_I\mathcal{O}$ with vertex (the closed subscheme defined by the irrelevant ideal) $V = \operatorname{Spec}\mathcal{O}/I$, the center of explosion, and with directrix $E = \operatorname{Proj} G_I\mathcal{O}$, the exceptional fibre $E = \pi^{-1}(V)$. Let $Y = \pi^{-1}(x)$ be the fibre of the closed point $x \in V$. We have natural projections $C \to V$ and $V - C \to E$, and the fibre Y_0 of C over x is a cone with directrix Y. Since V and C are affine, and $C - V$ is the complement of the zero section of a line bundle over E, we have a local cohomology exact sequence (where λ is in fact an homogeneous morphism)

$$\begin{array}{ccccccc} H^i_x(C,\mathcal{O}_C) & \longrightarrow & H^i_{Y_0}(C,\mathcal{O}_C) & \xrightarrow{\lambda} & H^i_{Y_0}(C-x,\mathcal{O}_C) & \longrightarrow & H^{i+1}_x(C,\mathcal{O}_C) \\ & & \| & & \| & & \\ & & \bigoplus_n H^i_x(V, I^n/I^{n+1}) & & H^i_{Y_0}(C-V,\mathcal{O}_C) & & \\ & & & & \| & & \\ & & & & \bigoplus_{n\in\mathbb{Z}} H^i_Y(E,\mathcal{O}_E(n)) & & \end{array}$$

   If $C = \operatorname{Spec} G_I\mathcal{O}$ is Cohen-Macaulay, then $H^i_x(C,\mathcal{O}_C) = 0$, $i < \dim_x C = d$, and

$$H^i_Y(E,\mathcal{O}_E(-n)) = 0,\ n > 0,\ i < d-1.$$

   Moreover, if C is Cohen-Macaulay, so is E because $C - V \to E$ is a line bundle (without the zero section) and, E being an hypersurface of $\bar{X}$, we see that $\bar{X}$ also is Cohen-Macaulay, with a dualizing sheaf $\omega_{\bar{X}}$ such that $D_{\bar{X}} \simeq \omega_{\bar{X}}[-d]$. Now the exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(1) \longrightarrow \mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

   induces local cohomology long exact sequences, where $n > 0$, $i < d$,

$$0 = H^{i-1}_Y(E,\mathcal{O}_E(-n)) \longrightarrow H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}(-n+1)) \longrightarrow H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}(-n))$$

   and $H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}(-n))^* = \big(R^{d-i}\pi_*\omega_{\bar{X}}(n)\big)^\frown = 0$, $n \gg 0$, $i < d$ (p. 337). By descending induction on n, we conclude that $H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}(-n)) = 0$, $n \geq 0$, $i < d$. In particular

$$H^i_Y(\bar{X},\mathcal{O}_{\bar{X}}) = 0,\ i < d,$$
$$R^i\pi_*\omega_{\bar{X}} = 0,\ i > 0.$$

6. If the ideal I is generated by a regular sequence, then $G_I\mathcal{O} = (\mathcal{O}/I)[x_1,\dots,x_r]$ is Cohen-Macaulay; hence $R^i\pi_*\omega_{\bar{X}} = 0$, $i > 0$.

7. Let X be an hypersurface (i.e., defined by a locally principal ideal) of an ambient regular variety Z. If we blow-up X along a regular center C, and we denote by $\mathfrak{p}$ and $\mathfrak{p}_0$ the ideals of C in X and Z respectively, then $\operatorname{Spec} G_{\mathfrak{p}}\mathcal{O}_X$ is an hypersurface of the regular variety $\operatorname{Spec} G_{\mathfrak{p}_0}\mathcal{O}_Z$; hence it is Cohen-Macaulay and again $R^i\pi_*\omega_{\bar{X}} = 0$, $i > 0$.

15.7 Grothendieck Topologies

Definition: A **presheaf** of sets on a category $\mathbf{C}$ is a contravariant functor $\mathcal{P}\colon \mathbf{C} \rightsquigarrow \mathbf{Ens}$, and **morphisms of presheaves** are just natural transformations. Hence, any morphism $f\colon Y \to X$ induces a map $f^*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ and, given a section $s \in \mathcal{P}(X)$, we put $s|_Y = f^*(s)$ when no confusion is possible.

Definition: A **sieve** of an object X of a category **C** is a family morphisms $Y \to X$ stable under specialization; i.e. a subfunctor of the functor of points $\mathrm{Hom}(-, X)$.

So any family of morphisms $\{U_i \to X\}$ generates a sieve $\mathcal{R}$:

$$\mathcal{R}(T) = \{\text{morphisms } T \to X \text{ factoring through some } U_i \to X\}.$$

If $\mathcal{P}$ is a presheaf of sets, then a morphism of presheaves $\mathcal{R} \to \mathcal{P}$ is defined by a compatible family of sections $s_i \in \mathcal{P}(U_i)$, in the sense that $f_i^*(s_i) = f_j^*(s_j)$ when a morphism $T \to X$ admits two factorizations (eventually $i = j$)

$$\begin{array}{ccc} T & \xrightarrow{f_i} & U_i \\ {\scriptstyle f_j}\downarrow & \searrow & \downarrow \\ U_j & \longrightarrow & X \end{array}$$

From now on, **C** *will be a category with fibred products* $X \times_S Y$.

Then the compatibility condition means that $p_1^*(s_i) = p_2^*(s_j)$, where p_1, p_2 are the natural projections of $U_i \times_X U_j$ onto the factors; i.e. we have an exact sequence

$$\mathrm{Hom}(R, \mathcal{P}) \longrightarrow \prod_i \mathcal{P}(U_i) \rightrightarrows \prod_{i,j} \mathcal{P}(U_i \times_X U_j)$$

Definitions: A **pretopology** on **C** is given by some sets of morphisms $\{U_i \to X\}_{i \in I}$ for any object X, named **covers** of X, so that

1. If $\{U_i \to X\}_{i \in I}$ is a cover, then $\{U_i \times_X Y \to Y\}_{i \in I}$ is a cover for any morphism $Y \to X$.
2. If $\{U_i \to X\}_{i \in I}$ is a cover and for any index i we have a cover $\{V_{ij} \to U_i\}_{j \in J_i}$, then $\{V_{ij} \to V_i \to X\}_{i \in I, j \in J_i}$ is a cover.
3. The identity $X \to X$ is a cover of any object X.

A presheaf of sets $\mathcal{F}$ is **separated** when the natural map $\mathcal{F}(X) \longrightarrow \mathrm{Hom}(R, \mathcal{F})$ is injective for any cover $\{U_i \to X\}$, i.e. when so is the map $\mathcal{F}(X) \to \prod_i \mathcal{F}(U_i)$, and it is a **sheaf** when $\mathcal{F}(X) \longrightarrow \mathrm{Hom}(R, \mathcal{F})$ is bijective, i.e. we have exact sequences

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \overset{p_1^*, p_2^*}{\rightrightarrows} \prod_{i,j} \mathcal{F}(U_i \times_X U_j)$$

Morphisms of sheaves are just morphisms of presheaves.

Theorem: *Let $\mathcal{F}$ be a separated presheaf (resp. sheaf) of sets. If a sieve $\mathcal{C}$ of X contains some cover $\{U_i \to X\}$, then the natural map $\mathcal{F}(X) \longrightarrow \mathrm{Hom}(\mathcal{C}, \mathcal{F})$ also is injective (resp. bijective).*

Proof: Let $\mathcal{R}$ be the sieve generated by $\{U_i \to X\}$, and consider the natural maps

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{f} & \mathrm{Hom}(\mathcal{C}, \mathcal{F}) \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & \mathrm{Hom}(\mathcal{R}, \mathcal{F}) \end{array}$$

If $\mathcal{F}$ is separated, then h is injective; hence so is f. Moreover g also is injective, since any morphism $Y \to X$ in $\mathcal{C}$ admits a cover $\{Y_i \to Y\}$ such that the morphisms $Y_i \to Y \to X$ are in $\mathcal{R}$, and $\mathcal{F}$ is separated.

If $\mathcal{F}$ is a sheaf, then h is moreover bijective; hence so is f.

Definitions: A sieve $\mathcal{C}$ of X is a **covering sieve** if it contains some cover of X. Two pretopologies on $\mathbf{C}$ are **equivalent** if both have the same covering sieves (any cover $\{V_\alpha \to X\}$ is refined by some cover $\{U_i \to X\}$ of the other pretopology, in the sense that any morphism $U_i \to X$ factors through some morphism $V_{\alpha_i} \to X$). A **topology** is a pretopology such that any set of morphisms $\{V_\alpha \to X\}$ refined by a cover also is a cover, so that any pretopology is equivalent to a unique topology (defined by all the sets $\{V_\alpha \to X\}$ generating a covering sieve; i.e. refined by a cover). A **situs** or **site** (with finite fibred products) is a category with finite fibred products endowed with a topology.

Given sites $\mathbf{C}, \mathbf{D}$, a covariant functor $f^{-1}\colon \mathbf{C} \rightsquigarrow \mathbf{D}$ preserving fibred products is **continuous** when $\{f^{-1}U_i \to f^{-1}X\}$ is a covering in $\mathbf{D}$ for any covering $\{U_i \to X\}$ in $\mathbf{C}$, so that, if $\mathcal{F}$ is a sheaf on $\mathbf{D}$, then the presheaf $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$ is in fact a sheaf, and any sheaf morphism $\mathcal{F} \to \mathcal{G}$ induces a natural sheaf morphism $f_*\mathcal{F} \to f_*\mathcal{G}$.

1. Let X be a topological space. The open sets, with the inclusion maps, form a category $\mathbf{Op}(X)$, and covers are defined to be surjective families $\{U_i \to U\}$; i.e. $U = \bigcup_i U_i$. Sheaves are just sheaves in the ordinary sense, because $U_i \times_U U_j = U_i \cap U_j$. Moreover, any continuous map $\phi\colon X \to Y$ between topological spaces induces a continuous functor $\phi^{-1}\colon \mathbf{Op}(Y) \to \mathbf{Op}(X)$ preserving the final object, and ϕ_* is the usual direct image of sheaves (p. 317).

   If $U \subset X$ is an open set, then the inclusion $\mathbf{Op}(U) \hookrightarrow \mathbf{Op}(X)$ is a continuous functor (not preserving the final object) and the corresponding direct image is just $\mathcal{F} \rightsquigarrow \mathcal{F}|_U$.

2. Any type of morphisms stable under base change and compositions, and including isomorphisms, defines a pretopology, where covers are families $\{R \to X\}$ of a single morphism of the considered type.

3. In the category of topological spaces, the surjective families $\{U_i \to X\}$ of open embeddings define a pretopology (hence a topology). Sheaves are just presheaves defining a classical sheaf on any topological space.

   Analogously the **Zariski topology** on the category of schemes **Sch** is defined by the pretopology given by the (set-theoretically) surjective families $\{U_i \to X\}$ of open embeddings.

4. The coarsest topology on $\mathbf{C}$ is the **trivial topology**: the total sieve is the unique covering sieve, so that any presheaf $\mathcal{P}$ is a sheaf. If a morphism $R \to X$ admits a section, it is a cover of X; hence the sequence $\mathcal{P}(X) \to \mathcal{P}(R) \rightrightarrows \mathcal{P}(R \times_X R)$ is exact.

   The finest topology is the **discrete topology**: any sieve is a covering sieve. The empty sieve covers any object X; hence $\mathcal{F}(X) = \mathrm{Hom}(\emptyset, \mathcal{F})$ is a one point set for any sheaf $\mathcal{F}$.

5. The **canonical topology** on $\mathbf{C}$ is defined by the families $\{U_i \to U\}$ such that we have exact sequences $\mathrm{Hom}(V, X) \to \prod_i \mathrm{Hom}(V_i, X) \rightrightarrows \prod_{i,j} \mathrm{Hom}(V_{ij}, X)$ for any object X and any base change $V \to U$; where $V_i = U_i \times_U V$, $V_{ij} = V_i \times_V V_j$. It is the finest topology on $\mathbf{C}$ such that the representable functors $\mathrm{Hom}(-, X)$ are sheaves.

6. The canonical topology on the category of sets **Ens** is defined by the surjective families $\{U_i \to X\}$.

7. Let G be a group. The canonical topology on the category of G-sets is defined by the surjective families $\{U_i \to X\}$. Now $G \to G/H$ is a cover, and $G \times_{G/H} G = \amalg_H G$, so that we have $\mathcal{F}(G/H) = \mathcal{F}(G)^H$ for any sheaf $\mathcal{F}$, where the action of $g \in G$ on $\mathcal{F}(G)$ is induced by the right multiplication by g^{-1} on G. Any sheaf is representable, $\mathcal{F}(U) = \mathrm{Hom}_G(U, \mathcal{F}(G))$.

8. Given a finite Galois extension $k \to L$ of group G, let $\mathbf{Triv}_{L/k}$ the category of finite k-algebras trivial over L. According to Galois theorem, the opposite category $\mathbf{Triv}_{L/k}^{\rm op}$ is equivalent to the category of finite G-sets. Hence, if we consider on $\mathbf{Triv}_{L/k}^{\rm op}$ the toplogy defined by the surjective families $\{\operatorname{Spec} B_i \to \operatorname{Spec} A\}$, we have that the category of sheaves is equivalent to the category of G-sets, where any sheaf $\mathcal{F}$ corresponds the the G-set $\mathcal{F}(L)$. Moreover, any sheaf is an inductive limit of representable sheaves.

9. Let S be a connected and locally connected space admitting a universal covering $\bar{S} \to S$. According to Galois theorem, the category $\mathbf{Rev}_S$ of coverings $X \to S$ is equivalent to the category of G-sets, where $G = \operatorname{Aut}(\bar{S}/S)$. Hence the canonical topology on $\mathbf{Rev}_S$ is defined by the surjective families $\{U_i \to X\}$, and any sheaf is representable. Any sheaf $\mathcal{F}$ is fully determined by the G-set $\mathcal{F}(\bar{S})$.

10. Let $\mathbf{C}$ be a situs. Given a covariant functor $f^{-1}\colon \mathbf{C}' \rightsquigarrow \mathbf{C}$ preserving finite fibred products, the families $\{U_i' \to X'\}$ such that $\{f^{-1}U_i' \to f^{-1}X'\}$ is a cover in $\mathbf{C}$ define on $\mathbf{C}'$ the **initial topology**: the coarsest topology such that f^{-1} is continuous.

    Now, given $S \in \operatorname{Ob}\mathbf{C}$, let $\mathbf{C}|_S$ be the category of S-objects $X \to S$ and S-morphisms. The natural functor $\mathbf{C}|_S \rightsquigarrow \mathbf{C}$, $(X \to S) \rightsquigarrow X$, preserves finite fibred products; hence it induces a topology on $\mathbf{C}|_S$, so that any sheaf $\mathcal{F}$ on $\mathbf{C}$ induces a sheaf $\mathcal{F}|_S$ on $\mathbf{C}|_S$. Any S-object T induces a natural continuous functor $\mathbf{C}|_T = (\mathbf{C}|_S)|_T \rightsquigarrow \mathbf{C}|_S$, and $(\mathcal{F}|_S)|_T = \mathcal{F}_T$.

    Let $\mathbf{R}$ be a **full subcategory** of $\mathbf{C}$ (some objects of $\mathbf{C}$, with the same morphisms and composition of morphisms, $\operatorname{Hom}_{\mathbf{R}}(X,Y) = \operatorname{Hom}_{\mathbf{C}}(X,Y)$) stable under fibred products of $\mathbf{C}$. The natural functor $\mathbf{R} \hookrightarrow \mathbf{C}$ preserves finite fibred products; hence it induces a topology on $\mathbf{R}$, so that any sheaf $\mathcal{F}$ on $\mathbf{C}$ induces a sheaf $(\mathcal{F}|_{\mathbf{R}})(X) = \mathcal{F}(X)$ on $\mathbf{R}$.

11. Given topologies T_i on $\mathbf{C}$, the families $\{U_j \to X\}$ which are covers for any T_i define a topology $\bigcap_i T_i$ on $\mathbf{C}$: the finest topology coarser than any T_i. We also have the topology $\bigvee_i T_i$ **generated** by the topologies T_i, the coarsest topology finer than any T_i: given a pretopology P_i of each T_i, the families $\{U_{j_1\dots j_n} \to \dots \to U_{j_1} \to X\}$, where $\{U_{j_1\dots j_r} \to U_{j_1\dots j_{r-1}}\}$ is a cover of some P_i, give a pretopology defining $\bigvee_i T_i$.

    A presheaf $\mathcal{F}$ is a sheaf (resp. separated) for $\bigvee_i T_i$ if and only if it is a sheaf (resp. separated) for any T_i. In fact, it is easy to check that the families $\{U_i \to X\}$ such that we have an exact sequence $\mathcal{F}(Y) \to \prod_i \mathcal{F}(Y_i) \rightrightarrows \prod_{ij} \mathcal{F}(Y_{ij})$ (resp. $\mathcal{F}(Y) \to \prod_i \mathcal{F}(Y_i)$ is injective) for any base change $Y \to X$ define a pretopology; hence a topology by the above theorem.

12. If a situs $\mathbf{C}$ admits a set $\{G_\alpha\}$ of **topological generators** (any object X admits a cover $\{G_{\alpha_i} \to X\}$), then the natural map $\operatorname{Hom}(\mathcal{F},\mathcal{G}) \to \prod_\alpha \operatorname{Hom}\big(\mathcal{F}(G_\alpha),\mathcal{G}(G_\alpha)\big)$ is injective for any two sheaves $\mathcal{F},\mathcal{G}$; hence $\operatorname{Hom}(\mathcal{F},\mathcal{G})$ is a set and the sheaves on $\mathbf{C}$ form a category $\widetilde{\mathbf{C}}$.

    Moreover, *a sheaf morphism $\mathcal{F} \to \bar{\mathcal{F}}$ is an isomorphism if and only if $\mathcal{F}(G_\alpha) \to \bar{\mathcal{F}}(G_\alpha)$ is bijective for any index α*: Just consider covers $\{G_{\alpha_i\alpha_j\beta} \to G_{\alpha_i} \times_X G_{\alpha_j}\}$, so that for any sheaf $\mathcal{G}$ we have an exact sequence $\mathcal{G}(X) \to \prod_i \mathcal{G}(G_{\alpha_i}) \rightrightarrows \prod_{ij\beta} \mathcal{G}(G_{\alpha_i\alpha_j\beta})$.

Recollement Theorem: *let $\mathcal{F},\mathcal{G}$ be sheaves on a situs $\mathbf{C}$, and fix a cover $\{U_i \to X\}$.*

1. *The "presheaf" $\underline{\operatorname{Hom}}(\mathcal{F},\mathcal{G})(X) := \operatorname{Hom}(\mathcal{F}|_X,\mathcal{G}|_X)$ is a "sheaf": given sheaf morphisms $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ such that $\phi_i = \phi_j$ on $U_{ij} = U_i \times_X U_j$ (i.e. on $\mathbf{C}|_{U_{ij}}$) ¡even when $i = j$!, there is a unique sheaf morphism $\phi\colon \mathcal{F}|_X \to \mathcal{G}|_X$ such that $\phi_i = \phi|_{U_i}$.* Hence, ϕ is an isomorphism when so is $\phi|_{U_i}$ for any index i.

2. *Given sheaves $\mathcal{F}_i$ on $\mathbf{C}|_{U_i}$ and isomorphisms $\phi_{ji}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$, such that $\phi_{ki} = \phi_{kj} \circ \phi_{ji}$ on $U_i \times_X U_j \times_X U_k$, then there is a sheaf $\mathcal{F}$ on $\mathbf{C}|_X$ with isomorphisms $\mathcal{F}|_{U_i} \simeq \mathcal{F}_i$ such that ϕ_{ji} is the composition $\mathcal{F}_i|_{U_{ij}} \simeq \mathcal{F}|_{U_{ij}} \simeq \mathcal{F}_j|_{U_{ij}}$*, and it is unique up to an isomorphism by (1).

Proof: Let $T \to X$ be a morphism in the sieve $\mathcal{R}$ of X generated by $\{U_i \to X\}$.

We have a well-defined map $\phi\colon \mathcal{F}(T) \to \mathcal{G}(T)$, independent of the factorization through a morphism $U_i \to X$, because $\phi_i = \phi_j$ on U_{ij}. Hence, if we consider $\mathcal{R}$ as a full subcategory of $\mathbf{C}|_X$, with the induced topology, we have a sheaf morphism $\phi\colon (\mathcal{F}|_X)|_{\mathcal{R}} \to (\mathcal{G}|_X)|_{\mathcal{R}}$, and (1) follows from the comparison lemma (p. 475); but let us give a direct proof:

For any morphism $Y \to X$ we have a cover $\{Y_i \to Y\}$, hence a commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}(Y) & \longrightarrow & \mathcal{F}(Y_i) & \rightrightarrows & \mathcal{F}(Y_{ij}) \\
 & & \downarrow \phi & & \downarrow \phi \\
\mathcal{G}(Y) & \longrightarrow & \mathcal{G}(Y_i) & \rightrightarrows & \mathcal{G}(Y_{ij})
\end{array}$$

with exact rows, inducing a map $\phi\colon \mathcal{F}(Y) \to \mathcal{G}(Y)$. So we obtain a sheaf morphism $\phi\colon \mathcal{F}|_X \to \mathcal{G}|_X$ such that $\phi_i = \phi|_{U_i}$, and it is clearly unique.

On the other hand, for any factorization $t\colon T \to U_i$ we have a set $\mathcal{F}_i(t)$ and a bijection $\phi_{t't}\colon \mathcal{F}_i(t) \simeq \mathcal{F}_j(t')$ for any other factorization $t'\colon T \to U_j$, so that $\phi_{t''t} = \phi_{t''t'}\phi_{t't}$ for any third factorization $t''\colon T \to U_k$ (in particular $\phi_{tt} = \mathrm{Id}$, since it is bijective and $\phi_{tt} = \phi_{tt}\phi_{tt}$); hence we may identify such sets, and we obtain a well-defined set $\mathcal{F}(T)$, endowed with bijections $\mathcal{F}(T) \simeq \mathcal{F}_i(t)$ such that $\phi_{t't}$ is just the composition $\mathcal{F}_i(t) \simeq \mathcal{F}(T) \simeq \mathcal{F}_j(t')$. Moreover, we have a well-defined map $\mathcal{F}(T) \to \mathcal{F}(T')$ for any X-morphism $T' \to T$, so that we have in fact a sheaf on $\mathcal{R}$, and (2) also follows from the comparison lemma; but let us give a direct proof:

For any morphism $Y \to X$, we define $\mathcal{F}(Y)$ to be the kernel of $\mathcal{F}(Y_i) \rightrightarrows \mathcal{F}(Y_{ij})$.

So we obtain a presheaf $\mathcal{F}$ on $\mathbf{C}|_X$, and it is a sheaf:

For any cover $\{V_\alpha \to Y\}$ we have a commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathcal{F}(Y) & \longrightarrow & \prod_i \mathcal{F}(Y_i) & \rightrightarrows & \prod_{ij} \mathcal{F}(Y_{ij}) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_\alpha \mathcal{F}(V_\alpha) & \longrightarrow & \prod_{\alpha i} \mathcal{F}(V_{\alpha i}) & \rightrightarrows & \prod_{\alpha ij} \mathcal{F}(V_{\alpha ij}) \\
\downarrow\downarrow & & \downarrow\downarrow & & \\
\prod_{\alpha\beta} \mathcal{F}(V_{\alpha\beta}) & \longrightarrow & \prod_{\alpha\beta i} \mathcal{F}(V_{\alpha\beta i}) & &
\end{array}$$

where the middle column is exact and $\prod_{ij} \mathcal{F}(Y_{ij}) \to \prod_{\alpha ij} \mathcal{F}(V_{\alpha ij})$ is injective (because $\mathcal{F}|_{U_i} \simeq \mathcal{F}_i$ is a sheaf). A diagram chasing shows that the first column is exact. So we obtain a sheaf $\mathcal{F}$ on $\mathbf{C}|_X$, with isomorphisms $\mathcal{F}|_{U_i} \simeq \mathcal{F}_i$ such that ϕ_{ji} is the composition $\mathcal{F}_i|_{U_{ij}} \simeq \mathcal{F}|_{U_{ij}} \simeq \mathcal{F}_j|_{U_{ij}}$.

15.7.1 The Faithfully Flat Topology

Lemma: *Let M be an A-module and $A \to B$ a faithfully flat morphism. We have an exact sequence*

$$M \longrightarrow M \otimes_A B \overset{d_1,d_2}{\rightrightarrows} M \otimes_A B \otimes_A B$$

where $\mathrm{d}_1(m \otimes b) = m \otimes 1 \otimes b,\ \mathrm{d}_2(m \otimes b) = m \otimes b \otimes 1.$

Proof: It is clear when $\operatorname{Spec} B \to \operatorname{Spec} A$ admits a section (see p. 197, or p. 467).

In general, after the base change $A \to B$, we obtain the sequence corresponding to the B-module $M_B = M \otimes_A B$ and the morphism $B \to B \otimes_A B$, $b \mapsto 1 \otimes b$, which admits the section $B \otimes_A B \to B$, $b_1 \otimes b_2 \mapsto b_1 b_2$. Since $A \to B$ is faithfully flat, we conclude.

Definition: Let A be a ring. The finite surjective families of flat morphisms $\{\operatorname{Spec} B_i \to \operatorname{Spec} B\}$ (i.e. $B \to \prod_i B_i$ is faithfully flat) define a pretopology on the category $\mathbf{Aff}|_A$ of affine A-schemes; hence a topology: the **fp topology** (faithfully flat topology). *A presheaf $\mathcal{F}$ is a sheaf if and only if it preserves finite direct products $\mathcal{F}(\prod_n B_n) = \prod_n \mathcal{F}(B_n)$ and the sequence $\mathcal{F}(B) \to \mathcal{F}(C) \rightrightarrows \mathcal{F}(C \otimes_B C)$ is exact for any faithfully flat morphism $B \to C$.*

Theorem: *If M is an A-module, then $\tilde{M}(B) = M_B = M \otimes_A B$ is a sheaf on $(\mathbf{Aff}|_A)_{\mathrm{fp}}$.*

Proof: Clearly $\tilde{M}(B_1 \times B_2) = \tilde{M}(B_1) \times \tilde{M}(B_2)$ and, if $B \to C$ is a faithfully flat morphism of A-algebras, then $M_B \to M_C \rightrightarrows M_{C \otimes_B C}$ is exact by the lemma.

Definition: On $(\mathbf{Aff}|_A)_{\mathrm{fp}}$ we have a sheaf of rings $\mathcal{O}(B) = B$, and $\tilde{M}$ is an $\mathcal{O}$-module. An $\mathcal{O}$-module $\mathcal{M}$ is **quasi-coherent** when it is defined by some A-module, clearly $\mathcal{M}(A)$.

Moreover $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_{\mathcal{O}}(\tilde{M}, \tilde{N})$; hence, by the recollement of sheaf morphisms

Corollary: *If M, N are A-modules, then $B \rightsquigarrow \operatorname{Hom}_B(M_B, N_B)$ is a sheaf.*

Theorem: *Any locally quasi-coherent $\mathcal{O}$-module $\mathcal{M}$ (i.e. the sheaves $\mathcal{M}|_{U_i}$ are quasi-coherent in some cover $\{U_i = \operatorname{Spec} B_i \to S = \operatorname{Spec} A\}$) is quasi-coherent.*

Proof: Let us prove that the natural sheaf morphism $\mathcal{M}(S)^\sim \to \mathcal{M}$ is an isomorphism on the cover $\{U_i \to S\}$; i.e. $\mathcal{M}(S) \otimes_A B_i = \mathcal{M}(U_i)$, both sheaves being quasi-coherent on U_i.

For any flat morphism $T = \operatorname{Spec} B \to S$ we have a commutative diagram with exact rows (the two last vertical morphisms being isomorphisms because $\mathcal{M}$ is quasi-coherent on U_i)

$$
\begin{array}{ccccc}
\mathcal{M}(S) \otimes_A B & \longrightarrow & \prod_i \mathcal{M}(U_i) \otimes_A C & \rightrightarrows & \prod_{ij} \mathcal{M}(U_{ij}) \otimes_A B \\
\downarrow & & \downarrow \wr & & \downarrow \wr \\
\mathcal{M}(T) & \longrightarrow & \prod_i \mathcal{M}(T_i) & \rightrightarrows & \prod_{ij} \mathcal{M}(T_{ij})
\end{array}
$$

so that $\mathcal{M}(S) \otimes_A B = \mathcal{M}(T)$. When $B = B_i$, we conclude that $\mathcal{M}(S) \otimes_A B_i = \mathcal{M}(U_i)$.

Corollary: *Let $U = \operatorname{Spec} B \to \operatorname{Spec} A$ be faithfully flat and let N be a B-module. If we have an isomorphism $\phi \colon p_1^* N \simeq p_2^* N$ such that $p_{13}^* \phi = p_{23}^* \phi \circ p_{12}^* \phi$, then there is an A-module M such that $N = M_B$,* (where $p_{ij} \colon U \times_S U \times_S U \to U \times_S U$ is the projection on the factors i, j).

Proof: By the recollement of sheaves, we have an $\mathcal{O}$-module $\mathcal{M}$ such that $\mathcal{M}|_U \simeq N^\sim$, and $\mathcal{M}$ is quasi-coherent by the theorem.

Theorem: *The functor of points* $\operatorname{Hom}_A(-, X)$ *of any A-scheme X is a sheaf on $(\mathbf{Aff}|_A)_{\mathrm{fp}}$.*

Proof: When X is affine, the result follows from the exact sequence of rings $B \to C \rightrightarrows C \otimes_B C$ for any faithfully flat morphism $B \to C$.

In general, if two morphisms $\phi, \psi \colon \operatorname{Spec} B \to X$ coincide on a cover $\operatorname{Spec} C \to \operatorname{Spec} B$, then they coincide topologically (covers are surjective). Hence $\operatorname{Spec} B = \bigcup_i V_i$, where V_i is a basic open set such that $\phi(V_i) = \psi(V_i)$ is contained in an affine open subset of X.

By the previous case, ϕ and ψ coincide on V_i, so that $\phi = \psi$, and $X^\bullet$ is separated.

Now, let $\phi\colon \operatorname{Spec} C \to X$ be a morphism such that $p_1^*\phi = p_2^*\phi\colon \operatorname{Spec}(C \otimes_B C) \to X$.

Then, set theoretically, ϕ is constant on the fibres of $\operatorname{Spec} C \to \operatorname{Spec} B$, because for any two points y_1, y_2 of a fibre there is a point $z \in \operatorname{Spec}(C \otimes_B C)$ such that $p_1(z) = x_1$, $p_2(z) = x_2$ (remark that $\kappa(y_1) \otimes_{\kappa(x)} \kappa(y_2) \neq 0$ when y_1 and y_2 are in the fibre of x).

By the following lemma, ϕ factors through a continuous map $\bar\phi\colon \operatorname{Spec} B \to X$.

Hence $\operatorname{Spec} B = \bigcup_i V_i$, where V_i is a basic open set such that $\bar\phi(V_i)$ is contained in an affine open subset of X. By the affine case, we obtain scheme morphisms $\bar\phi_i\colon V_i \to X$, compatible on the intersections $V_i \cap V_j$ because $\operatorname{Hom}_A(-, X)$ is separated, and we conclude.

Lemma: *If the image of a morphism of schemes $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is dense, then it contains any point $x \in \operatorname{Spec} A$ defined by a minimal prime $\mathfrak{p}$ of A. Moreover*

1. *If ϕ is a flat morphism and $y \in \operatorname{Spec} B$ is defined by a minimal prime of B, then $\phi(y)$ is defined by a minimal prime of A.*

2. *If ϕ is faithfully flat, then $\operatorname{Spec} A$ has the quotient topology (a subset $Z \subseteq \operatorname{Spec} A$ is closed if and only if $\phi^{-1}Z \subseteq \operatorname{Spec} B$ is closed).*

Proof: If $x \notin \operatorname{Im}\phi$, then $B_\mathfrak{p} = 0$; hence $f = 0$ in B for some $f \in A - \mathfrak{p}$, and $\phi^{-1}(U_f) = \emptyset$.

(1) Since $A_{\phi(y)} \to B_y$ is flat and local, it is faithfully flat; hence $y = \operatorname{Spec} B_y \to \operatorname{Spec} A_{\phi(y)}$ is surjective, so that $\operatorname{Spec} A_{\phi(y)} = \phi(y)$.

(2) Replacing Z by $\overline{Z}$, we may assume that Z is dense.

If $\phi^{-1}Z$ is closed (hence affine), applying (1) to $\phi^{-1}Z \to Z \subseteq \operatorname{Spec} A$ we see that Z contains any point of $\operatorname{Spec} A$ defined by a minimal prime.

Now, by (2), the closed set $\phi^{-1}Z$ contains any point of $\operatorname{Spec} B$ defined by a minimal prime; hence $\phi^{-1}Z = \operatorname{Spec} B$, and $Z = \operatorname{Spec} A$ because $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Corollary: *If X, Y are A-schemes, then $B \rightsquigarrow \operatorname{Hom}_B(X_B, Y_B)$ is a sheaf.*

Proof: It is clear that $\operatorname{Hom}_A(X, Y) = \operatorname{Hom}(X^\bullet, Y^\bullet)$, and the result follows from the recollement of sheaf morphisms.

The fpqc Topology:

A morphism of schemes $\phi\colon Y \to X$ is **quasicompact** when the inverse image of any compact open set also is compact (i.e. the inverse image of any affine open set is a finite union of affine open sets) and it is **faithfully flat** when it is **flat** (if $x = \phi(y)$, then $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-module) and set-theoretic surjective. If S is a scheme, the **fpqc topology** on the category $\mathbf{Sch}|_S$ of S-schemes is the topology generated by the Zariski topology and the topology defined by the faithfully flat quasicompact S-morphisms $Y \to X$.

Lemma: *If a presheaf $\mathcal{F}$ on $\mathbf{Sch}|_S$ is separated (resp. a sheaf) in the Zariski topology and $\mathcal{F}(A) \to \mathcal{F}(B)$ is injective (resp. $\mathcal{F}(A) \to \mathcal{F}(B) \rightrightarrows \mathcal{F}(B \otimes_A B)$ is exact) for any faithfully flat S-morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, then $\mathcal{F}$ is separated (resp. a sheaf) in the fpqc topology.*

Proof: We have to show that $\mathcal{F}$ is separated (resp. a sheaf) in the pretopology defined by the faithfully flat quasicompact S-morphisms $f\colon R \to X$.

Fix an affine open cover $X = \bigcup_i U_i$, and finite affine open covers $f^{-1}(U_i) = \bigcup_\alpha U_{i\alpha}$.

Now, the composition $\mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \to \prod_i \mathcal{F}(\amalg_\alpha U_{i\alpha})$ is injective and it factors through $\mathcal{F}(X) \to \mathcal{F}(R)$, so that $\mathcal{F}(X) \to \mathcal{F}(R)$ is injective and $\mathcal{F}$ is separated.

Moreover, if $s \in \mathcal{F}(R)$ is in the kernel of $\mathcal{F}(R) \rightrightarrows \mathcal{F}(R \times_X R)$, put $s_{i\alpha} = s|_{U_{i\alpha}}$.

Since $\mathcal{F}(\amalg_\alpha U_{i\alpha}) = \Pi_\alpha \mathcal{F}(U_{i\alpha})$ and $\amalg_\alpha U_{i\alpha}$ is affine, there is $s_i \in \mathcal{F}(U_i)$ such that $s_{i\alpha} = s_i|_{U_{i\alpha}}$.

Now, $s_i = s_j$ on $U_i \cap U_j = U_i \times_X U_j$, since both coincide on the cover $\{U_{i\alpha} \times_X U_{j\beta} \to U_i \times_X U_j\}$ and $\mathcal{F}$ is separated, so that there is $t \in \mathcal{F}(X)$ such that $s_i = t$ on U_i.

Hence $s = s_i = t$ on $U_{i\alpha}$, and $s = t|_R$ since $\mathcal{F}$ is separated. q.e.d.

The above results show that in the category of S-schemes $T \to S$ with the fpqc topology

1. *$\mathcal{O}(T) = \mathcal{O}_T(T)$ is a sheaf of rings, and any quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$ defines a sheaf of $\mathcal{O}$-modules* $\mathcal{M}(T) = (\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_T)(T)$ (also named **quasi-coherent**).

2. *If $\mathcal{M}, \mathcal{N}$ are quasi-coherent $\mathcal{O}_S$-modules, $T \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{O}_T, \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$ is a sheaf.*

3. *Any locally quasi-coherent $\mathcal{O}$-module also is quasi-coherent.*

4. *The functor of points $X^\bullet = \mathrm{Hom}_S(-, X)$ of any S-scheme X is a sheaf.*

5. *If X, Y are S-schemes, then $T \rightsquigarrow \mathrm{Hom}_T(X \times_S T, Y \times_S T)$ is a sheaf.*

6. *If a presheaf of sets is locally representable by affine morphisms, then it is representable by an affine morphism $X \to S$.*

   In fact $(f\colon X \to S) \rightsquigarrow f_*\mathcal{O}_X$ is an equivalence of the category of affine morphisms with the (opposite) category of quasi-coherent $\mathcal{O}_S$-algebras.

Cech Cohomology: Let $\mathcal{F}$ be an abelian presheaf on a situs **C**. For any cover $\{U_i \to X\}$ we have a complex of abelian groups $\check{C}^\bullet(\{U_i \to X\}, \mathcal{F})$, where $U_{i_0 \dots i_p} = U_{i_0} \times_X \dots \times_X U_{i_p}$,

$$\check{C}^p(\{U_i \to X\}, \mathcal{F}) = \textstyle\prod_{i_0, \dots, i_p} \mathcal{F}(U_{i_0 \dots i_p})$$

$$\mathrm{d}\colon \check{C}^p\mathcal{F} \longrightarrow \check{C}^{p+1}\mathcal{F},\ (\mathrm{d}s)_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_0 \dots \widehat{i_k} \dots i_{p+1}}|_{U_{i_0 \dots i_p}},$$

$$\check{H}^p(\{U_i \to X\}, \mathcal{F}) = H^p\big[\check{C}^\bullet(\{U_i \to X\}, \mathcal{F})\big].$$

In particular, $\check{H}^0(\{U_i \to X\}, \mathcal{F}) = \mathcal{F}(X)$ when $\mathcal{F}$ is a sheaf.

If $\{V_j \to X\}$ is a finer cover, we may fix X-morphisms $V_j \to U_{\theta(j)}$, hence a morphism of complexes $\phi\colon \check{C}^\bullet(\{U_i \to X\}, \mathcal{F}) \to \check{C}^\bullet(\{V_j \to X\}, \mathcal{F})$ and group morphisms

$$\check{H}^p(\{U_i \to X\}, \mathcal{F}) \longrightarrow \check{H}^p(\{V_j \to X\}, \mathcal{F}).$$

If we fix other X-morphisms $V_j \to U_{\bar{\theta}(j)}$ we obtain a **homotopic** morphism $\bar{\phi}$ (i.e. we have $\bar{\phi} - \phi = \mathrm{d}h + h\mathrm{d}$ for some group morphisms $h\colon \check{C}^{p+1}(\{U_i \to X\}, \mathcal{F}) \to \check{C}^p(\{V_j \to X\}, \mathcal{F})$) so that it induces the same morphism on the cohomology groups:

$$h(s)_{j_0 \dots j_p} = \sum_{k=0}^{p} (-1)^k s_{\theta(j_0) \dots \theta(j_k) \bar{\theta}(j_k) \dots \bar{\theta}(j_p)}|_{V_{j_0 \dots j_p}}$$

So we obtain an inductive system of groups and (assuming that any object X admits a cofinal set of covers) the **Cech cohomology** groups are defined to be

$$\check{H}^p(X, \mathcal{F}) = \varinjlim \check{H}^p(\{U_i \to X\}, \mathcal{F}).$$

Note: In multiplicative notation we have

$$\check{H}^1(\{U_i \to X\}, \mathcal{F}) = \big[(s_{ij}) \in \textstyle\prod_{ij} \mathcal{F}(U_{ij})\colon s_{ki} = s_{kj} s_{ji}\big]/\!\equiv,$$

where $(s_{ij}) \equiv (t_{ij})$ when $t_{ij} = s_i s_{ij} s_j^{-1}$ for some $(s_i) \in \prod_i \mathcal{F}(U_i)$.

When the sheaf of automorphisms of a sheaf $\mathcal{T}$ (of rings, modules,...) is commutative, then $\check{H}^1(\{U_i \to X\}, \underline{\mathrm{Aut}})$ classifies, up to isomorphisms, the sheaves on X locally isomorphic to $\mathcal{T}$ on the cover $\{U_i \to X\}$, and $\check{H}^1(X, \underline{\mathrm{Aut}})$ classifies the sheaves on X locally isomorphic to $\mathcal{T}$.

Theorem: $\mathrm{Pic}(S) = \check{H}^1_{\mathrm{fpqc}}(S, \mathbb{G}_m)$ *for any scheme* S.

Proof: Locally free sheaves of rank n in the fpqc topology are locally free in the Zariski topology (p. 219), and the sheaf of automorphisms of the trivial line sheaf is just the functor of points of the multiplicative group, $\mathbb{G}_m(X) = \mathcal{O}_X^*$.

Example: When a cover $X \to S$ admits a finite group of automorphisms G such that the morphism $G \times X := \amalg_G X \to X \times_S X$, $(g,x) \mapsto (x, gx)$, is an isomorphism (a Galois covering,...), then $\mathcal{F}(X)$ is a G-module and we have isomorphisms $G \times \overset{p}{\dots} \times G \times X = X \times_S \overset{p+1}{\dots\dots} \times_S X$, $(g_1, \dots, g_p, x) \mapsto (x, g_1 x, g_1 g_2 x, \dots, g_1 \dots g_p x)$, so that the Cech complex is

$$\check{C}^p(X/S, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Ens}}\big(G \times \overset{p}{\dots} \times G, \mathcal{F}(X)\big)$$
$$(\mathrm{d}f)(g_1, \dots, g_{p+1}) = g_1 f(g_2, \dots, g_{p+1}) + \sum_{k=1}^{p} (-1)^k f(\dots, g_k g_{k+1}, \dots) + (-1)^{p+1} f(g_1, \dots, g_p)$$

and we put $\check{H}^p(X/S, \mathcal{F}) = H^p\big(G, \mathcal{F}(X)\big)$ because these groups only depend on the G-module $\mathcal{F}(X)$. In multiplicative notation, we see that

$$H^1(G, A) = \big[(a_g) \in \textstyle\prod_G A \colon a_{gh} = a_g \cdot g(a_h)\big] / \equiv$$

where $a_g \equiv b_g$ when $b_g = a^{-1} a_g \cdot g(a)$ for some $a \in A$.

Hence $H^1(G, K^*) = \check{H}^1(K/k, \mathbb{G}_m) = 0$ when $k \to K$ is a Galois extension of group G.

When $G = (\sigma)$ is cyclic, a cocycle (a_g) is fully determined by an element $\beta = a_\sigma \in K^*$ of norm 1, so that $\beta = \sigma(\alpha)/\alpha$ for some $\alpha \in K^*$ (**Hilbert's theorem 90**).

15.7.2 Sheafification of a Presheaf

(1) Given a presheaf of sets $\mathcal{P}$, on $\mathcal{P}(X)$ we may consider an equivalence relation: $s \sim s'$ when both coincide on some cover $\{U_i \to X\}$.

Now $\mathcal{P}_s(X) := \mathcal{P}(X)/\sim$ is a separated presheaf, and the natural morphism $\mathcal{P} \to \mathcal{P}_s$ has a universal property: any morphism to a separated presheaf uniquely factors through $\mathcal{P}_s$.

(2) Given a separated presheaf of sets $\mathcal{S}$, for any object X we han an injective natural map

$$\mathcal{S}(X) \longrightarrow L\mathcal{S}(X) := \varinjlim \mathrm{Hom}(\mathcal{R}, \mathcal{S}),$$

where $\mathcal{R}$ runs over the covering sieves of X, and the maps $\mathrm{Hom}(\mathcal{R}, \mathcal{S}) \to \mathrm{Hom}(\mathcal{R}', \mathcal{S})$ are injective. Assume that this inductive limits is a set for any object X (for example, when the situs admits a set of topological generators).

Any morphism $Y \to X$ induces a natural map $L\mathcal{S}(X) \to L\mathcal{S}(Y)$, so that $L\mathcal{S}$ is a presheaf of sets, separated because so is $\mathcal{S}$, and we have a canonical injective morphism $\mathcal{S} \to L\mathcal{S}$.

Theorem: *Let $\mathcal{S}$ be a separated presheaf. If $L\mathcal{S}(X)$ is a set for any object X, then $L\mathcal{S}$ is a sheaf of sets, and any morphism $\mathcal{S} \to \mathcal{F}$ to a sheaf $\mathcal{F}$ uniquely factors trough $L\mathcal{S}$.*

$$\mathrm{Hom}(\mathcal{S}, \mathcal{F}) = \mathrm{Hom}(L\mathcal{S}, \mathcal{F}).$$

Proof: Let us consider a cover $\{U_i \to X\}$ and a $L\mathcal{S}$-compatible family of sections $s_i \in L\mathcal{S}(U_i)$. By definition s_i is given by a $\mathcal{S}$-compatible family of sections $s_{ij} \in \mathcal{S}(V_{ij})$ on a cover $\{V_{ij} \to U_i\}$.

On the cover $\{V_{ij} \to X\}$ the $L\mathcal{S}$-compatible sections $s_{ij} \in \mathcal{S}(V_{ij})$ are $\mathcal{S}$-compatible, because $\mathcal{S}$ is separated, and define the required section $s \in L\mathcal{S}(X)$.

Such section s is unique because $L\mathcal{S}$ is a separated presheaf.

Now let us see the universal property:

Any morphism $\mathcal{S} \to \mathcal{F}$ into a sheaf $\mathcal{F}$ induces natural maps

$$L\mathcal{S}(X) = \varinjlim \operatorname{Hom}(R, \mathcal{S}) \longrightarrow \varinjlim \operatorname{Hom}(R, \mathcal{F}) = \mathcal{F}(X)$$

so that it factors trough $L\mathcal{S}$. The factorization is unique because, by construction, any section $s \in L\mathcal{S}(X)$ comes, on a cover of X, from sections of $\mathcal{S}$.

Definition: When $L(\mathcal{P}_s)$ is a sheaf of sets (v.gr. when $\mathbf{C}$ admits a set of topological generators) $\mathcal{P}^\sharp = L(\mathcal{P}_s)$ is the **sheafification** of $\mathcal{P}$ or the **associated sheaf** to $\mathcal{P}$.

Any morphism $\mathcal{P} \to \mathcal{F}$ into a sheaf of sets $\mathcal{F}$ uniquely factors trough $\mathcal{P}^\sharp$, any section $s \in \mathcal{P}^\sharp(X)$ is defined on some cover of X by sections of $\mathcal{P}$, and two sections $s, s' \in \mathcal{P}(X)$ coincide in $\mathcal{P}^\sharp(X)$ if and only if both coincide on some cover of X.

Note: Even if a category $\mathbf{C}$ has not a final object p, the sections of any presheaf $\mathcal{P}\colon \mathbf{C} \rightsquigarrow \mathbf{Ens}$ on it may be defined to be the **projective limit** (it may be not a set!)

$$\varprojlim_{\mathbf{C}} \mathcal{P} = \varprojlim_{X \in \operatorname{Ob}\mathbf{C}} \mathcal{P}(X) := \operatorname{Hom}_{\text{presh}}(p, \mathcal{P})$$

where p denotes the constant presheaf defined by a one-point set, the functor of points of the hypothetical final object (analogously, we may define the projective limit of a covariant functor $\mathbf{C} \rightsquigarrow \mathbf{Ens}$). Hence, a section s of the projective limit is just a family of elements $s_X \in \mathcal{P}(X)$, compatible in the sense that $s_Y = f^*(s_X)$ for any morphism $f\colon Y \to X$.

The **inductive limit** of $\mathcal{P}$ over $\mathbf{C}$ is defined so that for any set T

$$\operatorname{Hom}\big(\varinjlim_{\mathbf{C}} \mathcal{P}, T\big) = \varprojlim_{X \in \operatorname{Ob}\mathbf{C}} \operatorname{Hom}\big(\mathcal{P}(X), T\big).$$

(the inductive limit may be not a set!). A section s of the inductive limit is just an element $s_X \in \mathcal{P}(X)$, identifying s_X with $f^*(s_X)$ for any morphism $f\colon Y \to X$.

Now, given a functor $f^{-1}\colon \mathbf{C} \rightsquigarrow \mathbf{D}$ and a presheaf of sets $\mathcal{P}$ on $\mathbf{C}$, for any object V in $\mathbf{D}$ we consider the category of pairs $(U, V \to f^{-1}U)$, where $U \in \operatorname{Ob}\mathbf{C}$, the morphisms of $(U, V \to f^{-1}U)$ into $(U', V \to f^{-1}U')$ being the morphisms $\phi\colon U \to U'$ such that $f^{-1}\phi$ is a V-morphism, and we put

$$(f^*\mathcal{P})(V) = \varinjlim_{V \to f^{-1}U} \mathcal{P}(U).$$

So we obtain a "presheaf" $f^*\mathcal{P}$ on $\mathbf{D}$. When it is a true presheaf of sets, for any presheaf $\mathcal{Q}$ on $\mathbf{D}$ we have the adjunction formula

$$\operatorname{Hom}(f^*\mathcal{P}, \mathcal{Q}) = \operatorname{Hom}(\mathcal{P}, f_*\mathcal{Q})$$

since any morphism $f^*\mathcal{P} \to \mathcal{Q}$ induces natural maps $\mathcal{P}(U) \to (f^*\mathcal{P})(f^{-1}U) \to \mathcal{Q}(f^{-1}U) = (f_*\mathcal{Q})(U)$, and any natural transformation $\mathcal{P}(U) \to \mathcal{Q}(f^{-1}U)$ induces a natural transformation

$$(f^*\mathcal{P})(V) = \varinjlim_{V \to f^{-1}U} \mathcal{P}(U) \longrightarrow \varinjlim_{V \to f^{-1}U} \mathcal{Q}(f^{-1}U) \longrightarrow \mathcal{Q}(V),$$

When f^{-1} is continuous and $\mathcal{F}$ is a sheaf on $\mathbf{C}$, if the "sheafification" of $f^*\mathcal{F}$ is a sheaf of sets (also denoted $f^*\mathcal{F}$), we also have $\mathrm{Hom}(f^*\mathcal{F},\mathcal{G}) = \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$ for any sheaf $\mathcal{G}$ on $\mathbf{D}$.

Examples: If $\mathbf{C}$ is a small category, then $f^*\mathcal{F}$ is always a presheaf of sets.

In the case of the full subcategory $f^{-1}\colon \mathcal{R} \hookrightarrow \mathbf{C}|_X$ defined by a crible $\mathcal{R} \subseteq X^\bullet$, we have $f_*\mathcal{Q} = \mathcal{Q}|_{\mathcal{R}}$, and $(f^*\mathcal{P})(U) = \mathcal{P}(U)$ when the X-object U is in $\mathcal{R}$, and $(f^*\mathcal{P})(U) = \emptyset$ otherwise, so that $f^*\mathcal{P}$ is a presheaf of sets.

In particular, if U is an open set in a topological space X, and we consider the inclusion $f^{-1}\colon \mathbf{Op}(U) \hookrightarrow \mathbf{Op}(X)$, for any sheaf $\mathcal{F}$ on U, we have that $f^*\mathcal{F}$ is just the sheaf of continuous sections of $\mathcal{F}^{\mathrm{et}} \to U \hookrightarrow X$.

Comparison Lemma: *Let $\mathbf{C}$ be a situs and $\mathbf{R}$ a full subcategory, stable under fibred products in $\mathbf{C}$, with the induced topology. If any object of $\mathbf{C}$ admits a cover by objects of $\mathbf{R}$ (and for any sheaf $\mathcal{G}$ on $\mathbf{R}$ the inverse image "presheaf" is a presheaf of sets), then the "categories" of sheaves on $\mathbf{C}$ and $\mathbf{R}$ are equivalent, where a sheaf $\mathcal{F}$ on $\mathbf{C}$ corresponds to $\mathcal{F}|_{\mathbf{R}}$:*

1. *We have $\mathrm{Hom}(\mathcal{F},\bar{\mathcal{F}}) = \mathrm{Hom}(\mathcal{F}|_{\mathbf{R}},\bar{\mathcal{F}}|_{\mathbf{R}})$ for any two sheaves $\mathcal{F},\bar{\mathcal{F}}$ on $\mathbf{C}$: given a morphism $\phi\colon \mathcal{F}|_{\mathbf{R}} \to \bar{\mathcal{F}}|_{\mathbf{R}}$, there is a unique morphism $f\colon \mathcal{F} \to \bar{\mathcal{F}}$ such that $\phi = f|_{\mathbf{R}}$.*

2. *Given a sheaf $\mathcal{G}$ on $\mathbf{R}$, we have $\mathcal{G} = \mathcal{F}|_{\mathbf{R}}$ for some sheaf $\mathcal{F}$ on $\mathbf{C}$.*

Proof: Let $f^{-1}\colon \mathbf{R} \hookrightarrow \mathbf{C}$, $f^{-1}V = V$, be the inclusion functor.

First we show that the "sheaf" $f^*\mathcal{G}$ is in fact a sheaf of sets:

The presheaf $\mathcal{P} = f^*\mathcal{G}$ fulfils that $\mathcal{P}(V) = \mathcal{G}(V)$ when $V \in \mathrm{Ob}\,\mathbf{R}$ and, since any cover of V in $\mathbf{C}$ may be refined by a cover in $\mathbf{R}$, we also have $\mathcal{P}_s(V) = \mathcal{G}(V)$. Now, any object U in $\mathbf{C}$ admits a covering crible $\mathcal{R}$ generated by a family $\{V_i \to U\}$ with $V_i \in \mathrm{Ob}\,\mathbf{R}$, so that any covering crible of U contains a covering crible $\mathcal{C}$ generated by a family $\{V_{i\alpha} \to V_i \to U\}$ with $V_{i\alpha} \in \mathrm{Ob}\,\mathbf{R}$. Hence, any $\mathcal{P}_s$-compatible family $s_{i\alpha} \in \mathcal{P}_s(V_{i\alpha})$ defines, $\mathcal{G}$ being a sheaf, sections $s_i \in \mathcal{P}_s(V_i)$, and we have $s_i = s_j$ on $V_i \times_U V_j$ because both coincide on the cover $\{V_{i\alpha} \times_U V_{j\beta} \to V_i \times_U V_j\}$ and $\mathcal{P}_s$ is separated. That is to say, $\mathrm{Hom}(\mathcal{R},\mathcal{P}_s) = \mathrm{Hom}(\mathcal{C},\mathcal{P}_s)$, so that $\mathcal{P}_s^\sharp(U) = \mathrm{Hom}(\mathcal{R},\mathcal{P}_s)$ is a set, and we see that the "sheaf" $f^*\mathcal{G} = \mathcal{P}_s^\sharp$ is a sheaf of sets.

Moreover, $(f^*\mathcal{G})|_{\mathbf{R}} = \mathcal{G}$, because when $U \in \mathrm{Ob}\,\mathbf{R}$, we may take $\mathcal{R} = U^\bullet$.

Finally, the natural map $f^*(\mathcal{F}|_{\mathbf{R}}) \to \mathcal{F}$ is an isomorphism because (p. 468) any object of $\mathbf{C}$ admits a cover by objects of $\mathbf{R}$, and $\big(f^*(\mathcal{F}|_{\mathbf{R}})\big)(V) = (\mathcal{F}|_{\mathbf{R}})(V) = \mathcal{F}(V)$ when $V \in \mathrm{Ob}\,\mathbf{R}$.

Corollary: *Let $\mathbf{C}$ be a situs and $\mathbf{R}$ a full subcategory, stable under fibred products in $\mathbf{C}$, with the induced topology. If $\mathbf{R}$ contains a set of topological generators of $\mathbf{C}$, then the categories of sheaves $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{R}}$ are equivalent, where a sheaf $\mathcal{F}$ on $\mathbf{C}$ corresponds to $\mathcal{F}|_{\mathbf{R}}$.*

Proof: By hypothesis we have full subcategories $\mathbf{G} \hookrightarrow \mathbf{R} \hookrightarrow \mathbf{C}$, where $\mathrm{Ob}\,\mathbf{G}$ is a set of topological generators of $\mathbf{C}$, that we may assume to be stable under fibred products.

By the comparison lemma, the categories $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{R}}$ are equivalent to $\widetilde{\mathbf{G}}$.

Example: Let G be a group and let G-**Sets** be the situs of G-sets. The object G is a topological generator; hence it defines a full subcategory fulfilling the above hypotheses, so that any sheaf $\mathcal{F}$ on G-**Sets** is fully determined by the G-set $\mathcal{F}(G)$. Analogously, given a finite Galois extension $k \to L$ of group G, the object $\mathrm{Spec}\,L$ is a topological generator of the category $\mathbf{Triv}_{L/k}^{\mathrm{op}}$, so that any sheaf $\mathcal{F}$ is fully determined by the G-set $\mathcal{F}(\mathrm{Spec}\,L)$.

Injective Resolutions:

Let $\mathbf{C}$ be a situs with a set of topological generators $\{G_\alpha\}$.

In the category of sheaves of abelian groups on **C**, images, quotients, direct sums and inductive limits are defined to be the sheafification of the corresponding presheaves, so that the usual universal properties and isomorphism theorem $\mathcal{F}/\mathrm{Ker}\, f = \mathrm{Im}\, f$ hold.

Moreover, inductive limits preserve exact sequences, and for any sheaf of abelian groups $\mathcal{F}$,

$$\mathrm{Hom}_{\mathrm{AbSh}}(\mathbb{Z}[G^\bullet_\alpha]^\sharp, \mathcal{F}) = \mathrm{Hom}_{\mathrm{AbPrsh}}(\mathbb{Z}[G^\bullet_\alpha], \mathcal{F}) = \mathrm{Hom}_{\mathrm{Prsh}}(G^\bullet_\alpha, \mathcal{F}) = \mathcal{P}(G_\alpha)$$

so that the category of sheaves of abelian groups admits a generator $U = \oplus_\alpha \mathbb{Z}[G^\bullet_\alpha]^\sharp$, and any subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ is fully determined by the set of all morphisms $U \to \mathcal{F}$ factoring through $\mathcal{F}'$. In particular, the subsheaves of an abelian sheaf $\mathcal{F}$ form a set.

Lemma: *An abelian sheaf $\mathcal{I}$ is injective if and only if the natural map* $\mathrm{Hom}(U, \mathcal{I}) \to \mathrm{Hom}(V, \mathcal{I})$ *is surjective for any subsheaf V of the generator U.*

Proof: If $\mathcal{F}'$ is a subsheaf of a sheaf $\mathcal{F}$, we must show that any morphism $f\colon \mathcal{F}' \to \mathcal{I}$ is the restriction of a morphism $\mathcal{F} \to \mathcal{I}$. Take a morphism $\rho\colon U \to \mathcal{F}$ which does not factor trough $\mathcal{F}'$, put $V = \rho^{-1}(\mathcal{F}')$ and consider the morphism $\phi\colon V \to \mathcal{I}$, $\phi = f\rho$. By hypothesis it admits an extension $\phi'\colon U \to \mathcal{I}$, and the argument of the ideal criterion (replacing the ring A by U and the ideal I by V) let us extend f to a morphism $\mathcal{F}' + \mathrm{Im}\, \rho$, and we conclude.

Theorem: *Any abelian sheaf $\mathcal{F}$ admits a natural monomorphism $\mathcal{F} \to \mathcal{I}$ into an injective sheaf $\mathcal{I}$, so that we have a funtorial injective resolution $\mathcal{F} \to \mathcal{I}^\bullet \mathcal{F}$.*

Proof: Let I be the set of all morphisms $f_i\colon V_i \to \mathcal{F}$, where V_i is a subsheaf of the generator U.

The natural morphisms $(1_i, -f_i)\colon V_i \to (\oplus_I U) \times \mathcal{F}$, where $1_i\colon V_i \to \oplus_I U$ is the inclusion into the i-th term, induce a natural morphism $\oplus_{i\in I} V_i \to (\oplus_I U) \times \mathcal{F}$.

Let $\mathcal{I}_1\mathcal{F}$ be the cokernel, so that we have a natural monomorphism $\mathcal{F} \to \mathcal{I}_1\mathcal{F}$ and, for any morphism $f_i\colon V_i \to \mathcal{F}$, we have a commutative square

$$\begin{array}{ccc} V_i & \longrightarrow & U \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle 1_i} \\ \mathcal{F} & \longrightarrow & \mathcal{I}_1\mathcal{F} \end{array}$$

Now, inductively, we define a sheaf $\mathcal{I}_\alpha\mathcal{F}$ for any ordinal number α, and a natural monomorphism $\mathcal{I}_\beta\mathcal{F} \to \mathcal{I}_\alpha\mathcal{F}$ whenever $\beta \leq \alpha$.

When $\alpha = \beta + 1$, put $\mathcal{I}_\alpha\mathcal{F} = \mathcal{I}_1(\mathcal{I}_\beta\mathcal{F})$; and put $\mathcal{I}_\alpha\mathcal{F} = \varinjlim_{\beta<\alpha} \mathcal{I}_\beta\mathcal{F}$ when α is a limit ordinal.

Let σ be the first ordinal number with infinite cardinal strictly greater than the cardinal $\aleph$ of the set of subsheaves of U, and let us see that $\mathcal{I}_\sigma\mathcal{F}$ is an injective sheaf (so conluding).

Let $f\colon V \to \mathcal{I}_\sigma\mathcal{F}$ be a morphism. If we show that $f(V) \subseteq \mathcal{I}_\alpha\mathcal{F}$ for some $\alpha < \sigma$, we conclude. Now, since σ is a limit ordinal, we have $\mathcal{I}_\sigma\mathcal{F} = \varinjlim_{\alpha<\sigma} \mathcal{I}_\alpha\mathcal{F}$, so that

$$V = f^{-1}\big(\varinjlim_{\alpha<\sigma} \mathcal{I}_\alpha\mathcal{F}\big) = \varinjlim_{\alpha<\sigma} f^{-1}(\mathcal{I}_\alpha\mathcal{F}).$$

If the chain of subsheaves $f^{-1}(\mathcal{I}_\alpha\mathcal{F})$ of V stabilizes at some index, the result is obvious.

Otherwise, there is a set $\Omega = \{\alpha_i\}$ of ordinal numbers $\alpha_i < \sigma$, with limit σ, and such that $|\Omega| \leq \aleph$, against the following result of the theory of ordinal numbers:

Lemma: *Let $\aleph$ be an infinite cardinal, and σ the first ordinal with $\aleph < |\sigma|$. If $\Omega = \{\alpha_i\}$ is a set of ordinal numbers $\alpha_i < \sigma$, with limit σ, then $\aleph < |\Omega|$.*

Proof: Otherwise $|\Omega| \leq \aleph$. Since $|\alpha_i| \leq \aleph$, the cardinal of the limit of Ω is bounded above by the cardinal of a union of $\aleph$ sets of cardinal $\aleph$; i.e. by $\aleph^2 = \aleph$. Absurd. q.e.d.

Definition: The right derived functors of a left exact functor F, defined on the category of abelian sheaves over a situs with a set of topological generators, are $(R^n F)(\mathcal{F}) = H^n[F(\mathcal{I}^\bullet \mathcal{F})]$.

In particular, we have the cohomology groups $H^n(X, \mathcal{F}) = H^n[(\mathcal{I}^\bullet \mathcal{F})(X)]$ of an object X with coefficients in an abelian sheaf $\mathcal{F}$.

Part VI

Exercises

1. Analysis I

1. If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are cardinal numbers, show that $0 + \mathfrak{a} = \mathfrak{a}$, $0 \cdot \mathfrak{a} = 0$, $1 \cdot \mathfrak{a} = \mathfrak{a}$, $\mathfrak{a}^0 = 1$ (in particular $0^0 = 1$), $\mathfrak{a}^1 = \mathfrak{a}$, $1^{\mathfrak{a}} = 1$, $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$, $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$, $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$, $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$, $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$, $(\mathfrak{a}\mathfrak{b})^{\mathfrak{c}} = (\mathfrak{a}^{\mathfrak{c}})(\mathfrak{b}^{\mathfrak{c}})$, $\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = (\mathfrak{a}^{\mathfrak{b}})(\mathfrak{a}^{\mathfrak{c}})$, $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\mathfrak{c}}$.

   Moreover, if $\mathfrak{a} \le \mathfrak{b}$, prove that $\mathfrak{a} + \mathfrak{c} \le \mathfrak{b} + \mathfrak{c}$ and $\mathfrak{a}\mathfrak{c} \le \mathfrak{b}\mathfrak{c}$.

2. If $\mathfrak{a}$ is a non null cardinal, show that $0^{\mathfrak{a}} = 0$.

   If $\mathfrak{a}_1 \le \mathfrak{a}_2$, $\mathfrak{b}_1 \le \mathfrak{b}_2$ are non null cardinals, show that $\mathfrak{a}_1^{\mathfrak{b}_1} \le \mathfrak{a}_2^{\mathfrak{b}_2}$. (Remark that $0 \le 1$, while $1 = 0^0 > 0^1 = 0$).

3. If $2 \le n \in \mathbb{N}$, prove that $n + \aleph_0 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$, $n\aleph_0 = \aleph_0$, $\aleph_0^n = \aleph_0$, $n2^{\aleph_0} = 2^{\aleph_0}$, $n^{\aleph_0} = 2^{\aleph_0}$, $2^{\aleph_0}2^{\aleph_0} = 2^{\aleph_0}$, $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, $\aleph_0 2^{\aleph_0} = 2^{\aleph_0}$, $\aleph_0^{\aleph_0} = 2^{\aleph_0}$.

4. Let F be the set of all finite subsets of $\mathbb{N}$. Prove that $|F| = \aleph_0$.

5. Prove that the rings of polynomials $\mathbb{Q}[x_1, \dots, x_n]$ are countable.

6. Let $\mathfrak{a} = |A|$ be an infinite cardinal. Prove that $\aleph_0 \le \mathfrak{a}$, $\aleph_0 + \mathfrak{a} = \mathfrak{a}$ and $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$. (*Hint*: By Zorn's lemma, A is a disjoint union of countable subsets, so that $\mathfrak{a} = \aleph_0\mathfrak{b}$, and $\mathfrak{a} + \mathfrak{a} = \aleph_0\mathfrak{b} + \aleph_0\mathfrak{b} = \aleph_0\mathfrak{b} = \mathfrak{a}$.)

   Also prove that $\aleph_0\mathfrak{a} = \mathfrak{a}$ and $\mathfrak{a}\mathfrak{a} = \mathfrak{a}$. (*Hint*: Consider pairs (B, ϕ) where $B \subseteq A$ and $\phi\colon B \to B \times B$ is a bijection. By Zorn's lemma there is a maximal pair (B, ϕ). If $\mathfrak{b} := |B| < \mathfrak{a}$, take a subset $C \subseteq B^c$ of cardinal $\mathfrak{b}$, so that we have a bijection $C \to (B \times C) \amalg (C \times B) \amalg (C \times C)$; hence a bijection $B \amalg C \to (B \times B) \amalg (B \times C) \amalg (C \times B) \amalg (C \times C) = (B \amalg C) \times (B \amalg C)$ extending ϕ.)

   If moreover $2 \le \mathfrak{b} \le 2^{\mathfrak{a}}$, prove also that $\mathfrak{b}^{\mathfrak{a}} = 2^{\mathfrak{a}}$.

7. If Y is a subset of a well-order X, show that Y is isomorphic to an initial ray of X.

8. Show that any well-order X coincides with the set of its incomplete initial rays, ordered by inclusion.

9. Let α, β be the ordinal numbers defined by two well-orders A, B respectively. Then $\alpha + \beta$ is defined to be the ordinal of the disjoint union $A \amalg B$, where $a < b$ for any $a \in A$, $b \in B$; and $\alpha\beta$ is defined to be the ordinal of $A \times B$, with the lexicographical order: $(a_1, b_1) < (a_2, b_2)$ when $a_1 < a_2$, or $a_1 = a_2$ and $b_1 < b_2$. Show that $\omega < \omega + 1$ while $1 + \omega = \omega$, and that $\omega < 2\omega = \omega + \omega$, while $\omega 2 = \omega$.

10. Let $x, y \in \mathbb{R}$. If $x \le y$, prove that $-y \le -x$. If $0 < x < y$, prove that $y^{-1} < x^{-1}$.

11. If a Cauchy sequence (q_n) represents $x \in \mathbb{R}$, show that $(|q_n|)$ represents $|x|$.

12. For all $x, y \in \mathbb{R}$, show that $|x + y| \le |x| + |y|$.

13. Let $z = a + bi$ be a complex number, $b \ne 0$, and $\rho = |z|$. Show that (the sign is the sign of b)

$$\sqrt{z} = \sqrt{\rho}\,\frac{\rho + z}{|\rho + z|} = \sqrt{\frac{\rho + a}{2}} \pm i\sqrt{\frac{\rho - a}{2}}.$$

14. If there is a map $f\colon X \to \mathbb{N}$ with finite fibres, show that $|X| \le \aleph_0$.

    Prove that the set of **algebraic** complex numbers (root of a non null polynomial with rational coefficients) is countable. Conclude that the set of **transcendent** (non-algebraic) real numbers is uncountable. (In particular, they exist!). (*Hint*: Exercise 5).

15. An **ordered ring** is a (conmutative) ring $A \ne 0$ with a total orden $\le$ such that $a \le b \Rightarrow a + c \le b + c$ and $0 \le a, 0 \le b \Rightarrow 0 \le ab$, for all $a, b, c \in A$. In an ordered ring, prove that $0 < 1$, $-1 < 0$ and $a \le b, 0 \le c \Rightarrow ac \le bc$. Moreover, exactly one of the following is true: $0 < a$, $0 < -a$ or $a = 0$.

    Prove that, up to a ring isomorphism, the ring $\mathbb{Z}$ of integer numbers is the unique ordered domain A such that the set of positive elements $A_+ = \{a \in A\colon 0 < a\}$ is a well-order. (*Hint*: The first positive element m is $m = 1$, since $m < 1$ implies that $m^2 < m$.)

    Prove that, up to a ring isomorphism, the field of real numbers $\mathbb{R}$ is the unique complete ordered field (in the sense that any bounded above set admits a supremum).

16. Prove that any compact connected set in $\mathbb{R}$ is a closed interval $[a, b]$, where $a \le b$.

17. Prove that $f\colon (-1,1)\to\mathbb{R}$, $f(x)=\frac{x}{1-x^2}$ is a homeomorphism. Conclude that any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

18. Show that a map $f\colon X\to Y$ between metric spaces is continuous at a point $x\in X$ if and only if for any $\varepsilon>0$ there exists $\delta>0$ such that $f\big(B(x,\delta)\big)\subseteq B(f(x),\varepsilon)$.

19. Prove that two continuous maps $f,g\colon X\to Y$ between metrizable spaces coincide whenever both coincide on a dense subset of X.

20. If p is a point of a metric space X, show that $f\colon X\to\mathbb{R}$, $f(x)=\mathrm{d}(p,x)$, is continuous.

21. If $f,h\colon X\to\mathbb{R}$ are continuous functions, prove that so are $\min(f,h)$ and $\max(f,h)$.

22. Prove that the cardinal of the ring of continuous functions $\mathcal{C}(\mathbb{R})$ is $2^{\aleph_0}$.

23. If X,Y are metric spaces, show that $\mathrm{d}\big((x,y),(x',y')\big):=\mathrm{d}(x,x')+\mathrm{d}(y,y')$ is a metric on $X\times Y$, and that we have a canonical isometry of the completion of $X\times Y$ onto $\widehat{X}\times\widehat{Y}$.

24. Let $f\colon U\to V$ be a bijection between two open subsets of $\mathbb{R}$. If f is differentiable, with non null derivative at any point, prove that so is the inverse $g\colon V\to U$, and that $g'(y)=\frac{1}{f'(x)}$, where $y=f(x)$.

    Moreover, show that $\int f(x)\mathrm{d}x+\int g(y)\mathrm{d}y=xf(x)=g(y)y$, (i.e.; if $F(x)$ is a primitive of $f(x)$, then $G(y)=g(y)y-F\big(g(y)\big)$ is a primitive of $g(y)$), and conclude that (draw a picture)

$$\int_{f(a)}^{f(b)} g(y)\mathrm{d}y=f(b)b-f(a)a-\int_a^b f(x)\mathrm{d}x.$$

25. Let $f,g\colon(a,b)\to\mathbb{R}$ be differentiable functions. If $|f'(x)|\leq g'(x)$, $\forall x\in(a,b)$, prove that $|f(y)-f(x)|\leq |g(y)-g(x)|$, $\forall x,y\in(a,b)$.

26. Let $f\colon(a,b)\to\mathbb{R}$ be a differentiable function. Prove that $f'(x)\geq 0$, $\forall x\in(a,b)$, if and only if f is an increasing function. Conclude that $f'(x)\leq 0$, $\forall x\in(a,b)$, if and only if f is a decreasing function.

27. Given a function $f\colon U-\{a\}\to\mathbb{R}$, we say that $c\in\mathbb{R}$ is the **limit** of $f(x)$ as $x\to a$ when for any $\varepsilon>0$ there exists $\delta>0$ such that $B(a,\delta)\subseteq U$ and $0<|x-a|<\delta\Rightarrow|f(x)-c|<\varepsilon$, and we put $\lim\limits_{x\to a}f(x)=c$ or we say that $f(x)\to c$ as $x\to a$. Prove the following assertions:

    (a) If it exists, the limit is unique.

    (b) The limit of $f(x)$ as $x\to a$ exists if and only if f admits an extension $\tilde{f}\colon U\to\mathbb{R}$ continuous at $x=a$. Moreover, $\tilde{f}(a)=\lim\limits_{x\to a}f(x)$.

    (c) A function $f\colon U\to\mathbb{R}$ is continuous if and only if $\lim\limits_{x\to a}f(x)=f(a)$, $\forall a\in U$.

    (d) A function $f\colon U\to\mathbb{R}$ is differentiable at $x=a$ if and only if the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{t\to 0}\frac{f(a+t)-f(a)}{t}$$

    exists, and in such a case such limit is just $f'(a)$.

    (e) If a continuous function f is differentiable at $x=a$, then $f(a+t)=f(a)+f'(a)t+\tilde{u}(t)t$ for some continuous function $\tilde{u}(t)$ defined on a neighborhood of $t=0$ and vanishing at $t=0$.

    (f) **L'Hospital's Rule:** Let $f,g\colon U\to\mathbb{R}$ be differentiable functions such that $f(a)=g(a)=0$. If g and g' do not vanish on $U-\{a\}$, and the limit of $\frac{f'(x)}{g'(x)}$ as $x\to a$ exists, then so does the limit of $\frac{f(x)}{g(x)}$ as $x\to a$, and both coincide. (*Hint*: Use Cauchy's theorem to prove that in a neighborhood $\frac{f(x)}{g(x)}=\frac{f'(\xi)}{g'(\xi)}$, where $0<|\xi-a|<|x-a|$, so that $\xi\to a$ when $x\to a$).

28. Let $f\colon U\to\mathbb{R}$ be a function of class $\mathcal{C}^n$, n≥ 2, such that $f''(a)=\ldots=f^{(n-1}(a)=0$.

    If n is even and $f^{(n}(a)>0$, prove that $x=a$ is a point of **strict convexity** (in a neighborhood of a we have $f(x)>f(a)+f'(a)(x-a)$ when $x\neq a$.)

    If n is even and $f^{(n}(a)<0$, prove that $x=a$ is a point of **strict concavity** (in a neighborhood of a we have $f(x)<f(a)+f'(a)(x-a)$ when $x\neq a$.)

If n is odd and $f^{(n}(a) > 0$, then in a neighborhood of a we have that $f(x) > f(a) + f'(a)(x-a)$ when $x > a$, and $f(x) < f(a) + f'(a)(x-a)$ when $x < a$.

If n is odd and $f^{(n}(a) < 0$, then in a neighborhood of a we have that $f(x) < f(a) + f'(a)(x-a)$ when $x > a$, and $f(x) > f(a) + f'(a)(x-a)$ when $x < a$.

29. Prove that the Taylor polynomial $T_a^n f$ is the unique polynomial $P(x)$ of degree $\leq n$ such that $P(a) = f(a)$, $P'(a) = f'(a)$, $P''(a) = f''(a), \ldots, P^{(n}(a) = f^{(n}(a)$.

30. Let f be a function of class $\mathcal{C}^n$ on an set $U \subseteq \mathbb{R}$ and $a \in U$. Prove that $f(x) = (T_a^n f)(x) + O(x)(x-a)^n$, where $O(x)$ is a continuous function on U such that $O(a) = 0$.

31. If we put $\int_a^a f \mathrm{d}x = 0$, and $\int_a^b f \mathrm{d}x = -\int_b^a f \mathrm{d}x$ when $b < a$, show that $\int_a^b f \mathrm{d}x + \int_b^c f \mathrm{d}x + \int_c^a f \mathrm{d}x = 0$ for any three real numbers a, b, c.

32. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Show that $\int_a^b f \mathrm{d}x = f(\xi)(b-a)$ for some $a < \xi < b$.

    If moreover $\int_a^b |f| \mathrm{d}x = 0$, prove that $f = 0$.

    Conclude that $\mathrm{d}(f,h) = \int_a^b |h-f| \mathrm{d}x$ defines a metric on the ring $\mathcal{C}([a,b])$.

33. Let $\ln\colon \mathbb{R}_+ \to \mathbb{R}$ be the inverse of the exponential $e^x\colon \mathbb{R} \to \mathbb{R}_+$. Prove that $(\ln x)' = \frac{1}{x}$, and determine a primitive of the function $\frac{1}{x}\colon \mathbb{R}_- \to \mathbb{R}$.

34. For any positive real number t, show that $\int_0^\infty e^{-tx} \mathrm{d}x = t^{-1}$. Differentiate under the integral sign to obtain $\int_0^\infty x e^{-tx} \mathrm{d}x = 1/t^2$, $\int_0^\infty x^2 e^{-tx} \mathrm{d}x = 2/t^3$ and in general $\int_0^\infty x^n e^{-tx} \mathrm{d}x = n!/t^{n+1}$. Then obtain Euler's formula

$$n! = \int_0^\infty x^n e^{-x} \mathrm{d}x.$$

35. Let us consider the integral $I = \displaystyle\int_0^\infty e^{-x^2/2} \mathrm{d}x$. For any $t \in \mathbb{R}$ set $f(t) = \displaystyle\int_0^\infty \frac{e^{-t^2(1+x^2)/2}}{1+x^2} \mathrm{d}x$.

    Show that $f(0) = \pi/2$ and $f(t) \to 0$ as $t \to \infty$. Differentiate under the integral sign to obtain

$$f'(t) = -e^{-t^2/2} \int_0^\infty e^{-(tx)^2/2} t \mathrm{d}x = -e^{-t^2/2} \int_0^\infty e^{-y^2/2} \mathrm{d}y = -I e^{-t^2/2},$$
$$f(a) - f(0) = \int_0^a f'(t) \mathrm{d}t = -I \int_0^a e^{-t^2/2} \mathrm{d}t.$$

    When $a \to \infty$, conclude that $I^2 = \pi/2$, so obtaining the gaussian integral $\displaystyle\int_{-\infty}^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}} \mathrm{d}t = 1$.

    Now, for any positive real number t, show that $\int_{-\infty}^\infty e^{-tx^2/2} \mathrm{d}x = \sqrt{2\pi/t}$, and differentiate under the integral sign to obtain the moments of the gaussian:

$$\int_{-\infty}^\infty x^{2n} e^{-tx^2/2} \mathrm{d}x = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)\sqrt{2\pi}}{t^{(2n+1)/2}}$$
$$\int_{-\infty}^\infty \frac{x^{2n} e^{-x^2/2}}{\sqrt{2\pi}} \mathrm{d}x = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

36. Consider the Euler interpolation $n! = \int_0^\infty x^n e^{-x} \mathrm{d}x$ of the factorial. Use the substitution $y = \sqrt{x}$ to show that $\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$.

37. Assume that there is a non constant polynomial $P(x) \in \mathbb{C}[x]$ without complex roots, so that we have a well-defined continuous function $\frac{1}{P(z)}\colon \mathbb{C} \to \mathbb{C}$. For any positive $r \in \mathbb{R}$ consider the integral

$$I(r) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{P(re^{i\theta})}.$$

    Differentiate under the integral sign to obtain that $I(r)$ is constant:

$$\frac{\partial}{\partial \theta} \frac{1}{P(re^{i\theta})} = -\frac{P'(re^{i\theta})}{P(re^{i\theta})^2} ire^{i\theta} = ir \frac{\partial}{\partial r} \frac{1}{P(re^{i\theta})}$$

$$I'(r) = \int_0^{2\pi} \frac{1}{ir} \frac{\partial}{\partial\theta} \frac{1}{P(re^{i\theta})} \mathrm{d}\theta = \frac{1}{irP(re^{i\theta})}\Big|_{\theta=0}^{\theta=2\pi} = 0.$$

Now, $\left|\int_0^{2\pi} \frac{1}{P(re^{i\theta})}\right| \leq \int_0^{2\pi} \frac{1}{|P(re^{i\theta})|}$ goes to 0 as $r \to \infty$. Hence $I(r) = 0$. Obtain a contradiction, since $I(r) \to 2\pi/P(0)$ as $r \to 0$, so obtaining a new proof of **D'Alembert's theorem**.

38. If $0 \neq z \in \mathbb{C}$, show that the argument of $z/\bar{z}$ doubles the argument of z.
    Show that there are infinite angles with rational sine and cosine.
    Show that the equation $x^2 + y^2 = z^2$ has infinite integer solutions with x, y, z coprime.
39. Prove that the centers of the exterior squares drawn on two sides of a triangle and the middle point of the third side determine a rectangle isosceles triangle.
40. Where does the following argument fail? $2 = e^{\ln 2} = e^{2\pi i \frac{\ln 2}{2\pi i}} = (e^{2\pi i})^{(\ln 2)/(2\pi i)} = 1^{(\ln 2)/(2\pi i)} = 1$.
41. Consider the **hyperbolic** functions $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Show that $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$, $\cosh^2 x - \sinh^2 x = 1$, $\cosh(x+y) = (\cosh x)(\cosh y) + (\sinh x)(\sinh y)$, $\sinh(x+y) = (\sinh x)(\cosh y) + (\cosh x)(\sinh y)$.

2. Linear Algebra

1. Does the operation $x * y = (x+y)/(1+xy)$ define a group structure on the interval $(-1, 1)$?
2. Given a group G, the **opposite group** G^{op} is defined to be the set G with the operation $a * b := ba$. Show that we have a natural group isomorphism $G \xrightarrow{\sim} G^{\mathrm{op}}$, $a \mapsto a^{-1}$.
3. Prove that the set $\mathrm{Aut}(G)$ of all automorphisms of a group G, with the composition of maps, is a group, that any element $a \in G$ defines an **inner automorphism**, $\tau_a \colon G \to G$, $\tau_a(x) = axa^{-1}$, and that so we obtain a morphism of groups $\tau \colon G \to \mathrm{Aut}(G)$, $\tau(a) = \tau_a$.
4. Prove that a group G is commutative if and only if the product $G \times G \to G$ is a group morphism.
5. Prove that no group is a union of two smaller subgroups.
6. Let H be a subgroup of a group G. If $aHa^{-1} \subseteq H$, $\forall a \in G$, prove that $aHa^{-1} = H$, $\forall a \in G$.
7. Let H_1 and H_2 be normal subgroups of a group G. Prove that $H_1H_2 := \{h_1h_2\}_{h_i \in H_i}$ is a normal subgroup of G.
8. If G is a group, prove that $\mathrm{ord}(ab) = \mathrm{ord}(ba)$, $\forall a, b \in G$. (*Hint*: $\mathrm{ord}(a) = \mathrm{ord}(bab^{-1})$.)
9. Prove that any subgroup of index 2 is a normal subgroup.
10. If $H_0 \subset H_1 \subset \ldots \subset H_n$ is a sequence of normal subgroups, show that $|H_n/H_0| = \prod_i |H_i/H_{i-1}|$.
11. Prove that any group of prime order is cyclic.
12. Prove that a finite group G is a cyclic group if and only if for any divisor d of $|G|$ there is a unique subgroup of G of order d.
13. Let G be a finite group of order n and $a \in G$. Prove that the equation $x^m = a$ admits a unique solution in G when n, m are coprime. (*Hint*: $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$.)
14. Show that the **center** $Z(G) = \{g \in G \colon gx = xg,\ \forall x \in G\}$ of a group G is a normal subgroup of G.
    If $G/Z(G)$ is a cyclic group, prove that G is an abelian group.
15. Prove that any group of order 4 is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
16. If H_1, H_2 are subgroups of a group G, we put $H_1H_2 = \{h_1h_2 \colon h_i \in H_i\}$.
    Prove that H_1H_2 is a subgroup if and only if $H_1H_2 = H_2H_1$. Conclude that H_1H_2 is a subgroup whenever H_1 is a normal subgroup of G.
    If G is a finite group, prove that $|H_1| \cdot |H_2| = |H_1H_2| \cdot |H_1 \cap H_2|$.
17. Show that A_4 has no subgroup of order 6. (The converse of Lagrange's theorem fails).

18. Prove that the symmetric group S_n has trivial center, $Z(S_n) = 1$, when $n \geq 3$.
19. Show that A_n is generated by the 3-cycles. (*Hint*: $(12)(23) = (123)$ and $(12)(34) = (123)(234)$).
20. Show that any non trivial normal subgroup of S_4 is $\{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$ or A_4.
21. If $n \neq 4$, prove that A_n is the unique non trivial normal subgroup H of S_n.
    (*Hint*: Let $\sigma = \sigma_1 \dots \sigma_r$ be the decomposition as a product of disjoint cycles, of decreasing order, of an element of H. If $\sigma_1 = (1, \dots, d)$, $d \geq 3$, take $\bar\sigma = (d, \dots, 3, 1, 2)\sigma_2^{-1} \dots \sigma_r^{-1}$, so that $\bar\sigma \in H$ and $(1, d, 2) = \bar\sigma\sigma \in H$. If $\sigma = (12)(34)\sigma_3 \dots \sigma_r$, take $\bar\sigma = (13)(24)\sigma_3^{-1} \dots \sigma_r^{-1}$, so that $(14)(23) = \bar\sigma\sigma \in H$ and conclude that $(154) = (14)(23) \cdot (23)(15) \in H$).
22. Prove that the sign defines the unique epimorphism $S_n \to \{\pm 1\} = S_2$, and show the existence of an epimorphism $S_4 \to S_3$. (*Hint*: A tetrahedron has 3 pairs of opposite edges).
    If $n > 4$, prove that there is no epimorphism $S_n \to S_m$, where $2 < m < n$.
23. Prove that the group of automorphisms of a cyclic group of order n is isomorphic to the group $(\mathbb{Z}/n\mathbb{Z})^*$ of invertible elements of the ring $\mathbb{Z}/n\mathbb{Z}$.
24. If d divides n, show that the number of elements of $\mathbb{Z}/n\mathbb{Z}$ of order d is $\phi(d)$.
    Conclude that $n = \sum_{d|n} \phi(d)$ for any natural number n. (We agree that $\phi(1) = 1$).
25. If A is a subset of a set X, there is an indicator function $a\colon X \to \mathbb{F}_2$ such that $a(A) = 1$, $a(A^c) = 0$. So we may identify $\mathcal{P}(X)$ with the ring of $\mathbb{F}_2$-valued functions on X, the null function corresponding to $\emptyset$ and the function 1 corresponding to X. Prove that
    (a) We have $a + a = 0$, $a^2 = a$ and, if b corresponds to a subset $B \subseteq X$, then
    $$\begin{aligned} &A^c = X - A \text{ corresponds to } 1 + a,\\ &A \cap B \text{ corresponds to } ab,\\ &A \cup B \text{ corresponds to } a + b + ab.\\ &A \bigtriangleup B = (A \cap B^c) \cup (A^c \cap B) \text{ corresponds to } a + b.\\ &A - B = A \cap B^c \text{ corresponds to } a + ab. \end{aligned}$$
    (b) The symmetric difference $A \bigtriangleup B$ defines a commutative group structure on $\mathcal{P}(X)$.
    (c) $A \cap (B \bigtriangleup C) = (A \cap B) \bigtriangleup (A \cap C)$ and $(A \bigtriangleup B) - C = (A - C) \bigtriangleup (B - C)$.
26. Let 0 and 1 represent the logical values *true* and *false* respectively, so that the propositional calculus may be viewed as a calculus with $\mathbb{F}_2$-valued functions, and tautologies are just logical operations corresponding to the zero function. Prove the following correspondence between logical operations and operations with $\mathbb{F}_2$-valued functions:

Logical Operation	Functional Operation
$\neg p$	$1 + p$
$p \vee q$	pq
$p \wedge q$	$p + q + pq$
$p \Rightarrow q$	$(1 + p)q$
$p \Leftrightarrow q$	$p + q$

    If $a_1, \dots, a_n \in \mathbb{F}_2$, check that $(p_1 + a_1 + 1) \cdots (p_n + a_n + 1)$ vanishes everywhere, except at $p_1 = a_1, \dots, p_n = a_n$. Determine a logical operator with a given truth table (*Post's problem*).
27. Prove that any finite integral ring A is a field. (*Hint*: Consider the group morphisms $A \xrightarrow{a\cdot} A$).
28. If A is a ring, prove that there is a unique ring morphism $\mathbb{Z} \to A$.
29. Let $\mathrm{End}(E)$ be the (non commutative) ring of all endomorphisms of an abelian group E. Prove that the structures of k-vector space on E correspond to the ring morphisms $k \to \mathrm{End}(E)$.
30. Prove that the direction of a linear subvariety is well-defined: if $x + V = y + W$, then $V = W$.
31. Let k be a field where $1 + 1 \neq 0$. If a non empty subset X of a k-vector space contains the line passing through any two points of X, prove that X is a linear subvariety.
    Does this statement hold when $k = \mathbb{F}_2$?

32. Let V_1, V_2 be two vector subspaces of a vector space E. Prove that $(V_1+V_2)/V_2 \simeq V_1/(V_1 \cap V_2)$. If moreover $V_1 \subseteq V_2$, prove that $(E/V_1)/(V_2/V_1) \simeq E/V_2$.

33. If $f\colon E \to E$ is a linear map and $f^2 = f$, prove that $E = (\operatorname{Ker} f) \oplus (\operatorname{Im} f)$.

34. Let V, W be two vector subspaces of a vector space E. Prove that W is a supplement of V in E if and only if the canonical projection $\pi\colon E \to E/V$ induces an isomorphism $W \simeq E/V$.

35. Let $f\colon E \to F$ be a linear map. Let $V \subset E$ be a vector subspace and put $U = \{e \in E\colon e \notin V\}$. If $f(V) = F$, prove that $f(U) = F$.

36. If $V_0 \subset V_1 \subset \ldots \subset V_n$ is a sequence of vector subspaces, show that $\dim(V_n/V_0) = \sum_i \dim(V_i/V_{i-1})$.

37. If $0 \to E_0 \to E_1 \to \ldots \to E_n \to 0$ is an exact sequence of finite-dimensional vector spaces, show that

$$\dim E_0 - \dim E_1 + \ldots + (-1)^n \dim E_n = 0.$$

38. Let $f\colon E \to E'$ be a linear map between finite dimensional vector spaces. Prove that the matrix of f in some bases has all the coefficients 0 except some ones in the diagonal, which are just 1.

39. Given linear maps $f, h\colon E \to E'$, prove that $\operatorname{Im}(f+h) \subseteq \operatorname{Im} f + \operatorname{Im} h$. Conclude that $\operatorname{rk}(A+B) \leq \operatorname{rk} A + \operatorname{rk} B$, where $A, B \in M_{m\times n}(k)$.

40. Given linear maps $h\colon E \to E'$, $f\colon E' \to E''$, prove the following statements:

    (a) $\operatorname{Im}(fh) \subseteq \operatorname{Im} f$, $\operatorname{Ker} h \subseteq \operatorname{Ker} fh$.

    (b) $\dim(\operatorname{Ker} fh) \leq \dim(\operatorname{Ker} f) + \dim(\operatorname{Ker} h)$. (*Hint*: Consider the map $h\colon \operatorname{Ker} fh \to \operatorname{Ker} f$.)

    (c) $\operatorname{Im} fh = \operatorname{Im} f$, when h is an isomorphism; and $\operatorname{Ker} fh = \operatorname{Ker} h$, when f is an isomorphism.

    Given matrices $A \in M_{m\times n}(k)$, $B \in M_{n\times r}(k)$, conclude that $\operatorname{rk} AB \leq \operatorname{rk} A$ (and the equality holds when B is invertible), $\operatorname{rk} AB \leq \operatorname{rk} B$ (and the equality holds when A is invertible), and $\operatorname{rk} AB \geq \operatorname{rk} A + \operatorname{rk} B - n$.

41. If two linear forms $\omega, \omega'\colon E \to k$ have equal kernel, show that $\omega' = \lambda\omega$ for some $0 \neq \lambda \in k$.

42. If $E_1 \xrightarrow{f} E_2 \xrightarrow{h} E_3$ is an exact sequence of linear maps, prove that so is $E_3^* \xrightarrow{h^*} E_2^* \xrightarrow{f^*} E_1^*$.

43. The **cokernel** of a linear map $f\colon E \to F$ is $\operatorname{Coker} f := F/\operatorname{Im} f$. Prove that $(\operatorname{Ker} f)^* = \operatorname{Coker} f^*$ and $(\operatorname{Coker} f)^* = \operatorname{Ker} f^*$. (*Hint*: $0 \to \operatorname{Im} f \to F \to \operatorname{Coker} f \to 0$ is exact).

44. If E, F are finite-dimensional k-vector spaces, show that the map $\operatorname{Hom}_k(E,F) \to \operatorname{Hom}_k(F^*, E^*)$, $f \mapsto f^*$, is an isomorphism.

45. If $E = V \oplus W$, show that we have a natural isomorphism $E^* = V^* \times W^*$.

46. An endomorphism $T\colon E \to E$ of a finite dimensional vector space is **elementary** when it fixes all the vectors in some hyperplane of E. Prove the following statements:

    (a) An endomorphism T is elementary if and only if there is a vector $e \in E$ and a linear form $\omega \in E^*$ such that $T(x) = x + \omega(x)e$, $\forall x \in E$. In such case T is invertible if and only if $\omega(e) \neq -1$, the inverse being $T^{-1}(x) = x - \frac{\omega(x)}{1+\omega(e)}e$.

    (b) Given an hyperplane H and a vector $e \notin H$, for any vector $v \in E$ there is an elementary endomorphism T such that $T(e) = v$ and $T(x) = x$, $\forall x \in H$.

    (c) If $n = \dim E$, then any endomorphism $T\colon E \to E$ is a product of n elementary endomorphisms. (*Hint*: Extend a basis $e_1, \ldots, e_k$ of $\operatorname{Ker} T$ to a basis of E such that $T(e_{k+1}) = e_{k+1}, \ldots, T(e_{k+s}) = e_{k+s}$, it may be that $s = 0$, and prove by induction on $n-k-s$ that T is a product of $n-s$ elementary endomorphisms: it is clear when $n-k-s = 0$, and in the general case consider an elementary endomorphism S fixing $e_{k+1}, \ldots, e_{k+s}$ and such that $S(T(e_{k+s+1})) = e_{k+s+1}$, so that $ST(e_{k+1}) = e_{k+1}, \ldots, ST(e_{k+s+1}) = e_{k+s+1}$ and ST is a product of $n-s-1$ elementary endomorphisms.)

47. Show that the trace $\mathrm{tr}(a_{ij}) = \sum_i a_{ii}$ of square matrices satisfies $\mathrm{tr}(AB) = \mathrm{tr}(BA)$; hence
$$\mathrm{tr}(A_1 \dots A_{n-1} A_n) = \mathrm{tr}(A_n A_1 \dots A_{n-1}).$$
Moreover, $\langle A|B\rangle := \mathrm{tr}(\bar{A}^t B)$ is a scalar product on the vector space $M_{n\times n}(\mathbb{K})$.

48. Show that the polynomial $\|te + \frac{v\cdot e}{|e\cdot v|} v\|^2 = \|e\|^2 t^2 + 2|e\cdot v|t + \|v\|^2$, where $t \in \mathbb{R}$ and e, v are two vectors in a Euclidean vector space, has discriminant ≤ 0, and obtain another proof of the **Cauchy-Schwarz inequality**.

49. Let e, v be non null vectors in a Euclidean vector space.
If $|\langle e|v\rangle| = \|e\| \cdot \|v\|$, prove that e, v are linearly dependent.
If $\|e + v\| = \|e\| + \|v\|$, prove that $v = \lambda e$, $\exists \lambda \in \mathbb{R}_+$.

50. If two linear subvarieties $X = p + V$, $Y = q + W$ of a vector space E are **perpendicular** (i.e. V and $W^\perp$ are incident), prove that $V \cap W = 0$ or $V + W = E$.

51. Prove the following statements in any real Euclidean vector space:
   (a) (*Pons asinorum*): Any isosceles triangle has two equal angles.
   (b) The sum of the squares of the sides of a warped quadrilateral *abcd* is the sum of the squares of the two diagonals *ac*, *bd* plus 4 times the square of the segment joining their middle points.
   (c) Let g, h, f be the barycenter, orthocenter and circumcenter of a triangle *abc*. Prove that the 3 middle points of the sides, the 3 foots of the heights, and the 3 middle points of the segments determined by the orthocenter and the vertices, are all inscribed in a circle, with radius one-half of the circumradius, centered at the midpoint between the orthocenter and the circumcenter, $\frac{h+f}{2} = \frac{3g+h}{4} = \frac{a+b+c+h}{4}$.

52. Let E be a Euclidean vector space. A linear map $f\colon E \to E$ is said to be an **isometry** if it preserves the scalar product: $e \cdot v = f(e) \cdot f(v)$, $\forall e, v \in E$. Prove the following statements:
   (a) Any isometry is a linear isomorphism.
   (b) If A is the matrix of f in any orthonormal basis of E, then $\bar{A}^t A = I$.
   (c) For any vector subspace $V \subseteq E$, there is an isometry $s_V\colon E \to E$ such that s_V is the identity on V and $-\mathrm{Id}$ on $V^\perp$.
   (d) In the real case, the determinant of the matrix A of f in any basis of E is ± 1, (and isometries of determinant 1 are named **rotations**).
   (e) In the real case, any bijection $f\colon E \to E$ preserving the origin, $f(0) = 0$, and distances, $|f(v) - f(e)| = |v - e|$, is an isometry. (*Hint*: It preserves the quadratic form $q(e) = e \cdot e$; hence the scalar product, so that $\big(f(e + v) - f(e) - f(v)\big) \cdot f(w) = 0$.)

53. Prove that the group R_T of rotations of a tetrahedron (rotations transforming the figure into itself) is isomorphic to A_4, and that the group I_T of isometries is isomorphic to S_4.
Prove that the group R_C of rotations of a cube (hence of the dual octahedron) is isomorphic to S_4, and the group I_C of isometries is isomorphic to $S_4 \times \{\pm 1\}$.
Prove that the group R_D of rotations of a dodecahedron (hence of the dual icosahedron) is isomorphic to A_5, and the group I_D of isometries is isomorphic to $A_5 \times \{\pm 1\}$.
Check that the four vertices of a tetrahedron and the four vertices of the symmetric tetrahedron (with respect to the center) determine a cube, while the middle points of the edges of a tetrahedron are the vertices of an octahedron.
However, prove that from a tetrahedron you can not define a dodecahedron in a natural way; from a cube you can not define a tetrahedron nor a dodecahedron; and from a dodecahedron you can not define a tetrahedron nor a cube. (*Hint*: If you naturally define a platonic solid B from another one A, show that you have injective group morphisms $R_A \to R_B$ and $I_A \to I_B$).

54. Let T be an endomorphism of a vector space E.

    If all vectors $e \in E$ are eigenvectors of T, prove that $T = \lambda \mathrm{Id}$ for some scalar λ.

    If $e \in E$ and $T(e)$ are linearly independent, show that $ST \neq TS$ for some $S \in \mathrm{End}_k(E)$.

    If T commutes with all the endomorphisms of E, prove that $T = \lambda \mathrm{Id}$ for some scalar λ.

55. If two selfadjoint endomorphisms S, T of an Euclidean vector space E commute, $ST = TS$, prove that E admits an orthonormal base of common eigenvectors.

56. If f is an isometry of an hermitian vector space E, show that any eigenvalue $\alpha \in \mathbb{C}$ of f has modulus 1, and that E admits an orthonormal base of eigenvectors of f. (*Hint*: If $f(V) = V$, then $f(V^\perp) = V^\perp$.)

57. Show that any linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $f(X) = AX$, admits an invariant vector subspace of dimension ≤ 2. (*Hint*: If f has no eigenvector, there exists $0 \neq X \in \mathbb{C}^n$ such that $AX = \alpha X$; hence $A\bar{X} = \bar{\alpha}\bar{X}$, and $X, \bar{X}$ are linearly independent because $\bar{\alpha} \neq \alpha$. Now $X + \bar{X}$, $\frac{1}{i}(X - \bar{X}) \in \mathbb{R}^n$ generate an invariant plane.)

    If f is an isometry of a real Euclidean vector space E, conclude that E admits a decomposition $E = V_1 \oplus \ldots \oplus V_n$ as a direct sum of invariant vector subspaces of dimension 1 or 2.

58. Let T be an endomorphism of a k-vector space of finite dimension E. Prove that the eigenvalues of $h_T\colon \mathrm{End}_k(E) \to \mathrm{End}_k(E)$, $h_T(S) = TS$, are just the eigenvalues of T.

59. If T is a nilpotent endomorphism of a vector space of dimension n, show that $T^n = 0$.

    If $\phi(x) = (x - \alpha_1)^{n_1} \ldots (x - \alpha_r)^{n_r}$ is the annihilator polynomial of an endomorphism T and we put $d_i := \dim \mathrm{Ker}\,(x - \alpha_i)^{n_i}$, prove that $(x - \alpha_1)^{d_1} \ldots (x - \alpha_r)^{d_r}$ annihilates T.

    Conclude that the degree of the annihilator polynomial of an endomorphism $T\colon E \to E$ always is bounded above by the dimension of E.

60. If two diagonalizable endomorphisms of a finite-dimensional vector space E commute, prove that both diagonalize in a common basis of E.

61. If two endomorphisms of a finite-dimensional complex vector space commute, prove that they have a common eigenvector.

62. Let G be a finite group of automorphisms of a finite-dimensional complex vector space E. Prove that any automorphism $\tau \in G$ is diagonalizable. (*Hint*: Look at the annihilator of τ).

63. Let J be an endomorphism of a real vector space $E \neq 0$ of dimension n. If $J^2 = -\mathrm{Id}$, prove that n is even and any vector is in an invariant plane. (*Hint*: E admits a complex structure).

64. Prove that a class $[Q(x)]$ is invertible in $k[x]/(P(x))$ if and only if the polynomials $P(x)$ and $Q(x)$ have no common root (in any extension of k).

    Let $d \geq 2$ be a natural number. Show that $1 + t = S(t)^d$ for some formal power series $S(t) \in \mathbb{C}[[t]]$, and prove that any invertible element of $\mathbb{C}[x]/(x^n)$ is a d-th power. Obtain that any invertible element in $\mathbb{C}[x]/(P(x))$ is a d-th power. (*Hint*: The Chinese remainder theorem).

    If A is an invertible matrix with complex coefficients, conclude the existence of a polynomial $Q(x) \in \mathbb{C}[x]$ such that $A = Q(A)^d$. (*Hint*: In $\mathbb{C}[x]/(\phi_A(x))$, the class $[x]$ is invertible).

65. Let $T^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ be the coordinates of a (p,q)-tensor T in some base $e_1, \ldots, e_n$ of E, and let us consider a new base $\bar{e}_1, \ldots, \bar{e}_n$ of E. Let $B = (b_{ij}) = (b^i_j)$ be the base change matrix, $\bar{e}_j = \sum_i b^i_j e_i$, and put $B^{-1} = (c_{ij}) = (c^i_j)$. Prove that the coordinates $\bar{T}^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ of T in the new base are just

$$\bar{T}^{j_1 \ldots j_q}_{i_1 \ldots i_p} = \sum_{k_1 \ldots k_p l_1 \ldots l_q = 1}^{n} b^{k_1 \ldots k_p}_{i_1 \ldots i_p} c^{j_1 \ldots j_q}_{l_1 \ldots l_q} T^{l_1 \ldots l_q}_{k_1 \ldots k_p}$$

66. If $\mathrm{char}\, k \neq 2$, show that any 2-covariant tensor on a finite-dimensional k-vector space decomposes uniquely as a sum of a symmetric tensor and an alternate tensor.

67. If E is a finite-dimensional k-vector space, show that $T^2 E^* = \mathrm{Hom}_k(E, E^*)$.

68. Let $f^*\colon T^p E^* \to T^p F^*$ and $f_*\colon T^p F \to T^p E$ be the linear maps induced by a linear map $f\colon F \to E$ between vector spaces of finite dimension, $p \geq 1$. Prove that

(a) f is injective $\Leftrightarrow$ f^* is surjective $\Leftrightarrow$ f_* is injective.

(b) f is surjective $\Leftrightarrow$ f^* is injective $\Leftrightarrow$ f_* is surjective.

69. When $\dim E = 3$, prove that any 2-form $\Omega_2 \in \Lambda^2 E^*$ is $\Omega_2 = \omega_1 \wedge \omega_2$ for some linear forms $\omega_1, \omega_2 \in E^*$. When $\dim E > 3$, and $\omega_1, \omega_2, \omega_3, \omega_4 \in E^*$ are linearly independent, show that $\Omega_2 = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4$ is not an exterior product $\Omega_2 = \omega \wedge \omega'$ of linear forms.

70. Prove that any non null vector $e \in E$ induces exact sequences (where $n = \dim E$)

$$0 \longrightarrow \Lambda^0 E \xrightarrow{e\wedge} \Lambda^1 E \xrightarrow{e\wedge} \Lambda^2 E \xrightarrow{e\wedge} \dots \xrightarrow{e\wedge} \Lambda^{n-1} E \xrightarrow{e\wedge} \Lambda^n E \longrightarrow 0$$
$$0 \longrightarrow \Lambda^n E^* \xrightarrow{i_e} \Lambda^{n-1} E^* \xrightarrow{i_e} \dots \xrightarrow{i_e} \Lambda^2 E^* \xrightarrow{i_e} \Lambda^1 E^* \xrightarrow{i_e} \Lambda^0 E^* \longrightarrow 0$$

71. Show that any linear map $f\colon F \to E$ induces a natural linear map $\Lambda^p f\colon \Lambda^p F \to \Lambda^p E$, and prove that $\Lambda^p f = 0$ if and only if $p > \dim(\operatorname{Im} f)$.

72. Let $A = (A_1, \dots, A_n) \in M_{n\times n}(k)$ be invertible. Prove **Cramer's formula** for the solution of a system of linear equations $AX = B$,

$$x_i = \frac{|A_1, \dots, B, \dots, A_n|}{|A|} \ , \quad \text{where the column } B \text{ is in the } i\text{-th place.}$$

Hint: $|A_1, \dots, B, \dots, A_n| = |A_1, \dots, \sum_j x_j A_j, \dots, A_n| = \sum_j x_j |A_1, \dots, A_j, \dots, A_n| = x_i |A|$.

73. Given a matrix $A = (a_{ij}) \in M_{n\times n}(k)$, let A_{ij} denote the minor obtained by removing the i-th row and the j-th column, multiplied by $(-1)^{i+j}$. Prove that

(a) We have a n-form $M_{n\times n}(k) = k^n \times \dots \times k^n \to k$, $A \mapsto a_{i1}A_{j1} + \dots + a_{in}A_{jn}$, where $1 \leq i, j \leq n$.

(b) When $i \neq j$, we have that $a_{i1}A_{j1} + \dots + a_{in}A_{jn} = 0$.

(c) When $i = j$, we have the following expansion of $|A|$ along the i-th row

$$|A| = a_{i1}A_{i1} + \dots + a_{in}A_{in}.$$

(d) If $A = (a_{ij})$ be a square matrix with coefficients in a ring, then

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & \dots & A_{n1} \\ \dots & \dots & \dots \\ A_{1n} & \dots & A_{nn} \end{pmatrix} = \begin{pmatrix} |A| & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & |A| \end{pmatrix}$$

(*Hint*: Any matrix is a specialization of the generic matrix (x_{ij}) with coefficients in the polynomial ring $\mathbb{Z}[x_{ij}]$, which is contained in a field).

(e) Let e be a vector of a vector space E and $\omega_1, \dots, \omega_p \in E^*$. We have

$$i_e(\omega_1 \wedge \dots \wedge \omega_p) = \textstyle\sum_i (-1)^{i-1} \omega_i(e) \omega_1 \wedge \dots \wedge \widehat{\omega}_i \wedge \dots \wedge \omega_p.$$

Obtain another proof of the expansion of $|A|$ along the first column:

$$|A| = a_{11}A_{11} + \dots + a_{n1}A_{n1}.$$

74. Given a $(n-1) \times n$ matrix A with coefficients in a field k, prove the following homogeneous Cramer's formula: a solution of the homogeneous system $AX = 0$ is $x_1 = A_1, \dots, x_n = A_n$, where A_j is the minor obtained by removing the j-th column of A, multiplied by $(-1)^{j-1}$. (*Hint*: Add any row of A as a first row, and expands the corresponding null determinant along the first row.)

75. Let $e_1, \dots, e_n$ be a basis of a real Euclidean vector space E, and let $\omega_1, \dots, \omega_n$ be the dual basis. Show that the volume forms of E are $\Omega_E = \pm\sqrt{\det(e_i \cdot e_j)}\, \omega_1 \wedge \dots \wedge \omega_n$.

(a) Show that the area S of the parallelogram determined by two vectors $e, v \in E$ is

$$S = \sqrt{\begin{vmatrix} e \cdot e & e \cdot v \\ v \cdot e & v \cdot v \end{vmatrix}} = \sqrt{\|e\|^2 \|v\|^2 - (e \cdot v)^2} = \|e\| \cdot \|v\| \cdot |\operatorname{sen} \alpha|,$$

where α is the angle between e and v, and obtain **Heron's formula** for the area $S := \frac{1}{2}|\Omega_E(e, v)|$ of a triangle with vertices $p, p+e, p+v$, sides of lengths $a = \|e\|$, $b = \|v\|$, $c = \|v - e\|$, and semiperimeter $s = \frac{1}{2}(a + b + c)$:

$S = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} = \sqrt{s(s-a)(s-b)(s-c)}$.

Hint: $e^2 = a^2$, $v^2 = b^2$ and $2e \cdot v = a^2 + b^2 - c^2$, so that $S = \sqrt{P(a^2, b^2, c^2)}$, where $P(a^2, b^2, c^2) = Q(a, b, c)(a - b - c) + R(b, c)$ is a homogeneous polynomial of degree 4 in a, b, c vanishing when $a = b + c$. Hence $R(b, c)$ vanishes for any two positive values of b and c, so that $R(b, c) = 0$ and $a - b - c$ divides $P(a^2, b^2, c^2)$. Conclude that, up to a constant factor, $P(a^2, b^2, c^2)$ coincides with $(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$.

(b) Prove **Tartaglia's formula** for the volume $V := \frac{1}{3!}|\Omega_E(e_1, e_2, e_3)|$ of a tetrahedron with vertices p_0, p_1, p_2, p_3 (where $e_i := p_i - p_0$):

$$V = \sqrt{\frac{1}{288}\begin{vmatrix} d_{00}^2 & d_{01}^2 & d_{02}^2 & d_{03}^2 & 1 \\ d_{10}^2 & d_{11}^2 & d_{12}^2 & d_{13}^2 & 1 \\ d_{20}^2 & d_{21}^2 & d_{22}^2 & d_{23}^2 & 1 \\ d_{30}^2 & d_{31}^2 & d_{32}^2 & d_{33}^2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}} \quad , \quad d_{ij} := \|p_j - p_i\|.$$

Hint: Remark that $d_{ij}^2 = (e_j - e_i)^2 = e_i^2 + e_j^2 - 2e_i \cdot e_j$ and $d_{0i}^2 = d_{i0}^2 = e_i^2$. Then, subtracting the 0-th row and column to the first, second and third rows and columns, show that

$$\begin{vmatrix} 0 & \dots & e_j^2 & \dots & 1 \\ \vdots & & & & \vdots \\ e_i^2 & & e_i^2 + e_j^2 - 2e_i \cdot e_j & & 1 \\ \vdots & & & & \vdots \\ 1 & \dots & 1 & \dots & 0 \end{vmatrix} = \begin{vmatrix} 0 & \dots & e_j^2 & \dots & 1 \\ \vdots & & & & \vdots \\ e_i^2 & & -2e_i \cdot e_j & & 0 \\ \vdots & & & & \vdots \\ 1 & \dots & 0 & \dots & 0 \end{vmatrix} = \det(2e_i \cdot e_j) = 8\det(e_i \cdot e_j).$$

(c) Obtain a formula for the volume $V := \frac{1}{n!}|\Omega_E(e_1, \dots, e_n)|$ of a hypertetrahedron with vertices $p_0, p_1, \dots, p_n$ (where $e_i := p_i - p_0$) in terms of the edges $d_{ij} := \|p_j - p_i\|$.

76. Let E be a real Euclidean vector space, $e_1, \dots, e_n \in E$ and $G(e_1, \dots, e_n) := \det(e_i \cdot e_j)$. Show that

(a) $G(e_1, \dots, e_n) \geq 0$, and $G(e_1, \dots, e_n) = 0$ if and only if $e_1, \dots, e_n$ are linearly dependent.

(b) The volume of the parallelepiped determined by a basis $e_1, \dots, e_d$ of E is $G(e_1, \dots, e_d)^{1/2}$.

(c) If $G(e_1, \dots, e_n) \neq 0$, the distance of $e \in E$ to the vector subspace $\mathbb{R}e_1 + \dots + \mathbb{R}e_n$ is

$$\sqrt{\frac{G(e, e_1, \dots, e_n)}{G(e_1, \dots, e_n)}}\,.$$

77. Let Ω_E be the volume form of a 3-dimensional Euclidean $\mathbb{R}$-vector space E. If $e, v \in E$, then the linear form $i_v i_e \Omega_E = \Omega_E(e, v, -)$ corresponds to a vector $e \times v$, named **cross product** of e and v, so that $(e \times v) \cdot u = \Omega_E(e, v, u)$. Prove that it is bilinear, $e \times v = -v \times e$, and that $e \times v \neq 0$ if and only if e and v are linearly independent, and in such a case $e \times v$ is orthogonal to both factors, the module of $e \times v$ is the area of the parallelogram determined by e and v, and $e, v, e \times v$ is a direct base.

Hint: The area form of the plane $\langle e, v\rangle$ is $i_u \Omega_E$, the area of the parallelogram determined by e and v is $|(i_u \Omega_E)(e, v)| = |\Omega_E(u, e, v)| = |(e \times v) \cdot u| = |\lambda u \cdot u| = \lambda$, and $\Omega_E(e, v, e \times v) = (e \times v) \cdot (e \times v) > 0$.

78. Let e_1, e_2, e_3 be a direct basis of an oriented Euclidean $\mathbb{R}$-vector space, and let e^1, e^2, e^3 be the dual basis, $e^i \cdot e_j = \delta_{ij}$. If $u = \sum_i u_i e_i$, $v = \sum_i v_i e_i$, show that

$$u \times v = \frac{1}{\Omega_E(e_1, e_2, e_3)} \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3 \right).$$

79. Prove the following properties of the cross product:

(a) $(e \times v) \cdot u = e \cdot (v \times u)$.

(b) $\|e \times v\|^2 = \|e\|^2 \|v\|^2 - (e \cdot v)^2$. (*Hint*: Use ex. 75.)

(c) $(e \times v) \times u = (e \cdot u)v - (v \cdot u)e$.
(*Hint*: Consider a direct orthonormal base u_1, u_2, u_3 such that $e \in \mathbb{R}u_1$, $v \in \mathbb{R}u_1 + \mathbb{R}u_2$).

(d) $(e \times u) \times (v \times w) = \Omega_E(e, v, w)u - \Omega_E(u, v, w)e = \Omega_E(e, u, w)v - \Omega_E(e, u, v)w$.

(e) **Cramer's Formula**: $\Omega_E(e_1, e_2, e_3)v = \Omega_E(v, e_2, e_3)e_1 + \Omega_E(e_1, v, e_3)e_2 + \Omega_E(e_1, e_2, v)e_3$.

(f) **Jacobi's Identity**: $(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$.

(g) $(u \times v) \cdot (\bar{u} \times \bar{v}) = \begin{vmatrix} u \cdot \bar{u} & v \cdot \bar{u} \\ u \cdot \bar{v} & v \cdot \bar{v} \end{vmatrix}$.

(*Hint*: Use formula (c), or remark that both terms are covariant tensors symmetric in $u, \bar{u}$ and $v, \bar{v}$, defining the same quadratic form.)

(h) $\Omega_E(e, u, v)\Omega_E(\bar{e}, \bar{u}, \bar{v}) = \begin{vmatrix} e \cdot \bar{e} & u \cdot \bar{e} & v \cdot \bar{e} \\ e \cdot \bar{u} & u \cdot \bar{u} & v \cdot \bar{u} \\ e \cdot \bar{v} & u \cdot \bar{v} & v \cdot \bar{v} \end{vmatrix}$.

(*Hint*: Use Cramer's formula for $\Omega_E(e, u, v)(e' \times u')$ and then the former formula.)

80. Let T be an endomorphism of a 3-dimensional oriented Euclidean $\mathbb{R}$-vector space E. If $e \cdot (Te) = 0$ for all $e \in E$, prove the existence of a vector $v \in E$ such that $T(e) = e \times v$ for all $e \in E$.

81. Given a 4-dimensional $\mathbb{R}$-vector space $\mathbb{H}$ with a scalar product $\langle\ |\ \rangle$, an orientation $\Omega_{\mathbb{H}}$ and a fixed vector $u \in \mathbb{H}$ of modulus 1, let us consider the orientation of $\mathbb{H}_0 := (\mathbb{R}u)^\perp$ given by the 3-form $i_u\Omega_{\mathbb{H}} = \Omega_H(u, -)$, and the $\mathbb{R}$-bilinear product $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$ given by the formula:

$$v \cdot w := \langle v|w\rangle u + v \times w \ ; \quad v, w \in \mathbb{H}_0.$$

(a) Given any oriented orthonormal base i, j, k of $\mathbb{H}_0$, show that

$$i^2 = j^2 = k^2 = -1 \quad , \quad ijk = -1 \ ,$$

so that $\mathbb{H}$ is a non-conmutative $\mathbb{R}$-algebra with unity $1 = u$, and it is just the **quaternion** algebra:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

(b) The conjugation of quaternions is defined by the symmetry with respect to the real line $\mathbb{R} = \mathbb{R}u$, so that the **conjugate** of $q = a + bi + cj + dk$ is $\bar{q} = a - bi - cj - dk$. Show that $\overline{q_1q_2} = \bar{q}_2\bar{q}_1$ and $q\bar{q} = |q|^2 = a^2 + b^2 + c^2 + d^2$, so that $|q_1q_2| = |q_1| \cdot |q_2|$ and $\mathbb{H}$ is a division ring. Hence the product of quaternions defines a group structure on the unit sphere $\mathbb{S}_3 := \{q \in \mathbb{H} \colon |q| = 1\}$.

(c) Show that $\{q \in \mathbb{H} \colon q^2 = -1\} = \mathbb{H}_0 \cap \mathbb{S}_3 = \{bi + cj + dk \colon b^2 + c^2 + d^2 = 1\}$.

(d) Prove that the center of $\mathbb{H}$ is $\mathbb{R}$. (*Hint*: Condition $qi = iq$ means that $c = d = 0$.)

(e) Show that the $\mathbb{R}$-algebra structure of $\mathbb{H}$ let us recover the scalar product and the orientation. (*Hint*: Any quaternion q defines a $\mathbb{R}$-linear map $h_q \colon \mathbb{H} \to \mathbb{H}$, $h_q(x) = qx$ and $|q|^4 = \det(h_q)$, $\bar{q} = \frac{1}{2}\mathrm{tr}(h_q) - q = |q|^2/q$, and the cross product is just $\mathbb{H}_0 \times \mathbb{H}_0 \xrightarrow{\cdot} \mathbb{H} \xrightarrow{\text{ortog. proj.}} \mathbb{H}_0$.)

82. Let u be a non null vector of modulus ω in a real Euclidean vector space of dimension 3. Prove that $v = u \times \vec{r}$ is the velocity of a body at the position $\vec{r}$, turning around us with angular velocity ω (radians per time unit) and axis u. If we consider the endomorphism $T \colon E \to E$, $T(\vec{r}) = u \times \vec{r}$, the equations of such motion, with initial position $\vec{r}_0$, are just

$$\vec{r} = e^{tT}\vec{r}_0 \quad , \quad \vec{r}\,' = T\big(e^{tT}\vec{r}_0\big) = T(\vec{r}).$$

When $\omega = 1$, we have $u \times (u \times v) = -v$ (remark that $\Omega_E(u, u \times v, v) = -\Omega_E(u, v, u \times v) < 0$).
Hence $T^3 = -T$ (give an alternative proof showing that $c_T(x) = x^3 + x$).
Conclude that $T^4 = -T^2$, $T^5 = T, \dots$ and that the rotation of t radians around the axis u is

$$\begin{aligned} e^{tT} &= 1 + tT + \frac{t^2}{2!}T^2 + \frac{t^3}{3!}T^3 + \frac{t^4}{4!}T^4 + \frac{t^5}{5!}T^5 + \frac{t^6}{6!}T^6 + \frac{t^7}{7!}T^7 + \dots \\ &= 1 + tT + \frac{t^2}{2!}t^2 - \frac{t^3}{3!}T - \frac{t^4}{4!}T^2 + \frac{t^5}{5!}T + \frac{t^6}{6!}T^2 - \frac{t^7}{7!}T - \frac{t^8}{8!}T^2 \dots \\ &= 1 + (\sin t)T + (1 - \cos t)T^2. \end{aligned}$$

When $t = 2\pi$, we obtain $e^{2\pi T} = 1$ (an equation close to $e^{2\pi i} = 1$, since in the complex plane ωiz is also the velocity of a body, placed at the position z, turning around us with angular velocity ω).

83. If $\alpha \in k$ is a root of multiplicity m of the characteristic polynomial $c_T(x)$ of an endomorphism T, prove that $1 \leq \dim V_\alpha \leq m$, where $V_\alpha = \operatorname{Ker}(\alpha \mathrm{Id} - T)$.

84. Let T be an endomorphism of a finite-dimensional vector space E. If $V \subset E$ is an invariant vector subspace, $T(V) \subseteq V$, prove the existence of a unique endomorphism $\bar{T}\colon E/V \to E/V$ such that $\bar{T}(\bar{e}) = T(e)$, $\forall e \in E$. If $\bar{c}(x)$ is the characteristic polynomial of $\bar{T}$, and $c'(x)$ is the characteristic polynomial of the restriction $T|_V \colon V \to V$, $T|_V(v) = T(v)$, prove that $c_T(x) = \bar{c}(x)c'(x)$.

    If T is diagonalizable, conclude that so are $T|_V$ and $\bar{T}$.

85. Let $c_T(x) = \prod_i (x - \alpha_i)$ be the characteristic polynomial of an endomorphism T of a k-vector space E. If $P(x) \in k[x]$, show that $c_{P(T)}(x) = \prod_i (x - P(\alpha_i))$. (*Hint*: By Kronecker's theorem, you may assume that T has an eigenvalue).

    Conclude that T is nilpotent if and only if $\operatorname{tr}(T^r) = 0$, $1 \leq r \leq \dim E$.

86. Prove that the characteristic (resp. annihilator) polynomial of an endomorphism T coincides with the characteristic (resp. annihilator) polynomial of the transpose endomorphism T^*.

3. Algebra I

1. Let k be a field. If $a \in k$, show that we have a ring isomorphism $k[x]/(x - a) \simeq k$, $[P] \mapsto P(a)$.

    Given different elements $a_1, \ldots, a_n \in k$, show that we have a ring isomorphism

$$k[x]/\big((x - a_1)\ldots(x - a_n)\big) \simeq k \oplus \ldots \oplus k,\ [P] \mapsto (P(a_1), \ldots, P(a_n)).$$

2. Let k be a field and $a_1, \ldots, a_n \in k$. Prove that $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal of $k[x_1, \ldots, x_n]$ and that the natural morphisms $k \to k[x_1, \ldots, x_n]/\mathfrak{m}$ is an isomorphism.

    If $P \in k[x_1, \ldots, x_n]$, show that the following generalized *Ruffini's rule* holds:

$$P(a_1, \ldots, a_n) = 0 \text{ if and only } P = \textstyle\sum_i (x_i - a_i)Q_i.$$

3. Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of a ring A, and $\mathfrak{p}$ a prime ideal. If $\mathfrak{ab} \subseteq \mathfrak{p}$, show that $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

4. Let p be a prime number, and let d be a divisor of $p - 1$. If p does not divide $n \in \mathbb{Z}$, prove that n is a d-th residue modulo p if and only if $n^{\frac{p-1}{d}} \equiv 1$ (mod. p).

5. Let $Q(x) = (x^2 - 2)(x^2 + 1)(x^2 + 2)$. Show that the reduction $\bar{Q}(x) \in \mathbb{F}_p[x]$ has some root in $\mathbb{F}_p$ at any prime number p, while $Q(x)$ has no rational root.

6. Show that the congruence $Q(x) = (2x - 3)(3x - 2) \equiv 0$ (mod. n) has integer solution for any natural number $n \geq 2$, while $Q(x)$ has no integer root. (*Hint*: The chinese remainder theorem).

7. Show that the number 17 is a quadratic residue modulo 2^n for any exponent $n \geq 1$.

    Let p be an odd prime. If an integer a is a quadratic residue modulo p, prove that a is a quadratic residue modulo p^n for any exponent $n \geq 1$.

    Show that $Q(x) = (x^2 - 17)(x^2 + 1)(x^2 + 17) \equiv 0$ (mod. n) has integer solution for any natural number $n \geq 2$, while $Q(x)$ has no rational root.

8. Prove that any ideal of a ring $A \times B$ is $\mathfrak{a} \times \mathfrak{b}$, where $\mathfrak{a}$ is an ideal of A and $\mathfrak{b}$ is an ideal of B. Moreover, we have a ring isomorphism $(A \times B)/(\mathfrak{a} \times \mathfrak{b}) = (A/\mathfrak{a}) \times (B/\mathfrak{b})$, $[(a, b)] \mapsto ([a], [b])$.

9. Prove that a polynomial $Q(x) = a_n x^n + \ldots + a_0 \in \mathbb{Z}[x]$ admits an irreducible factor of degree $\geq k$ in $\mathbb{Q}[x]$ when there is a prime p not dividing a_n nor a_k such that p divides $a_{k-1}, \ldots, a_0$ and p^2 does not divide a_0.

10. Prove that any ideal $\mathfrak{a} \neq 0$ of the ring $k[[t]]$ of formal power series with coefficients in a field k is $\mathfrak{a} = (t^n)$ for some natural number n.

11. Let A and B be rings such that any ideal of A and B is a principal ideal. Prove that any ideal of the ring $A \times B$ is a principal ideal.

12. If a ring A is not a field, show that the ring $A[x]$ is not a principal ideal domain. (*Hint*: Consider the ideal (a, x), where a is not a unit).

13. Let A be a domain. If a map $\delta'\colon A-\{0\}\to\mathbb{N}$ satisfies condition 1 of the definition of Euclidean ring (p. 75) and we put $\delta(a)=\min\delta'(af)$, $f\in A-\{0\}$, prove that (A,δ) is a Euclidean ring. (*Hint*: Put $b=afc+r$ with $\delta(r)<\delta(af)$).
14. If (A,δ) is a Euclidean ring, prove that $a\in A-\{0\}$ is invertible if and only if $\delta(a)=\delta(1)$.
Show that the invertible gaussian integers are ±1 and $\pm i$.
15. Prove that the following conditions on a prime number p are equivalent:
    (a) p is not a sum of two perfect squares.
    (b) p is irreducible in the ring $\mathbb{Z}[i]$.
    (c) $p\equiv 3$ (mod. 4). (*Hint*: We have a ring isomorphism $\mathbb{Z}[i]/p\mathbb{Z}[i]\simeq\mathbb{F}_p[x]/(x^2+1)$).
16. Prove that $a+bi$, $b\neq 0$, is irreducible in $\mathbb{Z}[i]$ if and only if a^2+b^2 is a prime number.
17. Let $n=p_1^{n_1}\dots p_r^{n_r}$, where $p_1,\dots,p_r$ are different prime numbers. Prove that n is a sum of two perfect squares if and only if the exponents n_i are even whenever $p_i\equiv 3$ (mod. 4).
18. Prove that no prime number p admits two decompositions as a sum of two perfect squares. That is to say, if $p=a^2+b^2=c^2+d^2$, then $a^2=c^2$ or $a^2=d^2$.
19. If p is an odd prime factor of n^2+1, show that $p\equiv 1$ (mod. 4).
Conclude the existence of infinite prime numbers of the form $4m+1$.
20. Let $n,m\in\mathbb{N}$ be coprime. Prove that any prime dividing n^2+m^2 is a sum of two perfect squares.
21. If $d\geq 3$ is odd, show that $\mathbb{Z}[\sqrt{-d}\,]$ is not a principal ideal domain. (*Hint*: $(1+\sqrt{-d}\,)(1-\sqrt{-d}\,)$ is even).
22. Show that $\mathbb{Z}\big[e^{\frac{2\pi i}{3}}\big]$ and $\mathbb{Z}[\sqrt{-2}\,]$ are Euclidean rings.
23. If A is a Euclidean ring, prove the existence of a maximal ideal $\mathfrak{m}$ such that any non-zero element of the residue field $A/\mathfrak{m}$ is represented by a unit. (*Hint*: Consider the maximal ideal generated by an irreducible element p of minimal norm $\delta(p)$).
24. Put $A=\mathbb{Z}\big[\frac{1+\sqrt{-19}}{2}\big]$ and show that $A^*=\{\pm1\}$. Conclude that A is not a Euclidean ring. (*Hint*: We have $A\simeq\mathbb{Z}[x]/(x^2-x+5)$, so that $A/2A$ and $A/3A$ are fields of 4 and 9 elements).
25. Put $A=\mathbb{R}[x,y]/(x^2+y^2+1)$ and show that $A^*=\mathbb{R}-\{0\}$. Conclude that A is not a Euclidean ring. (*Hint*: $A=\mathbb{R}[x]+\mathbb{R}[x]y$, and if $p(x)+q(x)y$ is a unit, then $(p+qy)(p-qy)=p^2+q^2(x^2+1)$ is invertible in $\mathbb{R}[x]$; hence it is constant).
26. Express $4a^2b^2-(a^2+b^2-c^2)^2$ as a polynomial in the elementary symmetric function in a,b,c. (*Hint*: See Heron's formula, p. 489).
27. Prove that $x_1^d+\ldots+x_n^d=P(s_1,\dots,s_{d-1})-(-1)^d ds_d$ for some polynomial P, where s_i is the i-th elementary symmetric function in $x_1,\dots,x_n$. (*Hint*: Consider the roots of x^d-1).
28. If $a,b\in\mathbb{Q}$, prove that $\mathbb{Q}(\sqrt{a}\,)=\mathbb{Q}(\sqrt{b}\,)$ if and only if b/a is a square in $\mathbb{Q}$.
29. Let us consider an intermediate field $k\subset L\subseteq k(x)$. Show that the degree $[k(x):L]$ is finite.
30. If $\alpha\in\mathbb{C}$ is a root of a unitary irreducible polynomial $P(x)\in\mathbb{Z}[x]$, prove that $\mathbb{Z}[\alpha]\simeq\mathbb{Z}[x]/\big(P(x)\big)$.
31. Prove that any finite extension of $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.
32. Let $k\to L$ be an extension. If $\alpha_1,\dots,\alpha_n\in L$ are algebraic over k, prove the rationalization of algebraic expressions: $k(\alpha_1,\dots,\alpha_n)=k[\alpha_1,\dots,\alpha_n]$.
33. Prove the following properties of the cyclotomic polynomials:

$$\Phi_{2n}(x)=\Phi_n(-x)\qquad\text{when } n \text{ is odd.}$$
$$\Phi_{p^n}(x)=\Phi_p\big(x^{p^{n-1}}\big)\qquad\text{when } p \text{ is prime.}$$
$$\Phi_{p^n r}(x)=\frac{\Phi_r\big(x^{p^n}\big)}{\Phi_r\big(x^{p^{n-1}}\big)}\qquad\text{when } r \text{ is not a multiple of the prime } p.$$

34. Prove that the real part of $e^{\frac{2\pi i}{5}}$ is $\frac{\sqrt{5}-1}{4}$. Give a ruler-and-compass construction of a regular pentagon, and of a regular 15-sided polygon, inscribed in a given circle.

35. Let $p = 2^k + 1$ be a prime number. In the multiplicative group $\mathbb{F}_p^*$, prove that the order of 2 is $2k$, so that $2k$ divides $p - 1 = 2^k$, and p is a **Fermat prime**: $p = 2^{2^n} + 1$.

36. A **division ring** D is an eventually non-commutative ring such that any non null element has inverse and a **division K-algebra** is a division ring with a ring morphism $K \to D$ such that $\lambda\alpha = \alpha\lambda$ for any $\lambda \in K$, $\alpha \in D$. Let D be a finite dimensional real division algebra. If D is not commutative, prove the following statements:
    (a) There exists $i \in D$ such that $i^2 = -1$, so that $\mathbb{R}[i] = \mathbb{C}$. (*Hint*: If $\alpha \in D$, then $\mathbb{R}[\alpha]$ is a finite extension of $\mathbb{R}$.)
    (b) Consider D as a complex vector space by multiplication on the left, so that $T\colon D \to D$, $T(x) = xi$, is $\mathbb{C}$-linear and $x^2 + 1$ annihilates T. Then $D = D^+ \oplus D^-$, where $D^+ = \{x \in D\colon xi = ix\}$, $D^- = \{x \in D\colon xi = -ix\}$.
    (c) $D^+ = \mathbb{C}$. (*Hint*: If $\alpha \in D^+$, then $\mathbb{C}[\alpha]$ is a finite extension of $\mathbb{C}$.)
    (d) If $0 \neq \beta \in D^-$, then $\beta D^- \subseteq D^+ = \mathbb{C}$, so that $D^- = \mathbb{C}\beta$. Moreover $\beta^2 \in \mathbb{R}$ and $\beta^2 < 0$. (*Hint*: $\beta^2 \in \mathbb{C} \cap \mathbb{R}[\beta] = \mathbb{R}$ and, if $\beta^2 > 0$, then β^2 has 3 square roots in the field $\mathbb{R}[\beta]$.)
    (e) There exists $j \in D^-$ such that $j^2 = -1$, and $D = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$ is just the **quaternion** algebra: $i^2 = j^2 = -1$, $ij = -ji$.

37. If $P(x) \in \mathbb{Q}[x]$ has degree d and $P(n) \in \mathbb{Z}$ for any integer $n \gg 0$, prove that
$$P(n) = m_d\binom{n}{d} + \ldots + m_2\binom{n}{2} + m_1\binom{n}{1} + m_0,$$
where $m_0, \ldots, m_d \in \mathbb{Z}$. Conclude that $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. (*Hint*: $\Delta\binom{n}{d} = \binom{n}{d-1}$).

38. Define $e^{\nabla}(s)$ and $e^{\Delta}(s)$ rigorously when $s = (n^3)$, and give sense to the following calculation:
$$\begin{aligned}\sum\nolimits_n \frac{n^3}{n!} &= \left(s + \frac{\nabla}{1!}\,s + \frac{\nabla^2}{2!}\,s + \ldots + \frac{\nabla^n}{n!}\,s + \ldots\right)_0 = \left(e^{\nabla}s\right)_0 \\ &= \left(e^{\Delta+1}s\right)_0 = e\left(e^{\Delta}s\right)_0 = e\left(s + \frac{\Delta}{1!}\,s + \frac{\Delta^2}{2!}\,s + \ldots + \frac{\Delta^n}{n!}\,s + \ldots\right)_0 \\ &= e\left(s + \frac{\Delta}{1!}\,s + \frac{\Delta^2}{2!}\,s + \frac{\Delta^3}{3!}\,s\right)_0 = e(0+1+3+1) = 5e\,.\end{aligned}$$

39. Give sense and rigor to the following calculation:
$$\begin{aligned}\int e^{2t}t^3\mathrm{d}t &= \frac{1}{D}\,e^{2t}t^3 = e^{2t}\frac{1}{D+2}\,t^3 = \frac{e^{2t}}{2}\frac{1}{1+\frac{1}{2}D}t^3 \\ &= \frac{e^{2t}}{2}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \ldots\right)t^3 = e^{2t}\left(\frac{t^3}{2} - \frac{3t^2}{4} + \frac{3t}{4} - \frac{3}{8}\right).\end{aligned}$$

40. If α is a complex root of $x^n + a_1x^{n-1} + \ldots + a_n \in \mathbb{C}[x]$, then $|\alpha| < 1 + \max\{|a_1|, \ldots, |a_n|\}$. (*Hint*: See p. 41).

41. Prove that any real root of $p(x) \in \mathbb{R}[x]$ is bounded above by $a \in \mathbb{R}$ when $p(x) = (x-a)(c_0x^n + \ldots + c_n) + r$ with $c_0, \ldots, c_n, r \geq 0$, and also when $p(a), p'(a), p''(a), \ldots \geq 0$.

42. Prove that $0 \leq E_a^b \dfrac{P}{P'} + E_a^b \dfrac{P''}{P'}$ for any polynomial $P \in \mathbb{R}[x]$.
    Obtain **Budan-Fourier theorem**: $\begin{bmatrix}\text{Number of roots of } P \\ \text{between } a \text{ and } b\end{bmatrix} \leq V_a^b(P, P', P'', \ldots, P^{(n)})$.

43. Let k be a field of characteristic $p > 0$. If $a \in k$, show that $x^p - a$ has a unique root of multiplicity p. Moreover $x^p - a$ is irreducible in $k[x]$ or it has a root in k.
    When $k = \mathbb{F}_p(t)$, show that $x^p - t$ is irreducible in $k[x]$.

44. Imagine that the gravitational force between punctual bodies in the complex plane $\mathbb{C}$ is proportional to the masses and inversely proportional to the distance (and fix the mass unit so that the gravitational constant is 1). Prove that

    (a) A body at $\alpha \in \mathbb{C}$ attracts a body at z (both of mass 1) with a force $F_z = (\bar{\alpha} - \bar{z})^{-1}$.

    (b) If we place bodies of mass 1 at the roots $\alpha_1, \ldots, \alpha_n$ of a polynomial $P(z)$ with simple roots, then the force F_z acting on a body of mass 1 at a point z is given by the formula

    $$-\bar{F}_z = \tfrac{1}{z-\alpha_1} + \ldots + \tfrac{1}{z-\alpha_n} = \tfrac{P'(z)}{P(z)}$$

    so that the gravitational force vanishes just at the roots of the derivative $P'(z)$.

    (c) **Gauss-Lucas theorem**: *The roots of $P'(z)$ are in the convex envelope of the roots of $P(z)$.*

    (d) Prove this theorem when $P(z)$ has multiple roots. (*Hint*: Assume that the mass placed at any root α_i is just the multiplicity of α_i in $P(z)$).

45. If A is a principal ideal domain, show that so is any localization A_S.

46. Prove that $(A_S)_T = A_{ST}$, where we put $ST = \{st \colon s \in S, t \in T\}$.

47. If $n \geq 2$, show that $\mathbb{Z}[ni]$ is not a unique factorization domain. (*Hint*: $\frac{ni}{n}$ is a root of $x^2 + 1$).

48. If $d \in \mathbb{Z}$ is not a square, but it is a square in $\mathbb{F}_p$ and the equation $x^2 - dy^2 = \pm p$ has no integer solution, show that $\mathbb{Z}[\sqrt{d}\,]$ is not a unique factorization domain.

49. Let $Q(x) = a_0 x^n + \ldots + a_n \in \mathbb{Z}[x]$. If $a, b \in \mathbb{Z}$ are coprime and a/b is a root of $Q(x)$, show that $Q(x)/(bx - a)$ has integer coefficients, so that $a - b$ divides $Q(1)$ and $a + b$ divides $Q(-1)$.

50. If the product $(x^n + \ldots)(x^m + \ldots)$ of two unitary polynomials with rational coefficients has integer coefficients, show that both factors have integer coefficients.

51. Show that the discriminant of a quadratic polynomial x^2+bx+c with coefficients in a field is $\Delta = b^2-4c$.

52. Show that the discriminant of a cubic $x^3 - px^2 + qx - r$ with coefficients in a field is

    $$\sigma_1 = p,\ \sigma_2 = p^2 - 2q,\ \sigma_3 = p^3 - 3pq + 3r,\ \sigma_4 = p^4 - 4p^2q + 4pr + 2q^2.$$

    $$\Delta = \begin{vmatrix} 3 & \sigma_1 & \sigma_2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_4 \end{vmatrix} = -4p^3r - 27r^2 + 18pqr - 4q^3 + p^2q^2.$$

53. Prove that the discriminant of a quartic $x^4 - px^3 + qx^2 - rx + s = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ with coefficients in a field is just the discriminant of the **cubic resolvent**

    $$(y - \vartheta_1)(y - \vartheta_2)(y - \vartheta_3) = y^3 - qy^2 + (pr - 4s)y - (s(p^2 - 4q) + r^2),$$
    $$\vartheta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4,\ \vartheta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4,\ \vartheta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3.$$

54. Prove that some complex root of $x^4 - x^3 + x^2 + 1$ is not a quadratic irrational. (*Hint*: the cubic resolvent is irreducible).

55. (*Resolution of cubics and quartics with radicals* (See also ex. 18 in p. 514): Prove the following assertions:

    (a) The roots of a quadratic polynomial $ax^2 + bx + c$ with coefficients in a field of characteristic $\neq 2$ are just $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. (*Hint*: Dividing by a you may assume that $a = 1$, and the substitution $x = y - \frac{b}{2}$ let us assume that $b = 0$.)

    (b) Given a cubic $x^3 + px + q$ with coefficients in a field of characteristic $\neq 2, 3$, Vieta's substitution $x = y - \frac{p}{3y}$ reduces it to $y^6 + qy^3 - \frac{p^3}{27}$, so that $y^3 = -q/2 \pm \sqrt{q^2/4 + p^3/27}$.

    (c) Given a quartic $x^4 + px^2 + qx + r$ with coefficients in a field of characteristic $\neq 2, 3$ we have

    $$x^4 + px^2 + qx + r = (x^2 + u)^2 - \big((2u - p)x^2 - qx + (u^2 - r)\big),$$

    and the subtrahend is a perfect square when it has null discriminant, $q^2 = 4(2u - p)(u^2 - r)$. Solving a cubic equation, we may obtain a value u such that our quartic is a product $P^2 - Q^2 = (P + Q)(P - Q)$ of two factors of degree 2. Obtain the roots of the quartic in terms of u and the coefficients.

56. Prove that any element of a field of rational functions $k(x_1, \dots, x_n)$ algebraic over k is in k.

57. Let k be a field. The polynomial $x^n + c_1x^{n-1} + \dots + c_{n-1}x + c_n$ with coefficients in the field of rational functions $\Sigma = k(c_1, \dots, c_n)$ is said to be the **generic** unitary polynomial of degree n with coefficients in k. Prove that it has simple roots and that it is irreducible in $\Sigma[x]$.

58. Prove that a cubic with real coefficients has a unique real root when the discriminant Δ is negative, and that it has three real roots when Δ is positive.

59. If Δ is the discriminant of $P(x)$, show that $P(a)^2\Delta$ is the discriminant of $(x-a)P(x)$.

60. Let $\Delta(a_1, \dots, a_n)$ be the discriminant of the polynomial $x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ with real coefficients. Prove that $n + (n-1)a_1 \frac{\partial\Delta}{\partial a_2} + \dots + a_{n-1}\frac{\partial\Delta}{\partial a_n} = 0$.
(*Hint*: The discriminant remains invariant when we replace α_i by $\alpha_i + t$).

61. Let $P(x,y) = 0$, $Q(x,y) = 0$ be an algebraic system of two equations with rational coefficients. If it only has a finite number of complex solutions, prove that the resultants $R(y)$ and $\bar{R}(x)$ do not vanish, so that the rational solutions may be determined in a finite number of steps.

62. Let us express the Bézout resultant as a determinant when $m = n$.
If $r < n$, then $(a_0x^r + \dots + a_r)Q - (b_0x^r + \dots + b_r)P$ has degree $< n$,

$$
\begin{aligned}
a_0Q - b_0P &= A_{11}x^{n-1} + A_{12}x^{n-2} + \dots + A_{1n} \\
(a_0x + a_1)Q - (b_0x + b_1)P &= A_{21}x^{n-1} + A_{22}x^{n-2} + \dots + A_{2n} \\
\dots\dots\dots\dots\dots\dots &\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

In $k[x]/(P)$ we have $h_Q(a_0) = A_{11}x^{n-1} + A_{12}x^{n-2} + \dots + A_{1n}$,

$$
\begin{aligned}
h_Q(a_0x + a_1) &= A_{21}x^{n-1} + A_{22}x^{n-2} + \dots + A_{2n}, \\
\dots\dots &\quad \dots\dots\dots\dots\dots\dots\dots\dots \\
h_Q(a_0x^{n-1} + \dots + a_{n-1}) &= A_{n1}x^{n-1} + A_{n2}x^{n-2} + \dots + A_{nn}.
\end{aligned}
$$

Use these equalities to conclude that

$$
\begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix} = \begin{vmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ ,.. & .. & \dots & .. \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{vmatrix} \cdot |h_Q| = a_0^n \cdot |h_Q| = R^b(P,Q).
$$

63. Let $A = \mathbb{R}[\cos t, \sin t]$ and $A_{\mathbb{C}} = \mathbb{C}[\cos t, \sin t]$ be the rings of real and complex **trigonometric polynomials**, considered as subrings of the ring of complex valued functions on $\mathbb{R}$. Prove that

    (a) $A = \mathbb{R}[x,y]/(x^2 + y^2 - 1)$, where $x = \cos t$, $y = \sin t$.

    (b) $A_{\mathbb{C}} = \mathbb{C}[e^{it}, e^{-it}] = \mathbb{C}[z, z^{-1}]$, where $z = e^{it}$; so that $A_{\mathbb{C}}$ is an Euclidean ring, the maximal ideals being $(z - \alpha)$ and the units being αz^n, where $0 \neq \alpha \in \mathbb{C}$, $n \in \mathbb{Z}$.

    (c) A is the ring of invariants of the automorphism τ of $A_{\mathbb{C}} = A \oplus Ai$ induced by the complex conjugation, $\tau\big(\sum_i a_i z^i\big) = \sum_i \bar{a}_i z^{-i}$. The non null elements $p(x,y) \in A$ are

    $$p(x,y) = a_n z^n + \dots + a_{-n}z^{-n} = z^{-n}a_n(z^{2n} + \dots + c_0) = z^{-n}a_nP_{2n}(z), \;\; a_n \neq 0,$$

    where $a_{-i} = \bar{a}_i$; that is to say, $P_{2n}(z)$ is a polynomial of even degree $2n$ such that $z = 1/\bar{\alpha}$ is a root of multiplicity m of $P_{2n}(z)$ when so is $z = \alpha$,

    $$P_{2n}(z) = (z - \alpha_1)\big(z - \tfrac{1}{\bar{\alpha}_1}\big)\dots(z - \alpha_r)\big(z - \tfrac{1}{\bar{\alpha}_r}\big)(z - \beta_1)\dots(z - \beta_s),$$

    (where $0 \neq \alpha_i \neq 1/\bar{\alpha}_i$ and $\beta_j = 1/\bar{\beta}_j$ for any indices i, j; hence $|c_0| = 1$), and $\bar{a}_n/a_n = c_0$.
    The complex roots $\beta = x + yi$ of $P_{2n}(z)$ such that $\beta = 1/\bar{\beta}$ correspond to the intersection points of the curve $p(x,y) = 0$ with the circle $x^2 + y^2 = 1$.
    The ideal generated by $p(x,y)$ in $A_{\mathbb{C}}$ is just $P_{2n}(z)A_{\mathbb{C}}$.

    (d) The units of A are the non null constant functions.

    (e) Up to a unit of $A_{\mathbb{C}}$, the irreducible elements of A are:

i. $(z-\alpha)(z-\frac{1}{\bar{\alpha}})$, where $0 \neq |\alpha| \neq 1$. These irreducibles define real lines $ax+by+c=0$ not intersecting the circle $x^2+y^2=1$ (i.e. $a^2+b^2<c^2$).

ii. $(z-\beta_1)(z-\beta_2)$, where $|\beta_1|=|\beta_2|=1$. These irreducibles define real lines $ax+by+c=0$ intersecting $x^2+y^2=1$ at $x+yi=\beta_1,\beta_2$ (eventually $\beta_1=\beta_2$ when $a^2+b^2=c^2$).

(f) Any factorization of $p(x,y) = z^{-n}a_nP_{2n}(z)$ in A as a product of irreducible elements has n factors, and $p(x,y)$ admits a different factorization (up to order and units) if and only if $P_{2n}(z)$ has 3 different roots of modulus 1, or two different non simple roots of modulus 1.

(g) The irreducible elements of A are $a\cos t + b\sin t + c$; where $a \neq 0$ or $b \neq 0$.

(h) Any complex maximal ideal $\mathfrak{m}$ of A (i.e. $\mathbb{C} = A/\mathfrak{m}$) is a principal ideal, generated by the equation of a line not intersecting the circle $x^2+y^2=1$.

No real maximal ideal $\mathfrak{m}$ (i.e. $\mathbb{R} = A/\mathfrak{m}$) is a principal ideal.

If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are real ideals of A, then $\mathfrak{m}_1\mathfrak{m}_2$ is a principal ideal, generated by the equation of a line intersecting the circle $x^2+y^2=1$ at two points (coincident when $\mathfrak{m}_1=\mathfrak{m}_2$).

64. Show that $(2-\cos t)^2 = (x-2)^2$, $(1-\cos t)^2 = (x-1)^2$ and $(\cos t)(\sin t) - \sin t = (x-1)y$ admit a unique factorization in A as a product of irreducible elements, up to order and units, while $(\cos t)^2 = x^2$ admits other factorization, defined by the other pair of lines $(1+y)(1-y)$ intersecting twice the circle at the points $x=0$, $y=\pm 1$,

$$(\cos t)^2 = (1+\sin t)(1-\sin t),$$

and $(\cos t)(\sin t) = xy$ admits two other factorizations in A, defined by the two other pairs of lines $(y+x+1)(y+x-1)$ and $(x-y-1)(y-x-1)$ intersecting the circle at $x=\pm1$, $y=\pm1$,

$$2(\cos t)(\sin t) = (\sin t + \cos t + 1)(\sin t + \cos t - 1) = (\cos t - \sin t - 1)(\sin t - \cos t - 1).$$

Find also the factorizations of $(\sin t)^2 + \cos t - 1 = y^2 + x - 1$.

4. Analysis II

1. In a direct product $\prod_i X_i$ of topological spaces, show that the closure of a direct product $\prod_i Y_i$ of subsets is $\prod_i \bar{Y}_i$. Hence, $\prod_i Y_i$ is closed if and only if Y_i is closed in X_i for any index i.

2. Prove that a metrizable space X admits a countable base of open sets if and only if it is **separable** (it admits a countable dense set).

3. Let X be a topological space. If we consider on X the equivalence relation $x \equiv y$ when $\bar{x} = \bar{y}$, prove that $X_0 = (X/\equiv)$ is a T_0-space, and that any continuous map $X \to Y$ into a T_0-space uniquely factors through X_0; i.e. $\mathrm{Hom}_{\mathrm{Top}}(X,Y) = \mathrm{Hom}_{\mathrm{Top}}(X_0,Y)$.

   If we put $x \equiv y$ when there is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that $x_i \in \bar{x}_{i+1}$ or $x_{i+1} \in \bar{x}_i$, prove that $X_1 = (X/\equiv)$ is a T_1-space, and that any continuous map $X \to Y$ into a T_1-space uniquely factors through X_1; i.e. $\mathrm{Hom}_{\mathrm{Top}}(X,Y) = \mathrm{Hom}_{\mathrm{Top}}(X_1,Y)$.

   If we put $x \equiv y$ when $f(x) = f(y)$ for any continuous map from X into a separated space, prove that $X_H = (X/\equiv)$ is a separated space, and that any continuous map $X \to Y$ into a separated space uniquely factors through X_H; i.e. $\mathrm{Hom}_{\mathrm{Top}}(X,Y) = \mathrm{Hom}_{\mathrm{Top}}(X_H,Y)$.

4. Let $R \subseteq X \times X$ be an equivalence relation in a topological space. Show that X/R is separated if and only if R is closed in $X \times X$.

5. Given equivalence relations R, R' on two topological spaces X, X', show that we have a homeomorphism $(X \times X')/(R \times R') = (X/R) \times (X'/R')$.

6. Let $\emptyset \neq V \subset \mathbb{R}$ be an open set, and put $X = (\mathbb{R} \times \{0,1\})/\equiv$, where $(x,0) \equiv (x,1)$ when $x \in V$. Show that X is a non separated topological manifold. (*Hint*: $(x,0)$ and $(x,1)$ do not have disjoint neighborhoods when $x \in \partial V$).

7. If a T_0 space X admits a countable base of open sets, prove that $|X| \leq 2^{\aleph_0}$.

8. Prove that any countable intersection of open subsets of a compact separated space is Baire.

9. Prove that the concept of differentiable map does not depend on the scalar product that we fix on the space of free vectors E.

10. Let $\varphi\colon U \to E'$ be a differentiable map and let f be a differentiable function on U. Show that $f\varphi\colon U \to E'$ is differentiable and $(f\varphi)_* = f(\varphi_*) + \mathrm{d}f \otimes \varphi$.

11. Show that any $\mathbb{R}$-multilinear map $E_1 \times \ldots \times E_n \to F$ is a map of class $\mathcal{C}^\infty$.

12. Give an alternative proof of Schwarz's theorem, using Barrow's rule an the differentiation rule under the integral sign to show that $\int_a^{a+\varepsilon}(\partial_1\partial_2 f)\mathrm{d}x_1 = \int_a^{a+\varepsilon}(\partial_2\partial_1 f)\mathrm{d}x_1$.

13. Use the mean value theorem to give an alternative proof of Schwarz's theorem: Near the origin, given δ_1, δ_2, show that there exist $|\chi_1| < |\delta_1|$, $|\chi_2| < |\delta_2|$ such that

$$\big[f(\delta_1,\delta_2) - f(\delta_1,0)\big] - \big[f(0,\delta_2) - f(0,0)\big] = \big(\partial_1 f(\chi,\delta_2) - \partial_1 f(\chi_1,0)\big)\delta_1 = \partial_2\partial_1 f(\chi_1,\chi_2)\delta_1\delta_2.$$

$$\big[f(\delta_1,\delta_2) - f(0,\delta_2)\big] - \big[f(\delta_1,0) - f(0,0)\big] = \partial_1\partial_2 f(\chi_1',\chi_2')\delta_1\delta_2, \text{ with } |\chi_1'| < |\delta_1|,\ |\chi_2'| < |\delta_2|.$$

Hence $\partial_2\partial_1 f(\chi_1,\chi_2) = \partial_1\partial_2 f(\chi_1',\chi_2')$, and conclude when $(\delta_1,\delta_2) \to (0,0)$.

14. If $f \in \mathcal{C}^m(U)$ and $a \in U$, show that $f^{(n)}\colon E\times \overset{n}{\ldots} \times E \to \mathbb{R}$, $f^{(n)}(v_1,\ldots,v_n) = (\partial_{v_1}\ldots\partial_{v_n} f)(a)$ is a symmetric tensor. Moreover, if $f^{(1)} = \ldots = f^{(m-1)} = 0$, prove that

    (a) If $f^{(m)}(v,\ldots,v) > 0$ for any vector $v \neq 0$, then f has a local minimum at $x = a$.

    (b) If $f^{(m)}(v,\ldots,v) < 0$ for any vector $v \neq 0$, then f has a local maximum at $x = a$.

    (c) If $f^{(m)}(v,\ldots,v) > 0$ for some vector v, and $f^{(m)}(e,\ldots,e) < 0$ for another vector e, then f has not an extremum at $x = a$.

15. Let $f_1,\ldots,f_n$ be functions of class $\mathcal{C}^m$ on an open set U of a n-dimensional affine space, and assume that $\mathrm{d}_p f_1,\ldots,\mathrm{d}_p f_n$ are linearly independent at any point $p \in U$. If these functions separate points in U (for any two points $x \neq y$ of U, we have $f_i(x) \neq f_i(y)$ for some index i) and we consider the map $\varphi = (f_1,\ldots,f_n)\colon U \to \mathbb{R}^n$, prove that $\varphi(U)$ is an open set and that $\varphi\colon U \to \varphi(U)$ is a $\mathcal{C}^m$-diffeomorphism.

16. Prove the key lemma along a linear subvariety: Let V be an open set in $\mathbb{R}^d$ and let U be a tubular open neighborhood of $V \times 0$ in $\mathbb{R}^d \times \mathbb{R}^n$ (if $(x,y) \in U$, then $(x,ty) \in U$, for any $0 \leq t \leq 1$). If $f \in \mathcal{C}^m(U)$ vanishes on $V \times 0$, then there are functions $f_1,\ldots,f_n \in \mathcal{C}^{m-1}(U)$ such that $f = f_1y_1 + \ldots + f_ny_n$.

17. Let $(X,\mathcal{A},\mu)$ be a measure space and $A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$ a decreasing sequence of sets in $\mathcal{A}$. If $\mu(A_1) < \infty$, prove that $\mu\big(\bigcap_n A_n\big) = \lim \mu(A_n)$.

18. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a family closed under finite intersections (in particular $\emptyset \in \mathcal{I}$) such that I^c is a finite disjoint union of elements of $\mathcal{I}$, for any $I \in \mathcal{I}$. Show that the finite disjoint unions of elements of $\mathcal{I}$ form an algebra. Conclude that the finite unions of disjoint (eventually unbounded) rectangles in $\mathbb{R}^n$ form an algebra.

19. If $A \subset \mathbb{R}^n$, prove the existence of a Borel set $B \supseteq A$ such that $m_e(A) = m_e(B)$.

20. Prove that the category of measurable spaces has direct products.

    Let X, Y be topological spaces and T a measurable space. If $f\colon T \to X$ and $h\colon T \to Y$ are measurable maps, and X, Y have a countable base of open sets, show that $f \times h\colon T \to X \times Y$ is measurable. Conclude that $X \times Y$ also is the direct product in the category of measurable spaces.

21. If $f, h\colon X \to [0,\infty]$ are measurable, prove that so is $f\cdot h$. (*Hint*: $[0,\infty]\times[0,\infty] \xrightarrow{\cdot} [0,\infty]$ is measurable).

22. Prove that the cardinal of $\mathcal{B}(\mathbb{R}^n)$ is $2^{\aleph_0}$.

23. Consider the real vector space with base the vertices e_1, e_2, e_3, e_4 of a regular tetrahedron, and let M_i be the matrix of the linear automorphism induced by the reflection on the i-th face. Show that the coefficients of M_i are in $\frac{1}{3}\mathbb{Z}$, and that the reduction S_i modulo 3 of $3M_i$ vanishes on such base, except $S_i(e_i) = -(e_1 + \ldots + \widehat{e}_i + \ldots + e_4$.

    Show that $S_iS_i = 0$ and that, when $i_j \neq i_{j+1}$ for any index j, we have

$$S_{i_k} \ldots S_{i_1} = \begin{cases} (-1)^{k-1} S_{i_1} & \text{when } i_k = i_1 \\ (-1)^k S_{i_k} S_{i_1} & \text{when } i_k \neq i_1 \end{cases}$$

(in particular $S_{i_k} \ldots S_{i_1} \neq 0$). If $R_1, \ldots, R_4$ are the reflections on the faces of a regular tetrahedron and a product $R_{i_k} \ldots R_{i_1}$ is a translation, conclude that $i_j = i_{j+1}$ for some index j. Hence the linear components r_i of such reflections generate a free group (with the only relations $r_i^2 = 1$) and the rotations $a = r_1 r_2$, $b = r_3 r_4$ generate a free group G, so that

$$G = G_a \amalg G_{a^{-1}} \amalg G_b \amalg G_{b^{-1}} \amalg \{\text{Id}\} = G_a \amalg aG_{a^{-1}} = G_b \amalg G_{b^{-1}},$$

where $G_x = \{x \ldots\}$. Now consider the natural action of G on a sphere S_2, let M be a subset of S_2 with a unique point in each orbit, and put $M_x = G_x M$. Conclude that we may cover a sphere $S_2 = GM = M_a \cup M_{a^{-1}} \cup M_b \cup M_{b^{-1}} \cup M$ with 5 subsets with countable intersections, so that, after composing with motions, we may form two spheres $S_2 = M_a \cup aM_{a^{-1}} = M_b \cup bM_{b^{-1}}$ of equal radius (**Banah-Tarski paradox**).

24. Prove that any Riemann-integrable bounded function $f\colon [a,b] \to \mathbb{R}$ is Lebesgue-measurable, and the Lebesgue integral of f coincides with the Riemann integral.

25. Let $C \subset [0,1]$ be the cantor set. Show that some subset $N \subset C$ is not a Borel set, and conclude that I_N is a Riemann-integrable function, but not a Borel-measurable function.

26. If a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is almost everywhere continuous, prove that the set of points where f is continuous is a Borel set.

27. Let f, g be two continuous functions with compact support in $\mathbb{R}^n$. Prove the following statements about the **convolution product**

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-t)g(t)\mathrm{d}t.$$

   (a) $f * g$ is a continuous function in $\mathbb{R}^n$ and $\operatorname{supp}(f * g) \subseteq (\operatorname{supp} f) \cup (\operatorname{supp} g)$.

   (b) The set $\mathcal{C}_0(\mathbb{R}^n)$ of continuous functions with compact support, with the convolution product and the usual sum, is a commutative ring (without unit).

   (c) If f is of class $\mathcal{C}^\infty$, so is $f * g$, and $\dfrac{\partial (f*g)}{\partial x_i} = \left(\dfrac{\partial f}{\partial x_i}\right) * g$.

28. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a family closed under finite intersections (in particular $\emptyset \in \mathcal{I}$) such that I^c is a countable disjoint union of elements of $\mathcal{I}$, for any $I \in \mathcal{I}$. Prove that the σ-algebra generated by $\mathcal{I}$ is the least family $\mathcal{I} \subseteq \mathcal{M}$ closed under complements and countable disjoint unions. (*Hint*: Use the argument of the monotone class theorem). If $I \subset \mathbb{R}^n$ is a rectangle, show that I^c is a countable disjoint union of rectangles. Conclude that $\mathcal{B}(\mathbb{R}^n)$ is the least family containing the rectangles and closed under complements and countable disjoint unions.

29. Give an alternative proof of the change of variables formula as follows:

   Show that the formula holds for $\varphi_1 \circ \varphi_2$ whenever it holds for φ_1 and φ_2.

   Prove that the formula holds when $n = 1$ (*Hint*: Barrow's formula shows that it holds on any finite interval) and then proceed by induction on n:

   (a) The formula holds when $\varphi(x_1, \ldots, x_n) = (y_1, \ldots, y_{n-1}, x_n)$. (*Hint*: Show that in this case $J_\varphi(p)$ is the jacobian at $p = (a_1, \ldots, a_n)$ of $\varphi\colon U \cap \{x_n = a_n\} \to V \cap \{x_n = a_n\}$, and conclude using Fubini's theorem).

   (b) In general, if $y_i = y_i(x_1, \ldots, x_n)$ are the equations of φ and we fix a point $p \in U$, show that the jacobian at p of $\varphi_1(x_1, \ldots, x_n) = (y_1, \ldots, y_{n-1}, x_n)$ is non zero, relabelling $x_1, \ldots, x_n$ if necessary. Hence there is an open neighborhood U' of p such that $\varphi_1\colon U' \to V' = \varphi_1(U')$ is a diffeomorphism, and $\varphi_2 = \varphi\varphi_1^{-1}\colon V' \to \varphi(U')$ is a diffeomorphism and $\varphi_2(y_1, \ldots, y_n) = \big(y_1, \ldots, y_{n-1}, z_n(y_1, \ldots, y_n)\big)$. Conclude that the formula holds for $\varphi|_{U'}\colon U' \to \varphi(U')$.

   (c) If $U = \bigcup_i U_i$ is an open cover and the formula holds for $\varphi|_{U_i}$ for any index i, prove that the formula holds for φ.

30. Consider $\mathbb{R}^n$ with the σ-algebra $\mathcal{L}$ of all measurable sets and the Lebesgue measure m, so that a function f is **$\mathcal{L}$-measurable** when $f^{-1}(B)$ is a measurable set for any Borel set $B \subseteq \mathbb{R}$, and prove the following results:

    (a) If $f\colon \mathbb{R}^n \to [0,\infty]$ is $\mathcal{L}$-measurable, $\int_{\mathbb{R}^n} f \mathrm{d}m = m\{(x,y) \in \mathbb{R}^n \times \mathbb{R}\colon 0 \leq y < f(x)\}$.

    (b) If $N \subset \mathbb{R}^{n+m}$ has null measure, then $m(N_y) = 0$ almost everywhere. (*Hint*: Take a Borel set $N \subseteq B$ with $m(B) = 0$, show that $\int h_B \mathrm{d}m = m(B) = 0$ and conclude that $m(B_y) = 0$ almost everywhere).

    (c) If $f\colon \mathbb{R}^{n+m} \to [0,\infty]$ is $\mathcal{L}$-measurable, then $h_f(y) = \int_{\mathbb{R}^n} f(x,y)\mathrm{d}m$ is $\mathcal{L}$-measurable and Fubini's formula holds for f. (*Hint*: Take a disjoint union $L = B \cup N$, where $m(N) = 0$ and B is a Borel set, show that $h_N(y) = m(N_y)$ is $\mathcal{L}$-measurable and $\int h_N \mathrm{d}m = 0 = \int I_N \mathrm{d}m$. Hence $h_L = h_B + h_N$ is $\mathcal{L}$-measurable and $\int h_L \mathrm{d}m = \int I_L \mathrm{d}m$, and conclude with the monotone convergence theorem).

    (d) If $f\colon \mathbb{R}^{n+m} \to \mathbb{R}$ is $\mathcal{L}$-integrable, and $f_y(x) = f(x,y)$ is $\mathcal{L}$-integrable for any $y \in \mathbb{R}^m$, then $h_f(y) = \int_{\mathbb{R}^n} f(x,y)\mathrm{d}m$ is $\mathcal{L}$-integrable and Fubini's formula holds for f.

    (e) If $f\colon V \to [0,\infty]$ is $\mathcal{L}$-measurable, then $(f \circ \varphi)|J_\varphi|$ is $\mathcal{L}$-measurable and the change of variables formula holds for f. (*Hint*: If $L = B \cup N$ is a disjoint union of a Borel set and a set of null measure, the formula holds for $I_L = I_B + I_N$, and conclude with the monotone convergence theorem).

    (f) If $f\colon V \to \mathbb{R}$ is $\mathcal{L}$-integrable, then $(f \circ \varphi)|J_\varphi|$ is $\mathcal{L}$-integrable and the change of variables formula holds for f.

31. Prove that any direct product of topological vector spaces, with the product topology, also is a topological vector space.

32. If we endow the spaces of continuous maps with the compact-open topology, prove that

    (a) If $f\colon X \to Y$ is continuous, so are $f_*\colon \mathcal{C}(T,X) \to \mathcal{C}(T,Y)$ and $f^*\colon \mathcal{C}(Y,T) \to \mathcal{C}(X,T)$.

    (b) $\mathcal{C}(T \times X, Y) = \mathcal{C}\big(T, \mathcal{C}(X,Y)\big)$ when T is a separated space and X is a locally compact separated space. Moreover, the natural map $\mathcal{C}(T,X) \times \mathcal{C}(X,Y) \to \mathcal{C}(T,Y)$ is continuous.

33. Let $q_1, \ldots, q_n$ be seminorms on E, and p a seminorm on $\mathbb{R}^n$. Show that $q(e) = p\big(q_1(e), \ldots, q_n(e)\big)$ is a seminorm on E, defining the topology induced by $\{q_1, \ldots, q_n\}$ when p is a norm.

    Let $E_1, \ldots, E_n$ be **seminormable** spaces (topological vector spaces such that the topology is defined by a seminorm). Given linear maps $f_i\colon E \to E_i$, show that the initial topology is defined by a seminorm. Conclude that $E_1 \times \ldots \times E_n$ also is seminormable.

34. If $q\colon E \to \mathbb{K}$ is a seminorm, show that $|q(y) - q(x)| \leq q(y - x)$, $\forall x, y \in E$. Conclude that q is continuous, when we consider on E the linear topology defined by q.

    Let $q'\colon E \to \mathbb{K}$ be another seminorm. If $q \leq cq'$ for some constant $c \in \mathbb{R}$, prove that q also is continuous when we consider on E the linear topology defined by q'.

35. Prove that two seminorms q', q on a vector space E define the same topology if and only if there exist $m, M \in \mathbb{R}_+$ such that $mq(e) \leq q'(e) \leq Mq(e)$, $\forall e \in E$.

36. Let $e_1, \ldots, e_n$ be an orthonormal base of a real Euclidean space E. If p is a seminorm on E and we put $c = \max\{p(e_1), \ldots, p(e_n)\}$, show that $p(e) \leq nc\|e\|$, $\forall e \in E$.

    When p is a norm, prove that p is bounded below on the unit sphere $\{e \in E\colon \|e\| = 1\}$ by a constant $m > 0$, so that $m\|e\| \leq p(e)$, $\forall e \in E$. Conclude that, in a finite dimensional real vector space, all the norms define the same linear topology.

37. Prove that the product $\mathcal{C}^m(U) \times \mathcal{C}^m(U) \to \mathcal{C}^m(U)$, $0 \leq m \leq \infty$, is continuous.

5. Algebra II

1. If G is a group, show that $G \to G^{\mathrm{op}}$, $g \mapsto g^{-1}$, is a group isomorphism, so that right actions of G on a set X correspond to left actions of G on X via the formula $gx = xg^{-1}$.

2. Show that the actions of a group G on a given set with n elements correspond to the group morphisms $G \to S_n$. Conclude that any finite group G of order n is isomorphic to a subgroup of S_n.
3. Show that $G = PGl(2, \mathbb{F}_5)$ is a subgroup of S_6 of index 6. Obtain an automorphism $\tau\colon S_6 \to S_6$ such that $\tau(G) = \{\sigma \in S_6 \colon \sigma(1) = 1\}$, and conclude that τ is not an inner automorphism of S_6. (*Hint*: Consider the action of S_6 on the set of conjugate subgroups of G).
4. If p is the least prime dividing the order of a finite group G, prove that any subgroup H of G of index p is a normal subgroup. (*Hint*: The kernel of the action of G on G/H has index p.)
5. If p is a prime number, prove that any group of order p^2 is abelian, and that any non-trivial normal subgroup of a non abelian group G of order p^3 contains the center $Z(G)$.
6. If H is a normal subgroup of a p-group G, prove that $H \cap Z(G) \neq 1$.
7. If a finite group G has a unique Sylow p-subgroup for any prime p, prove that G is solvable.
8. If $n \neq 4$, prove that A_n is a **simple** group (any normal subgroup is trivial). (*Hint*: If H is a non-trivial normal subgroup of A_n, it is not a normal subgroup of S_n, p. 485, so that it has a unique conjugate subgroup $H' = \tau H \tau^{-1}$, where σ is any odd permutation. Now $H' \cap H = \mathrm{Id}$ and $H'H = A_n$, because they are normal subgroup of S_n, so that $A_n = H' \times H$. Hence $|H|$ is even, and H contains a product $\sigma = \tau_1 \dots \tau_r$ of disjoint transpositions, and we get a contradiction $\sigma = \tau_1 \sigma \tau^{-1} \in H'$).
9. If $n \geq 5$, prove that A_n is the unique non trivial subgroup of S_n of index $< n$. Moreover, A_n has no non-trivial subgroup of index $< n$.
10. If H is a non-trivial subgroup of a simple group G, prove that $[G : H] \geq 5$.

    Let n be the number of Sylow p-subgroups of a simple group $G \neq \mathbb{Z}/p\mathbb{Z}$. Prove that $n \geq 5$.
11. Show that the A-module structures on a given abelian group M correspond to the ring morphisms $A \to \mathrm{End}(M)$.
12. If M is an A-module, show that we have a natural isomorphism $\mathrm{Hom}_A(A, M) = M$, $f \mapsto f(1)$.
13. Prove that $\mathrm{Hom}_A(N, \bigoplus_i M_i) = \bigoplus \mathrm{Hom}_A(N, M_i)$ whenever N is a finitely generated A-module.
14. If $M_0 \subset M_1 \subset \ldots \subset M_n$ is a sequence of submodules, show that $l(M_n/M_0) = \sum_i l(M_i/M_{i-1})$.
15. If $0 \to M_0 \to M_1 \to \ldots \to M_n \to 0$ is an exact sequence of A-modules of finite length, show that
$$l(M_0) - l(M_1) + \ldots + (-1)^n l(M_n) = 0.$$
16. Let $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ be a flag of an A-module M of finite length. Given a maximal ideal $\mathfrak{m}$ of A, prove that the number of indices such that $M_i/M_{i-1} \simeq A/\mathfrak{m}$ does not depend on the flag. (*Hint*: Use induction on n, considering a quotient M/N by a simple submodule).
17. Show that any abelian group of finite length is a finite group.
18. If Σ is the field of fractions of a domain A, show that Σ and Σ/A are divisible A-modules.

    Moreover Σ always is an injective A-module. (*Hint*: $\mathrm{Hom}_A(M, \Sigma) = \mathrm{Hom}_\Sigma(M_\Sigma, \Sigma)$.)
19. Let k be a field and let $A = k[x]/(p(x))$, where $p(x) \neq 0$. Show the existence of an isomorphism of A-modules $A \simeq A^* = \mathrm{Hom}_k(A, k)$, and conclude that A is an injective A-module.

    Moreover, any projective A-module is an injective A-module.
20. Let A be an integral ring. If there is an injective A-linear morphism $A^r \to A^n$, prove that $r \leq n$. (For the general case, see a problem in p. 525).
21. Prove the universal property of the polynomial ring $A[x_1, \ldots, x_n]$:
$$\mathrm{Hom}_{A\text{-alg}}(A[x_1, \ldots, x_n], B) = B^n \ , \ \phi \mapsto \big(\phi(x_1), \ldots, \phi(x_n)\big).$$
22. Prove that $\big(A[x_1, \ldots, x_n]/(p_1, \ldots, p_r)\big) \otimes_A B = B[x_1, \ldots, x_n]/(p_1, \ldots, p_r)$.
23. If E is a finite-dimensional k-vector space, show that there is a canonical isomorphism
$$T^q_p E = E^* \otimes_k \overset{p}{\ldots} \otimes_k E^* \otimes_k E \otimes_k \overset{q}{\ldots} \otimes_k E.$$

24. If E, F are finite-dimensional k-vector spaces, show that $\mathrm{Hom}_k(E,F) = E^* \otimes_k F$.

25. If E, F are finite-dimensional k-vector spaces, prove that $\dim(E \otimes_k F) = (\dim E)(\dim F)$, and that $\dim_k E = \dim_K(E \otimes_k K)$ for any extension $k \to K$.

26. Let E, F be k-vector spaces of dimensions $n = \dim E$, $m = \dim F$. Given endomorphisms $T\colon E \to E$, $S\colon F \to F$, prove that $\det(T \otimes S) = (\det T)^m (\det S)^n$. (*Hint*: $T \otimes S = (T \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes S)$.)

    Moreover, we have a natural isomorphism

$$\Lambda^{nm}(E \otimes_k F) = (\Lambda^n E) \otimes_k \overset{m}{\ldots} \otimes_k (\Lambda^n E) \otimes_k (\Lambda^m F) \otimes_k \overset{n}{\ldots} \otimes_k (\Lambda^m F).$$

27. Prove that $\mathbb{Z}[x]/\big(Q(x)\big) \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathbb{F}_p[x]/\big(\bar{Q}(x)\big)$, where $\bar{Q}(x)$ is the reduction of $Q(x)$ modulo p.

28. Show that there is an isomorphism of $\mathbb{C}$-algebras $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, $z_1 \otimes z_2 \mapsto (z_1 z_2, \bar{z}_1 z_2)$.

29. If $d = \text{g.c.d.}(m,n)$, show that $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$, $[a] \otimes [b] \mapsto [ab]$, is a ring isomorphism.

30. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of a ring A and let M be an A-module. If $\mathfrak{a} + \mathfrak{b} = A$, prove that we have an isomorphism of A-modules $M/\mathfrak{a}\mathfrak{b}M = (M/\mathfrak{a}M) \oplus (M/\mathfrak{b}M)$, $[m] \mapsto ([m],[m])$.

31. Let $A, B \neq 0$ be algebras over a field k. Prove that $A \to A \otimes_k B$, $a \mapsto a \otimes 1$, is injective.

32. Show that an extension $k \to L$ is trivial if and only if $L \otimes_k L$ is a field. (*Hint*: The product defines a morphism of k-algebras $L \otimes_k L \to L$).

33. If $\{M_i\}_{i \in I}$ is a chain of submodules of a module, show that $\varinjlim M_i = \bigcup_i M_i$ and $\varprojlim M_i = \bigcap_i M_i$.

34. Prove that any A-module is an inductive limit of finitely generated A-modules, and that any ring is an inductive limit of finitely generated $\mathbb{Z}$-algebras.

35. Prove that the inductive limit of an inductive system $\{M_i, \phi^i_j\}$ of A-modules is isomorphic to the quotient of $\oplus_i M_i$ by the submodule N generated by the elements $m_i - \phi^i_j(m_i)$, endowed with the natural morphisms $\phi_i\colon M_i \to (\oplus_i M_i)/N$.

36. Let M be an A-module and $f \in A$. If we consider the inductive system of A-modules $\{M_n\}_{n \in \mathbb{N}}$, where $M_n = M$ and $\phi^n_{n+1}(m) = fm$ for all $n \in \mathbb{N}$, show that $M_f = \varinjlim M_n$.

37. Let $\{X_i, \phi^i_j\}$, $\{Y_i, \varphi^i_j\}$ be inductive (resp. projective) systems of sets with the same index set. If we have maps $f_i\colon X_i \to Y_i$ commuting with the transition maps, $f_j \phi^i_j = \varphi^i_j f_i$, prove the existence of a unique map $f\colon \varinjlim X_i \to \varinjlim Y_i$ (resp. $f\colon \varprojlim X_i \to \varprojlim Y_i$) such that $f\phi_i = \varphi_i f_i$ (resp. $\varphi_i f = f\phi_i$).

38. Let $\{M'_i\}$, $\{M_i\}$, $\{M''_i\}$ be inductive systems of A-modules, with the same index set. If we have exact sequences $0 \to M'_i \to M_i \to M''_i \to 0$, where the morphisms commute with the transition morphisms, prove that they induce an exact sequence $0 \to \varinjlim M'_i \to \varinjlim M_i \to \varinjlim M''_i \to 0$.

39. Let $\{M'_i\}$, $\{M_i\}$, $\{M''_i\}$ be projective systems of A-modules, with the same index set. If we have exact sequences $0 \to M'_i \to M_i to M''_i$, where the morphisms commute with the transition morphisms, prove that they induce an exact sequence $0 \to \varprojlim M'_i \to \varprojlim M_i \to \varprojlim M''_i$.

    However, show that the natural morphism $\mathbb{Z} = \varprojlim \mathbb{Z} \to \varprojlim(\mathbb{Z}/2^n\mathbb{Z})$ is not surjective.

40. In the ring $\varprojlim(\mathbb{Z}/10^n\mathbb{Z})$, show that $1 + 999\ldots = 0$ and $\frac{1}{1-10} = 1 + 10 + 10^2 + \ldots = 111\ldots$.

41. Prove that $\varinjlim(M_i \oplus N_i) = (\varinjlim M_i) \oplus (\varinjlim N_i)$ and $\varprojlim(M_i \times N_i) = (\varprojlim M_i) \times (\varprojlim N_i)$.

    Moreover, prove that inductive limits commute with arbitrary direct sums, and that projective limits commute with arbitrary direct products.

42. Let $\{F_i\}_{i \in I}$ be the family of all finite subsets of $\mathbb{R}$, with the inclusion order. Show that the natural restriction map $\mathrm{Inj}(F_j, \mathbb{Z}) \to \mathrm{Inj}(F_i, \mathbb{Z})$ is surjective whenever $F_i \subseteq F_j$ (where $\mathrm{Inj}(F_i, \mathbb{Z})$ stands for the set of all injective maps $F_i \hookrightarrow \mathbb{Z}$), but the projective limit $\varprojlim \mathrm{Inj}(F_i, \mathbb{Z})$ is empty. (*Hint*: An element would define an injective map $\mathbb{R} \hookrightarrow \mathbb{Z}$).

43. Given an inductive system of rings $\{A_i\}_{i \in I}$, show that $\mathrm{rad}(\varinjlim A_i) = \varinjlim(\mathrm{rad}\, A_i)$.

44. Prove that any flat module P over a domain A is torsion free.

45. Prove that any inductive limit of flat A-modules is a flat A-module.

46. Prove that a direct sum $\oplus_i M_i$ of A-modules is flat if and only if so are the A-modules M_i.

47. If M, N are flat A-modules, prove that so is $M \otimes_A N$.

    If B is a flat A-algebra and M is a flat B-module, prove that M is a flat A-module.

48. If M is a finitely generated (resp. flat, projective, free) A-module, prove that M_B is a finitely generated (resp. flat, projective, free) B-module for any base change $A \to B$

49. Where does the following argument fail? Let k be a field. If a k-algebra A admits a morphism of k-algebras $A \to k$, then any A-module M is free, because $M = M \otimes_A A = (M \otimes_A k) \otimes_k A$, and $M \otimes_A k$ is a free k-module, so that $(M \otimes_A k) \otimes_k A$ is a free A-module.

50. Given any object M of a category, prove that the composition of morphisms defines a group structure on the set of automorphisms of M.

51. Show that a group G may be viewed as a category $\mathbf{C}$ with a unique object where any morphism is an isomorphism, and prove that the opposite category $\mathbf{C}^{\mathrm{op}}$ is defined by the opposite group G^{op}. Moreover, if $\mathbf{C}'$ is the category defined by another group G', show that the covariant functors $\mathbf{C} \rightsquigarrow \mathbf{C}'$ are just the group morphisms $G \to G'$.

52. Prove that the construction of the dual space $E \rightsquigarrow E^*$ defines an equivalence of the category of finite-dimensional k-vector spaces and linear maps with the opposite category.

53. Let $F\colon \mathbf{C} \rightsquigarrow \mathbf{C}'$ be a covariant functor such that the maps $F\colon \mathrm{Hom}_{\mathbf{C}}(M,N) \to \mathrm{Hom}_{\mathbf{C}'}(F(M),F(N))$ are bijective and any object $\mathbf{C}'$ is isomorphic to $F(M)$ for some object M of $\mathbf{C}$. If the objects of $\mathbf{C}'$ form a set (so that for any object M' of $\mathbf{C}'$ we may choose an object $G(M')$ of $\mathbf{C}$ and an isomorphism $F(G(M')) \simeq M'$) prove that F defines an equivalence of categories.

54. Let $\mathbf{D}$ be the discrete (identities are the only morphisms) category of pointed sets (X,x) and let us consider the covariant functors $S, p\colon \mathbf{D} \rightsquigarrow \mathbf{Sets}$ such that $S(X,x) = X$ and $p(X,x) = \{x\}$. Show that for any set X we have an injective map $X \to \mathrm{Hom}_{\mathrm{nat}}(p,S)$, so that the functorial morphisms $p \to S$ do not form a set. (In general, the "category" $\mathrm{Hom}(\mathbf{C},\mathbf{C}')$ of covariant functors of a category $\mathbf{C}$ into a category $\mathbf{C}'$ is not a true category, unless you impose some conditions guaranteeing that $\mathrm{Hom}_{\mathrm{nat}}(F,G)$ is a set for any two functors $F, G\colon \mathbf{C} \rightsquigarrow \mathbf{C}'$.)

55. (*Equivalences of categories are just homotopical equivalences.*) Define the direct product $\mathbf{C}\times\mathbf{C}'$ of two categories, and consider the unit interval category $\mathbf{I}$ with two objects $0,1$ and a unique isomorphism $\gamma\colon 0 \to 1$. Show that any functor $F\colon \mathbf{C}\times\mathbf{I} \rightsquigarrow \mathbf{C}'$ is defined by the functor $F_0(X) = F(X,0)$, the functor $F_1(X) = F(X,1)$ and a functorial isomorphism $\theta_X = F(\mathrm{Id}_X,\gamma)\colon F_0(X) \to F_1(X)$.

56. Let $U \to X$, $V \to X$ be two subspaces of a topological space X. Show that $U \times_X V = U \cap V$.

57. Prove that a topological space X is connected if and only if the functor $\mathrm{Hom}(X,-)$ preserves coproducts (so explaining why connected spaces are assumed to be non empty):

$$\mathrm{Hom}(X, Y \amalg Z) = \mathrm{Hom}(X,Z) \amalg \mathrm{Hom}(X,Z).$$

58. Let k be a field and fix a natural number n. Let $\mathbf{C}$ be the category of n-dimensional k-vector spaces and linear isomorphisms, and let us view the dual E^* as a covariant functor $F\colon \mathbf{C} \rightsquigarrow \mathbf{C}$, $F(E) = E^*$, $F(\tau) = (\tau^{-1})^*$. If $n \geq 2$ (or k has more than 3 elements) prove that there is no natural isomorphism $E \to E^*$ defined on the n-dimensional k-vector spaces; i.e., F is not isomorphic to the identity functor. (*Hint*: A natural isomorphism $E \to E^*$ defines a non-singular intrinsic covariant tensor T_2 on E).

    If $n = 1$ and $k = \mathbb{F}_2$ or $\mathbb{F}_3$, show the existence of a natural isomorphism $E \simeq E^*$.

59. Using Yoneda's lemma, prove that $\mathrm{Hom}_{\mathbf{C}}(X,Y) = \mathrm{Hom}_{nat}(X^\bullet, Y^\bullet)$, for any objects X, Y of $\mathbf{C}$, and answer the following questions:

(a) What natural maps $G \to G$ may be defined on groups? (What are the natural transformations from the forgetful functor **Groups** $\rightsquigarrow$ **Sets** into itself?)

What natural morphisms $G \to G$ may be defined on groups? (What are the natural transformations from the identity functor **Groups** $\rightsquigarrow$ **Groups** into itself?)

What natural maps $G \to G \times G$ may be defined on groups?

(b) Let k be a field. Determine the natural maps $A \to A \times A$ and operations $A \times A \to A$ that may be defined on k-algebras. Obtain the natural ring morphisms $A \to A$ that may be defined on k-algebras, when $\operatorname{char} k = 0$ and when $\operatorname{char} k = p > 0$.

(c) Find the natural maps $X \to X \times X$ and operations $X \times X \to X$ on topological spaces.

(d) Find the natural maps $M \to M \times M$ and operations $M \times M \to M$ on A-modules.

60. Let $F\colon A$-**mod** $\rightsquigarrow$ A-**mod** be a covariant A-linear functor and put $N = F(A)$. Define a natural morphism of A-modules $M \otimes_A N \to F(M)$, $m \otimes n \mapsto F(h_m)(n)$, (where $h_m\colon A \to M$, $h_m(a) = am$), and prove that it is an isomorphism of functors if and only if F preserves direct sums and for any exact sequence of A-modules $M' \to M \to M'' \to 0$, so is the sequence $F(M'') \to F(M) \to F(M') \to 0$. (*Hint*: Any A-module M admits a presentation $A^{(J)} \to A^{(I)} \to M \to 0$.)

Let $F\colon A$-**mod** $\rightsquigarrow$ A-**mod** be a contravariant A-linear functor and put $N = F(A)$. Define a natural morphism of A-modules $F(M) \to \operatorname{Hom}_A(M, N)$ and prove that it is an isomorphism of functors if and only if F transforms direct sums into direct products and for any exact sequence of A-modules $M' \to M \to M'' \to 0$, so is the sequence $0 \to F(M'') \to F(M) \to F(M')$. (*Hint*: By Yoneda's lemma any element $\xi \in F(M)$ defines a natural morphism $M^\bullet \to F$, hence a map $M = M^\bullet(A) \to F(A)$.)

61. In the category of sets, determine the direct product $\prod_{i\in I} X_i$ when $X_i = \emptyset$ for any index i (resp. when $I = \emptyset$, and when $X_i = I = \emptyset$). (*Hint*: The direct product of a family of sets $\{X_i\}_{i\in I}$ such that $X_i = X$ for any index i is $\operatorname{Hom}(I, X) = X^I$.)

Given a set S, determine the fibred product over S of a family of maps $\{f_i\colon X_i \to S\}_{i\in I}$ when $X_i = \emptyset$ for any index i (resp. when $I = \emptyset$, when $X_i = I = \emptyset$, and when $X_i = I = S = \emptyset$).

62. Show that the covariant functor $\emptyset\colon A$-**mod** $\rightsquigarrow$ **Sets** preserves kernels, direct products and projective limits (with non-empty set of indices), but not the final object, so that it is not representable.

63. Let **WOrd** be the category of well-orders, the morphisms $X \to Y$ being the isomorphisms onto an initial ray, so that $\operatorname{Hom}(X, Y)$ is a one-point set when $X \le Y$ and $\operatorname{Hom}(X, Y) = \emptyset$ otherwise. Prove that **WOrd** has not a final object, so that the constant contravariant functor $F\colon$ **WOrd** $\rightsquigarrow$ **Sets** is not representable, even if it is left exact and transforms inductive limits into projective limits. Show that the minimal pairs of F do not form a set.

64. Let $\mathbf{I}$ be a category (its objects are named indices or vertices, and its morphisms are named arrows). An **I-diagram** in a category $\mathbf{C}$ is a covariant functor $\mathbf{I} \rightsquigarrow \mathbf{C}$; i.e. an object X_i for any index $i \in \mathbf{I}$ and a morphism $\tilde{f}\colon X_i \to X_j$ for any arrow f from the vertex i to j, preserving compositions and identities. The index category $\mathbf{I}$ is viewed as the generic diagram with indices in $\mathbf{I}$, given by the identity functor $\mathbf{I} \rightsquigarrow \mathbf{I}$. A diagram is **small** (resp. **finite**) when the indices and arrows form a set (resp. a finite set).

The **limit** of an $\mathbf{I}$-diagram is an object $\lim_{i\in I} X_i$ endowed with compatible morphisms $\pi_i\colon \lim X_i \to X_i$, in the sense that $\pi_j = \tilde{f}\pi_i$ for any arrow f from the vertex i to j, and such that for any compatible family $(p_i\colon T \to X_i)_{i\in\mathbf{I}}$ there is a unique morphism $p\colon T \to \lim X_i$ such that $p_i = \pi_i p$, $\forall i \in I$. It is the universal left cone. The **colimit** $\operatorname{colim}_{i\in I} X_i$ is the universal right cone, i.e. a universal compatible family of morphisms $(\lambda_i\colon X_i \to \operatorname{colim} X_i)_{i\in\mathbf{I}}$, in the sense that $\lambda_i = \lambda_j \tilde{f}$.

(a) If the limit or the colimit of a diagram exists, prove that it is unique up to a canonical isomorphism.

(b) The limit of a small diagram in **Sets** is just $\{(x_i) \in \prod_i X_i\colon \tilde{f}(x_i) = x_j \text{ for any arrow } f\}$, and the colimit is the quotient of the disjoint union $\coprod_i X_i$ by the smallest equivalence relation such that $x_i \equiv \tilde{f}(x_i)$ for any $x_i \in X_i$ and any arrow f with origin at the i-th vertex. Hence, for any small diagram in an arbitrary category, we have

$$\operatorname{Hom}_{\mathbf{C}}(T, \lim X_i) = \lim \operatorname{Hom}_{\mathbf{C}}(T, X_i)\ ,\ \operatorname{Hom}_{\mathbf{C}}(\operatorname{colim} X_i, T) = \lim \operatorname{Hom}_{\mathbf{C}}(X_i, T).$$

(c) Show that direct products (hence the final object), kernels, fibred products and projective limits are limits, while coproducts (hence the initial object), cokernels and inductive limits are colimits.

(d) Given a finite diagram $(X_1, \ldots, X_n)$ with morphisms $(\phi_1 \colon X_{i_1} \to X_{j_1}, \ldots, \phi_m \colon X_{i_m} \to X_{j_m})$, show that the limit is just the kernel of the morphisms

$$f, h \colon \prod_{i=1}^n X_i \rightrightarrows \prod_{k=1}^m X_{j_k},\ f(x_i) = (\phi_1(x_{i_1}), \ldots, \phi_m(x_{i_m})),\ h(x_i) = (x_{j_1}, \ldots, x_{j_m}).$$

Moreover, the kernel of $f, h \colon X \rightrightarrows Y$ is just the fibred product of the graphics $1 \times f \colon X \to X \times Y$ and $1 \times h \colon X \to X \times Y$, and the fibred product of $f \colon X \to S$, $h \colon Y \to S$ is just the kernel of the morphisms $f\pi_1, h\pi_2 \colon X \times Y \to S$.

Conclude that a category has finite limits if and only if it has finite products and kernels (or fibred products), and that a covariant functor preserves finite limits if and only if it is left exact (preserves finite products and kernels).

Analogously, when $\mathbf{I}$ is a small category, the limit of a diagram $(X_i)_{i \in \mathbf{I}}$ is the kernel of a pair of morphisms $\prod_i X_i \rightrightarrows \prod_\phi X_j$, where ϕ runs over all the morphisms $\phi \colon X_i \to X_j$ in the diagram. Hence a category has small limits if and only it has products and kernels (or finite products, projective limits and kernels, or products and fibred products, or...), and a covariant functor preserves small limits if and only if it is left exact and preserves products (or projective limits). State the dual results.

65. Prove that the limit of all the objects and morphisms of a category $\mathbf{C}$, if it exists, is just the initial object[4]: $\lim\limits_{X \in \mathbf{C}} X = \emptyset$.

(*Hint*: Consider the morphisms $\pi_X \colon \lim X \to X$, so that $\pi_X = f\pi_Y$ for any morphism $f \colon Y \to X$; hence $\pi_X = \pi_X \pi_{\lim X}$, and $\pi_{\lim X} = \mathrm{Id}_{\lim X}$ by the uniqueness of the factorization through $\lim X$. Conclude that any morphism $f \colon \lim X \to X$ is $\pi_X = f\pi_{\lim X} = f$.)

Conversely, if $\mathbf{C}$ has an initial object $\emptyset$, prove that $\emptyset = \lim\limits_{X \in \mathbf{C}} X$.

(*Hint*: Clearly the family $\{i_X \colon \emptyset \to X\}_{X \in \mathbf{C}}$ is compatible, and we have $p_X = i_X p_\emptyset$ for any compatible family $\{p_X \colon T \to X\}_{X \in \mathbf{C}}$. If $f \colon T \to \emptyset$ is another factorization, then $p_\emptyset = i_\emptyset f = f$.)

66. If $\mathbf{I}$ has a final index ∞, show that the colimit of any $\mathbf{I}$-diagram is just $X_\infty = \mathrm{colim}_{i \in \mathbf{I}} X_i$.

67. Let $\Omega \subset \mathbf{I}$ be a subcategory such that any vertex X_i admits some morphism $g \colon U_\alpha \to X_i$ with $U_\alpha \in \Omega$, and assume that for any other morphism $h \colon U_\beta \to X_i$, $U_\beta \in \Omega$, we have a commutative square (with ϕ and φ in Ω)

$$\begin{array}{ccc} U_\gamma & \xrightarrow{\ \phi\ } & U_\alpha \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ U_\beta & \xrightarrow{\ h\ } & X_i \end{array}$$

Prove that $\lim_{i \in \mathbf{I}} X_i = \lim_{\alpha \in \Omega} X_\alpha$ for any $\mathbf{I}$-diagram.

Hint: Any Ω-compatible family $(p_\alpha \colon T \to U_\alpha)$ admits a unique extension to a $\mathbf{I}$-compatible family $(p_i \colon T \to X_i)_{i \in I}$, since p_i must be the composition of $p_\alpha \colon T \to U_\alpha$ with $\tilde{g} \colon U_\alpha \to X_i$. The above square shows that it does not depend on g and that the family $(p_i)_{i \in I}$ is $\mathbf{I}$-compatible, since for any two arrows $f \colon X_i \to X_j$, $h \colon U_\beta \to U_j$, replacing g by fg we have a commutative diagram:

$$\begin{array}{cccc} & U_\alpha & \xrightarrow{\ \tilde{g}\ } & X_i \\ {\scriptstyle\tilde{\phi}}\nearrow & & & \big\downarrow{\scriptstyle \tilde{f}} \\ U_\gamma & & & \\ {\scriptstyle\tilde{\varphi}}\searrow & & & \\ & U_\beta & \xrightarrow{\ \tilde{h}\ } & X_j \end{array}$$

Conclude that if a category $\mathbf{C}$ has small limits and there is a set of objects $\{U_i\}_{i \in I}$ such that any object X admits some morphism $U_i \to X$, then $\mathbf{C}$ has an initial object.

68. Let $\mathbf{C}$ be a category with small limits and $F \colon \mathbf{C} \rightsquigarrow \mathbf{Sets}$ a covariant functor preserving small limits.

(a) Show that the category of pairs also has small limits. (*Hint*: The limit of a small diagram of pairs $\big((X_i)_{\xi_i}\big)$ is just $(\lim X_i)_{(\xi_i)}$, where $(\xi_i) \in \lim F(X_i) = F(\lim X_i)$.)

[4]This formula states that the points $T \to \emptyset$ are just the compatible families of points $\{p_X \colon T \to X\}_{X \in \mathbf{C}}$!

(b) Prove a **Representability Theorem:** *If there is a set of pairs* $\{(U_i)_{\alpha_i}\}_{i\in I}$ *such that any other pair* X_ξ *admits some morphism of pairs* $(U_i)_{\alpha_i} \to X_\xi$, *then* F *is representable.*

(c) Prove that $F(Y) \to F(X)$ is injective when $Y \hookrightarrow X$ is a **monomorphism**[5] (*Hint*: If $\alpha, \beta \in F(Y)$ have equal image $\xi \in F(X)$, consider a left cone of the diagram of pairs $Y_\alpha \to X_\xi \leftarrow Y_\beta$.)

Let X_ξ be a pair, assume that the subobjects $Y_i \hookrightarrow X$ such that $\xi \in F(Y_i)$ form a set, and put $\bigcap_i Y_i := \lim Y_i$. Show that $\bigcap_i Y_i$ is a subobject of X and $\xi \in F(\bigcap_i Y_i)$, so that $(\bigcap_i Y_i)_\xi$ is a minimal pair dominating X_ξ.

(d) Prove another **Representability Theorem:** *If the subobjects of any object form a set, and the minimal pairs of* F *form a set, then* F *is representable.*

69. If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are maximal ideals of a ring A, prove that $A/(\mathfrak{m}_1^{n_1} \ldots \mathfrak{m}_r^{n_r}) = (A/\mathfrak{m}_1^{n_1}) \oplus \ldots \oplus (A/\mathfrak{m}_r^{n_r})$.

70. If a prime ideal $\mathfrak{p}$ of a ring A is contained in a finite union of prime ideals, $\mathfrak{p} \subseteq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$, show that $\mathfrak{p} \subseteq \mathfrak{p}_i$ for some index i. (*Hint*: If the closures of some points $x, x_1, \ldots, x_n \in \operatorname{Spec} A$ are not incident, show the existence of a function $f_i \in A$ vanishing at all these points, except for x_i. Conclude the existence of a function $f \in A$ such that $f(x) = 0$ and $f(x_i) \neq 0$ for any index i).

71. If a finite group G acts on a ring A, prove that the fibres of the natural map $\operatorname{Spec} A \to \operatorname{Spec} A^G$ are just the orbits of the action of G on $\operatorname{Spec} A$. (*Hint*: If $\mathfrak{p}, \mathfrak{q}$ are prime ideals in A such that $A^G \cap \mathfrak{p} = A^G \cap \mathfrak{q}$ and we take $f \in \mathfrak{p}$, then $N(f) = \prod_{\tau\in G} \tau(f) \in A^G \cap \mathfrak{p} \subseteq \mathfrak{q}$. Conclude that $\mathfrak{p} \subseteq \bigcup_{\tau\in G} \tau(\mathfrak{q})$ and apply the above problem).

72. Show that $\operatorname{Spec} A_x$ is the intersection of all the neighborhoods of x in $\operatorname{Spec} A$.

73. Let $f, h \in A$. Prove that $U_f \subseteq U_h$ if and only if h divides some power f^n.
Conclude that $U_f = U_h$ if and only if there is an isomorphism of A-algebras $A_f \simeq A_h$.

74. Prove that any element of a minimal prime ideal of a ring is a zero divisor.

75. If there is a function $f \in A$ such that $f^2 = f$ and $f \neq 0, 1$, show that $\operatorname{Spec} A$ is not connected.

76. Let I be an ideal of a ring A. Prove that $1 + I$ is a multiplicative system such that $\operatorname{Spec} A_{1+I}$ is the intersection of all open neighborhoods of $(I)_0$.

77. Prove that $\operatorname{Spec}(\varinjlim A_i) = \varprojlim(\operatorname{Spec} A_i)$.

78. If M is an A-module and $S \subset A$ a multiplicative system, show that $(\operatorname{Ann}_A(M))_S = \operatorname{Ann}_{A_S}(M_S)$.

79. Prove that a ring A is reduced if and only if so is A_x at any point $x \in \operatorname{Spec} A$.

80. Let $(\mathcal{O}, \mathfrak{m})$ be a local k-algebra. If M is an $\mathcal{O}$-module, show that $\dim_k M = l(M) \cdot [\mathcal{O}/\mathfrak{m} : k]$.

81. Let M, N be finitely generated modules over a local ring $\mathcal{O}$.
If $M \otimes_{\mathcal{O}} N = 0$, prove that $M = 0$ or $N = 0$.
If $M \otimes_{\mathcal{O}} N = \mathcal{O}$, prove that $M = N = \mathcal{O}$. (*Hint*: M and N are monogenous modules).

82. If M, N are finitely generated A-modules, show that $\operatorname{supp}(M \otimes_A N) = (\operatorname{supp} M) \cap (\operatorname{supp} N)$.

83. If M is a finitely generated A-module, show that $\operatorname{supp}(M/IM) = (I + \operatorname{Ann} M)_0$ for any ideal I.

84. Let $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the continuous map induced by a ring morphism $A \to B$.
If $J \subset B$ is an ideal, prove that the closure of $\phi\big((J)_0\big)$ is just $(A \cap J)_0$.
If M is a finitely generated A-module, show that $\operatorname{supp}(M_B) = \phi^{-1}(\operatorname{supp} M)$.

85. Let I be an ideal of a ring A and $x \in \operatorname{Spec} A$. Prove that $I_x = A_x$ if and only if $x \notin (I)_0$.

86. Prove that any closed open set $U \subseteq \operatorname{Spec} A$ is a basic open set: $U = U_f$ for some function $f \in A$. Moreover, $(A_f)_x = A_x$ when $x \in U$, and $(A_f)_x = 0$ otherwise. (*Hint*: Show that $U = (I)_0$ and $U^c = (J)_0$, where $I + J = A$, so that $U \subseteq (h)_0$ and $U^c \subseteq (f)_0$, where $f + h = 1$).

[5] In the sense that $Y^\bullet(T) \to X^\bullet(T)$ is injective for any object T, and we say that two monomorphisms define the same **subobject** of X when both define the same subfunctor of $X^\bullet$. Dually we define **epimorphisms** and **quotients** of X.

87. Let N', N be submodules of an A-module M. Prove that $N' \subseteq N$ if and only if $N'_x \subseteq N_x$, $\forall x \in \operatorname{Spec} A$.

88. If A is a domain, prove that in the field of fractions we have $A = \bigcap_x A_x$, $x \in \operatorname{Spec} A$.

89. Prove that any A-linear epimorphism $A^n \to A^n$ is an isomorphism.

90. Let M, N be two A-modules. If M is finitely generated and $M \simeq M \oplus N$, prove that $N = 0$.

91. Prove that an A-module M is flat if and only if M_x is a flat A_x-module at any point $x \in \operatorname{Spec} A$.

92. Let $\bar{k}$ be an algebraic closure of a field k. If $k \to K$ is an algebraic extension, prove the existence of an injective k-morphism $K \to \bar{k}$.

93. Let $k[\varepsilon] = k[t]/(t^2)$. Show that to give a morphism of k-algebras $A \to k[\varepsilon]$ is just to give a rational point $x \in \operatorname{Spec} A$ and a k-derivation $D\colon A \to A/\mathfrak{m}_x = k$.

94. Let A, B be two k-algebras. Prove that $\Omega_{A\otimes_k B/k} = \big(\Omega_{A/k} \otimes_k B\big) \oplus \big(A \otimes_k \Omega_{B/k}\big)$.

95. Let $p(x) \in A[x]$. If we put $B = A[x]/(p(x))$, show that $\Omega_{B/A} = \big(B/p'(x)B\big)\mathrm{d}x$.

96. If $A = k[\xi_1, \ldots, \xi_n]$ is a finitely generated k-algebra, show that $\Omega_{A/k} = A\,\mathrm{d}\xi_1 + \ldots + A\,\mathrm{d}\xi_n$.

97. If $A = \varinjlim A_i$, prove that $\Omega_{A/k} = \varinjlim \Omega_{A_i/k}$.

98. Let $B \to C$ a morphism of A-algebras. Prove that $0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$ is a split exact sequence if and only if any A-derivation $B \to M$ into a C-module M may be extended to an A-derivation $C \to M$.

99. Let A be a ring and let M be an $A[x]$-module. If $D\colon A \to M$ is a derivation, show that for any element $m \in M$ there is a unique derivation $\bar{D}\colon A[x] \to M$ extending D such that $\bar{D}(x) = m$.

100. Let $k \to k(\alpha)$ be an extension and let $D\colon k \to E$ be a derivation into a $k(\alpha)$-vector space E.

     If α is transcendent over k, show that for any vector $e \in E$ there is a unique derivation $\bar{D}\colon k(\alpha) \to E$ extending D such that $\bar{D}\alpha = e$.

     If α is algebraic over k and $p(x) = \sum_i a_i x^i \in k[x]$ is the irreducible polynomial of α over k, prove the following statements:

     (a) If $p'(x) \neq 0$, there is a unique derivation $\bar{D}\colon k(\alpha) \to E$ extending D.
     (b) If $p'(x) = 0$ and $\sum_i (Da_i)\alpha^i \neq 0$, there is no derivation $\bar{D}\colon k(\alpha) \to E$ extending D.
     (c) If $p'(x) = 0$ and $\sum_i (Da_i)\alpha^i = 0$, for any $e \in E$ there is a unique derivation $\bar{D}\colon k(\alpha) \to E$ extending D such that $\bar{D}\alpha = e$.

101. Let $k \to L$ be an extension. If $\operatorname{char} k = 0$, prove that $\alpha \in L$ is transcendent over k if and only if there is a k-derivation $D\colon L \to L$ such that $D\alpha \neq 0$.

102. If a ring A has a finite discrete spectrum $\operatorname{Spec} A = \{x_1, \ldots, x_n\}$, prove that the natural map $A \to A_{x_1} \oplus \ldots \oplus A_{x_n}$ is an isomorphism.

103. Prove that any algebraically closed field of null characteristic (resp. positive characteristic p) is the algebraic closure of the field $\mathbb{Q}(x_i)_{i\in I}$ (resp. $\mathbb{F}_p(x_i)_{i\in I}$) for some set of indeterminates $\{x_i\}_{i\in I}$. (*Hint*: Use Zorn's lemma to show the existence of a maximal set of algebraically independent elements over the prime field $\mathbb{Q}$ or $\mathbb{F}_p$).

     Conclude that any algebraically closed field of null characteristic and cardinality $2^{\aleph_0}$ is isomorphic to the field of complex numbers $\mathbb{C}$.

     Prove that algebraically closed fields are classified by the characteristic and the **transcendence degree** (the cardinal of a maximal set of algebraically independent elements) over the prime field.

104. If A is a finite k-algebra, prove that the following conditions are equivalent:

     (a) A is reduced, $\operatorname{rad} A = 0$.
     (b) Any A-module is injective.
     (c) Any A-module is projective.
     (d) Any A-module is flat.

105. Prove that a finite k-algebra A is separable if and only if A is a flat $A \otimes_k A$-module (with the module structure defined by the diagonal morphism $\mu\colon A \otimes_k A \to A$, $\mu(a \otimes b) = ab$).

106. Prove that two finite extensions $k \to L_1$ and $L_1 \to L_2$ are separable if and only if so is $k \to L_2$.
If $k \to L_1$ and $k \to L_2$ are separable finite extensions, show that so is any composite $k \to L_1 L_2$.
If $k \to L$ is a separable finite extension, prove that so is any composite $K \to KL$ with an arbitrary extension $k \to K$.

107. Let $\mathbb{Q} \to K$ be a finite extension of degree d, and let us consider a separable polynomial $P \in K[x]$. Show that $A = \mathbb{Q}[a] = \mathbb{Q}[y]/(Q)$ for almost all $a \in A = K[x]/(P)$. If $Q = Q_1 \dots Q_r$ is the irreducible factor decomposition in $\mathbb{Q}[y]$, prove that $P = P_1 \dots P_r$, where P_i is irreducible in $K[x]$ and $d(\deg P_i) = \deg Q_i$. Moreover, $P_i(x)$ is the annihilator of y, viewed as a K-endomorphism of $\mathbb{Q}[y]/(Q_i)$. In particular, $P(x)$ is irreducible in $K[x]$ if and only if so is $Q(y)$ in $\mathbb{Q}[y]$, and the irreducible factor decomposition in $K[x]$ may be reduced to the case $K = \mathbb{Q}$, solved in p. 73.

108. If $k \to L$ is a separable finite extension, prove that the following conditions are equivalent:
   (a) L is the decomposition field of some polynomial with coefficients in k.
   (b) Any composite of L with itself is isomorphic to L.
   (c) L is a Galois extension: $L \otimes_k L = \oplus L$.
   (d) Any polynomial $p(x) \in k[x]$ with a root in L has all the roots in L.

109. If $k \to L$ is a Galois extension, show that any composite $K \to KL$ is a Galois extension, where K is an arbitrary extension of k.
If $k \to L'$ also is a Galois extension, prove that $L' \otimes_k L \simeq \oplus L''$ for some Galois extension L'' of k.

110. Let G be a finite group of automorphisms of a field L. Prove that $k := L^G$ is field and $k \to L$ is a finite extension (hence a Galois extension of group G). (*Hint*: If $\alpha_1, \dots, \alpha_r \in L$, then $k \to k(G\alpha_1, \dots, G\alpha_r)$ is a finite extension, hence a Galois extension of degree $\leq |G|$ by Artin's theorem).

111. If a k-algebra A is trivial over an extension $k \to L$ and $\mathrm{Hom}_{k\text{-alg}}(A, L) = \{p_1, \dots, p_n\}$, show that we have an isomorphism of L-algebras $A \otimes_k L = L \oplus \overset{n}{\dots} \oplus L$, $a \otimes \lambda \mapsto \big(\lambda p_1(a), \dots, \lambda p_n(a)\big)$.

112. *Alternative proofs of Galois Theorem*: Let $k \to L$ be a Galois extension, put $G := \mathrm{Aut}_{k\text{-alg}}(L)$ and let us consider the functors $F(A) = \mathrm{Spec}\, A_L$ and $R(\Delta) = \mathrm{Hom}_G(\Delta, L)$ involved in Galois theorem.
   (a) 1. Show that $H = \mathrm{Aut}(L/L^H)$ for any subgroup $H \subseteq G$.
   Hint: The group H acts transitively on the spectrum of the trivial L-algebra $L \otimes_{L^H} L$ because $(L \otimes_{L^H} L)^H = L^H \otimes_{L^H} L = L$ is a local algebra.
   2. Show that the natural map $G \to \mathrm{Hom}_{k\text{-alg}}(L^H, L)$ is surjective, so that $F(L^H) = G/H$.
   Hint: Any injective morphism $B \to A$ between two finite k-algebras induces a surjective map $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$, so that the natural map $\mathrm{Spec}\,(L \otimes_k L) \to \mathrm{Spec}\,(L^H \otimes_k L)$ is surjective.
   3. Conclude that any finite G-set Δ is $\Delta \simeq F(A)$ for some finite k-algebra A trivial over L, and that the natural map $\mathrm{Hom}_{k\text{-alg}}(A, B) \to \mathrm{Hom}_G(F(B), F(A))$ is bijective for any two finite k-algebras A and B trivial over L.
   Hint: $\Delta = \amalg_i (G/H_i)$ and $\mathrm{Hom}_{k\text{-alg}}(L^{H'}, L^H) = F(L')^H = (G/H')^H = \mathrm{Hom}_G(G/H, G/H')$.
   (b) 1. Prove that $L = RF(L)$, $G = FR(G)$, and that F and R preserve coproducts and cokernels.
   Hint: R is representable, and F is just the composition of a base change $A \rightsquigarrow A_L$ with an equivalence of categories, see p. 148.
   2. Show that any finite G-set Δ admits a presentation $\amalg_i G \rightrightarrows \amalg_j G \to \Delta$.
   Hint: We have exact sequences $G \times H = \amalg_H G \rightrightarrows G \to G/H$, where $(g, h) \mapsto g$ and $(g, h) \mapsto gh$.
   3. Show that any finite k-algebra A trivial over L is a kernel $A \to A \otimes_k L \rightrightarrows A \otimes_k L \otimes_k L$ of morphisms between trivial L-algebras.
   Hint: The sequence $k \to L \rightrightarrows L \otimes_k L$ is exact.
   4. Conclude that $\Delta = FR(\Delta)$ and $A = RF(A)$ using the following commutative diagrams:

$$\begin{array}{ccccc} \amalg_i G & \rightrightarrows & \amalg_j G & \to & \Delta \\ \| & & \| & & \downarrow \\ FR(\oplus_i G) & \rightrightarrows & FR(\oplus_j G) & \to & FR(\Delta) \end{array} \qquad \begin{array}{ccccc} A & \to & \oplus_j L & \rightrightarrows & \oplus_i L \\ \downarrow & & \| & & \| \\ RF(A) & \to & RF(\oplus_j L) & \rightrightarrows & RF(\oplus_i L) \end{array}$$

(c) 1. Prove Grothendieck's characterization of the forgetful functor G-$\mathbf{Sets}_f \rightsquigarrow \mathbf{Sets}_f$ on the category G-$\mathbf{Sets}_f$ of finite G-sets:

If a covariant functor $F\colon \mathbf{C} \rightsquigarrow \mathbf{Sets}_f$ *is representable,* $F(X) = \operatorname{Hom}(P, X)$*, then the finite group* $G := \operatorname{Aut}(F) = \operatorname{Aut}(P)^{\mathrm{op}}$ *acts on* $F(X)$*,* $\tau f := f \circ \tau$*, and the induced functor* $\mathbf{C} \rightsquigarrow G$-$\mathbf{Sets}_f$ *is an equivalence of categories when the following conditions hold:*

(i) *The category* $\mathbf{C}$ *admits finite coproducts and cokernels and* F *preserves them.*

(ii) *The functor* F *is* ***conservative*** (a morphism f is an isomorphism when so is $F(f)$)

Hint: Consider the functor $R\colon G$-$\mathbf{Sets}_{\mathrm{f}} \rightsquigarrow \mathbf{C}$, $R(\Delta) = \big(\coprod_\Delta P\big)/G$, where $g \in G$ acts on $\coprod_\Delta P$ via the morphisms $P_x \xrightarrow{g} P_{gx} \xrightarrow{i} \coprod_\Delta P$, $x \in \Delta$.

The morphisms $P_x \xrightarrow{i} \coprod_\Delta P \to R(\Delta)$ and $\coprod_{F(X)} P \to X$ induce (the second being G-invariant) natural morphisms $t_\Delta\colon \Delta \longrightarrow F(R(\Delta))$, $t_X\colon R(F(X)) \longrightarrow X$.

Show that t_Δ is bijective because F preserves coproducts and cokernels, hence quotients by G.

Check that the map $F(X) \xrightarrow{t_{F(X)}} FRF(X) \xrightarrow{F(t_X)} F(X)$ is just the identity. Since $t_{F(X)}$ is bijective, so is $F(t_X)$; hence t_X is an isomorphism by (ii).

2. Obtain Galois theorem, showing that the functor $F(A) = \operatorname{Spec} A_L = \operatorname{Hom}_{k\text{-alg}}(A, L)$ fulfills the above conditions.

Hint: F is the composition of a base change $A \rightsquigarrow A_L$ with an equivalence of categories, p. 148.

113. Let $F\colon \mathbf{C} \rightsquigarrow \mathbf{Sets}_f$ be a covariant functor. Prove that there exists a finite group G, an equivalence of categories $\Phi\colon \mathbf{C} \rightsquigarrow G$-$\mathbf{Sets}_f$ and an isomorphism of functors $F \simeq i \circ \Phi$ (where $i\colon G$-$\mathbf{Sets}_f \rightsquigarrow \mathbf{Sets}_f$ is the forgetful functor) if and only if $\mathbf{C}$ admits finite coproducts and cokernels, and F is representable, exact and conservative. Moreover, in such case, G is isomorphic to the group of automorphisms of F.

114. Let $k \to L$ be a Galois extension, $\operatorname{char} k = 0$. If $H = \{\tau_1, \ldots, \tau_n\}$ is a subgroup of the Galois group, prove that the k-linear map $s\colon L \to L^H$, $s(\alpha) = \tau_1(\alpha) + \ldots + \tau_n(\alpha)$, is surjective.

115. Prove that any finite separable extension $k \to K$ only has a finite number of intermediate fields.

116. Let K_1, K_2 be Galois extensions of a field k. If we fix a composite L of K_1 and K_2, prove that $\bar{K} = K_1 \cap K_2$ is a Galois extension of k. Moreover, if $G_1, G_2, G, \bar{G}$ are the Galois groups of $K_1, K_2, L, \bar{K}$ over k, show that $G = G_1 \times_{\bar{G}} G_2 = \{(g_1, g_2) \in G_1 \times G_2\colon g_1|_{\bar{K}} = g_2|_{\bar{K}}\}$.

117. Where does the following argument fail? If $i\colon k \to K$ and $j\colon K \to L$ are Galois extensions, then $j \circ i\colon k \to L$ also is a Galois extension because

$$K \otimes_k L = (K \otimes_k K) \otimes_K L = (\oplus K) \otimes_K L = \oplus (K \otimes_K L) = \oplus L,$$
$$L \otimes_k L = L \otimes_K (K \otimes_k L) = L \otimes_K (\oplus L) = \oplus (L \otimes_K L) = \oplus L.$$

(*Hint*: If $k \to K$ is a Galois extension, then we have an isomorphism of K-algebras $K \otimes_k K = \oplus K$, $\alpha \otimes \lambda \mapsto (\tau_1(\alpha)\lambda, \ldots, \tau_1(\alpha)\lambda)$, where $\operatorname{Aut}_{k\text{-alg}}(K) = \{\tau_1, \ldots, \tau_n\}$).

118. Let $k \to L'$ and $L' \to L$ be Galois extensions. If any automorphism of L' over k may be extended to an automorphism of L over k, prove that $k \to L$ is a Galois extension.

119. If $p(x) \in k[x]$ is separable and $\operatorname{char} k \neq 2$, prove that the Galois group G of $p(x)$ over k is contained in the alternate subgroup A_n if and only if the discriminant Δ of $p(x)$ is a square in k.

In the case of an irreducible cubic, if Δ is a square, then $G = A_3$. Otherwise, $G = S_3$.

120. Let $p_4(x) \in k[x]$ be a separable reducible quartic. If $p_4(x)$ has no root in k, show that the Galois group is $G = \{\mathrm{id}, (12)(34)\}$ or $G = \{\mathrm{id}, (12), (34), (12)(34)\}$.

121. *Galois Group of Quartics*: Let $p_4(x) = x^4 + px^3 + qx^2 + rx + s \in k[x]$ be a separable irreducible quartic, $\operatorname{char} k \neq 2$. Let $L = k(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the decomposition field and $G \subseteq S_4$ the Galois group. Let $r(y)$ be the cubic resolvent (p. 495), with roots $\vartheta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\vartheta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$, $\vartheta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$, and let $d = [k(\vartheta_1, \vartheta_2, \vartheta_3) : k]$. Put $V = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$.

Prove that the Galois group of L over $k(\vartheta_1, \vartheta_2, \vartheta_3)$ is $G \cap V$, so that $|G \cap V| = 2, 4$ and $|G| = 2d, 4d$.

(a) If $d = 6$, then $G = S_4$.

(b) If $d = 3$, then $G = A_4$.

(c) If $d = 1$, then $G = V$.

(d) If $d = 2$, then $|G| = 4$ or 8. Up to conjugation there are only two possibilities:

$$G = D_8 = \{ \mathrm{id}\,,\, (1234)\,,\, (13)(24)\,,\, (1432)\,,\, (12)(34)\,,\, (14)(23)\,,\, (13)\,,\, (24) \}$$
$$G = C_4 = \{ \mathrm{id}\,,\, (1234)\,,\, (13)(24)\,,\, (1432) \}$$

and the cubic resolvent $r(y)$ has a unique root in k, namely $a = \alpha_1\alpha_3 + \alpha_2\alpha_4$. Moreover, $G = C_4$ if and only if the polynomials

$$(x - \alpha_1\alpha_3)(x - \alpha_2\alpha_4) = x^2 - ax + s$$
$$\big(x - (\alpha_1 + \alpha_3)\big)\big(x - (\alpha_2 + \alpha_4)\big) = x^2 + px + q - a$$

have the roots in the decomposition field of $r(y)$, which is just $k(\vartheta_1, \vartheta_2, \vartheta_3) = k(\sqrt{\Delta})$, where Δ is the discriminant of $r(y) = (y - a)p_2(y)$. Hence, in the case $d = 2$, we have

i. $G = C_4$, when $x^2 - ax + s$ and $x^2 + px + q - a$ decompose over $k\big(\sqrt{\Delta}\,\big)$.

ii. $G = D_8$ otherwise.

122. Give quartics with rational coefficients and Galois groups S_4, A_4, D_8, C_4 and V.

123. If an irreducible quartic with rational coefficients has just two real roots, prove that the Galois group over $\mathbb{Q}$ is S_4 or D_4.

124. Let $\alpha \in \mathbb{C}$ be a root of an irreducible quartic $p(x) \in \mathbb{Q}[x]$. If the Galois group of $p(x)$ is S_4 or A_4, show that there is no non-trivial intermediate field between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$.

125. Prove that the Galois group of the generic polynomial of degree n is the symmetric group S_n.

126. Let $\mathbb{F}_q$ be the finite field with q elements. Prove that any finite extension $\mathbb{F}_q \to L$ is a Galois extension, of cyclic Galois group, generated by the automorphism $F(\alpha) = \alpha^q$.

127. Let $p \neq q$ be two prime numbers. Prove that $x^{q-1} + \ldots + x + 1$ is irreducible in $\mathbb{F}_p[x]$ if and only if p generates the group $(\mathbb{Z}/q\mathbb{Z})^*$.

128. Let $Q \in \mathbb{Z}[x]$ be separable. Prove that the reduction $\bar{Q} \in \mathbb{F}_p[x]$ is separable at any prime p, up to a finite number. (*Hint*: $\bar{Q}$ is inseparable if and only if p divides the discriminant of Q).

129. Determine a polynomial $P(x) \in \mathbb{Q}[x]$ of degree 5 and Galois group $\mathbb{Z}/5\mathbb{Z}$ over $\mathbb{Q}$.
For any prime number p, prove the existence of $P(x) \in \mathbb{Q}[x]$ with Galois group $G \simeq \mathbb{Z}/p\mathbb{Z}$.

130. We have $\sqrt{2} = e^{2\pi i/8} + e^{-2\pi i/8}$, but prove that $\sqrt[3]{2}$ is not a rational function, with rational coefficients, of some complex roots of unity.

131. Let p be an odd prime. If ε_8 is a primitive 8-th root of unity over $\mathbb{F}_p$, prove that $\sqrt{2} = \varepsilon_8 + \varepsilon_8^{-1}$.
Show that 2 is a square in $\mathbb{F}_p$ if and only if $\varepsilon_8 + \varepsilon_8^{-1} = \varepsilon_8^p + \varepsilon_8^{-p}$.
Conclude that $\left(\frac{2}{p}\right) = 1$ if and only if $p \equiv \pm 1$ (mod. 8); that is to say, $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

132. Let $p > 3$ be a prime number and let ε_{12} be a primitive 12-th root of the unity over $\mathbb{F}_p$.
Show that $\sqrt{3} = \varepsilon_{12}^3 + 2\varepsilon_{12}^7$, and obtain that $\left(\frac{3}{p}\right) = 1$ if and only if $p \equiv \pm 1$ (mod. 12).

133. Determine the natural numbers n such that $\cos \frac{2\pi}{n}$ is a rational number.

134. Let $\mathbb{Q} \to L$ be a Galois extension of group G. If G is not a cyclic group, show that L is the decomposition field of a polynomial $q(x) \in \mathbb{Q}[x]$ without rational roots such that the reduction $\bar{q}(x) \in \mathbb{F}_p[x]$ has some root in $\mathbb{F}_p$ for all prime numbers p, up to a finite number.

135. Let G be the Galois group over $\mathbb{Q}$ of a polynomial $q(x) \in \mathbb{Q}[x]$ of degree n without rational roots. If any element $\tau \in G$ fixes some root of $q(x)$, show that $n > 4$ and $q(x)$ is not irreducible in $\mathbb{Q}[x]$.

136. Let G be the Galois group over $\mathbb{Q}$ of $x^n - a \in \mathbb{Q}[x]$. If any element $\tau \in G$ fixes some root of $q(x)$, and n is prime or $n \leq 7$, show that a is a n-th power in $\mathbb{Q}$.

137. Prove that any extension of $\mathbb{Q}$ contained in an extension of $\mathbb{Q}$ by quadratic radicals also is an extension of $\mathbb{Q}$ by quadratic radicals.

138. Let us consider an extension $k \to k(\alpha)$, $\alpha^n \in k$, where $n \neq 0$ in k and k contains the n-roots of unity. If $d > 0$ is the first exponent such that $\alpha^d \in k$, show that $x^d - \alpha^d$ is irreducible in $k[x]$.

139. Let p be a prime number. If $a \in k$ is not a p-th power in k, prove that $x^p - a$ is irreducible in $k[x]$. (*Hint*: If $\operatorname{char} k = p$, use ex. 43 in p. 494. Otherwise the existence of a factor of degree $m < p$ implies that a^m is a p-th power in k and conclude using Bézout's identity $nm + qp = 1$.)

140. Let k be a field of null characteristic containing the n-th roots of unity $\{\varepsilon, \varepsilon^2, \ldots, \varepsilon^n = 1\}$. If $k \to L$ is a cyclic extension of degree n and σ is a generator of the Galois group, prove that

   (a) The k-endomorphism σ of L is diagonalizable, of eigenvalues $\varepsilon, \varepsilon^2, \ldots, \varepsilon^n = 1$.

   (b) Any element $\alpha \in L$ decomposes uniquely as a sum $\alpha = \alpha_1 + \ldots + \alpha_n$, where $\sigma(\alpha_i) = \varepsilon^i \alpha_i$, so that $\alpha_i = \sqrt[n]{a_i}$ for some $a_i \in k$.

   (c) In fact $\alpha_i = \frac{1}{n} R(\alpha, \varepsilon^{-i})$, where $R(\alpha, \bar{\varepsilon}) := \alpha + \bar{\varepsilon}\sigma(\alpha) + \bar{\varepsilon}^2\sigma^2(\alpha) + \ldots + \bar{\varepsilon}^{n-1}\sigma^{n-1}(\alpha)$ is the **Lagrange's resolvent** of α by ε, so that $\alpha = \frac{1}{n}\sum_i R(\alpha, \varepsilon^i)$. (*Hint*: When $\bar{\varepsilon}^n = 1$, we have $(1 + \bar{\varepsilon}x + \ldots + \bar{\varepsilon}^{n-1}x^{n-1} = (\bar{\varepsilon}x)^n - 1 = x^n - 1$.)

141. In a finite k-algebra A, the **norm** $N(a)$ and **trace** $\operatorname{tr}(a)$ of $a \in A$ are defined to be the determinant and trace of the endomorphism $A \xrightarrow{a\cdot} A$. In the case of a Galois extension $k \to L$ of group G, prove that $N(\alpha) = \prod_{g \in G} g(\alpha)$ and $\operatorname{tr}(\alpha) = \sum_{g \in G} g(\alpha)$.

   Let $k \to L$ be a Galois extension of cyclic group $\{\sigma, \sigma^2, \ldots, \sigma^n = \mathrm{Id}\}$, and $\beta \in L$. Prove Hilbert's theorem 90:

   (a) **Multiplicative Hilbert's theorem 90:** We have $N(\beta) = 1$ if and only if $\beta = \alpha/\sigma(\alpha)$ for some $0 \neq \alpha \in L$. (*Hint*: Fix $\theta \in L$ such that $0 \neq \alpha := \theta + \beta(\sigma\theta) + \ldots + \beta \ldots (\sigma^{n-2}\beta)(\sigma^{n-1}\theta)$. Or show that the endomorphism $\tau := \beta\sigma$ has the eigenvalue 1: the annihilator is $x^n - 1$, since $\tau^n = N(\beta)\sigma^n$ and automorphisms are L-linearly independent. Or show that $\sigma - \beta$ has non null kernel since so it does in $L \otimes_k L = \oplus_G L$, $a \otimes b = (\sigma(a)b, \ldots, \sigma^n(a)b)$.)

   (b) **Additive Hilbert's theorem 90:** We have $\operatorname{tr}(\beta) = 0$ if and only if $\beta = \alpha - \sigma(\alpha)$ for some $\alpha \in L$. (*Hint*: Put $\alpha = \frac{1}{\operatorname{tr}\theta}\big(\beta(\sigma\theta) + (\beta + \sigma\beta)(\sigma^2\theta) + \ldots + (\beta + \ldots + \sigma^{n-2}\beta)(\sigma^{n-1}\theta)\big)$, the trace $\operatorname{tr}\colon L \to k$ being non null because L is a separable extension. Or, using that $\sigma^n - 1 = 0$, show that $\operatorname{Ker}(1 + \sigma + \ldots + \sigma^{n-1}) = \operatorname{Im}(\sigma - 1)$ because $\operatorname{Ker}(\sigma - 1) = k$.)

142. Let $q(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of prime degree p. If $q(x)$ only has 2 imaginary roots, show that the Galois group is S_p; hence $q(x)$ is not solvable by radicals when $p \geq 5$.

143. *The Metacyclic Group*: Let p be a prime number and let $\mathrm{Aff}_p(1)$ be the group of affinities $ax + b$ of an affine line over $\mathbb{F}_p$, viewed as a subgroup of order $p(p-1)$ of the group S_p of all permutations of the affine line. Let T be the subgroup of all translations $x + b$, so that it is the subgroup of order p generated by the cycle $(1, 2, \ldots, p)$.

   (a) $\mathrm{Aff}_p(1)$ is a solvable group. (*Hint*: $\mathrm{Aff}_p(1) \to \mathbb{F}_p^*$, $ax + b \mapsto a$, is a group morphism).

   (b) $\mathrm{Aff}_p(1)$ is the normalizer $N(T)$ of T in S_p. (*Hint*: The index of $N(T)$ in S_p is the number $\frac{\text{number of } p\text{-cycles}}{p-1} = \frac{(p-1)!}{p-1} = (p-2)!$ of subgroups of order p).

   (c) Let $G \subseteq S_p$ be a solvable transitive subgroup. Any non trivial normal subgroup $N \subset G$ is transitive. (*Hint*: The isotropy subgroups $I_1, \ldots, I_p$ of G are conjugated, hence so are the subgroups $I_1 \cap N, \ldots, I_p \cap N$, and all the orbits of the action of N have equal cardinal).

   (d) Any solvable transitive subgroup $G \subseteq S_p$ is, up to conjugation, a subgroup of $\mathrm{Aff}_p(1)$ containing T. Hence $|G| = pd$, where d divides $p - 1$. (*Hint*: If $1 \lhd H_1 \lhd \ldots \lhd H_n = G$ is a resolution, show that, up to conjugation, $H_1 = T$. Then prove inductively that T is the unique subgroup of of H_i of order p, so that $T \lhd H_i$).

   (e) Let $G \subseteq S_p$ be a solvable transitive subgroup. If $\tau \in G$ has 2 fixed points, then $\tau = id$.

   (f) A transitive subgroup $G \subset S_p$ is solvable if and only if $|G| = pq$ for some $q < p$. (*Hint*: Prove that, up to conjugation, T is the unique subgroup of G of order p).

   (g) The Galois group of an irreducible polynomial $q(x) \in \mathbb{Q}[x]$ of prime degree p is solvable if and only if the decomposition field of $q(x)$ is generated by any two roots.

(h) (*Galois*): If an irreducible polynomial $q(x) \in \mathbb{Q}[x]$ of prime degree p is solvable by radicals and has 2 real roots, then all the roots are real.

144. A real root α of a polynomial $p(x) \in \mathbb{Q}[x]$ is said to be expressible by **real radicals** if there are real numbers $a_1, \ldots, a_n$ such that $\alpha \in \mathbb{Q}(a_1, \ldots, a_n)$ and $a_i^{m_i} \in K_{i-1} := \mathbb{Q}(a_1, \ldots, a_{i-1})$ for some exponents m_i (that may be assumed to be prime numbers). If an irreducible cubic has three real roots, prove that none is expressible by real radicals. (*Hint*: The decomposition field of an irreducible cubic is $\mathbb{Q}(\sqrt{\Delta}, \alpha)$, where α is any root and Δ is the discriminant, so that we may assume that $a_1 = \sqrt{\Delta}$. Then consider the first index i such that K_{i+1} contains a root of the cubic and, using ex. 139, show that $K_i \to K_{i+1}$ is a Galois extension of degree 3, hence the decomposition field of $x^3 - a_i$ over K_i.)

145. Prove that $\pi_0^k(A \oplus B) = \pi_0^k(A) \oplus \pi_0^k(B)$ and $\pi_0^k(A \otimes_k B) = \pi_0^k(A) \otimes_k \pi_0^k(B)$.

146. If $k \to K$ is a finite extension and A is a finite K-algebra, prove that $[A : k]_s = [A : K]_s \cdot [K : k]_s$.

147. Prove that subalgebras, quotients and tensor products of purely inseparable k-algebras are purely inseparable k-algebras, and that the concept of purely inseparable k-algebra is geometric and local.

148. Let B be a subalgebra of a finite k-algebra A. If A is rational over some extension $k \to L$, prove that any morphism of k-algebras $B \to L$ may extended to a morphism of k-algebras $A \to L$.

149. If $k \to L$ is a finite extension, prove that the following conditions are equivalent:

(a) L is a normal extension of k (i.e., $L \otimes_k L$ is a rational L-algebra).

(b) If an irreducible polynomial $p(x) \in k[x]$ has a root in L, then it has all the roots in L.

(c) L is the decomposition field over k of some polynomial $p(x) \in k[x]$.

(d) Any composite of L with itself is isomorphic to L.

150. Show that any purely inseparable extension is a normal extension.

151. Show that any extension of degree 2 is a normal extension.

152. Prove that any finite normal extension $k \to L$ of group $G = \mathrm{Aut}(L/k)$ decomposes as a tensor product of a Galois extension and a purely inseparable extension, $L = \pi_0^k(L) \otimes_k L^G$.

153. Let $k \to L$ be a normal extension. If K is an intermediate field, prove that the group $\mathrm{Aut}(L/k)$ acts transitively on the set $\mathrm{Hom}_{k\text{-alg}}(K, L)$.

154. Let Tr be the trace metric of a finite k-algebra A. If $\mathrm{char}\, k = 0$, show that $A = \pi_0(A) \oplus (\mathrm{rad}\,\mathrm{Tr})$.

155. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the cuaternion algebra and let I be the ideal of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ generated by $i \otimes 1 + 1 \otimes i$. Show that I is a $(1 \otimes \mathbb{C})$-vector space with base

$$e_1 := i \otimes 1 + 1 \otimes i \quad , \quad e_2 := -k \otimes 1 + j \otimes i \ ,$$

so that we have an isomorphism of $\mathbb{R}$-algebras $\mathbb{H} \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(I) \simeq M_{2\times 2}(\mathbb{C})$, $q \mapsto (q \otimes 1)\cdot$, where each quaternion $q = a + bi + cj + dk$ corresponds to the matrix

$$q = \begin{pmatrix} a+bi & -c-di \\ c-di & a-bi \end{pmatrix} .$$

6. Projective Geometry

1. Determine the cardinal of a projective space of dimension n over a finite field of q elements.

2. Show that a subset $X \subseteq \mathbb{P}(E)$ is a linear subvariety if and only if it contains the line passing through any two different points of X.

3. **Pappus theorem:** In a projective plane, let A_1, A_2, A_3 and B_1, B_2, B_3 be two triples of collinear points. Prove that the three pairs of lines A_iB_j and B_jA_i, $i \neq j$, intersect at collinear points.

4. Let $\lambda = (P_1, P_2; P_3, P_4)$ be the cross-ratio of 4 points of a projective line $\mathbb{P}_1$ over a field k. Show that unordered sets of 4 different points of $\mathbb{P}_1$ are projectively classified by the sum of the double products of the 6 possible cross-ratios (up to a sign and a constant)
$$j(\lambda) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \in k \cup \{\infty\}.$$

5. Show that an homography τ is an **involution**, $\tau^2 = \mathrm{Id}$, if and only if the linear representant T is traceless, $\mathrm{tr}\, T = 0$.

6. If a bijection $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E')$ between projective lines preserves harmonic quadruples, prove that it is induced by a semilinear transformation $(\sigma, T)\colon (k, E) \to (k', E')$. (*Hint*: A bijection $k \to k'$ preserves squares if it preserves sums and inverses, $\lambda^2 = \lambda - (\lambda^{-1} + (1 - \lambda)^{-1})^{-1}$).

7. Let $\mathbb{P}(E)$ be a projective space of dimension n. When $n \geq 2$, show that straight lines may be defined in terms of projectivities. In fact, if $O_{P_1P_2}(P_3)$ denotes the orbit of P_3 under the action of the group of projectivities fixing P_1 and P_2, minus P_3, prove that 3 different points are collinear if and only if $O_{P_1P_2}(P_3) = O_{P_1P_3}(P_2) = O_{P_2P_3}(P_1)$. Conclude that any bijection $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E)$ preserving projectivities, $\tau PGl(E)\tau^{-1} = PGl(E)$, is a collineation. (*Hint*: In the case of non collinear points, we have $P \notin O_{P_1P_2}(P_3)$ and $P \in O_{P_1P_3}(P_2)$ when P is a third point in the line $P_1 + P_2$.)

   When $n = 1$, consider an affine reference P_0, P_1, P_∞ in $\mathbb{P}(E)$ and the image P'_0, P'_1, P'_∞ by a bijection $\tau\colon \mathbb{P}(E) \to \mathbb{P}(E)$ preserving homographies, so that $\tau\colon \mathbb{A}_1 = \mathbb{P}(E) - \{P_\infty\} \to \mathbb{P}(E) - \{P'_\infty\} = \mathbb{A}'_1$ is a bijection such that $\tau(0) = 0$, $\tau(1) = 1$. Prove that τ is a ring automorphism and conclude that τ is induced by a semilinear automorphism $T\colon E \to E$. (*Hint*: τ preserves the addition and the product because it preserves translations and homotheties with center at the origin.)

8. Prove the **Fundamental Theorem of Euclidean Geometry:** *An affinity between Euclidean spaces of dimension $n \geq 2$ is a similarity if and only if it preserves the perpendicularity of lines.*

9. Let us consider an affine space $\mathbb{A}_n$ of dimension $n \geq 2$ over a field k. When $k \neq \mathbb{F}_2$, prove that $X \subseteq \mathbb{A}_n$ is an affine subvariety if and only if it contains the line passing through any two different points of X. (*Hint*: Given two vectors e, v in a k-vector space, show that the equation $xe + z(yv - xe) = e + v$ admits a solution in k.) Conclude that a bijection $\mathbb{A}_n \to \mathbb{A}'_n$ onto another affine space of dimension n is a semiaffinity if and only if it preserves lines.

   When $k = \mathbb{F}_2$ (so that any pair of points is a line), show that $X \subseteq \mathbb{A}_n$ is an affine subvariety if and only if it contains the plane passing through any three different points of X. Conclude that a bijection $\mathbb{A}_n \to \mathbb{A}'_n$ onto another affine space of dimension n is an affinity if and only if it preserves planes.

10. Let $\mathbf{Aff}_n$ be the category of n-dimensional affine spaces and affinities over a fixed base field k, and $\mathbb{P}\mathbf{Aff}_n$ the category of pairs $(\mathbb{P}(E), H)$, where H is a hyperplane of a n-dimensional k-projective space $\mathbb{P}(E)$, and projectivities preserving the hyperplane. Show that we have an equivalence of categories $F\colon \mathbf{Aff}_n \rightsquigarrow \mathbb{P}\mathbf{Aff}_n$, $F(\mathbb{A}_n, V) = (\mathbb{P}(E), \pi(V))$, where E is the vector extension (p. 164) of $\mathbb{A}_n$.

    Let $\mathbf{Euclid}_n$ the category of n-dimensional Euclidean affine spaces $(\mathbb{A}_n, V, \langle \Omega_2 \rangle)$ and similarities, and $\mathbb{P}\mathbf{Euclid}_n$ the category of sequences $(\mathbb{P}(E), H, \mathcal{Q})$, where $\mathcal{Q}$ is a non singular quadric of index 0 in a hyperplane H of a n-dimensional projective space $\mathbb{P}(E)$, and projectivities preserving the absolute $\mathcal{Q}$. Show that we have an equivalence of categories $\mathbf{Euclid}_n \rightsquigarrow \mathbb{P}\mathbf{Euclid}_n$.

11. Let T', T be two totally isotropic subspaces of a metric vector space E. If $\dim T' > \dim T$, prove the existence of a vector $0 \neq e \in T'$ such that $\langle e \rangle \oplus T$ is totally isotropic. (*Hint*: The natural map $T'/(T' \cap T) \to (T/T' \cap T)^*$ is not injective.)

    Conclude (without using Witt's theorem) that all maximal totally isotropic subspaces of E have equal dimension).

12. The **signature** of a (symmetric) metric S on a finite dimensional real vector space E is defined to be the pair (r_+, r_-) where r_+ (resp. r_-) is the maximal dimension of a vector subspace of E where S is positive definite (resp. negative definite). In the real case, show that the dimension and signature classify metrics. Moreover, if A is the matrix of S in some basis of E, show that r_+ (resp. r_-) is the number of positive (resp. negative) roots of $|xI - A|$, counted with multiplicity.

    Given a symmetric $n \times n$ matrix (a_{ij}) with real coefficients and signature (n_+, n_-), prove that there are vectors $e_1, \dots, e_n \in E$ such that $S(e_i, e_j) = a_{ij}$ if and only if $n_+ \leq r_+$ and $n_- \leq r_-$.

13. Let E be a real vector space of dimension 2. Show that $q\colon \mathrm{End}(E) \to \mathbb{R}$, $q(T) = \det T$, is a quadratic form of signature $(2, 2)$.

14. Let Ω_2 be an alternate metric on a k-vector space E. If $e \notin \mathrm{rad}\,\Omega_2$, show that $e \cdot e' = 1$ for some vector $e' \in E$, so that $E = \langle e, e' \rangle \perp \langle e, e' \rangle^{\perp}$. Obtain, by induction on the dimension, a direct proof of the classification of 2-forms (p. 169), even when $\mathrm{char}\, k = 2$.

15. Let $\phi\colon E \to E^*$ be the polarity of a non-singular metric S on a vector space E. The **dual metric** S^* is defined to be the unique metric on E^* such that $S^*(\phi(e), \phi(v)) = S(e, v)$, i.e., $\phi(e) \cdot \phi(v) = e \cdot v$.

    If we multiply S by a non-null factor λ, show that the dual metric is multiplied by λ^{-1}, and we define $\mathcal{Q}^* = \langle S^* \rangle$ to be the **dual quadric** of the non-singular quadric $\mathcal{Q} = \langle S \rangle$.

    Prove that the points of $\mathcal{Q}^*$ are just the tangent hyperplanes to $\mathcal{Q}$, and that two points are conjugated with respect to $\mathcal{Q}$ if and only if the polar hyperplanes are conjugated with respect to $\mathcal{Q}^*$.

    Prove that $\phi^{-1}\colon E^* \to E$ is just the polarity of the non-singular metric S^*, and conclude that S is the dual metric of S^*, so that $\mathcal{Q}$ is just the dual quadric of $\mathcal{Q}^*$.

16. Let $\mathcal{Q} = \langle S \rangle$ be a quadric in a projective space $\mathbb{P}(E)$ of vertex $X = \pi(V)$ and let $\bar{S}$ be the projection of S by the epimorphism $E \to \bar{E} = E/V$. Since $\bar{S}$ is non-singular, it defines a dual metric $\bar{S}^*$ in $\bar{E}^* \hookrightarrow E^*$; hence a **dual quadric** $\langle \bar{S}^* \rangle$ in $\mathbb{P}(\bar{E}^*) = X^{\mathrm{o}} = \pi(V^{\mathrm{o}}) \hookrightarrow \mathbb{P}(E^*)$.

    Prove that the points of the dual quadric are just the tangent hyperplanes to $\mathcal{Q}$.

    Show that quadrics in $\mathbb{P}(E)$ with vertex X naturally correspond with non-singular quadrics in X^{o}.

17. Let $\mathcal{Q} = \langle S \rangle$ be a quadric and $P = \langle e \rangle$ a non-incident point. Put $\omega = i_e S$, and show that in the pencil $\lambda S + \mu \omega \otimes \omega$, the quadric $\mathcal{C}$ passing through P is represented by $\omega(e) S - \omega \otimes \omega$.

    Prove that P is in the vertex of $\mathcal{C}$, and that any line joining P to an intersection point of $\mathcal{Q}$ and $\mathcal{C}$ is tangent to $\mathcal{Q}$ (so that $\mathcal{C}$ deserves the name of tangent cone to $\mathcal{Q}$ from the point P).

18. (*Resolution of cubics and quartics*) When $\mathrm{char}\, k \neq 2, 3$, prove the following statements:

    (a) Let Δ be the discriminant of a cubic $x^3 - px^2 + qx - r \in k[x]$ with three different roots $\alpha_1, \alpha_2, \alpha_3$. A homography $\tau(x) = \frac{ax+b}{cx+d}$ cyclically permuting the roots, $\tau(\alpha_1) = \alpha_2$, $\tau(\alpha_2) = \alpha_3$, $\tau(\alpha_3) = \alpha_1$, may be determined as a rational function of p, q, r and $\sqrt{\Delta}$.

    Hint: A solution (ex. 74 in p. 489) of the 3 equations $a\alpha_i + b = (c\alpha_i + d)\alpha_{i+1}$, where $\alpha_4 := \alpha_1$, is

$$a = \begin{vmatrix} 1 & -\alpha_1\alpha_2 & -\alpha_2 \\ 1 & -\alpha_2\alpha_3 & -\alpha_3 \\ 1 & -\alpha_1\alpha_3 & -\alpha_1 \end{vmatrix} = \frac{9r - qp + \sqrt{\Delta}}{2} , \qquad b = -\begin{vmatrix} \alpha_1 & -\alpha_1\alpha_2 & -\alpha_2 \\ \alpha_2 & -\alpha_2\alpha_3 & -\alpha_3 \\ \alpha_3 & -\alpha_1\alpha_3 & -\alpha_1 \end{vmatrix} = q^2 - 3rp ,$$

$$c = \begin{vmatrix} \alpha_1 & 1 & -\alpha_2 \\ \alpha_2 & 1 & -\alpha_3 \\ \alpha_3 & 1 & -\alpha_1 \end{vmatrix} = 3q - p^2 , \qquad d = -\begin{vmatrix} \alpha_1 & 1 & -\alpha_1\alpha_2 \\ \alpha_2 & 1 & -\alpha_2\alpha_3 \\ \alpha_3 & 1 & -\alpha_1\alpha_3 \end{vmatrix} = \frac{qp - 9r + \sqrt{\Delta}}{2} .$$

    (b) Fix a projective parameter θ so that the fixed points of τ are $\theta = 0$ and $\theta = \infty$. Then $\tau(\theta) = \varepsilon_3 \theta$, where ε_3 is primitive cubic root of unity, and our cubic equation reduces to $\theta^3 = \beta \in k(\sqrt{\Delta}\,)$.

    (c) The roots of $x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$ are given by the intersection points of two conics

$$\begin{cases} 0 = y^2 + c_1xy + c_2y + c_3x + c_4 \\ y = x^2 \end{cases}$$

    If we find a singular conic in the pencil $(y^2 + c_1xy + c_2y + c_3x + c_4) + \lambda(y - x^2) = 0$, intersecting it with $y = x^2$ we obtain the roots of the quartic. Conclude that the roots of a quartic may be calculated by means of square roots and the roots of a cubic equation.

19. Prove the following statements in the hyperbolic plane $(\mathbb{P}_2, \mathcal{Q} = \langle \Omega \rangle)$:

    (a) Lines have infinite length.

    (b) For any two pairs r, s and r', s' of perpendicular lines, there is a projectivity $\tau\colon \mathbb{P}_2 \to \mathbb{P}_2$ such that $\tau(\mathcal{Q}) = \mathcal{Q}$, $\tau(r) = r'$ and $\tau(s) = s'$. (All right angles are equal to one another).

    (c) The height of any equilateral right triangle is bounded above by $\mathrm{arccosh}\,\sqrt{2}$.

(d) The circles with center at a point $O = \pi(e)$ are just the conics $\langle \Omega - \mu\omega \otimes \omega \rangle$ bitangent at the intersection points with the polar line of O, such that $0 < \mu \leq 1$ (where $\omega = i_e \Omega$ and we fix Ω so that $\Omega(e,e) = 1$).

20. In a projective line, show that $(x, y; p, q)(y, z; p, q) = (x, z; p, q)$.

Let x, y, z be non-collinear points in the hyperbolic plane, and let us consider the points at infinity $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ of the lines xy, yz and xz respectively. Prove the existence of collinear points $c_1 < p < x < z < q < c_2$ such that

$$1 < (x, z; c_2, c_1) < (x, z; q, p) = (x, y; a_2, a_1)(y, z; b_2, b_1),$$

and conclude that the hyperbolic distance satisfies the triangle inequality.

21. *Special Theory of Relativity*: Now we consider an inertial reference $(p_0; e_0, e_1, e_2, e_3)$ in a Minkowski spacetime $(\mathbb{A}_4, V; g, h = -c^2 g)$, and the dual basis $\omega_0, \omega_1, \omega_2, \omega_3$ of e_0, e_1, e_2, e_3. Recall that we put $e \cdot v = g(e, v)$. Prove the following statements:

(a) If two events are joined by a trajectory (a connected smooth curve where the time metric g is positive-definite) then they may be joined by an inertial (= straight line) trajectory.

(b) A vector e is timelike ($e \cdot e > 0$) if and only if p and $q = p + e$ occur at the same place for some inertial observer. In this case $\sqrt{e \cdot e}$ is the time interval between p and q for such inertial observer, and it bounds below the time interval between p and q for any other inertial observer.

(c) A vector e is spacelike ($e \cdot e < 0$) if and only if p and $q = p + e$ are simultaneous for some inertial observer. In this case $\sqrt{h(e,e)}$ is the distance between p and q for such inertial observer, and it bounds below the distance between the places where p and q occur for any other inertial observer.

(d) If a voyager moves away with velocity ve_1 during a time interval t, and then he returns with velocity $-ve_1$, check that the proper time of the voyager is $\tau = 2t\sqrt{1 - (v/c)^2} < 2t$.
If $v \ll c$, the difference between the observed time $2t$ and the proper time τ is $\approx t(v/c)^2$.

(e) If we consider a rod of length le_1 at rest in an inertial reference, check that its length for another inertial observer moving with speed ve_1 is $l' = l\sqrt{1 - (v/c)^2}$.
When $v \ll c$, lengths contract $\approx lv^2/2c^2$.

(f) Now assume that $c = 1$ (i.e. $h = -g$) and let us consider a non-zero 2-form F on V and the endomorphism $\tilde{F} \colon V \to V$, such that $F(e, v) = h(\tilde{F}(e), v)$. If in an inertial reference e_0, e_1, e_2, e_3 we have

$$F = \omega_0 \wedge (E_1\omega_1 + E_2\omega_2 + E_3\omega_3) - (B_1\omega_2 \wedge \omega_3 + B_2\omega_3 \wedge \omega_1 + B_3\omega_1 \wedge \omega_2),$$

check that the matrix of the endomorphism $\tilde{F}$ is just

$$A = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$|xI - A| = x^4 + \left(\textstyle\sum_i B_i^2 - \sum_i E_i^2\right)x^2 - \left(\textstyle\sum_i E_i B_i\right)^2 = (x^2 - \alpha^2)(x^2 + \beta^2).$$

Even if the electric and magnetic fields $\vec{E} = E_1e_1 + E_2e_2 + E_3e_3$, $\vec{B} = B_1e_1 + B_2e_2 + B_3e_3$ depend on the inertial observer, the invariants

$$\|\vec{B}\|^2 - \|\vec{E}\|^2 = B_1^2 + B_2^2 + B_3^2 - E_1^2 - E_2^2 - E_3^2 = \beta^2 - \alpha^2$$
$$\langle \vec{E} | \vec{B} \rangle^2 = (E_1B_1 + E_2B_2 + E_3B_3)^2 = \beta^2\alpha^2$$

do not depend, and these invariants classify the pair (g, F):

i. If $\alpha = 0$ and $\beta = 0$ (i.e., $\vec{E}$ and $\vec{B}$ are orthogonal vectors of equal length), there exists an inertial reference where $F = \omega_0 \wedge \omega_1 + \omega_1 \wedge \omega_3$.

ii. Otherwise, there exists an inertial reference where the vectors $\vec{E}$ and $\vec{B}$ are parallel, of lengths α and β respectively, and $F = \alpha\omega_0 \wedge \omega_1 + \beta\omega_2 \wedge \omega_3$.

Hence we have that $\vec{B} = 0$ for some inertial observer if and only if $\beta = 0$, and that $\vec{E} = 0$ for some inertial observer if and only if $\alpha = 0$.

(g) Show that the electric field $\vec{E}$ only depend on the velocity e_0 of the inertial observer, not on the spatial axes e_1, e_2, e_3. (*Hint*: Show that $\vec{E} = \tilde{F}(e_0)$, or that $i_{e_0} F = i_{\vec{E}} h$.)

(h) Once we fix an orientation of V, prove that there is a unique 4-form Ω_V (the hypervolume form) such that the hypervolume of any oriented inertial reference frame e_0, e_1, e_2, e_3 is 1. Moreover, show that the restriction of F to $E = (\mathbb{R}e_0)^\perp$ is just $-i_{\vec{B}}\Omega_E$, where $\Omega_E := i_{e_0}\Omega_V$, and conclude that the magnetic field $\vec{B}$ only depend on the velocity e_0 of the oriented inertial observer, not on the spatial axes e_1, e_2, e_3.

Now assume, for the sake of simplicity that $c = 1$ and that free vectors in spacetime form a vector space V of dimension 2, so that the time metric g has signature $(1, 1)$, and let $\{e_0, e_1\}$ be a basis of V where the matrix of the time metric g is $\mathrm{diag}(1, -1)$. Prove the foolowing facts:

(a) If e_0' is another vector such that $e_0' \cdot e_0' = 1$, prove that the coordinates of e_0' are $t = \cosh\alpha$, $x = \sinh\alpha$, where α is the length (with respect to $-g$) of the arc joining e_0 with e_0' in the hyperbola $t^2 - x^2 = 1$. Hence the apparent speed v of any body with velocity e_0' is just $v = \tanh\alpha := (\sinh\alpha)/(\cosh\alpha)$.

(b) Let us consider inertial observers with velocities e_0, e_0', e_0'' and let $v_{i/j}$ be the speed of the i-th observer for the j-th observer. If α is the length of the arc joining e_0 with e_0' and β is the length of the arc joining e_0' with e_0'', then $v_{2/1} = \tanh\alpha$ and $v_{3/2} = \tanh\beta$.

Obtain (ex. 1 in p. 484) that $v_{3/1} = \tanh(\alpha + \beta) = \dfrac{\tanh\alpha + \tanh\beta}{1 + (\tanh\alpha)(\tanh\beta)} = \dfrac{v_{3/2} + v_{2/1}}{1 + v_{3/2}v_{2/1}}$.

22. Given a n-dimensional affine space $(\mathbb{A}_n, V)$, let us consider a polynomial q of degree ≤ 2 on $\mathbb{A}_n$. If we have $q = \sum a_{ij}x_ix_j + \sum a_ix_i + c$ in an affine reference, prove that the leading term $\sum a_{ij}x_ix_j$ is a well-defined quadratic form on V, i.e. that it does not depend on the fixed affine reference. (*Hint*: Show that q is the restriction of a unique quadratic form on the vector extension $\mathbb{A}_n \hookrightarrow E_{n+1}$ defined in p. 164, and consider its restriction to $V \subset E_{n+1}$.)

23. Let us consider an Euclidean plane $(\mathbb{A}_2, V)$, where we have fixed the unit length (i.e. the scalar product on V). The 4-dimensional real vector space $E = \{d(x^2 + y^2) + ax + by + c\}$ of polynomial functions on $\mathbb{A}_2$ of degree ≤ 2 with leading term proportional to the (quadratic form of the) scalar product is named *the space of circles* since it includes equations $(x - x_0)^2 + (y - y_0)^2 - r^2 = 0$ of real circles, imaginary circles $(x - x_0)^2 + (y - y_0)^2 + r^2 = 0$, points $(x - x_0)^2 + (y - y_0)^2 = 0$, lines $ax + by + c = 0$ and the empty set $c = 0$. Show that in this space E the coefficient d defines a well-defined linear form (it does not depend on the Euclidean reference) and prove that there is a unique quadratic form $q\colon E \to \mathbb{R}$ such that $q\big((x - x_0)^2 + (y - y_0)^2 - r^2\big) = r^2$, so that the modulus of a real circle $C = (x - x_0)^2 + (y - y_0)^2 - r^2 = 0$ is just the radius $r = \sqrt{q(C)} = \sqrt{C \cdot C}$.

Show that the bilinear metric h corresponding to q has signature $(3, 1)$, so that $(E, \mathbb{R}h)$ is a Minkowski spacetime. Prove that equations of real circles are spacelike vectors ($C \cdot C > 0$), equations of imaginary circles are timelike vectors ($C \cdot C < 0$), equations of points are light vectors ($P \cdot P = 0$), while equations of lines are just vectors in the hyperplane $d = 0$ tangent to the light cone at the generatrix defined by the empty set $d = a = b = 0$. Moreover:

(a) Two real circles C_1, C_2 are orthogonal, $C_1 \cdot C_2 = 0$, when both circles intersect orthogonally.

(b) In the case of a real circle C_1 and an imaginary circle C_2 of radius $r_2 i$, we introduce the concentric real circle $\bar{C}_2$ of radius r_2, and condition $C_1 \cdot C_2 = 0$ means that C_1 intersects $\bar{C}_2$ in a pair of diametrally opposite points.

(c) A real circle C and a point P are orthogonal when C passes through P.

(d) A circle C and a line L are orthogonal when L passes through the center of C.

(e) Two lines are orthogonal, $L_1 \cdot L_2 = 0$, when both are perpendicular.

(f) No circle is orthogonal to the empty set, while so are all lines.

(g) If two different real circles C_1, C_2 intersect and ϕ is the angle between the tangent lines at a common point, then $C_1 \cdot C_2 = |C_1| \cdot |C_2| \cdot \cos\phi$.

(h) If we represent any real circle by the equation $x^2 + y^2 + ax + by + c = 0$ with $d = 1$, check that we have $-2C_1 \cdot C_2 = D^2 - r_1^2 - r_2^2$, where r_i is the radius of C_i and D is the distance between the centers of both circles. When C_2 is a point ($r_2 = 0$), up to a factor, $C_1 \cdot C_2$ is just the power of a point with respect to the circle C_1.

Finally, we represent real circles by the equation C of modulus 1 and $d > 0$,

$$C = \frac{(x - x_0)^2 + (y - y_0)^2 - r^2}{r}.$$

(a) Prove that the product $C_1 \cdot C_2$ of two real circles has the following geometrical meaning:

$C_1 \cdot C_2 > 1$	non-intersecting interior circles
$C_1 \cdot C_2 = 1$	interior tangent circles
$C_1 \cdot C_2 = \cos\phi$	where ϕ is the angle formed by the radius at a common point
$C_1 \cdot C_2 = -1$	exterior tangent circles
$C_1 \cdot C_2 < -1$	non-intersecting exterior circles

(b) There exist equations $e_1, \ldots, e_n \in E$ with a given configuration matrix $(h_{ij}) = (e_i \cdot e_j)$ if and only if this matrix is symmetric, of signature $\leq (3, 1)$.

(c) If we look for 4 mutually orthogonal circles, the configuration matrix $(C_i \cdot C_j)$ is the unit matrix, so that this configuration is impossible.

(d) If the configuration matrix $(h_{ij}) = (C_i \cdot C_j)$ of 4 real circles has rank 4, then C_1, C_2, C_3, C_4 is a basis of E and the inverse matrix (h^{ij}) is just the matrix of the dual metric in the dual basis. Hence, any isotropic 1-form, $\omega \cdot \omega = 0$, determines a relation between the radii and the centers of the circles: $0 = \sum_{i,j} h^{ij}\omega(C_i)\omega(C_j)$.

(e) Isotropic 1-forms are $\omega(C) = d$ and the complex 1-form $\omega'(C) = -\frac{1}{2}(a + bi)$. If $\kappa = r^{-1}$ is the **curvature** of a real circle C and $z \in \mathbb{C}$ is the center, then $\omega(C) = \kappa$ and $\omega'(C) = \kappa z$.

(f) The curvatures κ_i and centers z_i of 4 mutually exterior tangent circles satisfy

$$2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) = (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2,$$
$$2(\kappa_1^2 z_1^2 + \kappa_2^2 z_2^2 + \kappa_3^2 z_3^2 + \kappa_4^2 z_4^2) = (\kappa_1 z_1 + \kappa_2 z_2 + \kappa_3 z_3 + \kappa_4 z_4)^2.$$

24. Try to generalize these results to the space $\{d(x_1^2 + \ldots + x_n^2) + a_1 x_1 + \ldots + a_n x_n + c\}$ of equations of spheres in a n-dimensional Euclidean space.

25. Pair of points $p + q$ in a real projective line $\mathbb{P}_1 = \mathbb{P}(E)$, including coincident points $2p$ and pairs of complex conjugate points, are just points of the real plane $\mathbb{P}_2 = \mathbb{P}(S^2 E^*)$. Show that the pairs $2p$ of coincident points define a real non-singular quadric $\mathcal{Q}$ in $\mathbb{P}_2(S^2 E^*)$, and that

(a) The tangent line to $\mathcal{Q}$ at a point $2p$ is just the line L_p of all pairs $p + x$.

(b) The exterior points are pairs $p + q$ of different real points, the tangent lines to $\mathcal{Q}$ trough $p + q$ being L_p and L_q.

(c) The interior points are pairs of conjugate complex points (they form an hyperbolic plane!).

(d) The polar of a pair $p + q$ is the line of all pairs of harmonic conjugates with respect to p, q.

(e) Three pairs $p_1 + q_1, p_2 + q_2, p_3 + q_3$ are collinear if and only if there is an involutive homography τ such that $\tau(p_i) = q_i$, $i = 1, 2, 3$.

26. Prove that central conics have two axes, while parabolas have a unique axis.

27. Prove that any central conic (not a circumference) has two real focuses on an axis, and two imaginary focuses on the other axis, while parabolas have a unique proper focus (and an improper focus), and it is on the axis. Moreover, the points of the directrix of a parabola are just points with perpendicular tangent lines.

28. Given two points in a plane, prove that the points such that the sum of the distances to such points is a fixed constant $2a$ define an ellipse with focuses at the fixed points, and semi-major axis a.

29. Let us consider a line and an exterior point in a plane. Prove that the points such that the distances to the fixed point and line have a given proportion $e>0$ (named **eccentricity**) define an ellipse when $e<1$, a parabola when $e=1$ and an hyperbola when $e>1$, such that the fixed point is a focus and the line is the corresponding directrix. In polar coordinates (ρ,θ) with center at the focus and vertical directrix, this conic is $\rho(1+e\cos\theta)=p$, where p is named *semi latus rectum.* In an ellipse of semi-major axis a, semi-minor axis b and distance c between the center and the focus (so that $a^2=b^2+c^2$) show that

$$e=\frac{c}{a}=\sqrt{1-\frac{b^2}{a^2}}\ ,\quad ap=b^2.$$

30. Assume that the position $\vec{x}(t)$ of a planet with respect to the Sun fulfills the differential equation

$$\vec{x}''=-\frac{m\vec{x}}{|\vec{x}|^3}.$$

If $\vec{x}$ and $\vec{v}:=\vec{x}'$ are linearly independent at some instant, show that the angular momentum $\vec{h}=\vec{x}\times\vec{v}$, hence $h:=|\vec{h}|$, is constant along the trajectory, so that $\vec{x}$ lies in a plane.

Consider polar coordinates, so that $\vec{x}=(\rho\cos\theta,\rho\sin\theta,0)$. Prove that $h\mathrm{d}t=\rho^2\mathrm{d}\theta$ (*a line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time*), and conclude that

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}\theta}=-\frac{m}{h}(\cos\theta,\sin\theta,0)$$

so that the hodograph $\vec{v}=\frac{m}{h}(-\sin\theta,\cos\theta,0)+\vec{c}$ lies in a cercle of radious $R:=\frac{m}{h}$ (the center $\vec{c}$ being essentially the **Runge-Lenz** vector $\vec{h}\times\vec{c}$).

Put $e=|\vec{c}|/R$, and assume that $\vec{c}$ is in the positive y-axis, so that $\vec{v}=R(-\sin\theta,e+\cos\theta,0)$. Using that $\vec{h}=\vec{x}\times\vec{v}$, conclude that *the orbit of a planet is an ellipse with the Sun at a focus*:

$$\rho(1+e\cos\theta)=p\ ,\quad p:=h^2/m.$$

Finally, relate the area $A=\pi ab=\sqrt{a^3p}$ of the ellipse to the period T, and conclude that *the square of the period of a planet is proportional to the cube of the semi-major axis of its orbit*:

$$A=\int\frac{1}{2}\rho^2\mathrm{d}\theta=\int\frac{h}{2}\mathrm{d}t=\frac{hT}{2}\ ,\quad mT^2=4\pi^2a^3.$$

31. In the case of the positions $\vec{x}_1(t)$, $\vec{x}_2(t)$ of two punctual masses m_1, m_2 moving according to the newtonian gravitational attraction,

$$m_1\vec{x}_1''(t)=\frac{m_1m_2}{|\vec{x}_1-\vec{x}_2|^3}(\vec{x}_2-\vec{x}_1),\quad m_2\vec{x}_2''(t)=\frac{m_1m_2}{|\vec{x}_1-\vec{x}_2|^3}(\vec{x}_1-\vec{x}_2),$$

show that the barycenter $\frac{1}{m_1+m_2}(m_1\vec{x}_1+m_2\vec{x}_2)$ moves uniformly. In the case that $m_1\vec{x}_1+m_2\vec{x}_2=0$, so that the position of both particles is fully determined by the relative position $\vec{x}_1-\vec{x}_2$,

$$\vec{x}_1=\frac{m_2}{m_1+m_2}(\vec{x}_1-\vec{x}_2),\quad \vec{x}_2=-\frac{m_1}{m_1+m_2}(\vec{x}_1-\vec{x}_2),$$

prove that $\vec{x}(t):=\vec{x}_1(t)-\vec{x}_2(t)$ fulfills the differential equation $\vec{x}''=-\frac{(m_1+m_2)}{|\vec{x}|^3}\vec{x}$ and $\vec{x}(t)$ follows a conic with a focus at the barycenter, so that the trajectories of both particles are homothetic conics, the center of homothety being a common focus and the ratio being $-m_1/m_2$.

32. Let $T(M)$ be the torsion submodule of a module M over a domain A. If S is a multiplicative system of A, show that $T(M_S)=(TM)_S$. Conclude that M is torsion free if and only if so is M_x, $\forall x\in\operatorname{Spec}A$.

33. Put $B=k[x]/(p^n)$, where $p(x)\in k[x]$ is irreducible. Show that the annihilator of $B^*=\operatorname{Hom}_k(B,k)$ is p^n, and obtain the existence of a B-linear isomorphism $B\simeq B^*=\operatorname{Hom}_k(B,k)$.

If we have an exact sequence $0\to B\to M\to\bar{M}\to 0$ of finitely generated B-modules, prove that the dual sequence $0\to\bar{M}^*\to M^*\to B^*\to 0$ splits, and conclude that $M\simeq B\oplus\bar{M}$; so obtaining an elementary (without using the ideal criterion on injectivity) proof of the second decomposition theorem for $k[x]$-modules and the classification of endomorphisms.

34. Let T be an endomorphism of a finite-dimensional vector space E. Prove that E is a monogenous $k[x]$-module if and only if so is E^* (with the module structure defined by $T^*\colon E^* \to E^*$).

    Conclude that T and T^* have the same invariant factors.

35. Prove the following properties of the tensor algebra of a module:

    (a) $T^\bullet(M \oplus N) = (T^\bullet M) \otimes_A (T^\bullet N)$.

    (b) $T^\bullet(M/N) = (T^\bullet M)/(N)$, where (N) is the two-sided ideal generated by $N \subseteq M = T^1 M$.

    (c) $(T^\bullet_A M)_B = T^\bullet_B(M_B)$, for any base change $A \to B$.

36. Prove the following properties of the symmetric algebra of a module:

    (a) $S^\bullet(M \oplus N) = (S^\bullet M) \otimes_A (S^\bullet N)$.

    (b) $S^\bullet(A \oplus \overset{n}{\ldots} \oplus A) = A[x_1, \ldots, x_n]$.

    (c) $S^\bullet(M/N) = (S^\bullet M)/(N)$, where (N) is the ideal generated by $N \subseteq M = S^1 M$.

    (d) $(S^\bullet_A M)_B = S^\bullet_B(M_B)$, for any base change $A \to B$.

37. Let $L \simeq A^n$ be a free A-module of rank n. Prove that $\Lambda^p L$ is a free A-module of rank $\binom{p}{n}$, so that any endomorphism $T\colon L \to L$ induces an endomorphism $\Lambda^n\colon \Lambda^n L \to \Lambda^n L$ of a free module of rank 1: the product by an element $\det T \in A$, named **determinant** of T. Show that

    (a) We have $\det(T \otimes 1) = \det T$ for any ring morphism $A \to B$.

    (b) The determinant is multiplicative: $\det(T \circ S) = (\det T)(\det S)$.

    (c) T is an automorphism if and only if $\det T$ is invertible in A.

38. Let E be a finite-dimensional k-vector space. Show that $\Lambda^p E^* = (\Lambda^p E)^*$ is canonically isomorphic to the vector space $\mathrm{Alt}(E, \overset{p}{\ldots}, E; k)$ of p-covariant alternate tensors, that $(S^p E)^*$ is canonically isomorphic to the vector space $\mathrm{Sym}(E, \overset{p}{\ldots}, E; k)$ of p-covariant symmetric tensors, and that $S^2 E^*$ is canonically isomorphic to the vector space of **quadratic forms** (maps $q\colon E \to k$ such that $q(\lambda e) = \lambda^2 q(e)$ and $g(e, v) = q(e+v) - q(e) - q(v)$ is a bilinear map, so that in coordinates $q(x_1, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j$).

    When $\mathrm{char}\, k = 0$, show that the composition $\mathrm{Sym}(E, \overset{p}{\ldots}, E; k) \hookrightarrow T^p E^* \to S^p E^*$ is an isomorphism; but, when $\mathrm{char}\, k = 2$, there is no natural (i.e. functorial) isomorphism $\mathrm{Sym}(E, E; k) = S^2 E^*$. (*Hint*: When $\dim E = 2$, the natural map $S^2 E^* \to \mathrm{Sym}(E, E; k)$ has 1-dimensional image.)

39. If E is a finite-dimensional k-vector space, show that we have natural exact sequences

$$0 \longrightarrow \Lambda^2 E \longrightarrow E \otimes_k E \longrightarrow S^2 E \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Sym}(E, E; k) \longrightarrow \mathrm{Bil}(E, E; k) \xrightarrow{\ h\ } \mathrm{Alt}(E, E; k) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Alt}(E, E; k) \longrightarrow \mathrm{Bil}(E, E; k) \longrightarrow S^2 E^* \longrightarrow 0.$$

40. Let us consider the polarity $E \to E^*$ defined by a symmetric metric g, where $\dim_k E < \infty$.

    Prove that the induced linear maps $T^q_p E \to T^q_p E^* = (T^q_p E)^*$ and $\Lambda^p E \to \Lambda^p E^* = (\Lambda^p E)^*$ define symmetric metrics $T^q_p g$ and $\Lambda^p g$ on $T^q_p E$ and $\Lambda^p E$.

    If g is non singular, prove that so are $T^q_p g$ and $\Lambda^p g$, and that $T^p_q g$ is the dual metric of $T^q_p g$, via the natural isomorphism $T^p_q E = (T^q_p E)^*$.

    If $\mathrm{char}\, k = 0$ and we view $\Lambda^p E$ as a vector subspace of $T^p E$, formed by the alternate tensors, show that the restriction of $T^p g$ to $\Lambda^p E$ is just $p!(\Lambda^p g)$.

41. If T is an endomorphism of a vector space of dimension n, show that $c_T(x) = \sum_{i=0}^{n} (-1)^i \mathrm{tr}(\Lambda^i T) x^{n-i}$.

42. Classify the group of invertible elements in the ring $\mathbb{Z}[x]/(x^n)$.

43. Let I be the annihilator of a finitely generated module M over a principal ideal domain A. Show that the ring of all endomorphisms of M commuting with all the idempotent endomorphisms of M is A/I.

44. Let $d_1, \ldots, d_r$ be the invariant factors of a finite abelian group G. Prove that $|\mathrm{Aut}\, G| = \prod_{i=1}^{r} d_i^{2r-2i+1}$.

45. Let E be a finite dimensional k-vector space. Let $\phi_1, \ldots, \phi_r$ be the invariant factors of an endomorphism $T\colon E \to E$, and put $d_i = \deg \phi_i$. Prove that
    (a) $\dim_k\big(\mathrm{End}_{k[x]} E\big) = \sum_{i=1}^{r}(2r - 2i + 1)d_i$.
    (b) Any endomorphism of E commuting with the endomorphisms commuting with T is in $k[T]$.
    (c) E_T is monogenous if and only if the ring $\mathrm{End}_{k[x]} E$ is commutative.

46. Let T be an isometry, $(Te)\cdot(Tv) = e \cdot v$, of a non singular metric on a k-vector space, $\mathrm{char}\, k \neq 2$. Prove that T has as many elementary divisors $(x-a)^n$ as $\big(x - \frac{1}{a}\big)^n$.

47. Let S be a non-singular symmetric metric on a finite dimensional k-vector space E, $\mathrm{car}\, k \neq 2$. If S' is another symmetric metric on E and we put $e \cdot v = S(e,v)$, $e * v = S'(e,v)$, then the linear endomorphism $T\colon E \to E$ such that $(Te)\cdot v = e * v = e \cdot (Tv)$ defines a structure of $k[x]$-module on E. Prove the following statements:
    (a) The polarity $\phi\colon E \to E^*$, $\phi(e) = S(e,-)$, is an isomorphism of $k[x]$-modules.
    (b) The S-orthogonal of a submodule also is a submodule.
    (c) If V_1, V_2 are submodules and $V_1 \cdot V_2 = 0$, then $V_1 * V_2 = 0$.
    (d) The radical of S' is a submodule.
    (e) If another pair $(\bar S, \bar S')$ defines $\bar T\colon \bar E \to \bar E$, then a k-linear isomorphism $E_T \xrightarrow{\sim} \bar E_{\bar T}$ transforms (S, S') into $(\bar S, \bar S')$ if and only if it is $k[x]$-linear and transforms S into $\bar S$.
    (f) **First Theorem:** *Let $p = p_1^{n_1} \ldots p_s^{n_s}$ be the decomposition into irreducible factors of the annihilator polynomial of T. Then the decomposition $E = E_1 \perp \ldots \perp E_s$, $E_i = \mathrm{Ker}\, p_i^{n_i}$, is orthogonal for both metrics.* (*Hint*: The polarity $\phi\colon E \to E^* = E_1^* \oplus \ldots \oplus E_s^*$ is a $k[x]$-linear and $p_i^{n_i}$ annihilates $E_i^* = \{\omega \in E^*\colon \omega(E_j) = 0,\ j \neq i\}$. Hence $\phi(E_i) = E_i^*$, and $E_i \cdot E_j = 0$ when $j \neq i$.)
    (g) **Second Theorem:** *E is an orthogonal sum of homogeneous modules* (i.e. $\simeq A/p^n \oplus \ldots \oplus A/p^n$ with p irreducible). (*Hint*: Assume that $E = H_n \oplus \ldots \oplus H_1$, where H_i is homogeneous, with annihilator p^i. Show that H_n is non singular: since $\mathrm{rad}\, H_n$ is a submodule, otherwise there is $0 \neq p^{n-1}e \in \mathrm{rad}\, H_n$, so that $p^{n-1}e \cdot H_i = e \cdot p^{n-1}H_i = 0$, $i \leq n-1$, and $p^{n-1}e \in \mathrm{rad}\, E$. Hence $E = H_n \perp H_n^{\perp}$.)

    Now assume that E is homogeneous, with annihilator p^n, and consider the filtration

    $$E = E_n \supset E_{n-1} \supset \ldots \supset E_1 \supset E_0 = 0, \text{ where } E_i = p^{n-i}E.$$

    (a) $\dim E_i = i(\dim E_1)$. (*Hint*: Show that $p\colon E_i \to E_{i-1}$ is an epimorphism with kernel E_1).
    (b) $E_i \cdot E_j = 0$, when $i + j \leq n$. (*Hint*: Show that $p^{n-i}E \cdot p^{n-j}E = E \cdot p^{2n-i-j}E = 0$).
    (c) $E_1 = E_{n-1}^{\perp} = \mathrm{rad}\, E_{n-1}$. (*Hint*: Show that $E_1 \subset E_{n-1}^{\perp}$, and compare dimensions.)

    Third Theorem: *There are vector subspaces $F_i \subset E_i$ such that*
    (a) $E = F_n \oplus F_{n-1} \oplus \ldots \oplus F_1$.
    (b) *We have isomorphisms $p\colon F_i \to F_{i-1}$, $i \geq 2$.*
    (c) $F_i \cdot F_j = 0$, *when $i + j \neq n + 1$.*

    (*Hint*: When $n = 1$, put $F_1 = E_1$. When $n = 2$, show that $F_1 = E_1$ is totally isotropic, so that there is another totally isotropic subspace F_2 such that $E = F_2 \oplus F_1$. When $n \geq 3$, consider the projections $\bar S$, $\bar S'$, $\bar T$ of S, S' and T by $\pi\colon E \to \bar E \coloneqq E_{n-1}/E_1$, so that $\bar S$ is non singular and $\bar T$ is the endomorphism associated to the pair $(\bar S, \bar S')$. Show that the annihilator of $\bar T$ is p^{n-2}. Hence, by induction, you have a decomposition $\bar E = \bar F_{n-2} \oplus \ldots \oplus \bar F_1$, $p\colon \bar F_i \simeq \bar F_{i-1}$ and $\bar F_i \cdot \bar F_j = 0$, when $i + j \neq n - 2 + 1 = n - 1$. Now fix a subspace $F_{n-1} \subset E_{n-1}$ such that $\pi\colon F_{n-1} \simeq \bar F_{n-2}$ and put $F_1 = E_1$, $F_i = p^{n-i-1}F_{n-1}$, so that all the morphisms in the following commutative diagram

    $$\begin{array}{ccccccccc} F_{n-1} & \xrightarrow{p} & F_{n-2} & \xrightarrow{p} & \ldots & \xrightarrow{p} & F_2 & \xrightarrow{p} & F_1 \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & & & \downarrow{\scriptstyle\pi} & & \\ \bar F_{n-2} & \xrightarrow{p} & \bar F_{n-3} & \xrightarrow{p} & \ldots & \xrightarrow{p} & \bar F_1 & & \end{array}$$

are in fact isomorphisms. Using that $\pi\colon F_{n-2}\oplus\ldots\oplus F_2\to\bar{E}$ is an isometry, show that the subspaces $F_{n-1},\ldots,F_1$ satisfy the required conditions. Then find a totally isotropic subspace F_n such that $p\colon F_n\xrightarrow{\sim} pF_n$ and $\pi\colon pF_n\xrightarrow{\sim}\bar{F}_{n-2}$ (and put $F_{n-1}:=pF_n$), so that $F_n\cdot F_i=0$ when $i\neq 1$): fix a subspace U_n such that $p\colon U_n\xrightarrow{\sim}F_{n-1}$, and consider on $U_n\oplus F_2$ the <u>non singular</u> metric $e\circ v=e\cdot pv$; in fact, if R is the radical, you have

$$0=R\circ F_2=R\cdot pF_2=R\cdot F_1,$$
$$0=R\circ U_n=R\cdot pU_n=R\cdot F_{n-1},$$

these equalities stating that $R\subseteq E_{n-1}$ (hence $R\subseteq F_2$) and $\pi(R)\subseteq\operatorname{rad}\bar{E}=0$. Hence $R=0$ because $\pi\colon F_2\to\bar{F}_1$ is an isomorphism. Now show that F_2 is totally isotropic for this metric $\circ$

$$F_2\circ F_2=F_2\cdot pF_2=F_2\cdot F_1=0,\ (n\geq 3)$$

hence there is a totally isotropic subspace $V_n\subset U_n\oplus F_2$ (i.e. $V_n\cdot pV_n=0$) such that $U_n\oplus F_2=V_n\oplus F_2$. Moreover $\pi(pV_n)=\pi(pU_n)=\bar{F}_{n-2}$, and you may fix $F_{n-1}=pV_n$, so that $p\colon V_n\xrightarrow{\sim}F_{n-1}$, and $V_n\cdot F_{n-1}=0$. Now $V_n\oplus F_1$ is non singular for S and F_1 is totally isotropic; hence there is a totally isotropic subspace F_n with $F_n\oplus F_1=V_n\oplus F_1$. Check that $E=F_n\oplus E_{n-1}$, $p\colon F_n\xrightarrow{\sim}F_{n-1}$, it is isotropic, $F_n\cdot F_{n-1}=(V_n\oplus F_1)\cdot F_{n-1}=0$, and $F_n\cdot F_i=F_n\cdot pF_{i+1}=pF_n\cdot F_{i+1}=F_{n-1}\cdot F_{i+1}=0$ when $n+i\neq n+1$.)

(a) When k is algebraically closed, E is an orthogonal sum of monogenous submodules. (*Hint*: The radical of $p^{n-1}e\cdot v$ is $\operatorname{Ker}p^{n-1}=E_{n-1}$; hence there is $e\in F_n$ such that $p^{n-1}e\cdot e\neq 0$. Now $V=\langle e,pe,\ldots,p^{n-1}e\rangle$ is non singular, and a submodule because $p=x-\alpha$; hence $E=V\perp V^{\perp}$.)

(b) *When k is algebraically closed, pairs of symmetric metrics (the first non singular) are classified by the elementary divisors of T.*

48. Now assume that S is symmetric and non singular, and $S'=H\in\Lambda^2E^*$ is hemisymmetric:

$$(Te)\cdot v=e*v=e\cdot(-Tv)$$
$$qe\cdot v=e\cdot\bar{q}v,\text{ where }\bar{q}(x)=q(-x).$$

In this case T and $-T$ have equal annihilator $p(x)$, so that $p(-x)=\pm p(x)$. If $p=p_1^{n_1}\ldots p_s^{n_s}$ is the decomposition into irreducible factors, for any factor p_i we have $(\bar{p}_i)=(p_i)$, or $(\bar{p}_i)=(p_j)$, $(\bar{p}_j)=(p_i)$ with $j\neq i$. Prove the following statements:

(a) **First Theorem:** *In the decomposition $E=E_1\oplus\ldots\oplus E_s$, each term $E_i=\operatorname{Ker}p_i^{n_i}$ is (for S)*
1.- *non singular and orthogonal to the remaining terms, when $(\bar{p}_i)=(p_i)$,*
2.- *or totally isotropic, and $E_i\oplus E_j$ is non singular and orthogonal to the remaining terms, when $(\bar{p}_i)=(p_j)$.*
(*Hint*: The polarity $\phi\colon E\to E^*$ of S is a semilinear isomorphism, $\phi(qe)=\bar{q}\phi(e)$, so that it transforms E_i into the $\bar{p}_i$-primary component of $E^*=E_1^*\oplus\ldots\oplus E_s^*$. Hence $\phi(E_i)=E_i^*$ when $(\bar{p}_i)=(p_i)$, and $\phi(E_i)=E_j^*$, $\phi(E_j)=E_i^*$ when $(\bar{p}_i)=(p_j)$, so that $\phi(E_i\oplus E_j)=E_i^*\oplus E_j^*=(E_i\oplus E_j)^*$.)

(b) In case 1, the second and third decomposition theorems hold. (*Hint*: You may repeat the proofs of the symmetric case, $p^ne\cdot v=e\cdot\bar{p}^nv=\pm e\cdot p^nv$.)

(c) In case 2, the pair of metrics in $E_i\oplus E_j$ is unique up to an automorphism. (*Hint*: If $\tilde{S},\tilde{H}$ is other pair of metrics defining the same module structure, $\tilde{S}(Te,v)=\tilde{H}(e,v)$, and $\varphi\colon E\xrightarrow{\sim}E^*$ is the polarity of $\tilde{S}$, then $\tau=\phi^{-1}\varphi\colon E_i\to E_i$ is a module isomorphism such that $\tilde{S}(e_i,e_j)=\tau e_i\cdot e_j$; hence $\tau\oplus\operatorname{Id}\colon E_i\oplus E_j\to E_i\oplus E_j$ transforms S into $\tilde{S}$, and H into $\tilde{H}$.)

(d) *When k is algebraically closed, pairs of metrics symmetric and hemisymmetric (the first non singular) are classifies by the elementary divisors of T. If $\alpha\neq 0$, there are as many elementary divisors $(x-\alpha)^n$ as $(x+\alpha)^n$, and the number of elementary divisors equal to x^{2n} is even.*
If $p=x-\alpha$, then $\bar{p}=-(x+\alpha)$ and we are in case 2. Consider the sum $V\oplus\bar{V}$ of two monogenous submodules of annihilator $(x-\alpha)^n$ and $(x+\alpha)^n$. Fix a base $e,pe,\ldots,p^{n-1}e$ in V. The image by the polarity is a base of $\bar{V}^*$ that, in the reverse order, is dual of a base $\bar{e},\bar{p}\bar{e},\ldots,\bar{p}^{n-1}\bar{e}$ in $\bar{V}$,

$$(*)\qquad p^ie\cdot\bar{p}^j\bar{e}=e\cdot\bar{p}^{i+j}\bar{e}=\begin{cases}1 & \text{when } i+j=n-1\\ 0 & \text{when } i+j\neq n-1\end{cases}$$

$$(*) \qquad p^ie * \bar{p}^j\bar{e} = Tp^ie\cdot\bar{p}^j\bar{e} = (\alpha p^ie + p^{i+1}e)\cdot\bar{p}^j\bar{e} = \begin{cases} \alpha & \text{when } i+j=n-1 \\ 1 & \text{when } i+j=n-2 \\ 0 & \text{otherwise} \end{cases}$$

If $p = x$, and $\bar{p} = -x$, the second decomposition theorem let us reduce to the case of a homogeneous module of annihilator x^n. If n is even, the metric $p^{n-1}e\cdot v$ is hemisymmetric and $p^{n-1}e\cdot e = 0$, $\forall e \in F_n$; but $p^{n-1}e\cdot\bar{e} = 1$ for some $\bar{e} \in F_n$, and the monogenous submodules $V =< e, pe, \dots, p^{n-1}e >$, $\bar{V} =< \bar{e}, \bar{p}\bar{e}, \dots, \bar{p}^{n-1}\bar{e} >$ are totally isotropic, with null intersection, and $V \oplus \bar{V}$ is non singular. The former calculation shows that the matrices of the metrics are (*), with $\alpha = 0$. Since $E = (V\oplus\bar{V}) \perp (V\oplus\bar{V})^\perp$, we conclude.

If n is odd, the metric $p^{n-1}e\cdot v$ is symmetric, and the argument of the pair of symmetric metrics shows that the space is an orthogonal sum of monogenous submodules where the pair of metrics is fully determined. In fact, by the third decomposition theorem, there is a base $e, Te, \dots, T^{n-1}e$ such that $T^{n-1}e\cdot e \neq 0$ and, k being algebraically closed, you may assume that $T^{n-1}e\cdot e = 1$,

$$T^ie\cdot T^je = (-1)^jT^{i+j}e\cdot e = \begin{cases} (-1)^j & \text{when } i+j=n-1 \\ 0 & \text{when } i+j\neq n-1 \end{cases}$$

$$T^ie * T^je = T^{i+1}e\cdot T^je = \begin{cases} (-1)^j & \text{when } i+j=n-2 \\ 0 & \text{when } i+j\neq n-2 \end{cases}$$

49. If S is symmetric and non singular, for any 2-covariant tensor S' we have

$$Te\cdot v = S'(e,v) = e\cdot\bar{T}v$$

for some endomorphisms $T, \bar{T}\colon E \to E$. We have studied the cases $\bar{T} = T$ and $\bar{T} = -T$; but show that the above decomposition theorems may be extended to any involution relating $\bar{T}$ with T.

(a) If T is an isometry of S, then $Te\cdot v = e\cdot T^{-1}v$, so that E is a $k[x,x^{-1}]$-module and the polarity $\phi\colon E\to E^*$ is semilinear, of automorphism $p(x)\mapsto\bar{p}(x) = p(x^{-1})$.

(b) If S is hemisymmetric and non singular, then $\bar{T} = T$ when S' is hemisymmetric, and $\bar{T} = -T$ when S' is symmetric.

(c) Since any 2-covariant tensor T_2 decomposes uniquely as a sum $T_2 = S + H$ of a symmetric and a hemisymmetric tensors, obtain a classification, when k is algebraically closed, of the 2-covariant tensors with non singular symmetric or hemisymmetric component.

50. If S is a simple A-module, show that $S \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A.

51. Let A be a ring (eventually non-commutative). Use Zorn's lemma to prove the following statements about arbitrary A-modules:

(a) If $N \subset M$ is a submodule and $M = N + \sum_{i\in I} S_i$, where the submodules S_i are simple, then $M = N \oplus (\oplus_{j\in J}S_j$ for some subset $J \subseteq I$.

(b) Every semisimple module is a direct sum of simple modules.

(c) Every submodule of a semisimple module also is semisimple.

(d) Every semisimple module admits a unique decomposition as a direct sum (eventually infinite) of homogeneous submodules of different types.

52. Show that any left ideal of $\mathrm{End}_D(E)$ is the set of all endomorphisms vanishing on a certain vector subspace $V \subseteq E$, and that any right ideal of $\mathrm{End}_D(E)$ is the set of all endomorphisms with image contained in a certain vector subspace $V \subseteq E$.

53. Prove that a ring A is semisimple if and only if it is **artinian** (strictly decreasing sequences of ideals are finite) and any nilpotent ideal is null. (*Hint*: If I is a minimal ideal and $I^2 \neq 0$, then $I^2 = I$, so that $Ib = I$ for some $b \in I$, and $b = ab = a^2b$ for some $a \in I$. Show that $a^2 = a$, so that $A = Aa \oplus A(1-a)$. Iterate the argument with $A(1-a)$, and conclude because A is artinian.)

54. Let G be a group of automorphisms of a finite extension $k \to L$. Prove that the natural k-linear map $L\otimes_k k[G] \to \mathrm{End}_k(L)$ is an isomorphism if and only if $k \to L$ is a Galois extension of group G.

55. *Clifford Algebras*: Let q be a quadratic form on a n-dimensional k-vector space, char $k \neq 2$, and let $\langle\ ,\ \rangle\colon V \times V \to k$ be the corresponding symmetric bilinear map. The **Clifford** algebra of q is the quotient of the tensor algebra $T^\bullet V$ by the two-sided ideal generated by the elements $v \otimes v - q(v)$, $v \in V$, and it is endowed with a canonical map $V \hookrightarrow C(q)$.

   (a) We have $\langle e, v\rangle = \frac{1}{2}(e \cdot v + v \cdot e)$, for all $e, v \in V$.

   (b) Let A be a k-algebra. For any k-linear map $f\colon V \to A$ such that $f(v)^2 = q(v)$, there exists a unique morphism of k-algebras $\phi\colon C(q) \to A$ such that $\phi(v) = f(v)$, $v \in V$.

   (c) $C(q)$ admits an involutive automorphism such that $v \mapsto -v$ for all $v \in V$, and an involutive anti-automorphism $a = \sum \lambda_{i_1\dots i_p} v_{i_1} \dots v_{i_p} \mapsto a^t = \sum \lambda_{i_1\dots i_p} v_{i_p} \dots v_{i_1}$.

   (d) The natural map $\Lambda^\bullet V \hookrightarrow T^\bullet V \to C(q)$ is a linear isomorphism; hence $\dim C(q) = 2^n$, and any element $a \in C(q)$ defines p-vectors $a_p \in \Lambda^p V$. (*Hint*: Consider an orthogonal basis.)

   (e) The metric may be extended to the Clifford algebra, $\langle a, b\rangle = (a^t b)_0$, and $\langle ax, y\rangle = \langle x, a^t y\rangle$.

56. Let A be the Clifford algebra of a Minkowski space V, of signature $(1,3)$. Consider the matrices

$$v_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ v_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\ ,\ v_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}\ ,\ v_3 = \begin{pmatrix} ij & 0 \\ 0 & -ij \end{pmatrix}$$

and conclude that $A = \mathrm{End}_{\mathbb{H}} E$, where $E = \mathbb{H} \oplus \mathbb{H}$ is a right $\mathbb{H}$-vector space, named space of **spinors** (recall that $\mathbb{H}$ denotes the quaternions). Prove the following statements:

   (a) The volume form Ω is $\Omega = v_0 v_1 v_2 v_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\Omega^2 = -1$.

   (b) If Rtr is the real part of the trace, then $(a)_0 = \frac{1}{2}\mathrm{Rtr}(a)$, and $\langle v, v'\rangle = (vv')_0 = \frac{1}{2}\mathrm{Rtr}(vv')$.

   (c) Up to a real factor, there exists a unique $\mathbb{H}$-bilinear map $h\colon E \times E \to \mathbb{H}$, in the sense that $h(e_1 q_1, e_2 q_2) = \bar{q}_1 h(e_1, e_2) q_2$, such that $(ae) \cdot e' = e \cdot (a^t e')$. Moreover, $e \cdot u = \overline{u \cdot e}$, and

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ e \cdot e' = h(e, e') = (\bar{q}_1, \bar{q}_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \bar{q}_2 q_1' + \bar{q}_1 q_2'.$$

   (d) The metric h defines a $\mathbb{H}$-linear isomorphism $\bar{E} = E^*$ of left $\mathbb{H}$-vector spaces, hence a $\mathbb{R}$-linear isomorphism $E \otimes_{\mathbb{H}} \bar{E} = E \otimes_{\mathbb{H}} E^* = \mathrm{End}_{\mathbb{H}} E = A = \Lambda^\bullet V$, and we have a $\mathbb{R}$-bilinear metric $H_\bullet\colon E \times E \to \Lambda^\bullet V$ such that $H_\bullet(eq, u) = H_\bullet(e, u\bar{q})$. Hence we have a $\Lambda^\bullet V$-valued quadratic form $Q_\bullet(e) = H_\bullet(e, e) = e \otimes (e\cdot)$. In particular, any spinor $e \in E$ defines a vector $Q(e) = Q_1(e) \in V$, so that any spinor field may be viewed as a tangent vector field, and

$$\begin{aligned} &H_\bullet(e, u) = H_\bullet(u, e)^t, \\ &Q_\bullet(eq) = |q|^2 Q_\bullet(e), \\ &Q_2(e) = Q_3(e) = 0, \\ &Q_1(e) = Q_1(\Omega e) = \tfrac{1}{2}[e \otimes (e\cdot) + (\Omega e) \otimes (\Omega e \cdot)], \\ &Q_0(e) = \tfrac{1}{2}(e \cdot e). \end{aligned}$$

   (e) If T is a $\mathbb{H}$-linear endomorphism and $T(e) = e\alpha$, then $T(eq) = (eq)(q^{-1}\alpha q)$, so that eq has proper value $q^{-1}\alpha q$. Now, the proper values of Ω are the quaternions ε of square -1, and the subspace of proper vectors $S_\varepsilon = \{(q, q\varepsilon)\}$ is a 2-dimensional vector space over the field $\mathbb{R} + \mathbb{R}\varepsilon$.

   (f) Every spinor in S_ε is isotropic for h, while $S_i \cdot S_{-i} \subseteq \mathbb{C}$ and $S_i \cdot S_i \subseteq \mathbb{C}j$.

   (g) We have $E = S_i \oplus S_{-i}$, and the elements of A of even degree preserve S_i and S_{-i}, while the elements of odd degree interchange them.

   (h) If $e = s + s'$ is the decomposition of a spinor as a sum of proper spinors of values i and $-i$, then

$$\begin{aligned} &Q_0(e) = \mathrm{Re}(s \cdot s'), \\ &Q_4(e) = \mathrm{Im}(s \cdot s')\Omega, \\ &|Q(e)|^2 = |s \cdot s'|^2 = Q_0(e)^2 + Q_4(e)^2, \quad \text{where we see } Q_4 \text{ with values in } \mathbb{R}. \end{aligned}$$

   (i) If $e \neq 0$, then $Q(e) \neq 0$ always points to the future (changing the sign of h if necessary), and the following conditions are equivalent:

i. $Q(e)$ is an isotropic vector.
ii. $Q_0(e) = Q_4(e) = 0$.
iii. $s \cdot s' = 0$.
iv. e is a proper spinor, $\Omega e = e\varepsilon$.

(j) A vector $v \in V$ is isotropic if and only if $\operatorname{Ker} v = \{e\colon ve = 0\} \neq 0$, and in such case $\operatorname{Ker} v = e\mathbb{H}$, where $Q(e) = v$. A spinor $e \in E$ is proper if and only if $\operatorname{Ann} e = \{v \in V\colon ve = 0\} \neq 0$, and in such case $\operatorname{Ann} e = \mathbb{R}Q(e)$. Conclude that the projective space of the $(\mathbb{R}+\mathbb{R}\varepsilon)$-vector space S_ε is just the space of future light directions.

Using the following calculations in coordinates (where we see a pure quaternion $xi + yj + zij$ as a vector $xv_1 + yv_2 + zv_3$), prove that two spinors e', e represent the same vector, $Q(e) = Q(e')$, if and only if $e' = (a + b\Omega)eq$, where $a^2 + b^2 = 1 = |q|$.

$$Q_\bullet(e) = Q(q_1, q_2) = (q_1e_1 + q_2e_2) \otimes (\bar{q}_1\omega_2 + \bar{q}_2\omega_1) = \begin{pmatrix} q_1\bar{q}_2 & q_1\bar{q}_1 \\ q_2\bar{q}_2 & q_2\bar{q}_1 \end{pmatrix} \in \operatorname{End}_{\mathbb{H}} E,$$
$$Q_0(e) = \tfrac{1}{2}\mathrm{Rtr} Q_\bullet(e) = \tfrac{1}{2}(q_1\bar{q}_2 + q_2\bar{q}_1) \in \mathbb{R},$$
$$Q_1(e) = \tfrac{1}{2}[Q_\bullet(e) + \Omega Q_\bullet(e)\Omega] = \tfrac{1}{2}[(|q_1|^2 + |q_2|^2)v_0 + (q_1\bar{q}_2 - q_2\bar{q}_1)] \in V,$$
$$Q_4(e) = Q_\bullet(e) - Q_0(e) - Q(e) = \tfrac{1}{2}(|q_1|^2 - |q_2|^2)\Omega \in \Lambda^4 V.$$

7. Commutative Algebra

1. Let A be a ring of dimension 0. Prove that A is a finite direct sum of local rings, $A = A_{x_1} \oplus \ldots \oplus A_{x_n}$, where $\operatorname{Spec} A = \{x_1, \ldots, x_n\}$.

2. *Boolean algebras*: A **Boolean algebra** is a set A endowed with two associative and commutative operations $\wedge, \vee$ satisfying the following axioms (compare with definition in p. 229, where the operations are denoted $+$ and $\cdot$ respectively):
   1. Existence of neutral elements: $a \wedge 0 = a$, $a \vee 1 = a$.
   2. Distributive: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
   3. Absorption: $a \vee 0 = 0$, $a \wedge 1 = 1$.
   4. Idempotence: $a \vee a = a$, $a \wedge a = a$.
   5. Complement: If $a \in A$, there is $\bar{a} \in A$ such that $\bar{a} \wedge a = 1$ and $\bar{a} \vee a = 0$.

The fundamental examples are the algebra $\mathcal{P}(X)$ of all subsets of a set X (where $\wedge$ is the intersection and $\vee$ is the union, so that $0 = X$, $1 = \emptyset$ and $\bar{a}$ is the complement of a) and the statements of any formalized logical theory including the logical operations *and*, *or*, *no* (identifying equivalent statements, so that 0 are tautologies). Now, the implication $a \Rightarrow b$ is defined to be $\bar{a} \vee b$ and the equivalence $a \Leftrightarrow b$ is defined to be $(\bar{a} \vee b) \wedge (a \vee \bar{b})$. Prove the following statements:

(a) Any boolean algebra A is a ring with the operations $a + b := (a \Leftrightarrow b)$, $a \cdot b := a \vee b$, and any element is idempotent, $a^2 = a$ (such rings are named **Boole rings**).
Conversely, any Boole ring $(A; +, \cdot)$ is a boolean algebra with the operations $a \wedge b := a + b + ab$, $a \vee b := a \cdot b$, and $\bar{a} := 1 + a$. In particular, $(a \Rightarrow b) = (1 + a)b$.

(b) Any finitely generated ideal of a Boole ring is a principal ideal.

(c) Any Boole ring A has characteristic 2 and, if it is a domain, then $A = \mathbb{F}_2$.

(d) Any prime ideal $\mathfrak{p}$ of a Boole ring is a maximal ideal of residue field $A/\mathfrak{p} = \mathbb{F}_2$.

(e) If A is a Boole ring, the points $x \in \operatorname{Spec} A$ correspond to the ring morphisms $v_x\colon A \to \mathbb{F}_2$ (the coherent interpretations of the logical theory as true=0 or false=1 statements),
$$\operatorname{Spec} A = \operatorname{Hom}(A, \mathbb{F}_2).$$

(f) Any element $f \in A$ defines a function $f\colon X = \operatorname{Spec} A \to \mathbb{F}_2$, $f(x) = v_x(f)$, vanishing just on $(f)_0$. If we consider $\mathbb{F}_2$ with the discrete topology, these functions are continuous. So we obtain any continuous function $X \to \mathbb{F}_2$, and we have a ring isomorphism $A = \mathcal{C}(X, \mathbb{F}_2)$.

(g) A is isomorphic to a ring of subsets of X, the ring of all closed open subsets.

Show that any ideal I of a Boole ring is formed by all the continuos functions vanishing on the closed set $(I)_0$; that is to say, given a family of axioms $B \subset A$, the ideal I generated by B is formed by all the statements f which are true in any coherent interpretation satisfying the given axioms: $v(f) = 0$ whenever $v(b) = 0, \forall b \in B$. At first, this fact does not mean that the statement f may be derived from the given axioms and tautologies (the null element 0) only using the *Modus Ponens*; such statements forming the least subset B', containing B and 0, such that

$$\text{If } a \in B' \text{ and } (a \Rightarrow b) \in B', \text{ then } b \in B'.$$

Prove that the ideal of A generated by B contains B' (*Hint*: $(a \Rightarrow b) = b + ab$.)

In fact B' is the ideal generated by B (if a statement is true in any coherent interpretation, it may be obtained from the axioms and tautologies using the *Modus Ponens*):

(a) If $a \in A$ and $b \in B'$, since $b \Rightarrow (a \vee b)$ is null, we have $ab = (a \vee b) \in B'$.

(b) If $a, b \in B'$, since $a \Rightarrow (b \Rightarrow (a \Leftrightarrow b))$ is null, we have that $b \Rightarrow (a \Leftrightarrow b)$ is in B', and we conclude that $a + b = (a \Leftrightarrow b) \in B'$.

3. Prove that any proper element of a noetherian domain is a product of irreducible elements.
4. If A is a noetherian ring, show that so is any localization A_S.
5. If A is a noetherian ring, prove that $(\operatorname{rad} A)^n = 0$ for some $n \in \mathbb{N}$.
6. Show that the ring $\prod_\infty \mathbb{R} = \mathcal{C}(\mathbb{N})$ is not noetherian.
7. Prove that any surjective endomorphism $f \colon M \to M$ of a noetherian A-module is an isomorphism. (*Hint*: Consider the submodules $\operatorname{Ker} f^n$.)
8. Let I be the annihilator of a noetherian A-module M. Show that there is an injective morphism $A/I \to M \oplus \ldots \oplus M$ and conclude that A/I is a noetherian ring.
9. Let A be a noetherian ring. Prove that $\operatorname{Hom}_A(N, \varinjlim M_i) = \varinjlim \operatorname{Hom}_A(N, M_i)$ whenever N is a finitely generated A-module. Conclude that inductive limits of injective A-modules are injective A-modules.
10. Let $\{U_{f_i}\}$ be a basic open cover of $\operatorname{Spec} A$. Prove that an A-module M is noetherian if and only if M_{f_i} is a noetherian A_{f_i}-module for any index i.
11. Prove that any noetherian ring A of dimension 0 has finite length, finite and discrete spectrum, $\operatorname{Spec} A = \{x_1, \ldots, x_n\}$, and that $A = A_{x_1} \oplus \ldots \oplus A_{x_n}$.
12. Let A be a noetherian ring. Prove that finite length A-modules are just noetherian A-modules M such that $\operatorname{supp} M$ is a finite set of closed points.
13. If M is a module over a noetherian ring A, show that $M_f = \varinjlim \operatorname{Hom}_A(f^n A, M)$.
14. Let $A \neq 0$ be a ring. If there is an injective A-linear morphism $A^r \to A^n$, prove that $r \leq n$. (*Hint*: It is obvious when A has finite length. If A is noetherian, localize at a minimal prime. In the general case, consider the noetherian subring $\mathbb{Z}[a_{ij}]$, where (a_{ij}) is the matrix of the morphism.)
15. Let I be an ideal of a noetherian ring A. If $\mathfrak{m} = \operatorname{rad} I$ is a maximal ideal, prove that I is a $\mathfrak{m}$-primary ideal; hence any proper ideal containing some power $\mathfrak{m}^n$ is a $\mathfrak{m}$-primary ideal.
16. Let $A = k[x, y]$, $\mathfrak{p} = (x)$, $\mathfrak{m} = (x, y)$. Show that $I = \mathfrak{m}^2 \cap \mathfrak{p}$ is not a primary ideal, and that it admits another reduced primary decomposition, $I = (x^2, y) \cap \mathfrak{p}$. (*Hint*: $xy \in I$, $x \notin I$, $y \notin \operatorname{rad} I = \mathfrak{p}$.)
17. Let A be a noetherian UFD. If $p \in A$ is irreducible, show that $\mathfrak{p} = pA$ is a prime ideal, that the ideals $p^n A$ are $\mathfrak{p}$-primary, and that these are the only $\mathfrak{p}$-primary ideals of A.
18. Let $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be a reduced primary decomposition of an ideal I of a noetherian ring A, and put $\mathfrak{p}_i = \operatorname{rad} \mathfrak{q}_i$. Once we fix an associated prime $\mathfrak{p}$, prove that

    (a) The intersection of the primary components with radical contained in $\mathfrak{p}$ is just $I_\mathfrak{p} \cap A$, so that it does not depend on the decomposition. Hence, the intersection of the primary components with radical strictly contained in $\mathfrak{p}$ neither depend on the decomposition.

(b) If $I = \mathfrak{q}'_1 \cap \ldots \cap \mathfrak{q}'_r$ is another reduced primary decomposition, $\mathfrak{p}_i = \operatorname{rad} \mathfrak{q}'_i$, then we also have $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{i-1} \cap \mathfrak{q}'_i \cap \mathfrak{q}_{i+1} \cap \ldots \cap \mathfrak{q}_r$. (*Hint*: When $\mathfrak{p}_1, \ldots, \mathfrak{p}_i$ are just the associated primes contained in $\mathfrak{p}_i$, then $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_i = \mathfrak{q}'_1 \cap \ldots \cap \mathfrak{q}'_{i-1} \cap \mathfrak{q}'_i = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{i-1} \cap \mathfrak{q}'_i$.)

(c) If $\mathfrak{q}''_1, \ldots, \mathfrak{q}''_r$ are primary ideals, $\mathfrak{p}_i = \operatorname{rad} \mathfrak{q}''_i$, each one appearing in some reduced primary decomposition of I, then $I = \mathfrak{q}''_1 \cap \ldots \cap \mathfrak{q}''_r$.

(d) If $\mathfrak{p}^n \subseteq \mathfrak{q}_i$ (so that $\mathfrak{p} = \mathfrak{p}_i$) and we put $\mathfrak{q}_i^{(n)} := (I + \mathfrak{p}^n)_{\mathfrak{p}} \cap A$, then $I = \mathfrak{q}_1 \cap \ldots \mathfrak{q}_i^{(n)} \ldots \cap \mathfrak{q}_r$.

(e) If n_i is the first exponent such that $\mathfrak{q}_i^{(n_i)}$ appears in some reduced primary decomposition of I, then $I = \mathfrak{q}_1^{(n_1)} \cap \ldots \cap \mathfrak{q}_r^{(n_r)}$.

19. Let $B = \oplus_{n \geq 0} B_n$ be a **graded ring**, $B_i \cdot B_j \subseteq B_{i+j}$, (in particular B_0 is a subring). An ideal $I \subseteq B$ is **homogeneous** when $I = \oplus_n (B_n \cap I)$. If B is a noetherian ring, prove that

(a) If $\mathfrak{p}$ is a prime ideal, then $\mathfrak{p}^h = \oplus_n (B_n \cap \mathfrak{p})$ is a prime ideal.

(b) Any minimal prime ideal is a homogeneous ideal.

(c) If $\mathfrak{q}$ is a primary ideal, then $\mathfrak{q}^h = \oplus_n (B_n \cap \mathfrak{q})$ is a primary ideal. (*Hint*: Assume $\mathfrak{q}^h = 0$.)

(d) Any homogeneous ideal I admits a homogeneous primary decomposition, and the associated primes of I are homogeneous prime ideals.

(e) If any homogeneous element of positive degree is a zero divisor, then some associated prime contains the **irrelevant ideal** $\oplus_{n \geq 1} B_n$.

20. Let us consider an action of a group G on a ring A; i.e., a group morphism $G \to \operatorname{Aut}_{\text{rings}}(A)$. We say that the ring of invariants $A^G = \{a \in A \colon g \cdot a = a, \forall g \in G\}$ is stable under base change when, for any ring morphism $A^G \to B$, we have that the natural morphism $B \to (A \otimes_{A^G} B)^G$ is an isomorphism. In such case, if A is a noetherian ring, prove that so is A^G. (*Hint*: If I is an ideal of A^G and we put $I' = A^G \cap IA$, show that $I = I'$ using the base change $A^G \to A^G/I$.)

Moreover, if A is a graded finitely generated algebra over a field k and G acts by homogeneous k-automorphisms, conclude that A^G is a finitely generated k-algebra (*Hint*: A^G is a graded k-algebra and the irrelevant ideal is finitely generated.)

When G is a finite group of automorphisms of a $\mathbb{Q}$-algebra, show that the algebra of invariants is stable under base change (*Hint*: The arithmetic mean $m \colon A \to A^G$, $m(a) = \frac{1}{|G|} \sum_g ga$.)

21. If $\widehat{A}$ is the I-adic completion of a noetherian ring A, prove that $\operatorname{Spec}_{\mathrm{m}} \widehat{A} = \operatorname{Spec}_{\mathrm{m}} (A/I)$.

22. Let M be a finitely generated module over a noetherian ring A. Prove that $M = 0$ if and only if M has null completion at any point of $\operatorname{Spec} A$.

23. Let $(\mathcal{O}, \mathfrak{m})$ be a noetherian local ring. If $t \in \mathfrak{m}$ and $\dim \mathcal{O}/t\mathcal{O} = \dim \mathcal{O} - 1$, prove that

$$\operatorname{mult}(\mathcal{O}/t\mathcal{O}) \geq \operatorname{mult} \mathcal{O}.$$

24. Let $I \subseteq \mathfrak{m}$ be an ideal of a noetherian local ring $(\mathcal{O}, \mathfrak{m})$. If $\bar{f} \in I^r/I^{r+1}$, $r \geq 1$, is not a zero-divisor in the graded ring $G_I\mathcal{O}$, prove that

(a) $f \in I^r$ is not a zero-divisor in $\mathcal{O}$.

(b) $fI^n = I^{r+n} \cap f\mathcal{O}$ for any $n \geq 0$.

(c) $G_{\bar{I}}\bar{\mathcal{O}} = (G_I\mathcal{O})/(\bar{f})$, where $\bar{I}$ is the image of I in $\bar{\mathcal{O}} = \mathcal{O}/f\mathcal{O}$.

25. If $P_2(x) \in \mathbb{R}[x]$ has degree 2 and no real root, prove that $\mathbb{R}[x]/(P_2^m) \simeq \mathbb{C}[t]/(t^m)$, and that the trace metric (p. 149) of this $\mathbb{R}$-algebra has rank 2 and index 1. (*Hint*: Use the proposition of p. 207, and then consider the base $1, i, t, it, \ldots, t^{m-1}, it^{m-1}$.)

Show that the trace metric of $\mathbb{R}[x]/(x-a)^m \simeq \mathbb{R}[t]/(t^m)$ has rank 1 and sign $+$.

Prove that the rank of the trace metric Tr in $\mathbb{R}[x]/(P)$ is the number of different roots of $P(x)$, and the index is half the number of imaginary roots. Conclude that *all the roots of $P(x)$ are real if and only if* Tr *is non-negative*, $\operatorname{Tr}(a,a) \geq 0$.

Show that the matrix of Tr in the base $1, x, \ldots, x^{n-1}$ of $\mathbb{R}[x]/(P)$, where $n = \deg P(x)$, is just (the sums σ_r of powers of roots are given by the Newton and Girard formulae)

$$\mathrm{Tr} = \begin{pmatrix} n & \sigma_1 & \dots & \sigma_{n-1} \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \\ \dots & \dots & \dots & \dots \\ \sigma_{n-1} & \sigma_n & \dots & \sigma_{2n-2} \end{pmatrix}.$$

26. Let $A = \mathbb{C}[x,y]/(y^2 - x)$ and let $C = \operatorname{Spec} A$ be the complex plane curve of equation $x = y^2$. Prove that the complement of the point $x = 1, y = 1$ is a basic open set U_f, and that the first projection $\pi \colon U_f \to \mathbb{A}_1$ (given by the morphism of $\mathbb{C}$-algebras $\mathbb{C}[x] \to A_f$, $x \mapsto x$) is not a finite morphism, even if it is a surjective closed and open map with finite discrete fibres.

27. Let G be a finite group of automorphisms of a finitely generated k-algebra $A = k[\xi_1, \dots, \xi_n]$. Prove that A^G is a finitely generated k-algebra. (*Hint*: Consider the subalgebra $B \subseteq A^G$ generated by the coefficients of the polynomials $P_i(x) = \prod_g (x - g\xi_i)$.)

28. Let $A = k[\xi_1 \dots, \xi_n]$ be a finitely generated k-algebra. When k is infinite, show that some morphism $k[\xi'_1 \dots \xi'_{n-1}, \xi_n] \hookrightarrow A$ is finite, where $\xi'_i = \xi_i - \lambda_i \xi_n$, $\lambda_i \in k$ (i.e. some linear projection $\operatorname{Spec} A \to \mathbb{A}_d$ is finite).

29. If A is a finitely generated k-algebra and $k \to L$ is an extension, prove that $\dim A = \dim A_L$.

30. If A, B are finitely generated k-algebras, prove that $\dim A \otimes_k B = \dim A + \dim B$.

31. Prove that any affine algebraic variety with a finite number of closed points has dimension 0.

32. Let Σ be the field of fractions of an integral finitely generated k-algebra A. Prove that the dimension of A is the **transcendence degree** of Σ over k (the maximal number of algebraically independent elements in Σ).

33. Let $f \colon X \to Y$ be a morphism between algebraic varieties. If C is an irreducible closed set in X, prove that $\dim C \geq \dim \overline{f(C)}$.

34. Let $X = \operatorname{Spec} A$ be an irreducible algebraic variety of dimension n. If $f \in A$ is not invertible nor nilpotent, show that any irreducible component of $(f)_0$ has dimension $n - 1$.

35. Let $X = \operatorname{Spec} A$ be an irreducible algebraic variety. Prove that all maximal chains of irreducible closed subsets have equal length. (*Hint*: If $X \supset Y \supset \dots$ is a maximal chain, take $f \in A$ vanishing on Y. Prove that Y is an irreducible component of $(f)_0$ and apply induction on $\dim X$.)

    Prove that all irrefinable chains of irreducible closed subsets $Y \supset \dots \supset Z$, with fixed extremes, have equal length, namely $\dim Y - \dim Z$.

    If x is a closed point of X, conclude that $\dim X = \dim A_x$.

36. Let Y_1, Y_2 be irreducible closed sets in an irreducible algebraic variety X. If Y is an irreducible component of $Y_1 \cap Y_2$, show that $\operatorname{codim} Y \leq \operatorname{codim} Y_1 + \operatorname{codim} Y_2$, where $\operatorname{codim} Z = \dim X - \dim Z$.

37. Let $f \colon X \to Y$ be a morphism between irreducible algebraic varieties. If a closed point $y \in Y$ is in the image of f, prove that $\dim f^{-1}(y) \geq \dim X - \dim Y$.

38. Let Y, Z be two closed sets in an affine algebraic variety $X = \operatorname{Spec} A$ over an algebraically closed field k. If all the rational points of Y are in Z, prove that $Y \subseteq Z$.

39. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ be affine algebraic varieties over an algebraically closed field k. If X and Y are reduced (resp. connected, irreducible, integral) prove that so is $X \times_k Y$.

    (*Hint*: Take a basis $\{h_i\}$ of B as a k-vector space and prove that if $\sum_i f_i \otimes h_i \in A \otimes_k B$ is nilpotent, then so are the functions $f_i \in A$. On the other hand, if $X \times_k Y = C_1 \cup C_2$, where C_1 and C_2 are closed sets, and Y_r stands for the topological subspace of all rational points of Y, prove that $Y_i = \{y \in Y_r \colon X \times y \subseteq C_i\}$ is a closed set in Y_r.)

40. Let k be a field and let $A = \oplus_n A_n = k[\xi_1, \dots, \xi_d]$ be a graded k-algebra, $\deg \xi_i = 1$. Prove that $\dim A = \dim A_{\mathfrak{m}}$, where $\mathfrak{m} = (\xi_1, \dots, \xi_d)$, and that $\dim A - 1$ coincides with the degree of the Hilbert polynomial $H(n) = \dim_k A_n$.

41. Prove that any maximal ideal of $k[x_1, \dots, x_n]$ is generated by n elements.

42. If a system of polynomial equations $p_1(x_1,\dots,x_n)=0,\dots,p_r(x_1,\dots,x_n)=0$ with real coefficients has no complex solution, prove that $1=\sum_i q_ip_i$ for some $q_i\in\mathbb{R}[x_1,\dots,x_n]$.

43. Prove that any noetherian valuation ring is a discrete valuation ring.

44. Prove that any Dedekind domain with a finite number of maximal ideals is a principal ideal domain.

45. Let a be a non null element of a Dedekind domain A. Prove that any ideal of A/aA is a principal ideal. (*Hint*: A/aA is a direct product of local rings, see ex. 1.)

    Conclude that any ideal I of a A is generated by two elements, the first one being an arbitrary non null element of I.

46. Even if $\mathbb{R}[x,y]/(x^2+y^2+1)$ is not a Euclidean ring (p. 493), prove that it is a principal ideal domain. (*Hint*: Any imaginary point $x=\alpha, y=\beta$ is in some real line $ax+by=c$.)

47. Let d be a square-free integer number. Prove that $A=\mathbb{Z}[\sqrt{d}]$ is a Dedekind domain when $d\equiv 2$ or 3 (mod. 4), and that $A=\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ is a Dedekind domain when $d\equiv 1$ (mod. 4).

    Put $d=-19$, so that $A\simeq\mathbb{Z}[x]/(x^2-x+5)$. According to a fundamental result of number theory, any element of the Picard group $\mathrm{Pic}(A)$ is represented by some ideal I such that $|A/I|$ is bounded above by the square root $\sqrt{19}$ of the absolute value of the discriminant. Assuming this result, conclude that A is a principal ideal domain (even if it is not a Euclidean ring, p. 493).

48. Prove that the ring $A=\mathbb{R}[\cos t,\sin t]$ of trigonometric polynomials (see p. 496) is a Dedekind domain and $\mathrm{Pic}(A)=\mathbb{Z}/2\mathbb{Z}$, so that a non null ideal $\mathfrak{m}_1^{n_1}\dots\mathfrak{m}_r^{n_r}$ is a principal ideal if and only if the degree $\sum_i n_i[A/\mathfrak{m}_i:\mathbb{R}]$ of the corresponding divisor is even.

49. Let $(\mathcal{O},\mathfrak{m})$ be the local ring of an integral plane curve at a rational point x of multiplicity m. If $\mathcal{O}\to\mathcal{O}_1$ is the quadratic transformation at x, show that $\mathfrak{m}^{m-1}=\mathfrak{m}^{m-1}\mathcal{O}_1$.

50. Let A be a noetherian domain of dimension 1. Prove that $l(A/fhA)=l(A/fA)+l(A/hA)$ whenever $f,h\in A$ are nonzero.

51. Let C_1,C_2 be two plane curves over a field k of equations $P_1(x,y)=0$, $P_2(x,y)=0$. If they have no common irreducible component, prove the following statements:

    (a) $\dim_k k[x,y]/(P_1,P_2)=\sum\limits_{z\in C_1\cap C_2}(C_1\cap C_2)_z\cdot\deg z$.

    (b) Let us introduce the homogenization $\tilde{Q}(x_0,x_1,x_2)=x_0^dQ(\frac{x_1}{x_0},\frac{x_2}{x_0})$ of any polynomial $Q(x,y)$ of degree d. We say that the curves C_1,C_2 do not intersect at infinity when $x_0=0$, $x_1=0$, $x_2=0$ is the unique solution (in arbitrary extensions of k) of the system

    $$\left.\begin{aligned} x_0&=0\\ \tilde{P}_1(x_0,x_1,x_2)&=0\\ \tilde{P}_2(x_0,x_1,x_2)&=0\end{aligned}\right\}$$

    We put $k[x_0,x_1,x_2]=\oplus H_n$, $A=k[x_0,x_1,x_2]/(\tilde{P}_1,\tilde{P}_2)=\oplus_nA_n$, and we have exact sequences (where $\phi(\tilde{Q})=(\tilde{P}_2\tilde{Q},-\tilde{P}_1\tilde{Q})$, $\psi(\tilde{Q}_1,\tilde{Q}_2)=\tilde{P}_1\tilde{Q}_1+\tilde{P}_2\tilde{Q}_2$)

    $$0\longrightarrow H_{n-d_1-d_2}\xrightarrow{\ \phi\ }H_{n-d_2}\oplus H_{n-d_1}\xrightarrow{\ \psi\ }H_n\xrightarrow{\ \pi\ }A_n\longrightarrow 0.$$

    (c) $\dim_kA_n=\binom{n+2}{2}-\binom{n-d_1+2}{2}-\binom{n-d_2+2}{2}+\binom{n-d_1-d_2+2}{2}=d_1d_2$, when $n\geq d_1+d_2$.

    (d) We have an isomorphism (the union is considered in the ring A_{x_0})

    $$k[x,y]/(P_1,P_2)=\bigcup_nA_nx_0^{-n},\ \ Q(x,y)\mapsto Q\left(\frac{x_1}{x_0},\frac{x_2}{x_0}\right).$$

    (*Hint*: If $Q(\frac{x_1}{x_0},\frac{x_2}{x_0})=0$, then $\tilde{Q}(x_0,x_1,x_2)=x_0^dQ(\frac{x_1}{x_0},\frac{x_2}{x_0})=0$ in A_{x_0}; hence $x_0^r\tilde{Q}(x_0,x_1,x_2)$ is null in A, so that $x_0^r\tilde{Q}(x_0,x_1,x_2)=\tilde{Q}_1(x_0,x_1,x_2)\tilde{P}_1(x_0,x_1,x_2)+\tilde{Q}_2(x_0,x_1,x_2)\tilde{P}_2(x_0,x_1,x_2)$ and $Q(x,y)=\tilde{Q}_1(1,x,y)P_1+\tilde{Q}_2(1,x,y)P_2$.)

    (e) $\dim_k k[x,y]/(P_1,P_2)=\dim_k(A_nx_0^{-n})\leq\dim_kA_n=d_1d_2,\ n\gg 0$.

(f) If the curves do not intersect at infinity, then the radical of $k[x_0,x_1,x_2]/(x_0,\tilde{P}_1,\tilde{P}_2)$ is the maximal ideal (x_0,x_1,x_2). It follows that A_n is null in $k[x_0,x_1,x_2]/(x_0,\tilde{P}_1,\tilde{P}_2) = A/x_0A$ when $n \gg 0$, and we have epimorphisms $x_0\colon A_n \to A_{n+1}$; hence isomorphisms.

No element of A_n is annihilated by a power of x_0, and we have isomorphisms $A_n \to A_n x_0^{-n}$, $n \gg 0$, and $\dim_k k[x,y]/(P_1,P_2) = \dim_k A_n = d_1d_2$.

Obtain **Bézout's theorem**: *The number of common points, counted with degree and multiplicity, is* $\leq d_1d_2$ (and the equality holds when both curves do not intersect at infinity)

$$\sum_{z\in C_1\cap C_2} (C_1\cap C_2)_z \cdot \deg z \;\leq\; d_1d_2.$$

52. A singular point of a curve is a **cuspidal** point if it has a unique analytical branch. Let us consider a cuspidal point x of multiplicity m over an algebraically closed field. After a finite number h of quadratic transformations, the multiplicity will decrease, $m' < m$. Prove that the intersection number with a simple curve R is $(C\cap R)_x = km$ if both curves separate after $k \leq h$ quadratic transformations; but, if they do not separate after h quadratic transformations, then they intersect transversally and $(C\cap R)_x = hm + m'$. Show that there is a maximal contact $hm + m'$ with simple curves.

53. Let $C = \operatorname{Spec} A$ be a curve and let $\mathfrak{m} \subset A$ the maximal ideal of a point $x \in C$. Fix $f \in \mathfrak{m} = (f_1,\ldots,f_d)$ of minimal value at any discrete valuation centered at x, and without zeros at any other singular point of C. The functions f_j/f may be not integral over A, since f may have other zeros $z_1,\ldots,z_r$ on C; but take $g \in A$ not vanishing at any singular point of C, and vanishing at $z_1,\ldots,z_r$ with equal multiplicity than f. Prove that the functions f_jg/f are integral over A,

$$A \longrightarrow A_1 = A\left[\tfrac{f_1g}{f},\ldots,\tfrac{f_dg}{f}\right] \subseteq \bar{A}$$

If $A = A_1$, show that x is a simple point. (*Hint*: If $f_jg \in fA$, then $\mathfrak{m}A_x = fA_x$ since $g \in A_x^*$.)

54. If M is a finitely generated module over a domain A, prove the existence of a non empty basic open set U_f such that M_f is a free A_f-module.

55. Let M be a finitely generated A-module. If the natural morphism $I\otimes_A M \to M$ is injective for any finitely generated ideal $I \subset A$, show that M is a flat A-module.

56. Prove that any A-module is an inductive limit of finitely presented A-modules. (*Hint*: Consider the set of all finite families $m_1,\ldots,m_n \in M$ and relations $(a_{11},\ldots,a_{1n}),\ldots,(a_{r1},\ldots,a_{rn}) \in A^n$.)

    Conclude that $\operatorname{Hom}_A(M,-)$ preserves inductive limits if and only if M is finitely presented.

57. Prove that a ring morphism $A \to B$ is faithfully flat if and only if it is a flat morphism and the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

58. Let M be a finitely generated A-module. If any point $x \in \operatorname{Spec} A$ has a basic neighborhood U_f such that M_f is a free A_f-module, prove that M is a projective A-module.

59. If $\operatorname{Spec} A = \bigcup_i U_i$ is a finite basic open cover, prove that $A \to \oplus_i A_{U_i}$ is faithfully flat.

60. Show that any finite flat morphism $A \hookrightarrow B$ is faithfully flat. Hence, if A is a Dedekind domain and B is integral, any finite injective morphism $A \to B$ is faithfully flat.

61. If $\mathcal{O}$ is a local noetherian ring, prove that $\mathcal{O} \to \widehat{\mathcal{O}}$ is a faithfully flat morphism.

8. Topology

1. Let X be a topological space and $B \subseteq A(X)$ a base of the topology of X. Given a subspace $Y \subseteq X$, prove that $B|_Y := \{f|_Y : f \in B\} \subseteq A(Y)$ is a base of the topology of Y. Moreover, when $Y = z(f)$ is the zero-set of some function $f \in B$, show that we have a natural isomorphism $B/fB = B|_Y$.

2. Let I be an ideal of a semiring A and $\pi\colon A \to A/I$ the canonical projection. If J is an ideal of A, prove that $\pi^{-1}(\pi(J)) = I + J$.

3. Let Y be a subspace of a topological space X. Show that the subset $S \subseteq A(X)$ of all continuous functions $f\colon X \to \mathbb{K}$ without zeroes in Y is a multiplicative system of $A(X)$ and that $A(X)_S$ is just the semiring of germs of continuous functions along Y.

4. Let B be a base of open sets of a topological space X and $U \in B$. Prove that $S = \{1, U\}$ is a multiplicative system of B and B_S is isomorphic to the basis of U induced by B.

5. Prove that any ideal I of a semiring A is a multiplicative system of the dual semiring A^*, and that $(A^*)_I = (A/I)^*$. If $\mathfrak{p}$ is a prime ideal of A, conclude that $(A_\mathfrak{p})^* = A^*/(A - \mathfrak{p})$.

6. Show that an open subset U of X is not a zero divisor in $A(X)$ if and only if U is dense.

7. Prove that any prime ideal of a semiring contains a minimal prime ideal, so that the intersection of the minimal prime ideals is null.

   Conclude that the set of zero divisors is just the union of all minimal prime ideals.

8. Let A be a semiring. If $\operatorname{Spec} A = (f)_0 \amalg (h)_0$, prove that $A = A_f \times A_h = (A/hA) \times (A/fA)$.

9. Let $A \to B$ be a semiring morphism and $\mathfrak{p}$ a prime ideal of A. Prove that the fibre of the map $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ over the point defined by $\mathfrak{p}$ is $\operatorname{Spec} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, where $B_\mathfrak{p} := B_{f(A-\mathfrak{p})}$.

10. If $f\colon A \to B$ is an injective semiring morphism, show that $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

    (*Hint*: The morphism $A_\mathfrak{p} \to B_\mathfrak{p}$ is injective and, if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}B = B$ is absurd.)

    If $f\colon A \to B$ is a morphism, show that the image of $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is dense in $(\operatorname{Ker} f)_0$.

    (*Hint*: If $\operatorname{Ker} f = 0$, then any minimal prime ideal of A is in the image of f^*.)

11. Let $\mathcal{P}(X)$ be the lattice of subsets of a set X (i.e. the semiring of all functions $X \to \mathbb{K}$). Show that points $x \in X$ correspond to principal prime ideals $\mathfrak{p}_x = \{f \in \mathcal{P}(X)\colon f(x) = 0\}$ of $\mathcal{P}(X)$.

    Conclude that any semiring isomorphism $\mathcal{P}(X) \to \mathcal{P}(Y)$ is induced by a bijection $Y \to X$.

12. Show that the semiring $\mathbb{K}[x] := \{0, x, 1\}$, with the obvious operations $(1 + x = 1,\ x^2 = x)$ has the universal property of a "ring of polynomials"

    $$\operatorname{Hom}_{\text{srings}}(\mathbb{K}[x], A) = A,$$

    and conclude that $\operatorname{Spec} \mathbb{K}[x] = \mathbb{K}$.

    Show that the "affine space" $\mathbb{K}^n$ has the usual universal property $\operatorname{Hom}_{\text{top}}(X, \mathbb{K}^n) = A_X^n$, and the open subset $\mathbb{K}^n - \{0\}$ represents the functor

    $$\operatorname{Hom}_{\text{top}}(X, \mathbb{K}^n - \{0\}) = \{(f_1, \ldots, f_n) \in A_X^n\colon f_1 + \ldots + f_n = 1\} = \left\{ \begin{matrix} \text{open covers} \\ X = U_1 \cup \ldots \cup U_n \end{matrix} \right\}.$$

13. Describe the elements of the finite semiring $\mathbb{K}[x_1, \ldots, x_n] := \mathbb{K}[x] \otimes \overset{n}{\ldots} \otimes \mathbb{K}[x]$ and the natural homeomorphism $\mathbb{K}^n = \operatorname{Spec} \mathbb{K}[x_1, \ldots, x_n]$.

14. Given semiring morphisms $A \to B$, $A \to C$, define a semiring $B \otimes_A C$ with the universal property,

    $$\operatorname{Hom}_A(B \otimes_A C, D) = \operatorname{Hom}_A(B, D) \times \operatorname{Hom}_A(C, D),$$

    so that $\operatorname{Spec} B \otimes_A C = (\operatorname{Spec} B) \times_{\operatorname{Spec} A} (\operatorname{Spec} C)$.

15. Let A be a semiring and $X := \operatorname{Spec} A$, so that $f \in A$ defines a continuous function $f\colon X \to \mathbb{K}$, just vanishing on $(f)_0$. For any open set $U \subseteq X$, consider the semiring $\mathcal{O}_X(U)$ of all functions $U \to \mathbb{K}$ locally coinciding with a function defined by A. Prove the following statements:

    (a) So we obtain a sheaf of $\mathbb{K}$-valued functions, in the sense that, given any open cover $U = \bigcup_i U_i$, a function $f\colon U \to \mathbb{K}$ is in $\mathcal{O}_X(U)$ if and only if the restriction $f|_{U_i} \in \mathcal{O}_X(U_i)$ for any index i.

    (b) The natural map $A_f \to \mathcal{O}_X(U_f)$ is an isomorphism for any basic open set U_f, and $A = \mathcal{O}_X(X)$.

    (c) The map $\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ induced by a semiring morphism $A \to B$ is a morphism of semiringed spaces, in the sense that ϕ is continuous and $\phi^*(f) := f \circ \phi \in \mathcal{O}_{\operatorname{Spec} B}(\phi^{-1}U)$ for any $f \in \mathcal{O}_{\operatorname{Spec} A}(U)$.

    (d) Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is induced by a unique semiring morphism $A \to B$.

(e) We have a natural bijection $\mathrm{Hom}_{\mathrm{top}}(T, \operatorname{Spec} A) = \mathrm{Hom}_{\mathrm{srings}}(A, A_T)$ for any topological space T.

(f) Any continuous map $T \to \operatorname{Spec} A$ admits a unique extension to a morphism $\operatorname{Spec} A_T \to \operatorname{Spec} A$.

16. Let $I = [a, b]$ be a closed interval in the real line and let B be the base of all finite unions of points and closed subintervals. Let B_x be the localization of B at the maximal ideal $\mathfrak{m}_x$ of all closed sets containing a given point $x \in I$. If $a < x < b$, show that $B_x = \{0, 1, x, x^+, x^-\}$ (where x, x^+, x^- are the germs of x, $[x, b]$ and $[a, x]$ respectively) and that B_x has a unique maximal ideal (x) and two prime ideals (x^+) and (x^-).

    If $x = a$ or $x = b$, show that B_x has a maximal ideal and a prime ideal.

17. Show that any T_0 topological space is a dense subspace of a projective limit of finite topological spaces.

18. Let $B \subset A(\mathbb{R})$ be the base defined by the continuous functions $f \colon \mathbb{R} \to \mathbb{K}$ with compact support (and the constant function 1). Show that $\mathfrak{m} = \{f \in B \colon f \neq 1\}$ is the unique maximal ideal of B, so that $\mathfrak{p}_x = \{f \in B \colon f(x) = 0\}$ is not a maximal ideal, $\forall x \in \mathbb{R}$.

19. Let $B \subseteq A(X)$ be a base of a T_1 topological space X, and $x \in X$. Show that the prime ideal $\mathfrak{p}_x = \{f \in B \colon f(x) = 0\}$ is a maximal ideal if and only if for any function $h \in B$ such that $h(x) \neq 0$ there is some function $f \in \mathfrak{p}_x$ such that $f + h = 1$.

20. Prove that inductive limits and tensor products of boolean algebras are boolean algebras.

21. If some base of the topology of a T_0 space X is a Boolean algebra, show that X is **totally disconnected** (any connected subset is a point), and if $A(X)$ is a Boolean algebra, show that X is discrete.

22. If $Y \to X$ is a closed continuous map, prove that $\operatorname{Spec} A(Y) \to \operatorname{Spec} A(X)$ also is a closed map.

23. Prove that a topological space X is T_1 if and only if $X \subseteq \operatorname{Spec}_{\mathrm{m}} A(X)$.

24. Prove that any base of a compact separated space is a normal semiring.

25. Let B be a base of a locally compact space X. Prove that $X \cap \operatorname{Spec}_{\mathrm{m}} B$ is open in $\operatorname{Spec}_{\mathrm{m}} B$.

    Conclude that a normal T_1-space X is locally compact if and only if X is open in $\operatorname{Spec}_{\mathrm{m}} A(X)$.

26. Let A be a ring. If M is a noetherian A-module, prove that $\operatorname{supp} M$ is a noetherian subspace of $\operatorname{Spec} A$.

27. A topological space X is said to be **sober** if any irreducible closed subset is the closure of a unique point. Prove that any projective limit of noetherian sober spaces is a compact space.

28. Prove that semirings with homeomorphic spectra are isomorphic. Moreover, a semiring morphism $f \colon A \to B$ is an isomorphism if and only if the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism.

29. If A is a ring, prove that $\operatorname{Spec} A = \operatorname{Spec} B$ for some semiring B.

30. If X is a finite order, and X^* is the dual order, show that $|X|$ and $|X^*|$ are homeomorphic.

31. Let A be a ring. If any prime ideal of A is contained in a unique maximal ideal, prove that $\operatorname{Spec}_{\mathrm{m}} A$ is a separated space. (*Hint*: $(A_x)_y = 0$ when x, y are different closed points.)

    If $\operatorname{Spec}_{\mathrm{m}} A$ is a separated space, and the intersection of all maximal ideals of A is 0, prove that any prime ideal of A is contained in a unique maximal ideal.

32. Let A be a semiring such that the intersection of all maximal ideals is the zero ideal. If $\operatorname{Spec}_{\mathrm{m}} A$ is a separated space, prove that A is normal.

33. Prove that the spectrum of any ring of null dimension is a Stone space. If k is a field and X is a Stone space, show the existence of a k-algebra A of dimension 0 such that $X = \operatorname{Spec} A$.

    (*Hint*: X is a projective limite of discrete finite spaces, spectra of trivial finite k-algebras.)

34. Let X be a locally compact non-compact separated space. Prove that compact sets and complements of relatively compact open sets form a base B of closed sets of X, such that the complements of the relatively compact open sets form a maximal ideal, and $X \to X^* = \operatorname{Spec}_{\mathrm{m}} B$ is the **one point compactification** of X. Moreover, for any separated compactification $X \to K$ there is a unique continuous extension $f \colon K \to X^*$.

35. Prove that $\operatorname{Spec}_{\mathbb{R}} \mathcal{C}^*(X) = \operatorname{Spec}_{\mathrm{m}} \mathcal{C}^*(X)$.

36. Show that any non-empty open set in $\operatorname{Spec}\mathcal{C}^*(X)$ intersects $\operatorname{Spec}_{\mathrm{m}}\mathcal{C}^*(X)$.

Prove that any two disjoint closed sets in $\operatorname{Spec}_{\mathrm{m}}\mathcal{C}^*(X)$ have disjoint neighborhoods in $\operatorname{Spec}\mathcal{C}^*(X)$.

Prove that any prime ideal of $\mathcal{C}^*(X)$ is contained in a unique maximal ideal, and that the retraction $\operatorname{Spec}\mathcal{C}^*(X) \to \operatorname{Spec}_{\mathrm{m}}\mathcal{C}^*(X)$ so obtained is continuous.

37. Show that any real maximal ideal of $\mathcal{C}(\mathbb{R})$ is the ideal of all continuous functions vanishing at a certain point $a \in \mathbb{R}$. Conclude the existence of non-real maximal ideals in $\mathcal{C}(\mathbb{R})$.

38. Prove that any two disjoint closed sets in $\operatorname{Spec}_{\mathrm{m}}\mathcal{C}(X)$ have disjoint neighborhoods in $\operatorname{Spec}\mathcal{C}(X)$. (*Hint*: If $f_1 + f_2 = 1$ for two non-negative functions, and $h := f_1 - f_2$, $h_1 := |h| - h$, $h_2 := |h| + h$, then we have $h_1h_2 = 0$, $f_1 + h_1 > 0$ and $f_2 + h_2 > 0$.)

Prove that any prime ideal of $\mathcal{C}(X)$ is contained in a unique maximal ideal, and that the retraction $\operatorname{Spec}\mathcal{C}(X) \to \operatorname{Spec}_{\mathrm{m}}\mathcal{C}(X)$ so obtained is continuous.

39. Prove that any projective limit of finite non-empty sets is non-empty.

40. Let X be a metrizable space without isolated points. Considering a sequence (x_n) with a unique limit point x, show that we have a subspace $X_1 = \{x, x_1, x_2, \dots\}$ such that $\dim A(X_1) = 1$.

Now X_1 is the set of limit points of a sequence (y_n), and we put $X_2 = X_1 \cup \{y_1, y_2, \dots\}$. Show that $\dim A(X_2) = 2$, and so on. Conclude that $\dim A(X) = \infty$.

41. Let $\mathcal{C}(X)$ be the ring of real continuous functions on a metrizable space X. Prove that

   (a) If $S \subset \mathcal{C}(X)$ is the multiplicative system of all continuous functions without zeros on a given open subset $U \subseteq X$, we have a canonical ring isomorphism $\mathcal{C}(X)_S = \mathcal{C}(U)$.

   (b) If $\mathfrak{m} \subset \mathcal{C}(X)$ is the maximal ideal of all continuous functions vanishing at a given point $x \in X$, then $\mathcal{C}(X)_{\mathfrak{m}}$ is isomorphic to the ring $\mathcal{O}$ of germs at x of continuous functions. Hence, the localization morphism $\mathcal{C}(X) \to \mathcal{C}(X)_{\mathfrak{m}}$ is surjective, and the kernel is the ideal $\mathfrak{n}$ of all continuous functions with null germ at x.

   (c) $\mathfrak{m} = \mathfrak{m}^2$.

   (d) If x is not an isolated point, then $\mathfrak{m}\mathcal{O}$ is not a finitely generated ideal, so that $\mathcal{O}$ and $\mathcal{C}(X)$ are not noetherian rings. Moreover $\mathfrak{n} \neq \mathfrak{m}$, and $\mathcal{O}_x$ is a finitely generated flat $\mathcal{C}(X)$-module, but not a projective $\mathcal{C}(X)$-module.

   (e) Let $\mathfrak{a} \subseteq \mathcal{C}(X)$ be an ideal. The $\mathcal{C}(X)$-module $\mathcal{C}(X)/\mathfrak{a}$ is flat if and only if $\mathfrak{a} = \bigcap_{y \in Y} \mathfrak{n}_y$ for some closed subset $Y \subseteq X$. (*Hint*: Take $Y = (\mathfrak{a})_0 \cap X$. If $\mathcal{C}(X)/\mathfrak{a}$ is flat, then $\mathfrak{a}\mathcal{O}_x = 0$ when $x \in Y$, while $\mathfrak{a}\mathcal{O}_x = \mathcal{O}_x$ when $x \notin Y$.)

   (f) Any $\mathbb{R}$-derivation $D\colon \mathcal{C}(X) \to \mathcal{C}(X)/\mathfrak{m} = \mathbb{R}$ is null.

   (g) Any $\mathbb{R}$-derivation $D\colon \mathcal{C}(X) \to \mathcal{C}(X)$ is null.

   (h) If a germ $f \in \mathcal{O}$ is not constant, there is a prime ideal $\mathfrak{p} \subset \mathcal{O}$ such that $\bar{f} \in \kappa(\mathfrak{p}) = (\mathcal{O}/\mathfrak{p})_{\mathfrak{p}}$ is transcendent over $\mathbb{R}$, and there is a $\mathbb{R}$-derivation $D\colon \mathcal{O} \to \kappa(\mathfrak{p})$ such that $Df \neq 0$.

   (i) The kernel of the differential $\mathrm{d}\colon \mathcal{O} \longrightarrow \Omega_{\mathcal{O}/\mathbb{R}}$ is defined by the constant germs.

42. Let $\beta X_{\mathrm{dis}} = \operatorname{Spec}_{\mathrm{m}}\mathcal{P}(X) = \operatorname{Spec}\mathcal{P}(X)$ be the space of ultrafilters in a set X (where $\mathcal{P}(X)$ is considered as the semiring of closed sets in the discrete topology, so that $+ = \cap$ and $\cdot = \cup$), and let B a base of closed sets of a topology on X. Show that

   (a) Any ultrafilter $\mathcal{M}$ defines a prime ideal $\mathfrak{p} := B \cap \mathcal{M}$ in B and so we obtain a continuous surjective map $\operatorname{Spec}\mathcal{P}(X) \to \operatorname{Spec} B$.

   (b) An ultrafilter $\mathcal{M}$ converges to a point $x \in X$ if and only if $\mathfrak{p} \subseteq \mathfrak{p}_x := \{b \in B\colon x \in b\}$.

   (c) Let $I \subset B$ be a proper ideal. If the filter $I\mathcal{P}(X)$ converges to $x \in X$, then $I \subseteq \mathfrak{p}_x$. (*Hint*: If $b \notin \mathfrak{p}_x$, then the complement of b contains some $a \in I$; hence $a + b = 1$, and $b \notin I$.)

   (d) Let $\mathfrak{m} \subset B$ be a maximal ideal. Then $\mathfrak{m}\mathcal{P}_X$ converges to $x \in X$ if and only if $\mathfrak{m} = \mathfrak{p}_x$.

43. Show that any group G with a topology such that the product $G \times G \to G$ and the inverse $G \to G$ are continuous (a **topological group**) is a uniform space, when a base of entourages is defined by the sets $\tilde{U} = \{(x, y) \in G \times G\colon x^{-1}y \in U\}$, where U runs over the neighborhoods of 1 in G.

44. Prove that a Cauchy filter $\mathcal{F}$ in a separated uniform space X can not converge to two points $x \neq y$.

45. Let X be a uniform space. Show that the filter of neighborhoods $\mathcal{N}_x$ of a point $x \in X$ (hence any convergent filter) is a Cauchy filter. (*Hint*: $V[x] \times V[x] \subseteq U$ when V is symmetric and $V^2 \subseteq U$.)

    If a Cauchy ideal $I \subset A_X$ has a zero $x \in X$, prove that the Cauchy filter $I\mathcal{P}_X$ converges to x. Conclude that a point $x \in X$ is a zero of a Cauchy ideal $I \subset A_X$ if and only if $I\mathcal{P}_X$ converges to x.

    (*Hint*: If $A \in I$ is $\mathcal{E}$-small, then $x \in A$, so that $A \subseteq \mathcal{E}[x]$, and $\mathcal{E}[x] \in I\mathcal{P}_X$.)

46. Put $S_1 = \{z \in \mathbb{C}\colon |z| = 1\}$, and $X = S_1 - \{1\}$. Let I_+ (resp. I_-) be the ideal of A_X of all closed sets in X containing some arc $\{e^{ti};\ t \in (0, \varepsilon)\}$, $\varepsilon > 0$ (resp. $\varepsilon < 0$), and let $\mathfrak{m}_i$ be a maximal ideal of A_X containing I_i. Prove that $\mathfrak{m}_+$ and $\mathfrak{m}_-$ are different Cauchy maximal ideals containing a common $\mathcal{E}$-small set for every entourage $\mathcal{E}$ of X. Conclude that the topology of the space $\tilde{X}$ of all Cauchy maximal ideals of A_X is not the Zariski topology.

47. Let $\bar{E}$ be the completion of a topological vector space (E, p). Show that the addition, $E \times E \to E$ and the products $\lambda \times E \to E$, $\lambda \in \mathbb{K}$, are uniformly continuous, so that they extend to the completion, and the maps $\bar{E} \times \bar{E} \to \bar{E}$, $\mathbb{K} \times \bar{E} \to \bar{E}$, define a structure of topological vector space on $\bar{E}$.

48. A **net** in a topological space X is a map $x_\bullet\colon I \to X$, where I is a filtered ordered set, and we put $x_i := x_\bullet(i)$. Given an index $i \in I$, the subset $C_i = \{x_j\}_{i \leq j} \subseteq X$ is said to be a **queue** of the net. A net $x_\bullet$ **converges** to a **limit point** $x \in X$ when any neighborhood of x contains a queue, and x is an **adherent point** when for any neighborhood U of x and any index $i \in I$ there is some index $j \geq i$ such that $x_j \in U$. Prove that

    (a) The limit point of a net (if it exists) is unique when X is separated.

    (b) A subset $A \subseteq X$ is closed if and only if A contains every limit point (in X) of any net $x_\bullet\colon I \to A$. (*Hint*: If $x \in \overline{A}$, consider the filtered set $I = \{U\}$ of neighborhoods of x, and a net $x_\bullet$ such that $x_U \in U \cap A$, so that a limit point of $x_\bullet$ is x.)

    (c) A map $f\colon X \to Y$ between topological spaces is continuous if and only if it preserves limit points, in the sense that $f(\lim x_\bullet) \subseteq \lim(f(x_\bullet))$ for any net $x_\bullet$ in X, where $f(x_\bullet) := x_\bullet \circ f$.

    (d) A topological space X is compact if and only if any net $x_\bullet\colon I \to X$ has an adherent point.

    (*Hint*: If X admits an open cover $\{U_j\}_{j \in J}$ with no finite subcover, consider the filtered set I of all finite subsets of J and, for any index $i = \{j_1, \ldots, j_r\} \in I$ fix a point $x_i \notin U_{j_1} \cup \ldots \cup U_{j_r}$. If $x_\bullet$ has an adherent point $x \in X$, then $x \in U_j$ for some $j \in J$, and there is an index $i = \{j, \ldots, j_r\} \geq \{j\}$ such that $x_i \in U_j$; absurd since $x_i \notin U_j \cup \ldots \cup U_{j_r}$.)

    Now, if X is a uniform space X, we put $|y - x| < \mathcal{E}$ (where $\mathcal{E}$ is an entourage) when $(x, y) \in \mathcal{E}$ (so that if $|y - x| < n\mathcal{E}$ and $|z - y| < m\mathcal{E}$, then $|z - x| < (n + m)\mathcal{E}$); we say that a net $x_\bullet$ in X is a **Cauchy** net when for any entourage $\mathcal{E}$ there is some index i such that $|x_k - x_j| < \mathcal{E}$ for all $j, k \geq i$ (i.e. $C_i \times C_i \subseteq \mathcal{E}$); that X is complete when it is separated and any Cauchy net converges to a point (obviously unique); and that two Cauchy nets $x_\bullet\colon I \to X$, $y_\bullet\colon I' \to X$ are equivalent when for any entourage $\mathcal{E}$ there are indices $i \in I, i' \in I'$ such that $|y_{j'} - x_j| < \mathcal{E}$ for all $j' \geq i', j \geq i$. Prove that

    (a) Any convergent net is a Cauchy net.

    (b) Any uniformly continuous map $f\colon X \to Y$ preserves Cauchy nets: if $x_\bullet$ is a Cauchy net in X, then $f(x_\bullet)$ is a Cauchy net in Y.

    (c) Any adherent point of a Cauchy net is a limit point.

    (d) Any compact separated space is complete.

    (e) Any closed subspace of a complete space also is complete.

    (f) Any complete subspace of a separated uniform space is a closed subspace.

    (g) Any two equivalent Cauchy nets have equal limit points.

    Now let $\{\mathcal{E}_\gamma\}_{\gamma \in \Gamma}$ be a base of symmetric entourages of X, so that Γ is a filtered set with the order $\alpha < \beta \Leftrightarrow \mathcal{E}_\alpha \supseteq \mathcal{E}_\beta$. Prove that

(a) If A is a dense subspace of X, then any Cauchy net $x_\bullet\colon I \to X$ is equivalent to some Cauchy net $a_\bullet\colon \Gamma \to A$. (*Hint*: For any $\gamma \in \Gamma$ fix an index $i \in I$ such that $|x_k - x_j| < \mathcal{E}_\gamma$ for all $k, j \geq i$, and choose a point $a_\gamma \in A \cap \mathcal{E}_\gamma[x_i]$, so that $|x_k - a_\gamma| < 2\mathcal{E}_\gamma$ for all $k \geq i$.)

(b) Equivalence classes of Cauchy nets in X define a set.

(c) In a metric space X, any Cauchy net $x_\bullet\colon I \to X$ is equivalent to a Cauchy sequence $a_\bullet\colon \mathbb{N} \to X$.

(d) Let A be a dense subspace of a separated uniform space X. If any Cauchy net in A has a limit point in X, then X is complete.

Now consider the set $\widehat{X}$ of equivalence classes of Cauchy nets in X, the canonical map $i\colon X \to \widehat{X}$ (any point of X defines a net in X with indices in a one-element set) and, for any entourage $\mathcal{E}$ of X, the set $\widehat{\mathcal{E}} := \{([x_\bullet], [x'_\bullet]) \in \widehat{X} \times \widehat{X} \colon C \times C' \subseteq \mathcal{E}$ for some queues C, C' of $x_\bullet, x'_\bullet\}$. Prove that

(a) The sets $\widehat{\mathcal{E}}$ define a base of a uniformity in $\widehat{X}$.

(b) If X is separated, then $\widehat{X}$ is complete and $i\colon X \to \widehat{X}$ is a uniform map with dense image, defining a homeomorphism of X onto $i(X)$.

(c) If A is a subspace of a complete space X, then its closure is just $\overline{A} = \widehat{A}$.

(d) Let X, Y be separated uniform spaces. Any uniform map $f\colon X \to Y$ admits a unique uniform extension $\widehat{f}\colon \widehat{X} \to \widehat{Y}$. Hence, when Y is complete, any uniform map $f\colon X \to Y$ uniquely factors through a uniform map $\widehat{f}\colon \widehat{X} \to Y$.

(e) Conclude that $\widehat{X}$ coincides with the completion of X defined in p. 247.

49. Show that any locally compact space X is a ***k*-space** (the inductive limit of its compact subspaces).

If any point of a topological space X has a countable base of neighborhoods, show that X is a k-space.

(*Hint*: If $Y \cap K$ is a closed set in K for any compact set $K \subseteq X$ and $y_n \to x$, then $K = \{x, y_1, y_2, \ldots\}$ is compact, so that $x \in Y$.)

50. Let X be a topological space, and Y a uniform space. If $K \subseteq X$ is a compact set and $\mathcal{E}$ is an entourage in Y, let $W(K, \mathcal{E})$ be the set of pairs (f, h) in $\mathrm{Hom}_{\mathrm{Sets}}(X, Y)$ such that $\big(f(x), h(x)\big) \in \mathcal{E}$, $\forall x \in K$. Prove that the sets $W(K, \mathcal{E})$ form a base of entourages for a uniformity, the **compact-convergence** uniformity: $\mathrm{Hom}_{\mathrm{Sets}}(X, Y)_c$. When we consider X with the discrete topology, we obtain the **weak** or pointwise convergence uniformity. Show that $\mathrm{Hom}_{\mathrm{Sets}}(X, Y)_w = \prod_X Y$, with the product uniformity. When we consider X with the trivial topology, we obtain $\mathrm{Hom}_{\mathrm{Sets}}(X, Y)_u$, the set of all maps with the uniformity of **uniform convergence** on X.

A family $\mathcal{F} \subseteq \mathrm{Hom}_{\mathrm{Sets}}(X, Y)$ is **equicontinuous** if, given a entourage $\mathcal{E}$ of Y, any point $p \in X$ has a neighborhood U_p in X such that $\big(f(x), f(p)\big) \in \mathcal{E}$, $\forall x \in U_p$, $f \in \mathcal{F}$. Then prove that the closure $\bar{\mathcal{F}}$ in $\mathrm{Hom}_{\mathrm{Sets}}(X, Y)_w$ also is equicontinuous, and that the compact and weak convergence induce the same uniformity on $\mathcal{F}$. Conclude that $\bar{\mathcal{F}}$ also is the closure of $\mathcal{F}$ in $\mathcal{C}(X, Y)_c$.

Prove **Ascoli's Theorem**: If $\mathcal{F} \subseteq \mathrm{Hom}_{\mathrm{Sets}}(X, Y)$ is equicontinuous and $\mathcal{F}(x) = \{f(x);\ f \in \mathcal{F}\}$ is precompact (resp. has compact closure in Y) for any point $x \in X$, then $\mathcal{F}$ is precompact (resp. has compact closure) in $\mathcal{C}(X, Y)_c$.

If K is a compact separated space, prove that any compact family $\mathcal{F} \subset \mathcal{C}(K, Y)$ is equicontinuous.

(*Hint*: Given a point $p \in K$ and an entourage $\mathcal{E}$ in Y, there is a finite cover of $\mathcal{F}$ by sets $W(K_i, \mathcal{E})$, where K_i are compact neighborhoods of p.)

Prove that $\mathcal{F} \subseteq \mathrm{Hom}_{\mathrm{Sets}}(X, Y)$ is equicontinuous if and only if the map $\delta\colon X \to \mathrm{Hom}_{\mathrm{Sets}}(\mathcal{F}, Y)_u$, $\delta_x(f) = f(x)$, is continuous. When X is a k-space, conclude that $\mathcal{F}$ is equicontinuous if and only if so is the restriction $\mathcal{F}|_K$ to any compact set $K \subseteq X$.

If X is a separated k-space, prove the converse of Ascoli's theorem: If a family $\mathcal{F} \subseteq \mathcal{C}(X, Y)_c$ has compact closure (resp. is precompact) then $\mathcal{F}$ is equicontinuous and $\mathcal{F}(x)$ has compact closure (resp. is precompact) in Y, $\forall x \in X$.

51. Let X be a topological space, and $\mathcal{C}(X, Y)$ the set of continuous maps to a complete uniform space Y, with the compact-convergence uniformity. Prove that $\mathcal{C}(X, Y)$ is complete when X is compact; hence also when X is a k-space.

52. Let S be a smooth manifold. If $\pi\colon X \to S$ is a covering, prove the existence of a unique structure of smooth manifold on X such that π is a local diffeomorphism.

53. Prove Grothendieck's characterization of the forgetful functor G-**Sets** $\rightsquigarrow$ **Sets** on the category of G-sets (follow the proof of ex. 112c in p. 509):

    *Let $F\colon \mathbf{C} \rightsquigarrow$ **Sets** be a representable covariant functor, $F(Y) = \mathrm{Hom}_{\mathbf{C}}(X,Y)$. Then the group $G := \mathrm{Aut}_{\mathbf{C}}(X)^{\mathrm{op}}$ naturally acts on the sets $F(Y)$, and the induced functor $\mathbf{C} \rightsquigarrow G$-**Sets** is an equivalence of categories when the following conditions hold:*

    (a) *The category $\mathbf{C}$ admits coproducts and quotients by groups, and F preserves them.*

    (b) *The functor F is conservative.*

    (c) *Any endomorphism of F is an automorphism*: $G = F(X)$.

    Obtain another proof of the Galois theorem for coverings of a connected and locally connected space.

54. Let S be a connected and locally connected topological space. If G is a group of automorphisms of a connected covering $P \to S$ and $P/G = S$, show that it is a Galois covering of group G.

55. Let S be a locally connected space with a countable base of open sets. If $\pi\colon X \to S$ is a connected covering, prove that S admits a countable base $\{U_n\}$ of connected open sets where π is trivial, and that the family $\mathcal{B}$ of connected components of the open sets $\pi^{-1}(U_n)$ is a base of the topology of X.

    If $V \in \mathcal{B}$, show that $\{V' \in \mathcal{B}\colon V' \cap V \neq \emptyset\}$ is countable. (*Hint*: Otherwise, V intersects an uncountable family of disjoint open sets of some $\pi^{-1}(U_n)$.)

    Fix an open set $V_0 \in \mathcal{B}$, and prove that the family $\mathcal{B}'$ of open sets $V \in \mathcal{B}$ such that there is a finite sequence $V_0, V_1, \ldots, V_n = V$ with $V_{i-1} \cap V_i \neq \emptyset$ for any $i = 1, \ldots, n$ is countable.

    Prove that $\bigcup_{V \in \mathcal{B}'} V$ is a closed open set, so that $X = \bigcup_{V \in \mathcal{B}'} V$ and $\mathcal{B}' = \mathcal{B}$.

    Conclude that X admits a countable base of open sets and π has countable fibres.

56. Show that any continuous map $f\colon [0,1] \to [0,1]$ has a fixed point.

57. Prove that any continuous map $f\colon S_n \to \mathbb{R}$, $n \geq 1$, coincides on two antipodal points.

58. Prove that any non-surjective continuous map $f\colon S_2 \to S_2$ has a fixed point.

59. Prove that any 3×3 matrix with positive coefficients has a proper vector with positive coordinates (and positive proper value).

60. Let $S_2 = U_1 \cup U_2 \cup U_3$ be an open cover of a sphere. Prove that some open set U_i contains two antipodal points. (*Hint*: Consider a subordinated partition of unity (f_1, f_2, f_3), so that the image of the map $(f_1, f_2, f_3)\colon S_2 \to \mathbb{R}^3$ is in the plane $x + y + z = 1$.)

    Let $S_2 = Y_1 \cup Y_2 \cup Y_3$ be a closed cover of a sphere. Prove that some closed set Y_i contains two antipodal points. (*Hint*: Consider the map $(f_1, f_2)\colon S_2 \to \mathbb{R}^2$, $f_i(x) = \mathrm{d}(x, Y_i)$.)

61. Given two bounded measurable sets A_1, A_2 in a plane, prove the existence of a line dividing both in two sets of equal area. (*Hint*: Parametrize the half-planes P by a sphere S_2, and consider the map $(f_1, f_2)\colon S_2 \to \mathbb{R}^2$, $f_i(P) =$ area of $A_i \cap P$.)

62. Show that the fundamental group of a 1-dimensional connected polyhedron with v vertices and w wedges is a free[6] group on $w - v + 1$ elements.

    Conclude that any subgroup of a free group L_n of index m is a free group on $nm - m + 1$ elements.

63. Show that the fundamental group of a torus τ_g of genus g is a free group on $2g$ generators.

64. Let X, Y be topological manifolds of dimension $n \geq 3$. Show that the fundamental group of the connected sum $X \# Y$ is just the coproduct $\pi_1(X,p) * \pi_1(Y,q)$.

    Conclude the existence of a connected compact manifold X of dimension n with fundamental group a free group on a given finite number of generators.

[6]The free group L_n on n elements is the coproduct, in the category of groups, of n infinite cyclic groups.

9. Analysis III

1. Prove that a smooth manifold X is compact if and only if $X = \operatorname{Spec}_{\mathrm{m}} \mathcal{C}^\infty(X)$.

2. Let X be a smooth manifold and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{C}^\infty(X)$ defined by a point $p \in X$. Prove that $\mathcal{C}^\infty(X)_{\mathfrak{m}}$ is isomorphic to the ring of germs at p of smooth functions.

3. Let $\mathcal{O}$ be the local ring of germs at $x = 0$ of smooth functions on $\mathbb{R}$. If I is the ideal of $\mathcal{O}$ defined by the germs with null Taylor expansion at $x = 0$, prove that $xI = I$, and conclude that $\mathcal{O}$ is not a noetherian ring. Conclude that $\mathcal{C}^\infty(\mathbb{R}^n)$, $n \geq 1$, is not noetherian.

4. Prove that the ideal $\mathfrak{n}$ of $\mathcal{C}^\infty(\mathbb{R})$, defined by the smooth functions with null germ at $x = 0$, is not finitely generated.

5. Let $U \subseteq \mathbb{R}^n$ be an open set and let $p = (a_1, \dots, a_n) \in U$. Prove the following statements:

   (a) The diagonal ideal $\Delta = \{f(x,y) \in \mathcal{C}^\infty(U \times U) \colon f(x,x) = 0\}$ is $\Delta = (y_1 - x_1, \dots, y_n - x_n)$.

   (b) The maximal ideal $\mathfrak{m}_p = \{f(x) \in \mathcal{C}^\infty(U) \colon f(p) = 0\}$ of p is $\mathfrak{m}_p = (x_1 - a_1, \dots, x_n - a_n)$.

   (*Hint*: Using a partition of unity, you may assume that U is convex.)

6. If X is a topological space, prove that no element of $\mathcal{C}(X)$ is irreducible. (*Hint*: For any real-valued function f we have $f = f_+ - f_- = (\sqrt{f_+} + \sqrt{f_-})(\sqrt{f_+} - \sqrt{f_-})$, where $f_+ = \max(f, 0)$.)

   If $\mathcal{O}$ is the ring of germs at the origin p of smooth functions on $\mathbb{R}^2$, show that the coordinate function x_1 is irreducible; while the ideal (x_1) is not prime. (*Hint*: $\mathrm{d}_p x_1 \neq 0$.)

7. Show that in $\mathbb{R}^n$ there is a sequence of functions $\rho_m \geq 0$ of class $\mathcal{C}^\infty$ with compact support, such that $\int_{\mathbb{R}^n} \rho_m(x)\mathrm{d}x = 1$ for any index m, and, given $\varepsilon > 0$, we have $\operatorname{supp} \rho_m \subseteq B_\varepsilon(0)$ when $m \gg 0$. Conclude that any continuous function f on $\mathbb{R}^n$ with compact support is a uniform limit of functions $\rho_m * f$ of class $\mathcal{C}^\infty$. Moreover, given $\varepsilon > 0$, we have $\operatorname{supp}(\rho_m * f) \subseteq (\operatorname{supp} f) + B_\varepsilon(0)$ when $m \gg 0$.

8. Let D be a vector field on a smooth manifold X. Given a topological space Λ, and continuous maps $a \colon \Lambda \to X$, $t_0 \colon \Lambda \to \mathbb{R}$, prove the existence of an open set $U \subseteq \Lambda \times \mathbb{R}$ and a continuous family of solutions $\varphi_\lambda(t) \colon U \to X$ such that $\varphi_\lambda(t_0(\lambda)) = a(\lambda)$. Moreover, if Λ is a smooth manifold, then φ is of class $\mathcal{C}^m$ when so are $a(\lambda)$, and $t_0(\lambda)$. (*Hint*: Put $\varphi_\lambda(t) = \tau(t - t_0(\lambda), a(\lambda))$, where τ is the flow.)

9. Prove that any vector field $\sum_i f_i \partial_i$ on $\mathbb{R}^n$ with bounded components f_i is complete.

   If D is a vector field on $\mathbb{R}^n$, show that fD is complete for some positive function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

   If D is a vector field on a smooth closed submanifold $X \subset \mathbb{R}^n$, conclude that fD is complete for some positive function $f \in \mathcal{C}^\infty(X)$.

10. Let X be a smooth manifold. Show that any vector field D on an open set $U \subset X$ is $D = D^*/h$, where D^* is a vector field on X (vanishing on $X - U$) and $h \geq 0$ has no zeros on U. (*Hint*: If K is a compact set contained in a coordinate neighborhood, we have the seminorms $\| \sum_i f_i \partial_i \|_{K,r} = \sum_i \|f_i\|_{K,r}$ on the module of vector fields on X, so that we may repeat the proof of p. 261).

    When X is compact, prove that there is a positive smooth function $h \in \mathcal{C}^\infty(U)$ such that hD is complete. (*Hint*: D^* is complete, and integral curves of points of U do not reach $X - U$).

11. If $\tau_t(x_1, \dots, x_n) = (f_1(t,x), \dots, f_n(t,x))$ is a local uniparametric group in $\mathbb{R}^n$, show that the infinitesimal generator is $D = (\partial_t f_1|_{t=0})\,\partial_{x_1} + \ldots + (\partial_t f_n|_{t=0})\,\partial_{x_n}$.

    Obtain the infinitesimal generator of the most common uniparametric groups of $\mathbb{R}^2$:

Translations	$\tau_t(x,y) = (x + at, y + bt)$	$D = a\partial_x + b\partial_y$
Rotations	$\tau_t(x,y) = \begin{pmatrix} \cos\alpha t & -\sin\alpha t \\ \sin\alpha t & \cos\alpha t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	$D = \alpha(-y\partial_x + x\partial_y)$
Homotheties	$\tau_t(x,y) = (e^{at}x, e^{at}y)$	$D = a(x\partial_x + y\partial_y)$
Linear maps	$\tau_t X = e^{At} X$	$D = \sum_i (\sum_j a_{ij} x_j)\partial_i$

12. Let X be a totally ordered set, and consider the **order topology** generated by the open intervals

$$(a,b) := \{x \in X : a < x < b\} \ , \ (a,\infty) := \{x \in X : a < x\} \ , \ (-\infty, b) := \{x \in X : x < b\}.$$

Prove that the open intervals define a base of the order topology.

Let us consider the first uncountable ordinal ω_1, and the lexicographical order on the **long ray** $X = \omega_1 \times [0,1) = \{\lambda_i;\ \lambda \in [0,1),\ i \in \omega_1\}$. Prove the following statements:

(a) X does not admit a countable base of open sets. (*Hint*: Find an uncountable disjoint family of open sets).

(b) If $0 < i \in \omega_1$ then there is an order-preserving homeomorphism $[0_0, 0_i) \simeq [0,1)$. (*Hint*: Let $i \in \omega_1$ be the first index where the result fails. If $i = j+1$, then $[0_0, 0_i) = [0_0, 0_j) \cup [0_j, 0_{j+1}) = [0,1) \cup [1,2) = [0,2)$. Otherwise i is the limit of a sequence $0 = i_0 < i_1 < i_2 < \ldots$, then show that $[0_{i_n}, 0_{i_{n+1}}) = [n, n+1)$ and conclude that $[0_0, 0_i) = [0, \infty)$. Absurd).

(c) For any $b \in X$ there is an order-preserving homeomorphism $[0_0, b) = [0,1)$, and for any $a, b \in X$, $a < b$, there are order-preserving homeomorphisms $(a,b) = (0,1)$ and $[a,b] = [0,1]$.

(d) X is connected and path-connected.

(e) The **long line** $L = X - \{0_0\}$ is a connected separated topological manifold, but not σ-compact.

13. Let X be a connected and separated smooth manifold (eventually not σ-compact). If there is a smooth vector field D on X (a tangent vector D_x at each point $x \in X$ such that $(Df)(x) := D_x f$ is a smooth function for any $f \in \mathcal{C}^\infty(U)$) without zeroes, prove that X is σ-compact. (*Hint*: Show that the maximal integral curves of D define open subsets of X).

Prove that any connected and separated riemannian manifold (X, g) is in fact σ-compact. (*Hint*: Show that the maximal geodesic lines define open subsets of X).

14. Prove that $i_{[D,D']} = [D^L, i_{D'}]$ on p-forms: $i_{[D,D']}\omega_p = D^L(i_{D'}\omega_p) - i_{D'}(D^L\omega_p)$.

15. Let $\tau_t, \bar{\tau}_t$ be the flows of two vector fields $D, \bar{D}$ on a smooth manifold X and let $p \in X$. Prove that the curve $\gamma\colon I \to X$, $\gamma(t) = \bar{\tau}_{-t}\tau_{-t}\bar{\tau}_t\tau_t(p)$, is well-defined on an open interval $I = (-\varepsilon, \varepsilon)$, and that, for any smooth function f on X,we have $\lim\limits_{t\to 0} \frac{f\left(\gamma(t)\right) - f(p)}{t^2} = [D, \bar{D}]_p f$.

16. Let X be a smooth manifold. Prove that any homogeneous derivation of the algebra $\Omega^\bullet(X)$ of differential forms is a Lie derivative D^L for a unique vector field D on X.

17. Show that the vector fields D on $\mathbb{R}^3$ such that $D^L(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) = 0$ are spanned by the vector fields ∂_x, ∂_y, ∂_z, $x\partial_y - y\partial_x$, $x\partial_z - z\partial_x$ and $y\partial_z - z\partial_y$.

18. Let $\omega_1, \ldots, \omega_r$ be 1-forms on a smooth manifold X, linearly independent at any point. Show that a p-form Ω_p belongs to the ideal of $\Lambda^\bullet(X)$ spanned by $\omega_1, \ldots, \omega_r$ if and only if $\Omega_p \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0$.

19. Let ω_2 be a closed 2-form. If the radical $\{D \in \mathcal{T} : i_D\omega_2 = 0\}$ is a distribution (i.e., it has constant rank), show that it is integrable.

20. *First Order Partial Differential Equations*: Given a smooth function $F(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$, a classical solution of the partial differential equation $F = 0$ is a smooth function $z(x_1, \ldots, x_n)$ such that $F(x_1, \ldots, x_n, z, \partial_1 z, \ldots, \partial_n z) = 0$. Let us study the problem on germs at a fixed point of $\mathbb{R}^{2n+1}$.

Consider the 1-form $\omega = \mathrm{d}z - p_1\mathrm{d}x_1 - \ldots - p_n\mathrm{d}x_n$, and let us call (generalized) solution any germ of submanifold of dimension n where F and ω specialize to 0. Prove that classical solutions are just solutions where $x_1, \ldots, x_n$ define a local coordinate system.

Now assume that $\mathrm{d}F$ and ω are linearly independent, and consider the Pfaff system $\mathcal{P} = \langle \omega, \mathrm{d}F\rangle$. Show that the characteristic system of $\mathcal{P}$ is generated by the following vector field (so that $\mathcal{P}$ is projectable to the subring B of first integrals of D_F):

$$D_F = \sum_{i=1}^n \frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i} - \sum_{i=1}^n \left(p_i\frac{\partial F}{\partial z} + \frac{\partial F}{\partial x_i}\right)\frac{\partial}{\partial p_i} + \left(\sum_{i=1}^n p_i \frac{\partial F}{\partial p_i}\right)\frac{\partial}{\partial z}.$$

If we are at the origin of $\mathbb{R}^{2n+1}$, the hypersurface $x_n = 0$ is not tangent to D_F (i.e. $\frac{\partial F}{\partial p_n}(0) \neq 0$) and we choose $X_1, \ldots, X_{n-1}, Z, P_1, \ldots, P_n \in B$ specializing at $x_n = 0$ into the coordinates, prove that $\langle \omega, \mathrm{d}F\rangle = \langle \mathrm{d}Z - P_1\mathrm{d}X_1 - \ldots - P_{n-1}\mathrm{d}X_{n-1}, \mathrm{d}F\rangle$. (*Hint*: $F \in B$ and the proof of the projection theorem).

Conclude that the following equations (whenever they define a germ of submanifold of dimension n)

$$\begin{cases} F = 0, \\ Z = f(X_1, \ldots, X_{n-1}), \\ P_i = (\partial_i f)(X_1, \ldots, X_{n-1}), \quad i = 1, \ldots, n-1 \end{cases}$$

define a solution coinciding on $x_n = 0$ with a given smooth function $f(x_1, \ldots, x_{n-1})$. Prove that other solutions of the equation $F = 0$ are defined by the equations ($\lambda_1, \ldots, \lambda_n \in \mathbb{R}$)

$$\begin{cases} F = 0 \\ Z = \lambda_n \\ X_i = \lambda_i \quad i = 1, \ldots, n-1 \end{cases}$$

Lagrange-Charpit Method: When $n = 2$, let G be a first integral of the characteristic field D_F, and assume that $\mathcal{Q} = \langle \omega, \mathrm{d}F, \mathrm{d}G\rangle$ is a Pfaff system of rank 3. Prove that

(a) $D_G F = 0$, so that D_F and D_G are in the characteristic system of $\mathcal{Q}$. (*Hint*: $D_F G = -D_G F$).

(b) D_F and D_G are linearly independent. (*Hint*: So are $\omega, \mathrm{d}F, \mathrm{d}G$ and $i_{D_F}\mathrm{d}\omega = \mathrm{d}F - (\partial_z F)\omega$, $i_{D_G}\mathrm{d}\omega = \mathrm{d}G - (\partial_z G)\omega$).

(c) $\mathcal{Q}^{\mathrm{o}} = \langle D_F, D_G\rangle$ and $\mathcal{Q} = \langle \mathrm{d}F, \mathrm{d}G, \mathrm{d}H\rangle$, so that $\omega = u\mathrm{d}H + v\mathrm{d}F + w\mathrm{d}G$.

(d) The manifolds $F = 0, G = \lambda_1, H = \lambda_2$, are solutions of the considered equation.

Jacobi's Method: In the case of an equation $F(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$ without the unknown z, show that a classical solution $z(x_1, \ldots, x_n)$ defines a germ of submanifold $p_i = (\partial_i z)(x_1, \ldots, x_n)$ of $\mathbb{R}^{2n}$ of dimension n, where $\omega = \sum_i p_i \mathrm{d}x_i$ specializes to $\mathrm{d}z$, so that F and $\mathrm{d}\omega$ specialize to 0. If we call (generalized) solution any germ of submanifold of dimension n where F and $\mathrm{d}\omega$ specialize to 0, prove that classical solutions are just solutions where $x_1, \ldots, x_n$ define a local coordinate system.

Show that $(\mathbb{R}^{2n}, \omega_2 := \mathrm{d}\omega)$ is a symplectic manifold, so that for any function $f \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ we have a vector field D_f such that $\mathrm{d}f = i_{D_f}\omega_2$. Prove that $[D_f, D_g] = D_{\{f,g\}}$, where the **Poisson bracket** is defined to be $\{f, g\} := \omega_2(D_f, D_g) = D_f g = -D_g f$. (*Hint*: Show that $D_f^L \omega_2 = 0$; hence $(i_{[D_f,D_g]}\omega_2)(D) = \omega_2([D_f, D_g], D) = D_f(\omega_2(D_g, D)) - \omega_2(D_g, [D_f, D]) = D_f(Dg) - [D_f, D]g = D(D_f g) = D\{f, g\} = (\mathrm{d}\{f, g\})(D)$).

Put $F_1 = F$, and assume that $\mathrm{d}F_1$ does not vanish at the considered point. Show that there is a first integral F_2 of D_{F_1} such that $\mathrm{d}F_1, \mathrm{d}F_2$ are linearly independent. Then prove that the distribution $\langle D_{F_1}, D_{F_2}\rangle$ is involutive (*Hint*: $\{F_1, F_2\} = D_{F_1}F_2 = 0$), and show that there is a common first integral F_3 such that $\mathrm{d}F_1$, $\mathrm{d}F_2$, $\mathrm{d}F_3$ are linearly independent. Conclude that so we obtain germs $F_1, \ldots, F_n$, with linearly independent differentials, such that $0 = D_{F_i}F_j = \omega_2(D_{F_i}, D_{F_j})$.

Show that $D_{F_1}, \ldots, D_{F_n}$ define a distribution, totally isotropic for ω_2, whose incident is generated by $\mathrm{d}F_1, \ldots, \mathrm{d}F_n$. Conclude that the submanifolds $F = 0, F_2 = \lambda_2, \ldots, F_n = \lambda_n$ are solutions of the considered equation $F = 0$.

21. Let X be an oriented smooth manifold of dimension n and let ω be a differential $(n+1)$-form on $\mathbb{R} \times X$. Use Fubini's theorem to prove that

$$\int_{\mathbb{R}\times X} \omega = \int_{\mathbb{R}} \left(\int_{t\times X} i_{\partial_t}\omega \right) \mathrm{d}t.$$

22. **Gauss-Green Formula:** If $\Omega \subset \mathbb{R}^2$ is a compact region bounded by a curve C, show that

$$\iint_\Omega \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \int_C (f\mathrm{d}x + g\mathrm{d}y).$$

23. In $\mathbb{R}^2$, show that the area of a polygon with vertices $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n), v_{n+1} = v_1$ is just the absolute value of

$$\frac{1}{2}\sum_{i=1}^n x_i y_{i+1} y_i - x_{i+1} y_i = \frac{1}{2}\sum_{i=1}^n \begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}.$$

(*Hint*: The integral of $y\mathrm{d}x$ on the segment with end points (x_1, y_1) and (x_2, y_2) is $\frac{1}{2}(x_2 - x_1)(y_2 + y_1)$.)

24. If ω and θ are closed forms, prove that so is $\omega \wedge \theta$. If moreover θ is exact, so is $\omega \wedge \theta$.

25. Let $H = \sum_i x_i \partial_i$ be the vector field on $\mathbb{R}^n$ defined by the flow $\tau_t(x_1, \dots, x_n) = (e^t x_1, \dots, e^t x_n)$. Prove the following equalities for any p-form ω on $\mathbb{R}^n$:

$$\tau_t^*(H^L \omega) = \frac{\partial}{\partial t}(\tau_t^* \omega),$$

$$\omega = \int_{-\infty}^0 \frac{\partial}{\partial t}(\tau_t^* \omega)\mathrm{d}t = \int_{-\infty}^0 \tau_t^*(H^L \omega)\mathrm{d}t = H^L \left(\int_{-\infty}^0 (\tau_t^* \omega)\mathrm{d}t \right),$$

and conclude that the Lie derivative $H^L \colon \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^n)$ is surjective.

If $\mathrm{d}\omega = 0$, show that $\omega = H^L \tilde{\omega}$, where $\mathrm{d}\tilde{\omega} = 0$, so obtaining another proof of **Poincaré's lemma**:

$$\omega = H^L \tilde{\omega} = \mathrm{d} i_H \tilde{\omega} + i_H \mathrm{d}\tilde{\omega} = \mathrm{d}(i_H \tilde{\omega}).$$

26. Prove that $H^1_{DR}(S_2) = 0$.

Prove that $H^1_{DR}(S_1) = \mathbb{R}$. (*Hint*: Consider the differential $\mathrm{d}\theta$ of the argument).

27. If n is even, show that any vector field D on an open neighborhood of the sphere S_n is normal at some point of the sphere, $D_x \in (T_x S_n)^\perp$.

Given a smooth map $f \colon S_n \to S_n$, show that $f(x) = \pm x$ at some point $x \in S_n$.

Prove that there is no Pfaff system on rank 1 on S_n.

Given a Pfaff system P of rank 1 on an open neighborhood of S_n, show that P_x is normal to the sphere at some point $x \in S_n$.

Conclude that any smooth map $\mathbb{P}_{n,\mathbb{R}} \to \mathbb{P}_{n,\mathbb{R}}$ has a fixed point.

28. Let us consider an integrable Pfaff system $\mathcal{P} = \langle \omega \rangle$ on a smooth manifold X. Prove that

(a) We have $\mathrm{d}\omega = \omega \wedge \omega_1$ for some 1-form ω_1 on X.

(b) The 3-form $\omega_3 = \omega_1 \wedge \mathrm{d}\omega_1$ is closed.

(c) The cohomology class $[\omega_3] \in H^3_{DR}(X)$ only depends on the Pfaff system $\mathcal{P}$, not on ω_1 nor the generator ω.

29. *Incompressible Stationary Fluids*: Let us consider an incompressible stationary perfect fluid in the Euclidean space $\mathbb{R}^3$; i.e., the density ρ is constant, and the pressure $p(x,y,z)$ and mean velocity $\vec{v} = v_1(x,y,z)\partial_x + v_2(x,y,z)\partial_y + v_3(x,y,z)\partial_z$ don't depend on time and satisfy the equations

$$\begin{cases} \operatorname{div} \vec{v} = 0 & \text{(Mass conservation law)} \\ \vec{v}^{\nabla} \vec{v} = -\operatorname{grad}(p/\rho) & \text{(Law of motion)} \end{cases}$$

If we put $v^2 = |\vec{v}|^2$, and $\omega_{\vec{v}}$ is the 1-form corresponding to $\vec{v}$, prove that

(a) The 1-form corresponding to $\vec{v}^{\nabla}\vec{v}$ is just $i_{\vec{v}}\,\mathrm{d}\omega_{\vec{v}} + \mathrm{d}\big(\frac{v^2}{2}\big)$.

(b) The law of motion may be restated as $i_{\vec{v}}\,\mathrm{d}\omega_{\vec{v}} = -\mathrm{d}\big(\frac{p}{\rho} + \frac{v^2}{2}\big)$.

(c) The flow of $\vec{v}$ preserves $\mathrm{d}\omega_{\vec{v}}$; i.e., the Lie derivative $\vec{v}^L(\mathrm{d}\omega_{\vec{v}})$ vanishes.

(d) The integral of $\mathrm{d}\omega_{\vec{v}}$ on any bounded 2-dimensional region of the fluid remains constant, and so does the integral of $\omega_{\vec{v}}$ on the boundary (named circulation of $\vec{v}$ along the boundary).

(e) The flow of $\vec{v}$ also preserves the **rotational** of $\vec{v}$, which is defined to be the vector field $\operatorname{rot}\vec{v}$ such that $i_{\operatorname{rot}\vec{v}}\,\omega_3 = \mathrm{d}\omega_{\vec{v}}$, where ω_3 is the volume form of $\mathbb{R}^3$.

(f) The function $\frac{p}{\rho} + \frac{v^2}{2}$ is constant along any trajectory of the fluid.

(g) In the case of an irrotational fluid, $\mathrm{d}\omega_{\vec{v}} = 0$, the function $\frac{p}{\rho} + \frac{v^2}{2}$ is constant (on the whole space, not just on each trajectory).

30. *Plane Stationary Fluids*: Let us consider a complex 1-form on an open set $U \subseteq \mathbb{C}$,

$$\omega = f(z)\,\mathrm{d}z = (v_1 - v_2 i)(\mathrm{d}x + i\mathrm{d}y) = (v_1\mathrm{d}x + v_2\mathrm{d}y) + (-v_2\mathrm{d}x + v_1\mathrm{d}y)i,$$

and put $\vec{v} = v_1 + v_2 i = \overline{f(z)}$, so that $f(z) = |\vec{v}|e^{-i\theta}$, where θ is the angle between the x-axis and $\vec{v}$. Prove that

(a) ω is holomorphic if and only if $\mathrm{d}\omega_{\vec{v}} = 0$ and $\operatorname{div} \vec{v} = 0$. In such case, if we fix the pressure to be $p = \rho(a - \frac{v^2}{2})$, where a and ρ are constants, we have $i_{\vec{v}}\, \mathrm{d}\omega_{\vec{v}} = -\mathrm{d}\big(\frac{p}{\rho} + \frac{v^2}{2}\big)$.

(b) Once we fix the mass density ρ and the pressure $a\rho$ at the equilibrium points, holomorphic 1-forms ω in U correspond with incompressible stationary irrotational fluids, the real part of ω being the 1-form $\omega_{\vec{v}}$ of the fluid.

(c) The 1-form $\omega = \lambda z^{-1}\mathrm{d}z$, with λ real, defines a fluid whose trajectories are the half-lines with end point at the origin, which is a source when $\lambda > 0$ and a drain when $\lambda < 0$.

(d) The 1-form $\omega = i\lambda z^{-1}\mathrm{d}z$ defines a fluid whose trajectories are concentric circles, the sign of λ determining the sense of rotation.

31. Let X be a Riemann surface. If $\pi\colon Y \to X$ is a covering, show that Y admits a unique structure of Riemann surface such that π is a local analytic isomorphism.

32. Let f be an analytic function on a Riemann surface X. Show that any point $p \in X$ has a coordinate neighborhood (U, z) such that $z(p) = 0$ and $f|_U = a + z^n$ for some $n \in \mathbb{N}$, $a \in \mathbb{C}$.

33. Let p be a point of a Riemann surface X, and let $\mathcal{O}_p$ be the ring of germs at p of analytic functions defined on a neighborhood of p. Prove that

(a) $\mathcal{O}_p$ is a discrete valuation ring, of maximal ideal $\mathfrak{m}_p = z\mathcal{O}_p$, where z is any local coordinate at p such that $z(p) = 0$

(b) Any $\mathbb{R}$-derivation $D\colon \mathcal{C}^\infty_p \to \mathbb{R}$ induces a $\mathbb{C}$-derivation $\tilde{D}\colon \mathcal{O}_p \to \mathbb{C}$, $\tilde{D}(u + iv) = Du + iDv$, so that we have a $\mathbb{R}$-linear isomorphism $T_pX = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_p, \mathbb{C})$.

(c) For any non constant analytic map $f\colon X \to Y$, we have $\mathrm{ind}_p f = \dim_{\mathbb{C}}(\mathcal{O}_p/\mathfrak{m}_q\mathcal{O}_p)$; $q = f(p)$.

34. An **almost complex** structure on a smooth manifold X is a (1,1)-tensor field J (a field of endomorphisms) such that $J^2 = -\mathrm{Id}$, and it is a **complex** structure if at any point $p \in X$ there exist local coordinate systems $(x_1, y_1, \dots, x_n, y_n)$ where $J(\partial_{x_i}) = \partial_{y_i}$, $J(\partial_{y_i}) = -\partial_{x_i}$.

The automorphism $J \otimes 1\colon (T_pX)_{\mathbb{C}} \to (T_pX)_{\mathbb{C}}$ has the eigenvalues $\pm i$, so that we have a decomposition $(T_pX)_{\mathbb{C}} = T_p^{(1,0)} \oplus T_p^{(0,1)}$ where the complex vectors of type $(1,0)$ have eigenvalue i and the complex vectors of type $(0,1)$ have eigenvalue $-i$. We also have an automorphism $J^* \otimes 1\colon (T_p^*X)_{\mathbb{C}} \to (T_p^*X)_{\mathbb{C}}$, and a decomposition $(T_pX)_{\mathbb{C}} = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$, where the $(1,0)$-type 1-forms have eigenvalue i and the $(0,1)$-type 1-forms have eigenvalue $-i$. Moreover, $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are $\mathbb{C}$-dual to $T_p^{(1,0)}$ and $T_p^{(0,1)}$ respectively. Prove that

(a) An almost complex structure is a complex structure if and only if the complex Pfaff system $\Omega^{(1,0)}$ of $(1,0)$-forms is integrable, in the sense that locally $\Omega^{(1,0)} = \langle \mathrm{d}z_1, \dots, \mathrm{d}z_n\rangle$ for some complex smooth functions $z_1, \dots z_n$

(b) The complex distribution $\mathcal{T}^{(0,1)}$ of complex $(0,1)$-vector fields is involutive (closed under the Lie bracket) if and only if the following **Nijenhuis tensor** N vanishes:

$$N(D_1, D_2) = [JD_1, JD_2] - J[JD_1, D_2] - J[D_1, JD_2] - [D_1, D_2].$$

35. In this exercise we assume, without proof, the Newlander-Nirenberg theorem: *An almost complex structure is complex if and only if the Nijenhuis tensor vanishes.* If X is a surface, prove that the Nijenhuis tensor is always null. Moreover, if (X, g, ω_2) is an oriented riemannian surface, prove the following results:

(a) The endomorphism J attached to the area 2-form, $J(D) \cdot D' = \omega_2(D, D')$, defines an almost complex structure on X.

(b) X, endowed with the sheaf $\mathcal{O}$ of complex smooth functions f with $\mathbb{C}$-linear differential $\mathrm{d}f$, is a Riemann surface.

(c) Locally, there exist **isothermal coordinates** (x, y) such that $g = h^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ for some positive function h.

(d) If u is an harmonic function with non null differential at a given point $p \in X$, then in a certain open neighborhood U of p there is a complex coordinate $z = u + iv$. Moreover, u, v are isothermal coordinates on U.

(e) Two riemannian metrics g', g on an oriented surface induce the same complex structure if and only if they are **conformal**: $g' = h^2 g$ for some positive function h.

(f) Let (X, g_X) and (Y, g_Y) be two oriented riemannian surfaces. An oriented smooth map $\phi\colon X \to Y$ is analytic if and only if $\phi^* g_Y = f g_X$ for some non-negative smooth function f.

36. Prove that any Riemann surface admits a compatible riemannian metric.

37. Let (X, g_X) and (Y, g_Y) be two Riemann surfaces with a compatible metric. Prove that an oriented smooth map $\varphi\colon X \to Y$ is analytic if and only if $\varphi^* g_Y$ is proportional to g_X.

38. Let us consider the unit disk $\mathbb{D}$, with the Poincaré metric. Prove the following statements:

(a) The geodesic lines are the circles orthogonally intersecting $\partial\mathbb{D}$ (including the diameters of $\mathbb{D}$). Hence, given two different points $z_1, z_2 \in \mathbb{D}$, there is a unique geodesic line joining them.

(b) The length of any curve $\gamma\colon [a, b] \to \mathbb{D}$ joining two points $z_1, z_2 \in \mathbb{D}$ is bounded below by the length $\mathrm{d}(z_1, z_2)$ of the geodesic joining them. (*Hint*: Assume that $z_1 = 0$ and $z_2 \in [0, 1)$, and compare the length of $\gamma(t) = x(t) + y(t)i$ with the length of $x(t)$).

(c) The hyperbolic distance $\mathrm{d}(0, z)$ is an increasing function of $|z|$.

(d) If an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ is an isometry at a point, then it is a global isometry.

39. If a compact Riemann surface X admits a hyperbolic metric g_X, prove that it is unique and the analytic automorphisms of X are just the oriented isometries of g_X.

40. Show that the complex plane $\mathbb{C}$ does not admit a hyperbolic metric.

41. Let X be a simply connected Riemann surface and fix a point $p \in X$ and a vector $0 \neq D_p \in T_p X$. Show that we have a bijection

$$\begin{bmatrix} \text{pointed local analytic} \\ \text{isomorphisms } \varphi\colon X \to \mathbb{D}_r \end{bmatrix} = \begin{bmatrix} \text{hyperbolic metrics on } X \\ \text{such that } |D_p| = 2/r^2 \end{bmatrix} \quad , \quad \varphi \mapsto \varphi^*(g_r).$$

(*Hint*: Any metric of constant curvature $K = -1$ defines (p. 305) an oriented local isometry $X \to \mathbb{D}$, unique up to an automorphism of $\mathbb{D}$).

42. Prove that $z + \frac{1}{z}$ defines an analytic isomorphism of $\mathbb{D}$ with $\mathbb{P}_1 - [-2, 2]$.

Find isomorphisms of $\mathbb{D}$ with $\mathbb{C} - [0, \infty)$ and with $\mathbb{C} - \Big((-\infty, -1] \cup [1, \infty)\Big)$.

10. Differential Geometry I

1. Prove that any open ball in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

2. Define the obvious structure of smooth manifold of the real and complex projective spaces.

3. Let X be the non separated topological manifold of exercise 6 in p. 497, and put $U_0 = \mathbb{R}\times 0$, $U_1 = \mathbb{R}\times 1$, so that U_0 and U_1 have an obvious smooth structure. Show that $(X, \mathcal{O}_X)$ is a non-separated smooth manifold when, for any open set $U \subseteq X$, we put

$$\mathcal{O}_X(U) = \{f \in \mathcal{C}(U)\colon\ f|_{U_0\cap U} \in \mathcal{C}^\infty(U_0 \cap U) \text{ and } f|_{U_1\cap U} \in \mathcal{C}^\infty(U_1 \cap U)\}.$$

4. Prove that the category of smooth manifolds has finite products and coproducts.

5. Let $(x_1, \ldots, x_n)$ be a local coordinate system of a smooth manifold X at a point $p = (a_1, \ldots, a_n)$. Let $(\mathcal{O}_p, \mathfrak{m}_p)$ be the local ring of germs at p of smooth functions on X. If $j^r_p f$ stands for the Taylor expansion of f at p up to degree r, show that we have an isomorphism of $\mathbb{R}$-algebras

$$\mathcal{O}_p/\mathfrak{m}_p^{r+1} \simeq \mathbb{R}[x_1 - a_1, \ldots, x_n - a_n]/(x_1 - a_1, \ldots, x_n - a_n)^{r+1},\ f \mapsto j^r_p f.$$

Moreover, if $\mathfrak{m}$ is the maximal ideal of $\mathcal{C}^\infty(X)$ defined by p, then $\mathcal{C}^\infty(X)/\mathfrak{m}^{r+1} = \mathcal{O}_p/\mathfrak{m}_p^{r+1}$.

6. If $\varphi\colon X \to Y$ is a surjective smooth map, prove that $\dim X \geq \dim Y$. (*Hint*: Use Srad's theorem).

7. If $i\colon Y \to X$ is a closed smooth submanifold, prove that $i^*\colon \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ is surjective.

8. Let $i\colon Y\to X$ be a submanifold and $y\in Y$. Prove that the kernel of $i^*\colon \mathcal{O}_{X,y}\to\mathcal{O}_{Y,y}$ is an ideal generated by a finite number of germs with linearly independent differentials at y.

9. Prove that most local properties of smooth manifolds are in fact local with respect to surjective regular projections $\pi\colon R\to X$. For example:

   (a) A map $\phi\colon X\to Y$ into a smooth manifold Y is continuous (resp. smooth, a regular projection) if and only if so is $\phi\circ\pi$.

   (b) A subset $Y\subseteq X$ is closed (resp. open, a submanifold) if and only if so is $\pi^{-1}(Y)\subseteq R$.

   (c) If $\phi\colon Y\to X$ is a smooth map, then $Y\times_X R$ is a smooth submanifold of $Y\times R$ and it is the fibred product in the category of smooth manifolds. Moreover, ϕ is a local embedding (resp. regular projection) if and only if so is $\phi\times\mathrm{Id}\colon Y\times_X R\to X\times_X R=R$.

10. Let $R\subseteq X\times X$ be an equivalence relation on a smooth manifold X. If there exists a structure of (separated) smooth manifold on the quotient set $\bar X=X/R$ such that the canonical projection $\pi\colon X\to\bar X$ is a regular projection, show that R is a closed submanifold of $X\times X$ and that both projections $\pi_1,\pi_2\colon R\to X$ are regular projections. Moreover, $\dim(X/R)=2\dim X-\dim R$.

11. Two smooth maps $\phi\colon X\to S$, $\psi\colon Y\to S$ are said to be **transversal** at a point (x,y) of the topological fibred product $X\times_S Y$ when $\mathrm{Im}\,\phi_{*,x}+\mathrm{Im}\,\psi_{*,y}=T_sS$, where $s=\phi(x)=\psi(y)$. If ϕ and ψ are transversal at any point, prove that the fibred product $X\times_S Y$ exists in the category of smooth manifolds, and that $\dim(X\times_S Y)=\dim X+\dim Y-\dim S$.

12. Prove the following formula for the exterior differential of p-forms:
$$(\mathrm{d}\omega_p)(D_0,\dots,D_p)=\textstyle\sum_i(-1)^iD_i\omega_p(\dots,\widehat{D}_i,\dots)+\sum_{i<j}(-1)^{i+j}\omega_p([D_i,D_j],\dots,\widehat{D}_i,\dots,\widehat{D}_j,\dots).$$

13. (*Lagrange multipliers*): Let $Y=\{x\in X\colon f_1(x)=\ldots=f_r(x)=0\}$,where X is a smooth manifold and $f_1,\dots,f_r\in\mathcal{C}^\infty(X)$. If $\mathrm{d}_pf_1,\dots,\mathrm{d}_pf_r$ are linearly independent at any point $p\in Y$, prove that the restriction $h|_Y$ of $h\in\mathcal{C}^\infty(X)$ has null differential at p if and only if $\mathrm{d}_ph\wedge\mathrm{d}_pf_1\wedge\ldots\wedge\mathrm{d}_pf_r=0$.

14. Show that the following subgroups of $Gl(n,\mathbb{C})=\{A\in M_{n\times n}(\mathbb{C})\}$ are closed submanifolds:
$U(n)=\{A\bar A^t=I\}$, $Sl(n,\mathbb{C})=\{\det A=1\}$, $O(n,\mathbb{C})=\{AA^t=I\}$,
$Gl(n,\mathbb{R})=Gl(n,\mathbb{C})\cap M_{n\times n}(\mathbb{R})$, $Sl(n,\mathbb{R})=Sl(n,\mathbb{C})\cap Gl(n,\mathbb{R})$, $O(n,\mathbb{R})=O(n,\mathbb{C})\cap Gl(n,\mathbb{R})$.

15. Let $f\colon X\to Y$ be a smooth map and $y=f(x)$. Prove that $f_*\colon T_xX\to T_yY$ is surjective if and only if $f*\colon\mathcal{O}_{Y,y}\to\mathcal{O}_{X,x}$ is injective and $\mathfrak{m}_x^2\cap\mathcal{O}_{Y,y}=\mathfrak{m}_y^2$.

16. Let $f\colon Y\to X$ be a smooth map and $x=f(y)$. Prove that the following conditions are equivalent:

   (a) $f_*\colon T_yY\longrightarrow T_xX$ is injective.

   (b) $f^*\colon\mathcal{O}_{X,x}\longrightarrow\mathcal{O}_{Y,y}$ is surjective

   (c) $f^*\colon\mathcal{O}_{X,x}\longrightarrow\mathcal{O}_{Y,y}$ is surjective, the kernel I is a finitely generated ideal and $\mathfrak{m}_x^2\cap I=\mathfrak{m}_xI$.

17. Prove the key lemma along a smooth subvariety: If Y is a submanifold of codimension r of a smooth manifold X, any point $q\in Y$ has a coordinate neighborhood $(U;u_1,\dots,u_n)$ in X such that the ideal of all smooth function $f\in\mathcal{C}^\infty(U)$ vanishing on $Y\cap U$ is just $(u_1,\dots,u_r)$.

18. Let Γ_{ij}^k be the Christoffel symbols of a linear connection ∇ in a local coordinate system $(x_1,\dots,x_n)$. Check that the torsion Tor_∇ and curvature R of ∇ are
$$\mathrm{Tor}_\nabla=\sum_{i,j,k}T_{ij}^k\mathrm{d}x_i\otimes\mathrm{d}x_j\otimes\partial_k,\quad T_{ij}^k=\Gamma_{ij}^k-\Gamma_{ji}^k,$$
$$R=\sum_{i<j}\sum_{kl}R_{ij,k}^l(\mathrm{d}x_i\wedge\mathrm{d}x_j)\otimes\mathrm{d}x_k\otimes\partial_l,\quad R_{ij,k}^l=\partial_i\Gamma_{jk}^l-\partial_j\Gamma_{ik}^l+\sum_h(\Gamma_{jk}^h\Gamma_{ih}^l-\Gamma_{ik}^h\Gamma_{jh}^l).$$

19. Let ∇ be a linear connection on a smooth manifold and let p be a point of X. Let $\sigma_D(t)$ be the geodesic line tangent to $D\in T_pX$ at $t=0$. Prove that

   (a) The **exponential map** $\exp\colon T_pX\to X$, $\exp(D)=\sigma_D(1)$, is a well-defined smooth map on a neighborhood U of the origin in T_pX.

(b) The tangent linear map $\exp_*\colon T_pX = T_pU \to T_pX$ is the identity, and the exponential map defines a diffeomorphism of an open neighborhood V of the origin in T_pX onto an open neighborhood of p in X.

(c) If we consider a local coordinate system on a neighborhood of p, corresponding to linear coordinates in V, then $\Gamma^k_{ij}(p) + \Gamma^k_{ji}(p) = 0$.

(d) A linear connection is symmetric if and only if any point $p \in X$ admits a coordinate neighborhood where the Christoffel symbols vanish at the given point, $\Gamma^k_{ij}(p) = 0$.

(e) If ∇ is symmetric, then $\mathrm{d}\omega$ is the hemisymmetrization of $\nabla\omega$ for any p-form ω.

20. If a symmetric linear connection ∇ on a smooth manifold X preserves a volume form, $\nabla(\mathrm{d}X) = 0$, prove that $D^L(\mathrm{d}X) = (C^1_1\nabla D)\mathrm{d}X$ for any vector field D.

21. If $\sigma\colon I \to X$ is a geodesic line, and $\phi\colon I' \to I$ is a diffeomorphism such that $\sigma\big(\phi(t)\big)$ also is a geodesic line, show that $\phi(t) = at + b$ for some $a, b \in \mathbb{R}$.

22. Prove that the Cartan connection ∇ preserves the time and space metrics, $\nabla g = 0$, $\nabla h = 0$, and that in any inertial coordinate system, $\nabla\partial_t = -\mathrm{d}t \otimes \vec{F}$, $\nabla\partial_i = 0$, $\nabla(\mathrm{d}t) = 0$, $\nabla(\mathrm{d}x_i) = F_i\mathrm{d}t \otimes \mathrm{d}t$. Moreover, the curvature tensor R and the Ricci tensor R_2 of ∇ are just

$$R = \textstyle\sum_i(\mathrm{d}t \wedge \mathrm{d}F_i) \otimes (\partial_i \otimes \mathrm{d}t)\ , \quad R_2 = -\textstyle\sum_i(\partial_i F_i)\mathrm{d}t \otimes \mathrm{d}t.$$

23. Let ∇ be a symmetric linear connection on a smooth manifold X and let ω be a 1-form on X not vanishing at any point. If $\nabla(\omega \otimes \omega) = 0$, show that $\nabla\omega = 0$ and $\mathrm{d}\omega = 0$.

24. Let (t, x, y, z) be inertial coordinates in the Minkowski spacetime. Fix the year as the time unit, and the light-year as the length unit, so that $c = 1$ and the gravitational acceleration on the earth surface (the most comfortable acceleration for us, $980\,\mathrm{cm/s^2}$) is ≈ 1. Check that

$$t = \sinh\tau,\ x = \cosh\tau - 1,\ y = 0,\ z = 0,$$

is a trajectory of proper time τ. Calculate the unitary tangent vector, $U \cdot U = 1$, and the acceleration $U^\nabla U$, and check that it is a trajectory of acceleration $g = 1$, starting at rest at $\tau = 0$. After traveling $\tau = 2.4$ years, we reach Proxima Centauri (≈ 4.5 light-years), after $\tau = 11$ years, the center of the Milky Way (≈ 29.000 light-years), after $\tau = 15.5$ years, the Andromeda Galaxy (≈ 2.7 million light-years) and after $\tau = 25$ years we arrive at the most remote observable parts of the Universe (≈ 35 billion light-years); even if we never reach the light speed!

25. Let us consider in $\mathbb{R}^6$ the cone $t^2 - x^2 - y^2 - z^2 - u^2 - v^2 = 0$, defining a quadric $\mathcal{Q}$ in $\mathbb{P}_5$. Prove that the intersection of the cone with the hyperplane $u - v = 1$, endowed with the metric

$$g = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - \mathrm{d}u^2 - \mathrm{d}v^2$$

induced by the cone, is isometric to the Minkowski space via the coordinates (t, x, y, z).

So $\mathcal{Q}$ is a compactification of the Minkowski space. Check that different light directions are compactified with different points, while all non-light directions are compactified with a unique point.

26. Let T_3 and T^1_2 be tensors on a riemannian manifold (X, g) satisfying the following conditions:

(a) $T_3(D_1, D_2, D_3) = T_3(D_1, D_3, D_2)$.

(b) $T^1_2(D_1, D_2, \omega) = -T^1_2(D_2, D_1, \omega)$.

Prove the existence of a unique linear connection ∇ on X such that $\nabla g = T_3$ and $\mathrm{Tor}_\nabla = T^1_2$. (*Hint*: Repeat the proof of the fundamental theorem of riemannian geometry).

27. Let g be a covariant symmetric metric of constant rank on a smooth manifold X. If there exists a symmetric linear connection ∇ such that $\nabla g = 0$, prove that the radical $\mathcal{V}$ of g is an involutive distribution. Moreover g is projectable by $\mathcal{V}$; i.e., $V^L g = 0$ whenever V is a vector field in $\mathcal{V}$. (*Hint*: $D^\nabla V$ is in $\mathcal{V}$ for any vector field D).

28. Let h be a contravariant symmetric metric of constant rank on a smooth manifold X, let $\mathcal{V}$ be the incident of the radical of h, and let $\langle\,,\,\rangle$ denote the metric induced by h on the distribution $\mathcal{V}$. Given a linear connection ∇ on X, prove that $\nabla h = 0$ if and only if

(a) $D^\nabla V \in \mathcal{V}$, whenever $V \in \mathcal{V}$.

(b) $D\langle V_1, V_2\rangle = \langle D^\nabla V_1, V_2\rangle + \langle V_1, D^\nabla V_2\rangle$, whenever $V_1, V_2 \in \mathcal{V}$.

(*Hint*: If we put $V_1 = h(\omega_1)$, then $(D^\nabla \omega_1)(V_2) = D(\langle V_1, V_2\rangle) - \langle V_1, D^\nabla V_2\rangle)$.

29. Let us consider the Euclidean space $\mathbb{R}^3$. Using that the classical newtonian potential $N(\rho)$ of a function with compact support $\rho \in \mathcal{C}^\infty(\mathbb{R}^3)$ is a solution, commuting with partial derivatives, of Poisson's equation $\Delta u = 4\pi\rho$, give a new proof of **Poincaré's lemma** in germs:

(a) For any p-form ω with compact support in $\mathbb{R}^3$, define a p-form $\Delta^{-1}\omega$, commuting with d and δ, such that $\Delta\Delta^{-1}\omega = \omega$.

(b) Prove Poincaré's lemma in $\mathbb{R}^3$ for closed germs, $\mathrm{d}\omega = 0$,

$$\omega = \Delta\Delta^{-1}\omega = \mathrm{d}\delta\Delta^{-1}\omega + \delta\Delta^{-1}\mathrm{d}\omega = \mathrm{d}(\delta\Delta^{-1}\omega).$$

(c) In $\mathbb{R}^3$, given a closed $(p+1)$-form, $\mathrm{d}\omega = 0$, and a coclosed $(p-1)$-form, $\delta\omega' = 0$, prove that at any point there is a germ of a p-form α such that $\mathrm{d}\alpha = \omega$, $\delta\alpha = \omega'$.

(*Hint*: $\omega + \omega' = \Delta\Delta^{-1}(\omega+\omega') = (\mathrm{d}+\delta)(\mathrm{d}+\delta)\Delta^{-1}(\omega+\omega') = (\mathrm{d}+\delta)(\delta\Delta^{-1}\omega + \mathrm{d}\Delta^{-1}\omega') = (\mathrm{d}+\delta)\alpha$).

30. Prove the following formulae in the Euclidean space of dimension 3:

$$\begin{gathered}
\mathrm{rot}(\mathrm{grad}\, f) = 0\ ,\ \mathrm{div}(\mathrm{rot}\, D) = 0\ ,\ \mathrm{grad}(fh) = h(\mathrm{grad}\, f) + f(\mathrm{grad}\, h)\\
\mathrm{rot}(fD) = (\mathrm{grad} f) \times D + f(\mathrm{rot}\, D)\ ,\ \mathrm{div}(fD) = (\mathrm{grad}\, f)\cdot D + f(\mathrm{div}\, D)\\
\mathrm{rot}(\mathrm{rot}\, D) = \mathrm{grad}(\mathrm{div}\, D) - \Delta D\\
\mathrm{div}(D_1 \times D_2) = (\mathrm{rot}\, D_1)\cdot D_2 - D_1 \cdot (\mathrm{rot}\, D_2)\ ,\ \mathrm{grad}(D_1\cdot D_2) = (\nabla D_1)\cdot D_2 + (\nabla D_2)\cdot D_1\\
\mathrm{rot}(D_1 \times D_2) = D_2\cdot(\nabla D_1) - (\mathrm{div}\, D_1)D_2 - D_1\cdot(\nabla D_2) + (\mathrm{div}\, D_2)D_1.
\end{gathered}$$

31. Let ∇ be the Levi-Civita connection of the Euclidean metric $g = \sum_i \mathrm{d}x_i^2$ in $\mathbb{R}^n$. If Γ_{ij}^k are the Christoffel symbols of ∇ in a local coordinate system $(U; u_1, \ldots, u_n)$, prove that

(a) $\nabla^2 f = \nabla(\nabla f) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}\, \mathrm{d}x_i \otimes \mathrm{d}x_j$, for any smooth function $f \in \mathcal{C}^\infty(U)$.

(b) $\nabla^2 u_k = -\sum_{ij} \Gamma_{ij}^k\, \mathrm{d}u_i \otimes \mathrm{d}u_j$.

(c) If $\Gamma_{ij}^k = 0$, then $u_i = b_i + \sum_j a_{ij}x_j$, where $A = (a_{ij})$ is invertible. (U connected).

(d) If $g = \sum_i \mathrm{d}u_i^2$, then $u_i = b_i + \sum_j a_{ij}x_j$, where $A^tA = I$. (U connected).

(e) Any diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ preserving ∇ is an affinity.

(f) Any diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ preserving g is a motion.

32. Let $\mathcal{C}^\infty$**Aff** be the category of pairs (X, ∇), where ∇ is a linear connection on a smooth manifold X, and diffeomorphism preserving the linear connection. Show that the category **Aff** of real affine spaces and affinities is equivalent to the subcategory of $\mathcal{C}^\infty$**Aff** defined by the pairs (X, ∇) isomorphic to some $\mathbb{R}^n$ with the Levi-Civita connection of the Euclidean metric $\sum_i \mathrm{d}x_i^2$.

If we consider the category **Riemann** of riemannian manifolds and diffeomorphisms preserving the metric, show that the category of Euclidean spaces $(\mathbb{A}_n, V, \Omega_2)$ with fixed unit length is equivalent to the subcategory of **Riemann** defined by the riemannian manifolds isomorphic to some $\mathbb{R}^n$ with the Euclidean metric $\sum_i \mathrm{d}x_i^2$.

(*Hint*: If you are willing to assume that the objects form a set, you may use problem 53 in p. 503 in both statements. Otherwise some additional work is required).

33. Prove that any harmonic coordinate system $(u_1, \ldots, u_n)$ in $\mathbb{R}^n$, such that $g^{ii} = g(\mathrm{d}u_i, \mathrm{d}u_i)$ goes to some constant c_i at infinity, is $u_i = b_i + \sum_j a_{ij}x_j$. (*Hint*: $g^{ii} = \sum_j \big(\frac{\partial u_i}{\partial x_j}\big)^2$).

34. Let $F = F_1\partial_1 + F_2\partial_2 + F_3\partial_3$ be a force vector field in the Euclidean space $(\mathbb{R}^3, g = \sum_i \mathrm{d}x_i^2)$, and let us consider the trajectory $\sigma\colon I \to \mathbb{R}^3$ of a punctual particle satisfying the newtonian law of motion $mT^\nabla T = F$, where $T = \sigma_*(\partial_t)$ is the velocity and the mass $m \in \mathbb{R}_+$. If we consider the work 1-form $\omega_F = g(F, -)$ and the momentum 1-form $\theta = g(mT, -)$, show that $T^\nabla\theta = \omega_F$.

If the forces are conservative, $\mathrm{d}\omega_F = 0$, prove that the forces derive from a potential, $\omega_F = \mathrm{d}u$ for some smooth function $u \in \mathcal{C}^\infty(\mathbb{R}^3)$. Moreover,

(a) The energy $e = \frac{m}{2}T \cdot T - u$ is constant along the trajectory of the particle.

(b) The punctual particle follows a geodesic line of the metric $\bar{g} = 2m(e+u)g$.

(c) Let $\mathcal{L} = \frac{m}{2}T \cdot T + u$ be the lagrangian. If we fix two points $p_1 = \sigma(t_1)$, $p_2 = \sigma(t_2)$ of the trajectory and a vector field D vanishing at both, and we extend T to a neighborhood so that $[D,T] = 0$, then we have **Lagrange's equations** $T(mT \cdot D) - D\mathcal{L} = 0$, not involving ∇, and the **principle of least action**

$$0 = \int_{t_1}^{t_2} (D\mathcal{L})\mathrm{d}t.$$

35. Let (X, g, ω_X) be an oriented semiriemannian manifold of dimension n. Given a p-vector $D_1 \wedge \ldots \wedge D_p$, we put $i_{D_1\wedge\ldots\wedge D_p}\omega_X := i_{D_p}\ldots i_{D_1}\omega_X$, and we define the **Hodge star** operator $*\colon \Omega^p(X) \to \Omega^{n-p}(X)$ by the formula $*\omega_p = i_{\omega_p}\omega_X$, where a p-form ω_p is viewed as a p-vector by means of the isomorphisms $g\colon T_xX \to T_x^*X$ induced by the metric g. If the metric g has s negative signs, prove that $*1 = \omega_X$, $*\omega_X = (-1)^s$, $**\omega_p = (-1)^{p(n-p)+s}\omega_p$, $g(\omega_p, \bar{\omega}_p) = (-1)^s g(*\omega_p, *\bar{\omega}_p)$, $\omega_p \wedge *\bar{\omega}_p = g(\omega_p, \bar{\omega}_p)\omega_X$, $*\delta\omega_p = (-1)^{p+1}\mathrm{d}*\omega_p$.

Now we consider on the space of p-forms with compact support $\Omega_c^p(X)$ the symmetric metric

$$\langle \omega_p, \bar{\omega}_p \rangle = \int_X g(\omega_p, \bar{\omega}_p)\omega_X.$$

Prove that $\langle \omega_p, \bar{\omega}_p \rangle = \int_X \omega_p \wedge *\bar{\omega}_p$, $\langle \mathrm{d}\omega_p, \omega_{p+1} \rangle = -\langle \omega_p, \delta\omega_{p+1} \rangle$, and $\langle \Box\omega_p, \bar{\omega}_p \rangle = \langle \omega_p, \Box\bar{\omega}_p \rangle$.

36. Put $K(D_1, D_2) = R_{2,2}(D_1, D_2, D_2, D_2)$, where $R_{2,2}$ is the Riemann-Christoffel tensor of a riemannian metric. Prove that

$$R_{2,2}(D_1, D_2, D_3, D_4) = \frac{1}{6}\left.\frac{\partial^2}{\partial t \partial s}\right|_{t=0,s=0} \big(K(D_1 + tD_3, D_2 + sD_4) - K(D_1 + tD_4, D_2 + sD_3)\big)$$

and conclude that the curvature tensor R vanishes at a point p whenever all the sectional curvatures K_Π at p are zero.

37. A **symplectic manifold** is a smooth manifold X, endowed with a closed non-singular 2-form ω_2. If D is a tangent vector field, we put $h(D) = i_D\omega_2$, and we say that D is a **hamiltonian** vector field when $D^L\omega_2 = 0$.

(a) Prove that D is a hamiltonian vector field if and only if $h(D)$ is a closed 1-form.

(b) If D_1, D_2 are hamiltonian vector fields, prove that so is $[D_1, D_2]$ and that, in fact, it corresponds to an exact 1-form. (*Hint*: $i_{[D,\bar{D}]} = [i_D, \bar{D}^L]$).

(c) The cotangent bundle $\pi\colon T^*X \to X$ of a smooth manifold has a structural 1-form $\theta_{\omega_x} = \pi^*(\omega_x)$; hence a 2-form $\omega_2 = \mathrm{d}\theta$. Prove that (T^*X, ω_2) is a symplectic manifold.

(d) In a riemannian manifold (X, g) we have a canonical isomorphism $TX \to T^*X$, $D \mapsto i_DG$; hence the tangent bundle TX inherits a 1-form θ and a 2-form $\omega_2 = \mathrm{d}\theta$. Prove that the geodesic vector field Z is the hamiltonian vector field corresponding to $e(D_x) = \frac{1}{2}D_x \cdot D_x$, in the sense that $i_Z\omega_2 = -\mathrm{d}e$. Moreover, $\theta(Z) = 2e$.

38. **Meusnier's Theorem:** Show that all curves passing through a point with common tangent have equal normal curvature.

39. Let $q(x,y) = d(x^2+y^2) - 2ax - 2by + c$ and let us consider, on the open subset of $\mathbb{R}^2$ where $q(x,y) \neq 0$, the riemannian metric $g = 4q(x,y)^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$.

(a) When $q(x,y) = x^2 + y^2 - 1$, the metric g has constant negative curvature (the unit disk, with this metric, is the **Poincaré's disk**).

(b) When $q(x,y) = x^2 + y^2 + 1$, the metric g has constant positive curvature (it is the stereographic projection, onto the equator plane, of the metric of the unit sphere).

(c) When $q(x,y) = x^2 + y^2$, the metric g has null curvature (it is just the inversion of the Euclidean metric with respect to the unit circle).

(d) When $q(x,y)=x$, the metric g has constant negative curvature (the semiplane $x>0$, with this metric, is the **Poincaré's half-plane**).

(e) When $q(x,y)=1$, the metric g is the Euclidean metric.

(f) In general, the riemannian metric g has constant curvature $-R_2(q)$, where R_2 is the quadratic form on the space of circles (p. 516) and the geodesics of g are just real circles and lines orthogonal to $q(x,y)$ with respect to R_2.

(*Hint*: If $0<f(x,y)$, the curvature of $f^{-2}(\mathrm{d}x^2+\mathrm{d}y^2)$ is $K=f\Delta f-\|\mathrm{grad}\, f\|^2$. To find the geodesics, use that $2(T^\nabla T)\cdot N = T(T\cdot N)+T(N\cdot T)-N(T\cdot T)$ when $[T,N]=0$).

40. If $q(x_1,\ldots,x_n)=d(x_1^2+\ldots+x_n^2)-2a_1x_1-\ldots-2a_nx_n+c$, prove that $4q^{-2}(\mathrm{d}x_1^2+\ldots+\mathrm{d}x_n^2)$ has constant curvature $K=cd-(a_1^2+\ldots+a_n^2)$. (*Hint*: If $f(x_1,\ldots,x_n)$ is positive, the sectional curvatures of $f^{-2}(\mathrm{d}x_1^2+\ldots+\mathrm{d}x_n^2)$ are $K_{ij}=f(f_{ii}+f_{jj})-\|\mathrm{grad}\, f\|^2$, where $f_{ii}=\partial_i\partial_i f$).

41. Prove that the second fundamental form of a surface $f(x,y,z)=0$ in $\mathbb{R}^3$ is $\phi_2=-\dfrac{\nabla(\mathrm{d}f)}{\|\mathrm{grad}\, f\|}$.

42. Let $D_0,\ldots,D_n$ be a local orthonormal basis of vector fields in a riemannian manifold X.

(a) Show that the laplacian of a smooth function u is just $\Delta u=\sum_i\big(D_i^2u-(D_i^\nabla D_i)u\big)$.

(b) Let ϕ the Weingarten endomorphism of an hypersurface $\bar{X}\to X$, and let us extend the unitary normal vector N to a neighborhood so that $N^\nabla N=0$. Check that the relation of Δ and the laplacian $\bar{\Delta}$ of $\bar{X}$ is given by the formula

$$\bar{\Delta}u=\Delta u+(\mathrm{tr}\phi)(Nu)-N(Nu)\ ,\quad u\in\mathcal{C}^\infty(X).$$

(*Hint*: Fix a local orthonormal basis $D_1,\ldots,D_n$ in $\bar{X}$, extend it to a neighborhood so that $N^\nabla D_i=0$, and check that $N,D_1,\ldots,D_n$ is an orthonormal basis).

(c) In the Euclidean space $X=\mathbb{R}^3$, prove that a surface $\bar{X}$ is **minimal** ($\mathrm{tr}\phi=0$) if and only if the cartesian coordinates x,y,z are harmonic functions on $\bar{X}$. (*Hint*: The condition $N^\nabla N=0$ states that $N^2x=N^2y=N^2z=0$).

43. Show that any Lie group G admits a unique linear connection ∇ such that the left invariant vector fields are parallel. Moreover:

(a) The morphisms of Lie groups $\sigma\colon\mathbb{R}\to G$ are the geodesic lines passing through the neutral element $e\in G$ at $t=0$.

(b) $\exp\colon\mathfrak{g}=T_eG\to G$ is just the exponential map of ∇ at e.

(c) ∇ has null curvature, but $\mathrm{Tor}_\nabla(D,D')=-[D,D']=0$ for all $D,D'\in\mathfrak{g}$.

(d) ∇ is complete.

44. Prove that a connected Lie group G is abelian if and only if $[D',D]=0$ for all $D',D\in\mathfrak{g}$.

45. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Show that left invariant linear connections on G, in the sense that $L_x(D^\nabla D')=(L_xD)^\nabla(L_xD')$, correspond to bilinear maps $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$.

46. Given a quaternion $q\in\mathbb{H}^*:=\mathbb{H}-\{0\}$, the inner automorphism $\tau_q\colon\mathbb{H}\to\mathbb{H}$, $\tau_q(x)=qxq^{-1}$ preserves the scalar product and the orientation of $\mathbb{H}$ because it preserves the product (see problem 81 in p. 491), so that it preserves $\mathbb{H}_0:=\mathbb{R}^\perp$. Show that we have exact sequences

$$0\longrightarrow\mathbb{R}^*\longrightarrow\mathbb{H}^*\longrightarrow SO(3)\longrightarrow 0,$$
$$0\longrightarrow\{\pm1\}\longrightarrow S_3\longrightarrow SO(3)\longrightarrow 0,$$

so that the universal covering of $SO(3)$ is S_3, and the fundamental group of $SO(3)$ is $\mathbb{Z}/2\mathbb{Z}$.

Moreover, let us consider a quaternion $q=a+e\neq\pm1$ in S_3 and an oriented orthonormal base i,j,k in $\mathbb{H}_0=\mathbb{R}i\oplus\mathbb{C}j$ with $i:=e/|e|$, so that $q=a+bi$, $b>0$. Since $1=|q|=a^2+b^2$, we have $q=\cos\theta+i\sin\theta$ for some angle θ, and show that

$$\tau_q(xi+yj+zk)=\tau_q\big(xi+(y+zi)j\big)=xi+(a+bi)^2(y+zi)j.$$

Conclude that $\tau_q\colon\mathbb{H}_0\to\mathbb{H}_0$ is just a rotation with axis $\mathbb{R}i=\mathbb{R}e$ and angle 2θ.

47. Show that the isomorphism $\mathbb{H}\otimes_\mathbb{R}\mathbb{C}\simeq M_{2\times2}(\mathbb{C})$ given in problem 155 of p. 512 induces an isomorphism of Lie groups $S_3\simeq SU(2)$, so that we have a covering $SU(2)\to SO(3)$ of degree 2.

11. Algebraic Geometry I

1. Prove that any open subset $U \hookrightarrow X$ is the étalé space of a sheaf of sets $U^\bullet$ on X. Moreover, for any sheaf of sets $\mathcal{F}$ on X we have $\mathrm{Hom}(U^\bullet, \mathcal{F}) = \mathcal{F}(U)$.

2. If X is a connected topological space, show that the category of constant sheaves of sets over X is equivalent to the category of sets.

3. Prove that the category of coverings $X \to S$ of a topological space S is equivalent to the category of locally constant sheaves of sets over S.

4. Prove that a continuous map $\pi\colon P \to X$ is a **local homeomorphism** (i.e. any point $p \in P$ has an open neighborhood V such that $\pi\colon V \to \pi(V)$ is a homeomorphism onto an open subset $\pi(V)$ of X) if and only if $\pi\colon P \to X$ and the diagonal map $\Delta\colon P \to P \times_X P$ are open maps.

5. Prove that the étalé space $\pi\colon \mathcal{P}^{\mathrm{et}} \to X$ of a presheaf of sets $\mathcal{P}$ is a local homeomorphism, and that the construction of the étalé space defines an equivalence of categories
$$\begin{bmatrix}\text{Sheaves of}\\ \text{sets over } X\end{bmatrix} \rightsquigarrow \begin{bmatrix}\text{Local homeomorphisms}\\ P \longrightarrow X\end{bmatrix}$$

6. Show that, in the category of étalé spaces over a topological space X, the final object is the identity map $X \to X$, and prove that the open embeddings $i\colon U \hookrightarrow X$ are just the étalé spaces such that the unique morphism to the final object is a monomorphism.

   Conclude that two T_0 topological spaces are homeomorphic if and only if their categories of sheaves of sets are equivalent.

7. Show that a morphism of abelian sheaves $\mathcal{F} \to \mathcal{G}$ over a topological space X is null if and only if so is the induced morphism $\mathcal{F}_x \to \mathcal{G}_x$ at any point $x \in X$.

8. Prove that a sequence $\mathcal{F}' \to \mathcal{F} \rightrightarrows \mathcal{F}''$ (resp. $\mathcal{F}' \rightrightarrows \mathcal{F} \to \mathcal{F}''$) is exact in the category of presheaves of sets on a topological space X if and only if so is the sequence $\mathcal{F}'(U) \to \mathcal{F}(U) \rightrightarrows \mathcal{F}''(U)$ (resp. $\mathcal{F}'(U) \rightrightarrows \mathcal{F}(U) \to \mathcal{F}''(U)$) in the category of sets for any open set $U \subseteq X$.

   Prove that a sequence $\mathcal{F}' \to \mathcal{F} \rightrightarrows \mathcal{F}''$ (resp. $\mathcal{F}' \rightrightarrows \mathcal{F} \to \mathcal{F}''$) is exact in the category of sheaves of sets on a topological space X if and only if so is the sequence $\mathcal{F}'_x \to \mathcal{F}_x \rightrightarrows \mathcal{F}''_x$ (resp. $\mathcal{F}'_x \rightrightarrows \mathcal{F}_x \to \mathcal{F}''_x$) in the category of sets for any point $x \in X$. (*Hint*: Consider the sheaf associated to the presheaf of sets $U \rightsquigarrow \mathrm{Coker}\,(\mathcal{F}'(U) \rightrightarrows \mathcal{F}(U))$ and look at the fibres).

9. Prove that a sequence $\mathcal{F}' \to \mathcal{F} \rightrightarrows \mathcal{F}''$ is exact in the category of sheaves of sets on a topological space X if and only if so is the sequence of sets $\mathcal{F}'(U) \to \mathcal{F}(U) \rightrightarrows \mathcal{F}''(U)$ for any open set $U \subseteq X$.

10. Prove that a sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of abelian sheaves on X is exact if and only if the sequence $0 \to \mathrm{Hom}(\mathcal{F}'', \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}', \mathcal{G})$ is exact for any sheaf of abelian groups $\mathcal{G}$ on X.

    Prove that a sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of abelian sheaves on X is exact if and only if the sequence $0 \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}') \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}'')$ is exact for any sheaf of abelian groups $\mathcal{G}$ on X.

11. Let ∇ be a flat connection on a smooth manifold X and let $\mathcal{F}$ be the sheaf of bases of parallel vector fields. Show that the étalé space $\mathcal{F}^{\mathrm{et}} \to X$ is a covering. When X is simply connected, conclude that it is **parallelizable** (there exists a global base of parallel vector fields).

12. Let $f(z)$ be an analytic function on a connected open set $U \subset \mathbb{C}$, let $\mathcal{O}$ be the sheaf of analytic functions on $\mathbb{C}$, and let us consider the connected component X of the étalé space of $\mathcal{O}$ containing the germ of f at a point of U. Show that X admits a unique structure of Riemann surface such that the natural map $\pi\colon X \to \mathbb{C}$ is a local isomorphism. Moreover, we have an analytic section $i\colon U \to X$ of π and an analytic continuation F of $f(z)$ on X (an analytic function F on X extending f).

    Show that for any other unramified analytic map $\pi'\colon X' \to \mathbb{C}$, endowed with an analytic section $i'\colon U \to X'$ of π' and an analytic continuation F' of $f(z)$ on X', there exists a unique analytic map $h\colon X' \to X$ such that $\pi' = \pi \circ h$, $i = i' \circ h$ and $f' = F \circ h$.

13. Let $\mathcal{P}_1, \mathcal{P}_2$ be two presheaves over a topological space X. Prove that the associated sheaf of the presheaf $U \rightsquigarrow \mathcal{P}_1(U) \otimes_{\mathbb{Z}} \mathcal{P}_2(U)$ is just $\mathcal{P}_1^\sharp \otimes_{\mathbb{Z}} \mathcal{P}_2^\sharp$.

14. In the category of sheaves on a topological space X, show that $\mathrm{Hom}(\mathcal{F}\otimes\mathcal{G},\mathcal{L}) = \mathrm{Hom}(\mathcal{F},\underline{\mathrm{Hom}}(\mathcal{G},\mathcal{L}))$.

15. Let $(X,\mathcal{O}_X)$ be a ringed space. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, show that $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{M}) = \mathcal{M}$.
If $\mathcal{E},\mathcal{E}'$ are two locally free $\mathcal{O}_X$-modules of finite rank, prove that
$$\mathcal{E}\otimes_{\mathcal{O}_X}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}\otimes_{\mathcal{O}_X}\mathcal{E}),$$
$$\mathcal{E}^*\otimes_{\mathcal{O}_X}\mathcal{E}' = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}').$$
If $0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0$ is an exact sequence of locally free $\mathcal{O}_X$-modules of finite rank, prove that the images of the natural maps $\Lambda^p\mathcal{E}'\otimes_{\mathcal{O}_X}\Lambda^{n-p}\mathcal{E}\to\Lambda^n\mathcal{E}$ define a filtration of $\Lambda^n\mathcal{E}$, and the associated graded module is just $\bigoplus_p(\Lambda^p\mathcal{E}'\otimes_{\mathcal{O}_X}\Lambda^{n-p}\mathcal{E}'')$.

16. If a topological space X is discrete, show that any sheaf on X is a flasque sheaf.
If a topological space X has a dense point, show that any constant presheaf on X is a flasque sheaf.

17. Let $\mathcal{F}$ be a sheaf on a topological space X and $x\in X$. Show that $\varinjlim_{x\in U} H^p(U,\mathcal{F}) = 0$, $p\geq 1$.

18. If $\mathcal{F},\mathcal{G}$ are sheaves on a topological space X, show that $H^p(X,\mathcal{F}\oplus\mathcal{G}) = H^p(X,\mathcal{F})\oplus H^p(X,\mathcal{G})$.
If $\mathcal{F}$ is a sheaf on a disjoint union $X\oplus Y$, show that $H^p(X\oplus Y,\mathcal{F}) = H^p(X,\mathcal{F})\oplus H^p(Y,\mathcal{F})$.

19. If M is a torsion free A-module, prove that $H^1(X,M)$ is a torsion free A-module.

20. Let R be a flat $\mathbb{Z}$-module and let $\mathcal{F}$ be a sheaf on a topological space X. Prove that

   (a) The presheaf $(\mathcal{F}\otimes_{\mathbb{Z}}R)(U) = \mathcal{F}(U)\otimes_{\mathbb{Z}}R$ is a sheaf, and it is flasque when so is $\mathcal{F}$.

   (b) $0\to\mathcal{F}\otimes_{\mathbb{Z}}R\to(\mathcal{C}^\bullet\mathcal{F})\otimes_{\mathbb{Z}}R$ is a flasque resolution of $\mathcal{F}\otimes_{\mathbb{Z}}R$.

   (c) $H^p(X,\mathcal{F}\otimes_{\mathbb{Z}}R) = H^p(X,\mathcal{F})\otimes_{\mathbb{Z}}R$; hence, $H^p(X,R) = H^p(X,\mathbb{Z})\otimes_{\mathbb{Z}}R$.

21. If $f\colon X\to S$ is a continuous map, and $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''$ is an exact sequence of sheaves on X, prove that the sequence $0\to f_*\mathcal{F}'\to f_*\mathcal{F}\to f_*\mathcal{F}''$ also is exact.

22. Let $\phi\colon\operatorname{Spec}B\to\operatorname{Spec}A$ be a morphism of schemes, defined by a ring morphism $A\to B$.
If M is an A-module, show that $\phi^*(\widetilde{M})$ is the sheaf defined by the B-module $M_B = M\otimes_A B$.
If N is a B-module, show that $\phi_*(\widetilde{N})$ is the sheaf defined by the A-module N.

23. Let us consider a morphism of ringed spaces $\Phi\colon Y=\operatorname{Spec}B\to X=\operatorname{Spec}A$, defined by a continuous map $\phi\colon\operatorname{Spec}B\to\operatorname{Spec}A$ and a sheaf morphism $\psi\colon\mathcal{O}_X\to\mathcal{O}_Y$. If Φ preserves quasi-coherent modules (i.e. $\phi_*\widetilde{M}$ is a quasi-coherent $\widetilde{A}$-module, for any B-module M) show that Φ is a morphism of locally ringed spaces. (*Hint*: Let us consider the ring morphism $\psi\colon A\to B$. We have to show that a function $f\in A$ vanishes at a point $x=\phi(y)$ if and only if $\psi(f)$ vanishes at $y\in Y$. If $f(x)\neq 0$, then $\psi(f)(y)\neq 0$ because Φ is a morphism of ringed spaces. If $f(x)=0$, then the open set $\phi^{-1}(U_f)$ does not contain the closure of y, i.e. the support of $B/\mathfrak{p}$, where $\mathfrak{p}$ is the prime ideal of y. Hence $0=\widetilde{B/\mathfrak{p}}(\phi^{-1}U_f) = (\phi_*\widetilde{B/\mathfrak{p}})(U_f)$, and $(\phi_*\widetilde{B/\mathfrak{p}})(U_f) = (B/\mathfrak{p})_f = B_f/\mathfrak{p}_f$ because $\phi_*\widetilde{B/\mathfrak{p}}$ is quasi-coherent. Now, $B_f/\mathfrak{p}_f = 0$ means that $\psi(f)(y)=0$.)

24. Let us consider the continuous map $\phi\colon Y=\operatorname{Spec}\mathbb{Q}\to X=\operatorname{Spec}\mathbb{Z}$, $\phi(Y)=2$, and the sheaf morphism $\psi\colon\mathcal{O}_X\to\mathcal{O}_Y$ corresponding to the inclusion map $\phi^*\mathcal{O}_X=\mathbb{Z}_2\to=\mathcal{O}_Y=\mathbb{Q}$, so that $\Phi=(\phi,\psi)\colon\operatorname{Spec}\mathbb{Q}\to\operatorname{Spec}\mathbb{Z}$ is a morphism of ringed spaces. Show that Φ is not a morphism of locally ringed spaces.

25. Let **Rings** be the category of rings and ring morphisms, **Aff** the category of affine schemes and morphisms of locally ringed spaces, and **C** the category of affine schemes and morphisms of ringed spaces. Since any morphism of locally ringed spaces is a morphism of ringed spaces, the spectrum defines a contravariant functor Spec **Rings** $\rightsquigarrow$ **C**. Since any morphism of ringed spaces $Y\to X$ induces a ring morphism $\mathcal{O}_X(X)\to\mathcal{O}_Y(Y)$, we also have a contravariant functor **C** $\rightsquigarrow$ **Rings**, $X\rightsquigarrow\mathcal{O}_X(X)$. For any ring A we have a natural ring isomorphism $A=\mathcal{O}_{\operatorname{Spec}A}(\operatorname{Spec}A)$, and for any affine scheme X we have a natural isomorphism of locally ringed spaces $X=\operatorname{Spec}\mathcal{O}_X(X)$, hence an isomorphism of ringed spaces. We conclude that the functor Spec **Rings** $\rightsquigarrow$ **C** defines an antiequivalence of categories. Where does this argument fail?

26. Let $\mathbb{A}_2 = \operatorname{Spec} k[x,y]$. Show that the complement U of the origin is not an affine scheme. (*Hint*: $\Gamma(U, \mathcal{O}_{\mathbb{A}_2}) = k[x,y] = \Gamma(\mathbb{A}_2, \mathcal{O}_{\mathbb{A}_2})$).

27. Let U, V be two affine open sets in an affine scheme $X = \operatorname{Spec} A$. Prove that $U \cap V$ also is an affine scheme and that the natural morphism $\mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is an isomorphism.

28. Prove that the underlying topological space of a noetherian scheme is a projective limit of finite topological spaces. (*Hint*: It is a spectral space, p. 239).

29. Let X be a scheme and let $\mathcal{O}_{X_{\text{red}}}$ be the sheaf of rings defined by the presheaf $U \rightsquigarrow \mathcal{O}_X(U)_{\text{red}}$. Show that $X_{\text{red}} = (X, \mathcal{O}_{X_{\text{red}}})$ is a reduced closed subscheme of X with the following universal property: We have $\operatorname{Hom}(Y, X) = \operatorname{Hom}(Y, X_{\text{red}})$ for any reduced scheme Y.

30. Let X be a scheme. Prove that a sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ of quasi-coherent sheaves is exact if and only if so is, for any quasi-coherent sheaf $\mathcal{N}$, the sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M}') \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M}'').$$

Prove that a sequence $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ of quasi-coherent sheaves is exact if and only if so is, for any quasi-coherent sheaf $\mathcal{N}$, the sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}'', \mathcal{N}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}'', \mathcal{N}).$$

31. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{M}$ is **quasi-coherent** if any point admits an open neighborhood U and a presentation (I, J are arbitrary sets of indices)

$$\oplus_J \mathcal{O}_X|_U \longrightarrow \oplus_I \mathcal{O}_X|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0.$$

Prove that this definition coincides with the definition of p. 324 when X is a scheme.

An $\mathcal{O}_X$-module $\mathcal{M}$ is of **finite type** if any point admits an open neighborhood U and an epimorphism

$$\mathcal{O}_X^n|_U \longrightarrow \mathcal{M}|_U \longrightarrow 0,$$

where n is a natural number, and $\mathcal{M}$ is **coherent** if it is of finite type and, for any open set V and any morphism $f\colon \mathcal{O}_X^n|_V \to \mathcal{M}|_V$, the kernel of f is of finite type. Prove that this definition coincides with the definition of p. 324 when X is a noetherian scheme.

32. Let X be a noetherian scheme. If $\mathcal{M}$ is a coherent sheaf and $\mathcal{N}$ is a quasi-coherent sheaf, prove that $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is quasi-coherent. (*Hint*: $\operatorname{Hom}_A(M,N)_S = \operatorname{Hom}_{A_S}(M_S, N_S)$ when M is a finitely generated module over a noetherian ring A).

33. If $\mathcal{M}$ is a coherent sheaf on a noetherian scheme X and $\mathcal{N}$ is an $\mathcal{O}_X$-module, show that

$$\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})_x = \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{N}_x).$$

34. Let $f\colon Y \to X$ be a morphism of noetherian schemes. Prove the following statements:

    (a) If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, then $f^*\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module.

    (b) If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, then $f^*\mathcal{M}$ is a coherent $\mathcal{O}_Y$-module.

    (c) If $\mathcal{N}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $f_*\mathcal{N}$ is a quasi-coherent $\mathcal{O}_X$-module. (*Hint*: When X is affine, cover Y with a finite number of affine open sets V_i, and cover $V_i \cap V_j$ with a finite number of affine open sets V_{ijk}.)

35. Let A be a noetherian ring and put $X = \operatorname{Spec} A$. If M is a finitely generated A-module, show that

$$\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) = \operatorname{Hom}_A(M, N)^{\sim}.$$

36. A morphism of schemes $f\colon X \to S$ is said to be **affine** if any point $s \in S$ has an open affine neighborhood $U = \operatorname{Spec} A$ such that $f^{-1}(U) = \operatorname{Spec} B$ is affine. If $f\colon X \to S$ is an affine morphism, prove that,

    (a) If $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, then $f_*\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module and

$$H^p(X, \mathcal{M}) = H^p(S, f_*\mathcal{M}), \quad p \geq 0.$$

(b) If $g\colon Y \to S$ is another affine morphism, then $\underline{\mathrm{Hom}}_S(X,Y)(U) = \mathrm{Hom}_U(X|_U, Y|_U)$ is a sheaf of sets on S, and the natural morphism $\underline{\mathrm{Hom}}_S(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S\text{-alg}}(f_*\mathcal{O}_X, g_*\mathcal{O}_Y)$ is an isomorphism of sheaves.

(c) The natural map $\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(f_*\mathcal{O}_X, g_*\mathcal{O}_Y)$ is bijective, so that a S-morphism $X \to Y$ is an isomorphism if and only so is the induced morphism $f_*\mathcal{O}_X \to g_*\mathcal{O}Y$.

(d) Conclude that $f^{-1}(V)$ is affine for any affine open set $V \subseteq X$. In fact, if $X \to \mathrm{Spec}\, A$ is an affine morphism, then X is an affine scheme, so that the category of affine schemes over $\mathrm{Spec}\, A$ is equivalent to the opposite category of A-algebras.

37. If $V \subset U$ are affine open sets in a scheme X, prove that the restriction morphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is flat, and the natural morphism $\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{M}(V)$ is an isomorphism for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$.

38. If $x_1, \ldots, x_n$ are closed points of an affine curve C, prove that $C - \{x_1, \ldots, x_n\}$ is affine.

39. If x is a closed point of a non singular complete curve X, show that the line sheaf L_{-x} is just the sheaf of ideals of the closed subscheme $\mathrm{Spec}\,\kappa(x) \to X$.

40. Let $\mathcal{M}$ be a torsion-free coherent sheaf of rank r on a non singular complete curve X. If D is a divisor of the line sheaf $\Lambda^r\mathcal{M}$, prove that $\chi(X,\mathcal{M}) = r \cdot \chi(X,\mathcal{O}_X) + \deg D$.

41. Let $(X,\mathcal{O}_X)$ be a ringed space. If $\mathcal{M}$ is an $\mathcal{O}_X$-module and $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, prove that the sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M},\mathcal{I})$ is flasque. (*Hint*: The sheaf $\mathcal{M}_U$ is an $\mathcal{O}_X$-module).
Conclude that $\mathrm{Ext}^n_{\mathcal{O}_X}(E,\mathcal{M}) = H^n(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(E,\mathcal{M}))$ when E is a locally free $\mathcal{O}_X$-module.

42. If E is locally free sheaf on $\mathbb{P}_1$, prove the following statements:

(a) There is an exact sequence $0 \to \mathcal{O}(-1) \to E(n) \to \bar{E} \to 0$, where $H^0(\mathbb{P}_1, E(n)) = 0$ and $\bar{E}$ is locally free.

(b) $H^0(\mathbb{P}_1, \bar{E}) = 0$, so that $\bar{E} = \oplus_i \mathcal{O}(-m_i)$, $m_i \geq 1$, by induction on the rank.

(c) $\mathrm{Ext}^1_{\mathcal{O}}(\bar{E}, \mathcal{O}(-1)) = 0$, so that the exact sequence $0 \to \mathcal{O}(-1) \to E(n) \to \bar{E} \to 0$ splits.

(d) Conclude that E is a direct sum of line sheaves, $E \simeq \oplus_i \mathcal{O}(n_i)$

Moreover, show that $\mathrm{Ext}^1_{\mathcal{O}}(E,T) =$ for any torsion coherent sheaf T, so that any coherent sheaf on $\mathbb{P}_1$ is a direct sum of line sheaves and a torsion sheaf.

43. Let Σ be the field of rational functions on a complete non singular curve X over a field k.
If $x \in X$ is a rational point and $\mathfrak{m}_x = t\mathcal{O}_x$, show that $\Sigma/\mathcal{O}_x = kt^{-1} \oplus \ldots \oplus kt^{-n} \oplus \ldots$
Given closed points $x_1, \ldots, x_r$ of an affine open set U and Laurent series $f_i \in \Sigma/\mathcal{O}_{x_i}$, $1 \leq i \leq r$, show the existence of a rational function $f \in \Sigma$ with Laurent expansion f_i at x_i and no other pole in U.

44. Let $\pi\colon X \to X'$ be a morphism of degree d between non singular complete curves. A closed point $x \in X$ is said to be a **ramification** point when $\mathfrak{m}_{x'}\mathcal{O}_{X,x} = \mathfrak{m}_x^r$, $r \geq 2$, and $r_x = r$ is defined to be the **ramification index** of π at x.

(a) Show that $d = \sum_{\pi(x)=x'} r_x$ at any point $x' \in X'$.

(b) If the base field k has null characteristic, prove that $l\big(\Omega_{\mathcal{O}_x/\mathcal{O}_{x'}}\big) = r_x - 1$.

45. Let C be a complete non singular curve over a field k of null characteristic. Prove that

(a) Any non-constant k-morphism $C \to \mathbb{P}_{1,k}$ without ramification points is an isomorphism.

(b) If there is a non constant k-morphism $\mathbb{P}_{1,k} \to C$, then the genus of C is 0.

(c) Any subfield of $k(t)$ strictly containing k is $k(q(t))$ for some rational function $q(t)$.

(d) Let C' be another complete non singular curve. If there is a non constant k-morphism $C' \to C$ without ramification points and C has genus $g = 1$, then the genus of C' is $g' = 1$.

46. Prove that the canonical series K of a complete non singular curve C has no fixed point; that is to say, $h^0(K) > h^0(K - x)$ for any closed point $x \in C$.

47. Let C be a complete non singular curve over a field k (algebraically closed in the field of rational functions Σ_C, as we always assume). If D is an effective divisor, show that $h^0(D) \leq 1 + \deg D$, and that the equality holds if and only if C has genus $g = 0$.

48. Let D', D be two effective divisors on a complete non singular curve C. Prove that the morphism
$$\mathbb{P}(H^0(C, L_D)) \times \mathbb{P}(H^0(C, L_{D'})) \longrightarrow \mathbb{P}(H^0(C, L_{D+D'})),\ (D_1, D_2) \mapsto D_1 + D_2,$$
has finite fibres, and obtain that $h^0(D) + h^0(D') \leq h^0(D + D') + 1$.
If D is a **special** divisor, $H^1(C, L_D) \neq 0$, conclude that $h^0(D) \leq 1 + \frac{1}{2}\deg D$.

49. Let $\pi\colon \bar{X} \to X$ be a non constant morphism between non singular complete curves over an algebraically closed field k. Since $(\Omega_X, \mathrm{Res}_X)$ represents the functor $\mathcal{M} \rightsquigarrow H^1(X, \mathcal{M})^*$ and we have the residue map $\mathrm{Res}_{\bar{X}}\colon H^1(X, \pi_*\Omega_{\bar{X}}) = H^1(\bar{X}, \Omega_{\bar{X}}) \to k$, then we have a morphism of $\mathcal{O}_X$-modules $\mathrm{tr}\colon \pi_*\Omega_{\bar{X}} \to \Omega_X$, the **trace** morphism, such that $\mathrm{Res}_X\big(\mathrm{tr}(\theta)\big) = \mathrm{Res}_{\bar{X}}(\theta)$, $\forall \theta \in H^1(X, \pi_*\Omega_{\bar{X}}) = H^1(\bar{X}, \Omega_{\bar{X}})$.
If $\pi^{-1}(x) = \{\bar{x}_1, \ldots, \bar{x}_n\}$, prove that for any meromorphic 1-form $\bar{\omega}$ on $\bar{X}$ we have
$$\mathrm{Res}_x\big(\mathrm{tr}(\bar{\omega})\big) = \mathrm{Res}_{\bar{x}_1}\bar{\omega} + \ldots + \mathrm{Res}_{\bar{x}_n}\bar{\omega}.$$

50. Show the existence of a natural morphism of schemes $\mathrm{Proj}\, A \to \mathrm{Spec}\, A_0$, $\mathfrak{p} \mapsto A_0 \cap \mathfrak{p}$, for any graded ring $A = A_0 \oplus A_1 \oplus \ldots$.

51. Let **Aff** be the category of affine spaces and affinities over a field k, and let $\mathbf{Aff_{sch}}$ be the category of pairs $(\mathbb{P}, H)$, where H is a hyperplane of a k-projective space $\mathbb{P} \simeq \mathrm{Proj}\, k[x_0, \ldots, x_d]$, and k-isomorphisms preserving the hyperplanes. Show that $F\colon \mathbf{Aff} \rightsquigarrow \mathbf{Aff_{sch}}$, $F(\mathbb{A}_n, V) = (\mathrm{Proj}\, S^\bullet E^*, (\omega)_0)$ is an equivalence of categories, where E is the vector extension of $\mathbb{A}_n$ and $\omega \in E^* = F$ is the linear form defined by the constant function 1. (*Hint*: If you are willing to assume that the objects form a set, you may use problem 53 in p. 503. Otherwise you must define $\mathbb{A}_n$ to be the set of rational points of $\mathbb{P} - H$ and V to be the kernel of the transpose of $k = \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \to \Gamma(\mathbb{P}, L_H) = F$).

52. Let C be a complete non-singular conic over a field k. If C has no rational point, prove that any closed point $x \in C$ has even degree $[k : \kappa(x)]$, and that $\deg\colon \mathrm{Pic}\, C \to 2\mathbb{Z}$ is an isomorphism.

53. Let X be a closed subscheme of $\mathbb{P}_{n,k}$ defined by an homogeneous ideal $(p_1, \ldots, p_n)$ of $k[x_0, \ldots, x_n]$, $d_i = \deg p_i$. If $\dim X = 0$, prove **Bézout's theorem**: $\dim_k H^0(X, \mathcal{O}_X) = d_1 \ldots d_n$.

54. Let $X = \mathrm{Proj}\, k[x_0, \ldots, x_d]/(P_n)$ be an hypersurface of degree n in $\mathbb{P}_d$, $d \geq 2$. Prove that
   (a) $H^0(X, \mathcal{O}_X) = k$, so that X is connected.
   (b) $\dim H^{d-1}(X, \mathcal{O}_X) = \binom{n-1}{d}$.
   (c) $H^p(X, \mathcal{O}_X) = 0$ when $p \neq 0, d-1$.

55. **Max Noether's Theorem:** Let $P_n(x_0, x_1, x_2) = 0$ and $Q_m(x_0, x_1, x_2) = 0$ be two plane curves with no common irreducible component, and let $\mathcal{I}$ be the sheaf of ideals of the intersection. If a curve $R_d(x_0, x_1, x_2) = 0$ passes through any common point and it is locally in the ideal of the intersection, in the sense that $R_d \in \Gamma(\mathbb{P}_2, \mathcal{I}(d))$, prove that $R_d \in (P_n, Q_m)$.

56. Let X be a projective variety over a field. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, prove the existence of a polynomial $H(n)$ with rational coefficients such that $H(n) = \chi(X, \mathcal{M}(n))$. (*Hint*: Induction on the dimension of $\mathrm{supp}\, \mathcal{M}$).

57. Now we put $R = k[x_0, \ldots, x_d]$, we consider the irrelevant ideal $\mathfrak{m} = (x_0, \ldots, x_d)$, and M will be a finitely generated graded R-module. We say that M is a free graded module if it admits a basis of homogeneous elements, $M \simeq \oplus_i R(n_i)$. Prove the following statements:
   (a) If $\mathfrak{m}M = M$, then $M = 0$.
   (b) If $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$, then M is a free graded R-module.
   (c) M admits a free graded resolution $0 \to L_{d+1} \to \ldots \to L_0 \to M \to 0$.
   (d) Let $A = k[x_1, \ldots, x_d] = R/(x_0 - 1)$ and let $\bar{M}$ be a finitely generated A-module. We have $\bar{M} \simeq M \otimes_R A$ for some finitely generated graded R-module M. (*Hint*: Applying the base change $R \to A$ to graded morphisms between free graded R-modules we may obtain any morphism between free A-modules, just homogenize the matrix).

(e) $\operatorname{Tor}_p^R(R/(x_0-1),\bar{M})=0$, $p\geq 1$.

(f) $\bar{M}$ admits a free resolution $0\to\bar{L}_{d+1}\to\ldots\to\bar{L}_0\to\bar{M}\to 0$.

(g) The K-group of finitely generated A-modules is $K(k[x_1,\ldots,x_d])=\mathbb{Z}$.

58. Let $(\mathcal{O},\mathfrak{m})$ be an integral noetherian local ring of dimension 1, and put

$$R_{\mathfrak{m}}\mathcal{O}=\mathcal{O}\oplus\mathfrak{m}\oplus\mathfrak{m}^2\oplus\ldots$$
$$G_{\mathfrak{m}}\mathcal{O}=(\mathcal{O}/\mathfrak{m})\oplus(\mathfrak{m}/\mathfrak{m}^2)\oplus\ldots$$

Prove that $\operatorname{Proj}G_{\mathfrak{m}}\mathcal{O}$ is finite, and that there exists $\xi\in\mathfrak{m}^d$ not vanishing at any point of it.

Prove that the blow-up $C_1=\operatorname{Proj}R_{\mathfrak{m}}\mathcal{O}$ is an affine scheme, $C_1=\operatorname{Spec}\mathcal{O}_1$.

Moreover, if $\mathfrak{m}^d=(\xi,\xi_1,\ldots,\xi_r)$, then $\mathcal{O}_1=\mathcal{O}\left[\frac{\xi_1}{\xi},\ldots,\frac{\xi_r}{\xi}\right]$ is a finite $\mathcal{O}$-algebra.

59. Let k be a noetherian ring. Prove the following statements:

(a) If M is a finitely generated k-module and N is a finitely generated flat k-module, then there is a k-scheme $\mathbf{Hom}_k(M,N)$ such that for any k-scheme X we have

$$\operatorname{Hom}_{\mathcal{O}_X}(M\otimes_k\mathcal{O}_X,N\otimes_k\mathcal{O}_X)=\operatorname{Hom}_k\big(X,\mathbf{Hom}_k(M,N)\big).$$

(b) If A is a finite flat k-algebra, the functor $X\rightsquigarrow\operatorname{Hom}_{\mathcal{O}_X\text{-alg}}(A\otimes_k\mathcal{O}_X,A\otimes_k\mathcal{O}_X)$ is representable by a k-scheme $\mathbf{End}_{k\text{-alg}}(A)$,

$$\operatorname{Hom}_{\mathcal{O}_X\text{-alg}}(A\otimes_k\mathcal{O}_X,A\otimes_k\mathcal{O}_X)=\operatorname{Hom}_k\big(X,\mathbf{End}_{k\text{-alg}}(A)\big),$$

and the functor $X\rightsquigarrow\operatorname{Aut}_{\mathcal{O}_X\text{-alg}}(A\otimes_k\mathcal{O}_X,A\otimes_k\mathcal{O}_X)$ is representable by an open subscheme $\mathbf{Aut}_{k\text{-alg}}(A)$ of $\mathbf{End}_{k\text{-alg}}(A)$,

$$\operatorname{Aut}_{\mathcal{O}_X\text{-alg}}(A\otimes_k\mathcal{O}_X,A\otimes_k\mathcal{O}_X)=\operatorname{Hom}_k\big(X,\mathbf{Aut}_{k\text{-alg}}(A)\big).$$

60. Determine $\mathbf{Aut}_{k\text{-alg}}(A)$ when $k=\mathbb{Q}$, $A=\mathbb{Q}(\sqrt[n]{2})$; and also when $k=\mathbb{F}_2(t)$, $A=k(\sqrt{t})$.

61. Let C be an **elliptic** curve (complete non singular curve of genus 1) over an algebraically closed field k. Prove the following statements (where all the considered points are closed):

(a) C is isomorphic to a non singular plane cubic.

(b) Given two points $x,y\in C$, we have $y=\tau(x)$ for some involution $\tau\colon C\to C$. (*Hint*: Construct a morphism $\pi\colon C\to\mathbb{P}_1$ of degree 2 such that $\pi(x)=\pi(y)$).

(c) Given morphisms $\pi_1,\pi_2\colon C\to\mathbb{P}_1$ of degree 2, prove the existence of automorphisms $\tau\colon C\to C$ and $\sigma\colon\mathbb{P}_1\to\mathbb{P}_1$ such that the following square commutes:

$$\begin{array}{ccc} C & \xrightarrow{\ \tau\ } & C \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ \mathbb{P}_1 & \xrightarrow{\ \sigma\ } & \mathbb{P}_1 \end{array}$$

(d) If $\operatorname{char}k\neq 2$, any morphism $C\to\mathbb{P}_1$ of degree 2 ramifies at 4 points $P_1,P_2;P_3,P_4\in\mathbb{P}_1$ and, if we put $\lambda=(P_1,P_2;P_3,P_4)$, the elliptic curve is classified by the j-invariant (p. 513)

$$j(\lambda)=\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}\cdot$$

(e) If we fix a point $x\in C$, any line sheaf of degree 0 is L_{y-x} for a unique point $y\in C$.

62. Let C be a non singular plane cubic over an algebraically closed field. Prove that the tangent lines at three collinear points intersect C at collinear points.

Moreover, if a straight line passes through two **inflexion** points (points where the tangent intersects with multiplicity greater than 2) of C, so is the third

63. Let $U=\operatorname{Spec}A=C-\{p_1,\ldots,p_r\}$ be an affine open set of a complete non singular curve C over an algebraically closed field k. Prove that the following conditions are equivalent:

(a) The ring A is a principal ideal domain.

(b) $\mathrm{Pic}(C) = \sum_i \mathbb{Z}p_i$.
(c) The curve C has genus 0.

Conclude that the following conditions are equivalent:

(a) The ring $k[x,y]/(p(x,y))$ is a principal ideal domain.
(b) The polynomial $p(x,y)$ is irreducible and $p(x,y)=0$ is a non singular curve of genus 0.
(c) The ring $k[x,y]/(p(x,y))$ is a Euclidean ring.

64. Show that no real maximal ideal of $\mathbb{R}[x,y]/(x^2+y^2-1)$ is a principal ideal.

65. Let X be a non singular complete curve of genus $g \geq 2$ over an algebraically closed field k. Prove that

(a) The canonical divisors of X define a closed embedding $X \hookrightarrow \mathbb{P}_{g-1}$ if and only if for any pair of closed points p,q we have $h^0(L_{K-p-q}) = g-2$, or equivalently, $h^0(L_{p+q}) = 1$.
(b) This canonical embedding identifies X with a curve of degree $2g-2$ in $\mathbb{P}_{g-1}$, well defined up to projectivities (so reducing the classification of curves of genus g to the projective classification of such canonical curves), except when X is **hyperelliptic** (it admits a projection $\pi\colon X \to \mathbb{P}_1$ of degree 2).
(c) If X is hyperelliptic and $\mathrm{char}\, k \neq 2$, this projection $\pi\colon X \to \mathbb{P}_1$ of degree 2 ramifies at $2g+2$ points, so that $\Sigma_X = k\big(x, \sqrt{P(x)}\,\big)$ for some polynomial $P(x)$ of degree $2g+2$. (*Hint*: Hurwitz's formula).

66. Let $X = \mathrm{Proj}\, k[x_0,x_1,x_2]/(P_n)$ be a plane curve of degree n. Show that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{P_n\cdot} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and obtain that $\pi := \dim_k H^1(X,\mathcal{O}_X) = \dim_k H^2(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(-n)) = \binom{n-1}{2}$.

When k is algebraically closed, if m_y denote the multiplicity of any point y of the blow-up tree A_x (p. 216) at a singular point x, conclude that the geometric genus g of X is

$$g = \binom{n-1}{2} - \sum_{x\in X}\sum_{y\in A_x}\binom{m_y}{2}.$$

67. Show that the dualizing sheaf $\mathcal{D}_X$ of a complete curve X is a torsion free coherent sheaf of rank 1. (*Hint*: Given a non-singular closed point x, show that $S(n) = \dim_k H^0(X, \mathcal{D}_X \otimes_{\mathcal{O}_X} L_{nx})$ goes to ∞ as a polynomial of degree 1). Conclude that, if ω_X is a torsion free coherent sheaf of rank 1 such that $h^0(X,\omega_X) = h^1(X,\mathcal{O}_X)$ and $h^1(X,\omega_X) = h^0(X,\mathcal{O}_X)$, then $\omega_X \simeq \mathcal{D}_X$.

12. Algebraic Topology I

1. Define a natural bijection between the set of subsheaves of a sheaf of sets $\mathcal{F}$ and the set of open sets in $\mathcal{F}^{et}$.

2. Let $\mathcal{C}$ be the sheaf of real valued continuous functions on the real line $X=\mathbb{R}$. Show that the natural map $\oplus_{\mathbb{N}}\mathcal{C}(X) \to (\oplus_{\mathbb{N}}\mathcal{C})(X)$ is not surjective, and that the natural maps $(\prod_{\mathbb{N}}\mathcal{C})x \to \prod_{\mathbb{N}}\mathcal{C}_x$ are not injective.

   Let $\mathcal{O}$ be the sheaf of analytic functions on the complex plane $X=\mathbb{C}$. Show that the natural maps $(\prod_{\mathbb{N}}\mathcal{O})x \to \prod_{\mathbb{N}}\mathcal{O}_x$ are not surjective.

3. Let U be an open set in a topological space X and put $Y = X-U$. If $\mathcal{F}$ is a sheaf on X, show that $\mathcal{F}_U = \mathbb{Z}_U \otimes_{\mathbb{Z}} \mathcal{F}$ and $\mathcal{F}_Y = \mathbb{Z}_Y \otimes_{\mathbb{Z}} \mathcal{F} = i_*(\mathcal{F}|_Y)$, where $i\colon Y \to X$ is the natural map.

4. Consider the closed set $Y=[0,\infty)$ in the real line $X=\mathbb{R}$. Show that the étalé space $(\mathbb{Z}_Y)^{et}$ is not a Hausdorff space.

5. Let $F_1(t,x_1,x_2,x_3)\partial_1 + F_2(t,x_1,x_2,x_3)\partial_2 + F_3(t,x_1,x_2,x_3)\partial_3$ be a smooth vector field on the complement $U \subset \mathbb{R}^4$ of a finite number of non-intersecting trajectories (smooth sections of the first projection $t\colon \mathbb{R}^4 \to \mathbb{R}$). If it is a conservative force field, in the sense that $\partial_j F_i = \partial_i F_j$, $i,j=1,2,3$, prove that $F_i = \partial_i u$, $i=1,2,3$, for some smooth function $u(t,x_1,x_2,x_3)$ on U.

6. Given a closed cover $X = Y_1 \cup Y_2$, show that $\chi(Y_1 \cup Y_2) = \chi(Y_1) + \chi(Y_2) + \chi(Y_1 \cap Y_2)$, in case that all the considered Euler-Poincaré characteristics are finite.

   If X, Y are connected compact manifolds of dimension n, conclude that $\chi(X\#Y) = \chi(X) + \chi(Y) - 2$ when n is even, and $\chi(X\#Y) = \chi(X) + \chi(Y)$ when n is odd.

7. Let ω_p be a closed differential p-form on the sphere S_n. If $p < n$, show that ω_p is exact. If $p = n$, show that ω_p is exact if and only if $\int_{S_n} \omega_p = 0$.

8. If A is a boolean algebra, prove that any sheaf on $X = \operatorname{Spec} A$ is acyclic.

9. Let $f\colon S_1 \to S_1$ be the covering $f(z) = z^2$. Show that the sheaf $f_*\mathbb{Z}$ is not constant, even if any stalk of $f_*\mathbb{Z}$ is isomorphic to $\mathbb{Z}^2$.

10. If I, J are two ideals of a ring A, show that

    (a) $\operatorname{Tor}_1^A(A/I, A/J) = (I \cap J)/IJ$.

    (b) $\operatorname{Tor}_2^A(A/I, A/J) = \operatorname{Tor}_1^A(I, A/J)$.

    (c) $\operatorname{Tor}_{n+1}^A(A/I, A/J) = \operatorname{Tor}_n^A(I, A/J) = \operatorname{Tor}_{n-1}(I, J)$, $n \geq 2$.

11. Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$, where $d = \text{m.c.d.}(n, m)$, and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$.

12. Prove that the Tor functors preserve inductive limits: $\operatorname{Tor}_n^A(\varinjlim M_i, N) = \varinjlim \operatorname{Tor}_n^A(M_i, N)$.

13. Prove that an A-module M is flat when $\operatorname{Tor}_1^A(M, A/I) = 0$ for any ideal I of A. (*Hint*: It is enough to show that $\operatorname{Tor}_1^A(M, N) = 0$ when $N = An_1 + \ldots + An_r$ is finitely generated. Then use induction on r and the exact sequence $0 \to An_1 \to N \to N/An_1 \to 0$.)

14. Prove that $\operatorname{Tor}_1^A(M, A/I) = 0$ if and only if the natural morphism $I \otimes_A M \to IM$ is an isomorphism.

15. If $A \to B$ is a flat morphism, show that $\operatorname{Tor}_n^A(M, N) \otimes_A B = \operatorname{Tor}_n^B(M_B, N_B)$.

    In particular, the Tor functors localize, $\operatorname{Tor}_n^A(M, N)_S = \operatorname{Tor}_n^{A_S}(M_S, N_S)$.

16. Using the ideal criterion, prove that an A-module M is injective when $\operatorname{Ext}^1_A(M, A/I) = 0$ for any ideal I of A.

17. When A is noetherian, prove that an A-module M is injective when $\operatorname{Ext}^1_A(M, A/\mathfrak{p}) = 0$ for any prime ideal $\mathfrak{p}$ of A. (*Hint*: Any finitely generated A-module N admits a filtration $0 = N_0 \subset N_1 \subset \ldots \subset N_r = N$ such that $N_i/N_{i-1} \simeq A/\mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i$.)

18. Prove that $\operatorname{Ext}^n_A(M, N) \otimes_A B = \operatorname{Ext}^n_B(M_B, N_B)$, when A is noetherian, M is finitely generated and $A \to B$ is a flat morphism. (*Hint*: If L is a finitely generated free A-module, then $\operatorname{Hom}_A(L, N) \otimes_A B = \operatorname{Hom}_B(L_B, N_B)$.)

19. Let $0 \to M \to R^\bullet$ be a resolution of an A-module, and $F\colon A\textbf{-mod} \rightsquigarrow B\textbf{-mod}$ a left-exact additive functor. If $0 \to M \to A^\bullet$ is a F-acyclic resolutions, and $f, g\colon R^\bullet \to A^\bullet$ are morphisms of complexes such that we have commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{\sim} & R^\bullet \\ {\scriptstyle\wr}\downarrow & \swarrow{\scriptstyle f} & \\ A^\bullet & & \end{array} \qquad\qquad \begin{array}{ccc} M & \xrightarrow{\sim} & R^\bullet \\ {\scriptstyle\wr}\downarrow & \swarrow{\scriptstyle g} & \\ A^\bullet & & \end{array}$$

prove that the induced morphisms $f, g\colon H^n[F(R^\bullet)] \to H^n[F(A^\bullet)]$ coincide. (*Hint*: Both coincide with the composition of $DR\colon H^n[F(R^\bullet)] \to R^nF(M)$ with the inverse of $DR\colon H^n[F(A^\bullet)] \xrightarrow{\sim} R^nF(M)$).

Conclude that, in the case of the functorial injective resolution $0 \to M \to I^\bullet M$, the natural morphism $DR\colon H^n[F(I^\bullet M)] \to R^nF(M)$ is the identity.

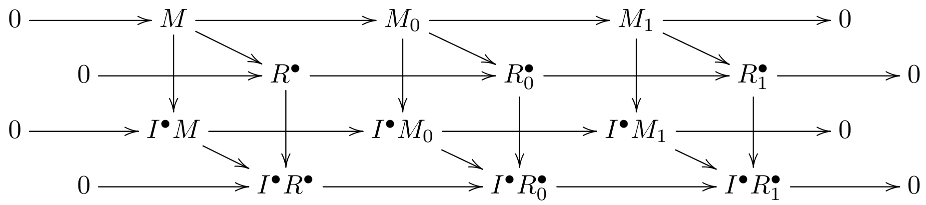
Given an exact sequence $0 \to M \to M_0 \to M_1 \to 0$, resolutions $0 \to M_0 \to R_0^\bullet$, $0 \to M_1 \to R_1^\bullet$ and a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^\bullet & \longrightarrow & R_0^\bullet & \longrightarrow & R_1^\bullet & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & 0 \end{array}$$

prove that we have a commutative square

$$\begin{array}{ccc} H^n[F(R^\bullet)] & \xrightarrow{\delta} & H^{n+1}F(R_1^\bullet)] \\ {\scriptstyle DR}\downarrow & & \downarrow{\scriptstyle DR} \\ R^nF(M) & \xrightarrow{\delta} & R^{n+1}F(M_1) \end{array}$$

so that the connecting morphism $\delta\colon R^nF(M_1) \to R^{n+1}F(M)$ is independent of the F-acyclic resolutions used to define it. (*Hint*: Apply F to the following commutative diagram with exact rows:)



State and prove analogous results when M is replaced by a bounded below complex $K^\bullet$.

Given an exact sequence of complexes $0 \to K^\bullet \xrightarrow{i} K_0^\bullet \xrightarrow{p} K_1^\bullet \to 0$, prove that we have a natural quasi-isomorphism $p\colon \mathrm{Cone}^\bullet i \xrightarrow{\sim} K_1^\bullet$ such that the connecting morphism $\delta\colon H^n(K_1^\bullet) \to H^{n+1}(K^\bullet)$ is just the opposite of the morphism $H^n(K_1^\bullet) = H^n(\mathrm{Cone}^\bullet i) \to H^n(K^\bullet[1]) = H^{n+1}(K^\bullet)$. Use this fact to prove again that the connecting morphism $\delta\colon \mathbf{R}^nF(K_1^\bullet) \to \mathbf{R}^{n+1}F(K^\bullet)$ is independent of the F-acyclic resolutions used to define it.

20. If $f\colon Y \to X$ is a continuous map, show that $f^*\mathbb{Z}_X = \mathbb{Z}_Y$.

21. If $f\colon Y \to X$ is a continuous map and $\mathcal{F}$ is a sheaf on X, show that $f^*\mathcal{F}$ is the associated sheaf of the presheaf $V \rightsquigarrow \varinjlim_{f(V)\subseteq U} \mathcal{F}(U)$.

22. Let $\bar{X}$ be a topological space X endowed with the discrete topology, and let $j\colon \bar{X} \to X$ be the obvious continuous map. Show that $\mathcal{C}^0\mathcal{F} = j_*j^*\mathcal{F}$ for any sheaf $\mathcal{F}$ on X.

23. Prove that the cohomology groups $H^1(X,\mathbb{Z})$, $H^1_c(X,\mathbb{Z})$ and $H^1_Y(X,\mathbb{Z})$ are always torsion-free. (*Hint*: The exact cohomology sequence of the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$).

24. If $Y \subseteq X$ is a locally closed subspace and $\mathcal{F}$ is a sheaf on Y, prove that there is a unique sheaf $\mathcal{F}^X$ on X such that $\mathcal{F}^X|_Y = \mathcal{F}$ and $\mathcal{F}^X|_{X-Y} = 0$, namely $\mathcal{F}^X(U) = \{s \in \mathcal{F}(U\cap Y)\colon \mathrm{supp}\,(s) \text{ is closed in } U\}$.

25. If $j\colon Y \to X$ is a closed subspace and $\mathcal{F}$ is a sheaf on X, show that $\mathcal{F}_Y = j_*j^*\mathcal{F}$. If $j\colon U \to X$ is an open subset of a σ-compact space X and $\mathcal{F}$ is a sheaf on X, show that $\mathcal{F}_U = j_!j^*\mathcal{F}$.

26. **Excision:** Let $Y \subseteq X$ be a closed subset and let $\mathcal{F}$ be a sheaf on X. If U is a neighborhood of Y, show that $H^p_Y(X,\mathcal{F}) = H^p_Y(U,\mathcal{F}|_U)$.

27. Let $\mathcal{O}$ be a sheaf of rings on a topological space X. If $\mathcal{M}$ is an $\mathcal{O}$-module, show that the cohomology groups $H^p(X,\mathcal{M})$, $H^p_c(X,\mathcal{M})$ and $H^p_Y(X,\mathcal{M})$ have a natural structure of $\mathcal{O}(X)$-module.

28. Let $j\colon U \to X$ be an open subset of a topological space X and let $i\colon Y = X-U \to X$ be the complementary closed set. If $\mathcal{O}$ is a sheaf of rings on X, show that $\mathcal{O}_U$ and $\mathcal{O}_Y$ are $\mathcal{O}$-modules, and prove that for any $\mathcal{O}$-module $\mathcal{M}$ we have natural isomorphisms $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_U,\mathcal{M}) = \mathcal{M}(U)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O}_U,\mathcal{M}) = j_*j^*\mathcal{M}$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_Y,\mathcal{M}) = \Gamma_Y(X,\mathcal{M})$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O}_Y,\mathcal{M}) = \underline{\Gamma}_Y\mathcal{M}$.

29. Let $(X,\mathcal{O})$ be a ringed space. If $\mathcal{M},\mathcal{N}$ are $\mathcal{O}$-modules, then $\underline{\mathrm{Tor}}_n^{\mathcal{O}}(\mathcal{M},\mathcal{N})$ will denote the associated sheaf of the presheaf $U \rightsquigarrow \mathrm{Tor}_n^{\mathcal{O}(U)}\big(\mathcal{M}(U),\mathcal{N}(U)\big)$, and it is endowed with a natural structure of $\mathcal{O}$-module. If $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is an exact sequence of $\mathcal{O}$-modules, prove that we have a long exact sequence of $\mathcal{O}$-modules

$$\ldots \longrightarrow \underline{\mathrm{Tor}}_1^{\mathcal{O}}(\mathcal{M}'',\mathcal{N}) \xrightarrow{\delta} \mathcal{M}'\otimes_{\mathcal{O}}\mathcal{N} \longrightarrow \mathcal{M}\otimes_{\mathcal{O}}\mathcal{N} \longrightarrow \mathcal{M}''\otimes_{\mathcal{O}}\mathcal{N} \longrightarrow 0.$$

30. Let $(X, \mathcal{O})$ be a ringed space. Show that the product

$$H^p(X,\mathcal{O}) \otimes H^q(X,\mathcal{O}) \xrightarrow{\cup} H^{p+q}(X,\mathcal{O}\otimes\mathcal{O}) \longrightarrow H^{p+q}(X,\mathcal{O})$$

defined by the cup product and the natural morphism $\mathcal{O}\otimes\mathcal{O}\to\mathcal{O}$ defines a structure of anticommutative graded $\mathcal{O}(X)$-algebra on $H^\bullet(X,\mathcal{O}) = \oplus_p H^p(X,\mathcal{O})$. Moreover, if $\mathcal{M}$ is an $\mathcal{O}$-module, the following product defines a structure of graded $H^\bullet(X,\mathcal{O})$-module on $H^\bullet(X,\mathcal{M}) = \oplus_p H^p(X,\mathcal{M})$:

$$H^p(X,\mathcal{O}) \otimes H^q(X,\mathcal{M}) \xrightarrow{\cup} H^{p+q}(X,\mathcal{O}\otimes\mathcal{M}) \longrightarrow H^{p+q}(X,\mathcal{M})$$

31. Let us consider the complex $I_\bullet$ such that $I_1 = \mathbb{Z}x$, $I_0 = \mathbb{Z}v_0 \oplus \mathbb{Z}v_1$ and $\mathrm{d}x = v_1 - v_0$. Prove that the morphisms $s_i\colon K^\bullet \to I_\bullet \otimes K^\bullet$, $s_i(k) = v_i \otimes k$, and $p\colon I_\bullet\otimes K^\bullet \to K^\bullet$, $p(v_i\otimes k) = k$, $p(x\otimes k) = 0$, are quasi-isomorphisms for any complex $K^\bullet$.

    Moreover, given two morphisms of complexes $f_0, f_1\colon K^\bullet \to L^\bullet$, prove that morphisms of complexes $h\colon I_\bullet \otimes K^\bullet \to L^\bullet$ such that $f_0 = h\circ s_0$, $f_1 = h\circ s_1$ (i.e. **homotopies** between f_0 and f_1) correspond to families of morphisms $\{\tau_n\colon K^n \to L^{n-1}\}_{n\in\mathbb{Z}}$ such that $f_1 - f_0 = \mathrm{d}\tau_n + \tau_{n+1}\mathrm{d}\colon K^n \to L^n$. (*Hint*: Put $h(v_i\otimes k) = f_i(k)$, $h(x\otimes k) = \tau(k)$.)

32. Show that the tangent bundle TX and the cotangent bundle T^*X of a smooth manifold X are always orientable manifolds.

33. If M is a finitely generated module over a local ring $(\mathcal{O}, \mathfrak{m})$, prove that all minimal generating systems of M have the same number of elements $\mathrm{g}(M) = \dim_{\mathcal{O}/\mathfrak{m}}(M/\mathfrak{m}M)$. If M is a finitely generated A-module, show that the function $f(x) = \mathrm{g}(M_x)$ is semicontinuous on $\operatorname{Spec} A$; i.e., $\{x\in\operatorname{Spec} A\colon \mathrm{g}(M_x) < n\}$ is an open subset for any $n\in\mathbb{N}$. Moreover, the following conditions are equivalent:

    (a) M is a projective A-module.

    (b) Any point of $\operatorname{Spec} A$ has a basic neighborhood U_f such that M_f is a free A_f-module.

    (c) M is a flat A-module and the function $f(x) = \mathrm{g}(M_x)$ is locally constant on $\operatorname{Spec}_{\mathrm{m}} A$.

34. Let $E\to X$ be a real vector bundle of rank n over a compact separated space X. Prove that the $\mathcal{C}(X)$-module of all continuous sections of E is a finitely generated projective $\mathcal{C}(X)$-module.

35. Prove that a real line bundle $L\to X$ over a (separated) topological manifold X is trivial if and only if for any continuous map $\gamma\colon S_1\to X$ we have that the line bundle $\gamma^*L\to S_1$ is trivial.

36. Let τ_g be a torus of genus g. Prove that the cup product defines a non-singular alternate metric $H^1(\tau_g,\mathbb{Z})\times H^1(\tau_g,\mathbb{Z}) \longrightarrow H^2(\tau_g,\mathbb{Z}) = \mathbb{Z}$.

    Let π_g be the connected sum of g real projective planes. Prove that the cup product defines a non-singular metric $H^1(\pi_g,\mathbb{F}_2)\times H^1(\pi_g,\mathbb{F}_2)\to H^2(\pi_g,\mathbb{F}_2) = \mathbb{F}_2$.

37. Prove that g is the maximum number of disjoint circles in τ_g (and in π_g) with connected complement.

38. Let Y be a closed set of a topological space X. If $\mathcal{F}$ is a flasque sheaf on X, prove that so is $\underline{\Gamma}_Y\mathcal{F}$.

39. Let $j\colon U\to X$ be an open subset of a topological manifold. If Y is a normally orientable closed submanifold of X, prove that $p_U(Y\cap U) = j^*\big(p_X(Y)\big)$.

40. Check that the topological intersection number of a real conic with a tangent line in $\mathbb{R}^2$ is 0, while the topological intersection number of a conic with a tangent line in the complex plane $\mathbb{C}^2$ is 2.

41. If X, Y are topological varieties, prove that $\mathbb{T}_{X\times Y} = (\pi_1^*\mathbb{T}_X)\otimes_{\mathbb{Z}}(\pi_2^*\mathbb{T}_Y)$.

42. If $j\colon Y\to X$ is a closed submanifold of a topological manifold X, prove that $\mathbb{T}_Y = \mathbb{T}_{Y/X}\otimes_{\mathbb{Z}} j^*\mathbb{T}_X$.

43. Prove that any topological manifold of dimension 1 is orientable. Conclude that any non-compact connected smooth curve is diffeomorphic to $\mathbb{R}$. (*Hint*: Consider the integral curves of a vector field without zeros).

44. Define a direct image $i_*\colon H^p(Y,\mathbb{T}_Y)\to H^{p+d}(X,\mathbb{T}_X)$ for any closed submanifold $i\colon Y\to X$ of constant codimension d, and prove that $(ij)_* = i_*j_*$ for any other closed submanifold $j\colon Z\to Y$ of constant codimension.

45. Let $(X, \mathcal{O}_X)$ be a smooth manifold of dimension n. Prove that $\mathbb{T}_X \otimes_{\mathbb{Z}} \mathcal{O}_X = \Omega^n_X$.
    If $\mathfrak{p}$ is the sheaf of ideals of a closed smooth submanifold $Y \to X$ of codimension d, and $\mathcal{N} = (\mathfrak{p}/\mathfrak{p}^2)^*$ is the sheaf of smooth sections of the normal bundle N, show that $\mathbb{T}_{Y/X} \otimes_{\mathbb{Z}} \mathcal{O}_Y \simeq \Lambda^d \mathcal{N}$.
    Conclude that Y is normally orientable in X if and only if N admits a smooth orientation.
46. Let $I = [a, b]$ be a closed interval and put $U = (a, b)$. Show that the dualizing complex $\mathcal{D}_I$ has null cohomology sheaves, except $\mathcal{H}^{-1}(\mathcal{D}_I) = \mathbb{Z}_U$.
47. Determine the cohomology sheaves $\mathcal{H}^p(\mathcal{D}_X)$ when X is the union of a plane and an intersecting line.
48. Let Y be a closed set in a sphere S_n, $n \geq 2$. Prove the following statements:
    (a) If Y is homeomorphic to a ball, $Y \simeq B_r$, then $S_n - Y$ is connected.
    (b) If Y is homeomorphic to sphere, $Y \simeq S_r$, then $r \leq n$, and $Y = S_n$ when $r = n$.
    (c) If $Y \simeq S_{n-1}$, then $S_n - Y$ has two connected components.
    (d) If $Y \simeq S_r$, $r < n - 1$, then $S_n - Y$ is connected.
49. Let Y be a closed set in $\mathbb{R}^n$, $n \geq 2$. If $Y \simeq S_{n-1}$, show that $\mathbb{R}^n - Y$ has two connected components, one bounded and the other not. If $f \colon B_n \to \mathbb{R}^n$ is an injective continuous map, prove that f maps the interior of B_n onto the bounded connected component of $\mathbb{R}^n - f(S_{n-1})$.
50. Prove that any continuous injective map $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is an open map.
51. Let X be a compact oriented smooth manifold of dimension n. Prove that we have a perfect pairing (identifying each factor with the dual of the other factor)
$$H^p(X, \mathbb{R}) \otimes_{\mathbb{R}} H^{n-p}(X, \mathbb{R}) \longrightarrow \mathbb{R}\ , \quad \langle \omega, \eta \rangle = \int_X \omega \wedge \eta.$$
52. Let X be a smooth manifold of dimension n. Prove that X admits a metric g of constant signature $(1, n-1)$ if and only if X admits a distribution of rank 1. (*Hint.*: Consider the endomorphism attached to g and a riemannian metric, and the eigenvectors with positive eigenvalue.)
    Conclude that the sphere S_4 admits no Lorentz metric. (*Hint.*: $\chi(S_4) \neq 0$).
53. Let us consider the roots $c(E) = (1 + \alpha_1) \dots (1 + \alpha_r)$ and $c(F) = (1 + \beta_1) \dots (1 + \beta_s)$ of two complex vector bundles E, F. Prove that $c(E \otimes F) = \prod_{i,j}(1 + \alpha_i + \beta_j)$, $c(\mathrm{Hom}(F, E)) = \prod_{i,j}(1 + \alpha_i - \beta_j)$, $c(\Lambda^p E) = \prod_{i_1 < \dots < i_p}(1 + \alpha_{i_1} + \dots + \alpha_{i_p})$, $c(S^p E) = \prod_{n_1 + \dots + n_r = p}(1 + n_1\alpha_1 + \dots + n_r\alpha_r)$.
54. If X is a closed smooth hypersurface in $\mathbb{R}^n$, prove that $w_i(X) = 0$. (*Hint*: $w_i(X) = w_1(X)^i$ and X is orientable).
55. Prove that $\mathbb{P}_{n,\mathbb{R}}$ admits a tangent vector field without zeroes if and only if n is odd.
56. Let Y be a d-dimensional connected compact topological manifold with boundary. Prove that the connecting morphism $\delta \colon H^d(\partial Y, \mathbb{F}_2) \to H^{d+1}(Y - \partial Y, \mathbb{F}_2)$ is an isomorphism, and that the restriction morphism $j^* \colon H^d(Y, \mathbb{F}_2) \to H^d(\partial Y, \mathbb{F}_2)$ is null.
    Given a sequence $I = (n_1, \dots, n_r) \in \mathbb{N}^r$, if we put $n = n_1 + 2n_2 + \dots + rn_r$, we have a Stieffel-Whitney number $w_I(X) = w_1(X)^{n_1} \dots w_r(X)^{n_r} \in H^n(X, \mathbb{F}_2) = \mathbb{Z}/2\mathbb{Z}$ for any connected compact smooth manifold X of dimension n. Prove that all Stieffel-Whitney numbers are 0 when X is the boundary of a connected compact smooth manifold Y. (*Hint*: Prove that $w_i(X) = j^* w_i(Y)$, using the exact sequence $0 \to TX \to j^*(TY) \to N_{X/Y} \to 0$).
57. Any complex vector bundle $V \to X$ of rank r has an underlying oriented real vector bundle $V_{\mathbb{R}} \to X$ of rank $2r$, and a conjugate vector bundle $\bar{V} \to X$, where the new complex structure is $\alpha \cdot v = \bar{\alpha} v$. Moreover, any real vector bundle $E \to X$ of rank r defines a complex vector bundle $E_{\mathbb{C}} = E \oplus iE \to X$ of rank r. Prove the following statements:
    (a) $E_{\mathbb{C}} = \overline{E_{\mathbb{C}}}$.
    (b) $(V_{\mathbb{R}})_{\mathbb{C}} = V \oplus \bar{V}$.
    (c) If E is an oriented vector bundle of rank r, the natural isomorphism $(E_{\mathbb{C}})_{\mathbb{R}} = E \oplus E$ preserves the orientation if and only if $\binom{r}{2}$ is even.

(d) $c_i(\bar{V}) = (-1)^i c_i(V)$.

(e) $2c_i(E_{\mathbb{C}}) = 0$ when i is odd.

58. The **Pontrjagin classes** of a real vector bundle $E \to X$ are defined to be

$$p_i(E) = (-1)^i c_{2i}(E_{\mathbb{C}}) \in H^{4i}(X, \mathbb{Z}),$$

and $p(E) = \sum_i p_i(E)$ is the total Pontrjagin class. For a smooth manifold X, we put $p_i(X) = p_i(T_X)$. Prove the following properties of these classes:

(a) $p(f^*E) = f^*\big(p(E)\big)$ for any continuous map $f \colon Y \to X$.

(b) $2\big(p(E \oplus F) - p(E) \cdot p(F)\big) = 0$.

(c) $p_i(V_{\mathbb{R}}) = \sum_j (-1)^j c_{i-j}(V) c_{i+j}(V)$ for any complex vector bundle V.

(d) $p_i(\mathbb{P}_{n,\mathbb{C}}) = \binom{n+1}{i} x^i$, where x is the cohomology class of an hyperplane.

59. Prove that the category of sheaves on a finite topological space X is equivalent to the category of systems with indices in the (eventually non-filtered) ordered set X (families of abelian groups $\{\mathcal{F}_x\}_{x \in X}$ endowed with morphisms $\rho^x_y \colon \mathcal{F}_x \to \mathcal{F}_y$ when $x \leq y$, so that $\rho^x_x = \mathrm{Id}$ and $\rho^x_z = \rho^y_z \rho^x_y$ whenever $x \leq y \leq z$). Moreover, show that $\Gamma(X, \mathcal{F}) = \varprojlim_{x \in X} \mathcal{F}_x$.

60. *Derived Functors of the projective limit*: Endow $\mathbb{N}$ with the open sets $U_n = \{0, 1, \ldots, n\}$.

Show that any sheaf $\mathcal{F}$ on $\mathbb{N}$ defines a projective system $F_n = \mathcal{F}(U_n)$, where $F_{n+1} \to F_n$ are the restriction morphisms, and that the global sections are $\Gamma(\mathbb{N}, \mathcal{F}) = \varprojlim F_n$.

Prove that so we obtain an equivalence of the category of sheaves of abelian groups on $\mathbb{N}$ with the category of projective limits of abelian groups with indices in $\mathbb{N}$. Moreover:

(a) $\mathcal{F}$ is flasque if and only if the morphisms $F_{n+1} \to F_n$ are surjective for all $n \in \mathbb{N}$.

(b) The derived functors $\varprojlim^p$ of $\varprojlim$ vanish when $p > 1$.

(*Hint*: In the Godement resolution $0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{G} \to 0$, check that $\mathcal{G}$ is flasque).

(c) Any projective system (F_n) with a finite number of non-null terms is acyclic, $\varprojlim^1(F_n) = 0$. (*Hint*: $\Gamma(\mathbb{N}, \mathcal{C}^0\mathcal{F}) = \Gamma(\mathbb{N}, \mathcal{G})$).

61. Let $\{X_n\}$ be an inductive system of topological spaces satisfying one of the following conditions:

(a) The morphism $X_n \to X_{n+1}$ is an open embedding for all $n \in \mathbb{N}$.

(b) X_n is a compact separated space, and $X_n \to X_{n+1}$ is a closed embedding, for all $n \in \mathbb{N}$.

Prove that for any sheaf $\mathcal{F}$ on $X = \varinjlim X_n$ we have a convergent spectral sequence

$$E_2^{p,q} = \varprojlim{}^p \big[H^q(X_n, \mathcal{F}|_{X_n})\big] \;\Rightarrow\; H^{p+q}(X, \mathcal{F}).$$

(*Hint*: $\Gamma(X, \mathcal{F}) = \varprojlim \Gamma(X_n, \mathcal{F}|_{X_n})$ and the projective system is surjective when $\mathcal{F}$ is flasque).

Calculate the cohomology groups $H^p(\mathbb{R}_\infty, \mathbb{Z})$, $H^p(\mathbb{P}_{\infty,\mathbb{R}}, \mathbb{F}_2)$, $H^p(\mathbb{P}_{\infty,\mathbb{C}}, \mathbb{Z})$ and $H^p(S_\infty, \mathbb{Z})$; where we put $\mathbb{R}_\infty = \varinjlim \mathbb{R}^n$, $\mathbb{P}_{\infty,\mathbb{R}} = \varinjlim \mathbb{P}_{n,\mathbb{R}}$, $\mathbb{P}_{\infty,\mathbb{C}} = \varinjlim \mathbb{P}_{n,\mathbb{C}}$ and $S_\infty = \varinjlim S_n$.

13. Analysis IV

1. In $\mathbb{R}^n$, put $B = B(0,1)$, $S = S(0,1)$ and $U = B - \{0\}$, so that $\partial U = S \cup \{0\}$. Show that the continuous function $f \colon \partial U \to \mathbb{R}$ such that $f(S) = 0$, $f(0) = 1$, admits no continuous extension $u \colon \bar{U} \to \mathbb{R}$ harmonic on U.

2. Prove that a function $u \in \mathcal{C}^2(U)$ is subharmonic if and only if $\Delta u \geq 0$.

3. Show that the function $|x|^a$ is subharmonic when $a \geq 1$.

If $u > 0$ is an harmonic function, show that u^a, $a \geq 1$, and $-\ln u$ are subharmonic.

4. Let $f \in \mathcal{C}^2(\mathbb{R})$ such that $f'' \geq 0$. If u is an harmonic function, prove that $f(u)$ is subharmonic.
5. Let $\Omega \subset \mathbb{R}^2$ be a relatively compact connected open set. If there is a closed segment I such that $\bar{\Omega} \cap I$ is just a point y_0, prove that y_0 is a regular point.
6. Let $U \subset \mathbb{R}^d$ be a relatively compact connected open set. If any continuous function $f \in \mathcal{C}(\partial U)$ has an extension $u \in \mathcal{C}(\bar{U})$ harmonic on U, show that any boundary point $y_0 \in \partial U$ is a regular point. (*Hint*: Consider $f(x) = |x - y_0|$).
7. Show that any two cylinders $\mathbb{C}/\mathbb{Z}e$, $\mathbb{C}/\mathbb{Z}v$ are isomorphic Riemann surfaces.
8. Show that any analytic automorphism $\tau \colon \mathbb{D} \to \mathbb{D}$ of finite order has a fixed point. (*Hint*: Brouwer's fixed point theorem).
9. If the universal covering of a connected open set $U \subset \mathbb{C}$ is isomorphic to $\mathbb{C}$, prove that $U = \mathbb{C} - p$ for some point $p \in \mathbb{C}$.
10. Let $T = \mathbb{C}/(\mathbb{Z}e + \mathbb{Z}v)$ be a torus. Prove that any analytic morphism $\mathbb{C} \to T$ is constant or surjective.
11. If a Riemann surface X admits a non-constant analytic morphism $\mathbb{C} \to X$, show that X is isomorphic to $\mathbb{C}$, $\mathbb{P}_1$, $\mathbb{C} - 0$ or a torus.
12. Let N be the closure of 0 in a topological vector space E. Prove that E/N is separated, and that any linear continuous map $f \colon E \to F$ into a separated topological vector space F uniquely factors through the canonical projection $\pi \colon E \to E/N$.
13. Prove that arbitrary direct products of locally convex spaces are locally convex.

    Moreover, the topology of any locally convex space E is the initial topology of some family of linear maps $E \to B_i$ to **Banach** spaces (complete normable spaces). In particular, if E is separated, it is isomorphic to a vector subspace of a direct product of Banach spaces.
14. If E is a seminormed space, prove that a set is bounded if and only if it is contained in a ball. Moreover, a locally convex space is seminormable if and only if it admits a bounded neighborhood of 0.
15. Prove that a linear map f of a seminormable space E to a topological vector space F is continuous if and only if f transforms the unit ball of E onto a bounded set of F.
16. Let E, F be seminormed spaces. Prove that a linear map $f \colon E \to F$ is continuous if and only if $\|f\| := \sup_{\|e\| \leq 1} \|f(e)\| < \infty$, and that so we obtain a seminorm on the space $L(E,F)$ of continuous linear maps. Moreover, $L(E,F)$ is separated (resp. complete) when so is F.
17. Prove that any finite-dimensional real vector space E admits a unique separated linear topology: the initial topology of the linear maps $E \to \mathbb{R}$. Conclude that, for any vector subspace $V \subseteq E$, there is a unique linear topology on E such that $\bar{0} = V$: the initial topology of the canonical projection $\pi \colon E \to E/V$, when we consider on E/V the unique separated linear topology.
18. Prove that countable direct products and projective limits of Fréchet spaces are Fréchet.
19. If two closed subspaces V_1, V_2 of a Fréchet space E are algebraic supplements ($V_1 \cap V_2 = 0$, $V_1 + V_2 = E$), prove that they are topological supplements (the natural map $V_1 \times V_2 \to E$ is an isomorphism).
20. If $0 \to E_1 \to E \to E_2 \to 0$ is a topologically exact sequence of metrizable locally convex spaces, prove that so is the induced sequence of Fréchet spaces $0 \to \widehat{E}_1 \to \widehat{E} \to \widehat{E}_2 \to 0$. (*Hint*: The topology of $E_2 = E/E_1$ is induced by $\widehat{E}/\widehat{E}_1$, so that we have a continuous section $s \colon \widehat{E}_2 \to \widehat{E}/\widehat{E}_1$ of $p \colon \widehat{E}/\widehat{E}_1 \to \widehat{E}_2$. The continuous linear bijection $s + \mathrm{Id} \colon \widehat{E}_2 \times (\operatorname{Ker} p) \to \widehat{E}/\widehat{E}_1$ is an isomorphism; hence $s(\widehat{E}_2)$ is closed and $\operatorname{Ker} p = 0$).
21. Prove that a continuous linear map f from a separated locally convex space E onto a Fréchet space F is a homomorphism if and only if it is a weak homomorphism. (*Hint*: Any closed equicontinuous family is weakly compact and, when F is Fréchet, any weakly compact set in F' is equicontinuous).
22. Let E, F, G be Fréchet spaces, $f \colon E \to G$ a continuous linear map, and $h \colon F \to G$ a compact linear map. If $f(E) + h(F) = G$, show that f is a homomorphism onto a closed subspace of G of finite codimension.

23. Let $f\colon E \to F$ be a weak homomorphism between separated locally convex spaces, such that every compact disk in F is the image by f of a compact disk in E. If $h\colon E \to F$ is a compact linear map, prove that $f+h$ is a weak homomorphism onto a closed subspace of F of finite codimension. (*Hint*: Follow the proof of Schwartz theorem).

24. If $f\colon E \to F$ is an isomorphism of separated locally convex spaces, and $h\colon E \to F$ is a compact linear map, show that $f+h\colon E \to F$ is a homomorphism onto a closed subspace of F of finite codimension, and the kernel is finite dimensional.

25. Let E be a complex topological vector space.

    If E is separated and finite-dimensional, show that E is isomorphic to $\mathbb{C}^n$.

    If $U \subset E$ is a convex open set, and V a linear subvariety not meeting U, show that there is a closed hyperplane containing V and not meeting U.

    If E is separated and locally convex, and $0 \neq e \in E$, show that $\omega(e) \neq 0$ for some continuous linear form $\omega\colon E \to \mathbb{C}$.

    If E is locally convex, show that any closed vector subspace is weakly closed.

26. If ω is a closed smooth complex 1-form on an open set $U \subseteq \mathbb{C}$, prove that there is an analytic function f on U such that $[f(z)\mathrm{d}z] = [\omega] \in H^1(U,\mathbb{C})$.

27. Show that the sheaf Ω_X of analytic 1-forms on a Riemann surface X admits the acyclic resolution
$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X^{1,0} \xrightarrow{\ \mathrm{d}\ } \Omega_X^{1,1} \longrightarrow 0,$$
where $\Omega_X^{1,1}$ denotes the sheaf of complex smooth 2-forms. Moreover, if $D \subseteq \mathbb{C}$ is an open disk, we have an exact sequence
$$0 \longrightarrow \Omega(D) \longrightarrow \Omega^{1,0}(D) \xrightarrow{\ \mathrm{d}\ } \Omega^{1,1}(D) \longrightarrow 0.$$

28. Given an open cover $X = \bigcup_i U_i$ of a Riemann surface, and meromorphic functions $f_i \in \mathcal{M}(U_i)$ such that $f_j - f_i \in \mathcal{O}(U_i \cap U_j)$, the additive **Cousin's problem** asks for a meromorphic function f on X such that $f - f_i \in \mathcal{O}(U_i)$; i.e. whether the natural map $H^0(X, \mathcal{M}_X) \to H^0(X, \mathcal{M}_X/\mathcal{O}_X)$ is surjective. Show that the answer is positive when $H^1(X,\mathcal{O}_X) = 0$. The multiplicative Cousin's problem assumes that f_j/f_i is a non-vanishing analytic function and asks for a meromorphic function f on X such that f/f_i is a non-vanishing analytic function on U_i; i.e. whether the natural map $H^0(X, \mathcal{M}_X^*) \to H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ is surjective. Show that the answer is positive when $H^1(X, \mathcal{O}_X^*) = 0$.

29. Let $X = \bigcup_i U_i$ be an open cover of a topological space X, and $\mathcal{F}$ a sheaf on X. Prove the existence of a natural morphism $\phi\colon \check{H}^1(\mathfrak{U},\mathcal{F}) \hookrightarrow H^1(X,\mathcal{F})$.

    (*Hint*: Consider the Godement resolution $0 \to \mathcal{F} \to C^\bullet\mathcal{F}$, and a Cech 1-cycle (s_{ij}). Inductively we construct $f_i \in (C^0\mathcal{F})(U_i)$ such that $s_{ij} = f_j - f_i$. Now $\mathrm{d}f_j = \mathrm{d}f_i + \mathrm{d}s_{ij} = \mathrm{d}f_i$, and we have a section $\omega \in (C^1\mathcal{F})(X)$ such that $\mathrm{d}\omega = 0$; hence a cohomology class $\phi([s_{ij}]) = [\omega] \in H^1(X,\mathcal{F})$. If we fix other $f_i' \in (C^0\mathcal{F})(U_i)$ such that $s_{ij} = f_j' - f_i'$, then $f_j' - f_j = f_i' - f_i$, and we have $f \in (C^0\mathcal{F})(X)$ such that $f = f_i' - f_i$; hence $\omega' = \omega + \mathrm{d}f$ and $[\omega'] = [\omega]$. Finally, when (s_{ij}) is a boundary, $s_{ij} = s_j - s_i$, we take $f_i = s_i$, so that $\omega = 0$.)

    Prove that $\phi\colon \check{H}^1(\mathfrak{U},\mathcal{F}) \hookrightarrow H^1(X,\mathcal{F})$ always is injective.

    (*Hint*: If $\omega = \mathrm{d}f$, $f \in (\mathcal{C}^0\mathcal{F})(X)$, then $\mathrm{d}(f_i - f) = 0$, so that $s_i := f_i - f \in \mathcal{F}(U_i)$. Then $f = f_i - s_i = f_j - s_j$, and $s_{ij} = f_j - f_i = s_j - s_i$ is a boundary).

    Conclude that $\check{H}^1(\mathfrak{U},\mathcal{F}) = H^1(X,\mathcal{F})$ when $H^1(U_i,\mathcal{F}) = 0$, $\forall i \in I$.

    (*Hint*: Given $[\omega] \in H^1(X,\mathcal{F})$, we have $\omega = \mathrm{d}f_i$, for some $f_i \in (\mathcal{C}^0\mathcal{F})(U_i)$. Then $\mathrm{d}(f_j - f_i) = \omega - \omega = 0$, so that $s_{ij} := f_j - f_i \in \mathcal{F}(U_i \cap U_j)$, and $s_{ij} + s_{jk} = f_j - f_i + f_k - f_j = f_k - f_i = s_{ik}$).

30. Let $\Omega^{(1,1)}$ be the sheaf of complex smooth 2-forms on a compact Riemann surface X. Prove that we have $\Omega^{(1,1)} = \Omega^{(1,0)} \otimes_{\mathcal{O}_X} \Omega^{(0,1)}$, and $H^1(X,\Omega_X) = \Omega^{(1,1)}(X)/\mathrm{d}\Omega^{(1,0)}(X)$.

    Conclude that we have a non-null linear map $\int_X\colon H^1(X,\Omega_X) \to \mathbb{C}$, $[\omega_2] \mapsto \int_X \omega_2$, and use it to prove that Ω_X is isomorphic to the dualizing sheaf $\mathcal{D}$. (*Hint*: Consider a local coordinate z centered at a point $x \in X$ and a function $f(r^2)$, $r^2 = z\bar{z}$, with compact support, such that $f(0) = 1$, $f'(r^2) \leq 0$. Show that $\omega = f(r^2)\frac{\mathrm{d}z}{z} \in (\Omega^{(1,0)} \otimes \mathcal{L}_x)(X)$ and

$$\mathrm{d}\omega = f'(r^2)\mathrm{d}\bar{z}\wedge\mathrm{d}z \in \Omega^{(1,1)}(X) \subset (\Omega_X^{(1,1)}\otimes\mathcal{L}_x)(X),$$

so that $\xi(\mathrm{d}\omega)=0$, while $\int_X \mathrm{d}\omega = \int_X f'(r^2)\mathrm{d}\bar{z}\wedge\mathrm{d}z = 2i\int_X f'(r^2)\mathrm{d}x\wedge\mathrm{d}y \neq 0$).

14. Differential Geometry II

1. Let X be a (eventually non σ-compact) smooth manifold. Prove that the module $\mathcal{D}(X)$ of smooth tangent vector fields is just the module of $\mathbb{R}$-derivations of the structural sheaf[7] $\mathcal{C}_X^\infty$; i.e., of morphisms of sheaves $D\colon \mathcal{C}_X^\infty \to \mathcal{C}_X^\infty$ such that $D\colon \mathcal{C}^\infty(U)\to\mathcal{C}^\infty(U)$ is an $\mathbb{R}$-derivation for any open set $U\subseteq X$.

   Prove that the sheaf of 1-forms Ω_X^1 on X is $\underline{\mathrm{Hom}}_{\mathcal{C}_X^\infty}(\mathcal{D}_X,\mathcal{C}_X^\infty)$.

   Show that the sheaf of (p,q)-tensor fields is $\Omega_X^1\otimes_{\mathcal{C}_X^\infty} \overset{p}{\ldots} \otimes_{\mathcal{C}_X^\infty}\Omega_X^1\otimes_{\mathcal{C}_X^\infty}\mathcal{D}_X\otimes_{\mathcal{C}_X^\infty} \overset{q}{\ldots} \otimes_{\mathcal{C}_X^\infty}\mathcal{D}_X$.

2. Let A be a k-algebra, $\mathrm{char}\, k=0$, and put $\mathcal{D}=\mathrm{Der}_k(A,A)$. A **connection** on an A-module M is a group morphism $\nabla\colon\mathcal{D}\to\mathrm{End}_k(M)$ such that $(aD)^\nabla = a(D^\nabla)$ and $D^\nabla(am)=(Da)m+a(D^\nabla m)$. Prove the following statements:

   (a) A natural connection on $M=A$ is $D^\nabla a = Da$. Moreover, given connections (M_i,∇_i), so are
   $\oplus_i M_i,\quad D^\nabla(\sum_i m_i)=\sum_i D^{\nabla_i}m_i$.
   $M_1\otimes_A\ldots\otimes_A M_n,\quad D^\nabla(m_1\otimes\ldots\otimes m_n)=\sum_i m_1\otimes\ldots\otimes D^{\nabla_i}m_i\otimes\ldots\otimes m_n$.
   $\mathrm{Hom}_A(M_1,M_2),\quad (D^\nabla f)(m)=D^{\nabla_2}\big(f(m)\big)-f(D^{\nabla_1}m)$.
   $M^*=\mathrm{Hom}_A(M,A),\quad (D^\nabla\omega)(m)=D(\omega(m))-\omega(D^\nabla m)$.
   $\mathrm{Mult}_A(M_1,\ldots,M_n;M'),\ (D^\nabla T)(m_1,\ldots,m_n)=D^{\nabla'}(T(m_1,\ldots,m_n))-\sum_i T(\ldots,D^{\nabla_i}m_i,\ldots)$.

   (b) Consider the A-module of A-multilinear maps $T_p\colon\mathcal{D}\times \overset{p}{\ldots}\,\mathcal{D}\to M$ and the **Lie derivative**

   $$(D^L T_p)(D_1,\ldots,D_p)=D^\nabla(T_p(D_1,\ldots,D_p))-\textstyle\sum_i T_p(D_1,\ldots,[D,D_i],\ldots,D_p).$$

   Then, $(D_1+D_2)^L=D_1^L+D_2^L$, $D^L(T_p+T_p')=D^LT_p+D^LT_p'$, $D^L(aT_p)=(Da)T_p+a(D^LT_p)$, $(\lambda D)^L=\lambda(D^L)$, $\forall\lambda\in k$, and $i_{[D,\bar{D}]}=i_D\circ\bar{D}^L-\bar{D}^L\circ i_D=[i_D,\bar{D}^L]$.

   (c) There is a unique k-linear operator d on the M-valued p-forms such that $D^L=i_D\circ\mathrm{d}+\mathrm{d}\circ i_D$:

   $$\mathrm{d}\omega_p(D_0,\ldots,D_p)=\big(\textstyle\sum_i(-1)^iD_i^\nabla\omega_p(\ldots\hat{D}_i\ldots)+\sum_{i<j}(-1)^{i+j}\omega_p([D_i,D_j]\ldots\hat{D}_i\ldots\hat{D}_j\ldots)\big).$$

   (d) If we have an A-bilinear map $M\times M'\to M''$, compatible with the connections in the sense that $D^{\nabla''}(m\cdot m')=(D^\nabla m)\cdot m'+m\cdot(D^{\nabla'}m')$, then $\mathrm{d}(\omega_p\wedge\omega_q)=(\mathrm{d}\omega_p)\wedge\omega_q+(-1)^p\omega_p\wedge(\mathrm{d}\omega_q)$. (*Hint*: Proceed by induction on $p+q$, using $D^L=i_D\circ\mathrm{d}+\mathrm{d}\circ i_D$).

   (e) Consider the **curvature** $\mathrm{End}_A(M)$-valued 2-form $K(D_1,D_2)=D_1^\nabla D_2^\nabla-D_2^\nabla D_1^\nabla-[D_1,D_2]^\nabla$. Then, on the p-forms we have:
      i. $[D_1^L,D_2^L]=[D_1,D_2]^L+K(D_1,D_2)\circ$
      ii. $[D^L,\mathrm{d}]=(i_DK)\wedge$
      iii. $\mathrm{d}^2=K\wedge$
      iv. $\mathrm{d}K=0$ (**Bianchi's Identity**).

   (*Hint*: The second and third identities, by induction on p. The last one, using Jacobi's identity, the formula $\mathrm{d}K(D_0,D_1,D_2)=\sum\big(D_0^\nabla K(D_1,D_2)-K([D_0,D_1],D_2)\big)$, where the summation extends over the three circular permutations, and the derivation law of endomorphisms $D_0^\nabla K(D_1,D_2)=D_0^\nabla\circ K(D_1,D_2)-K(D_1,D_2)\circ D_0^\nabla$).

   (f) If $h\in\mathrm{Hom}_A\big(\mathcal{D},\mathrm{End}_A(M)\big)$, then $\nabla'=\nabla+h$ also is a connection on M, and so we obtain any connection on M. Moreover:
      i. $D^{L'}=D^L+h(D)$,
      ii. $\mathrm{d}'=\mathrm{d}\wedge h$,
      iii. $K'=K+h\wedge h+\mathrm{d}h$.

[7] In fact this is the sensible definition of vector field when there is not enough global functions: non-separated smooth manifolds, analytic manifolds, schemes over a field, etc., so that any global vector field induces, by its very definition, a vector field on any open subset.

3. If ∇ is a linear connection on a smooth manifold X check that ∇T is just the differential $\mathrm{d}T$, where the tensor field T is viewed as a section of the sheaf of (p,q)-tensors on X.

4. Prove the following formulae of the differential calculus with $\mathcal{E}$-valued p-forms:
$[i_{D_1}, D_2^L] = i_{[D_1,D_2]}$, $[D_1^L, D_2^L] = [D_1, D_2]^L + R(D_1, D_2)$, $[D^L, \mathrm{d}] = (i_D R)\wedge$.

5. Let R be the curvature 2-form of a linear connection on a smooth manifold X of dimension n. Prove that the ordinary 2-form $\omega_2(D, D') = \mathrm{tr}(R(D, D'))$ is just the curvature of the induced linear connection on the sheaf Ω_X^n of volume forms.

6. A riemannian manifold (X, g) is **isotropic** at $p \in X$ when the metric $R_{2,2}$ induced (p. 301) by the Riemann-Christoffel tensor on $\Lambda^2 T_p X$ is proportional to the metric $\Lambda^2 g$,
$$R_{2,2}(D_1, D_2; D_3, D_4) = K\big((D_1 \cdot D_3)(D_2 \cdot D_4) - (D_1 \cdot D_4)(D_2 \cdot D_3)\big),$$
and X is isotropic when so it is any point (but the factor K may depend on the point). In an isotropic manifold, use Bianchi's differential identity to prove that
$$0 = (D_0 K)(\Lambda^2 g)(D_1, D_2, D_3, D_4) - (D_1 K)(\Lambda^2 g)(D_0, D_2, D_3, D_4) + (D_2 K)(\Lambda^2 g)(D_0, D_1, D_3, D_4).$$
Considering an orthonormal basis (and $D_3 = D_1$, $D_4 = D_2$) conclude that any isotropic connected manifold of dimension $n \geq 3$ has constant curvature (**Schur's Theorem**).

7. Let $\mathrm{d}X$ be a volume form on a smooth $(1+n)$-dimensional manifold, and let $\Pi_n = C_1^1(\mathrm{d}X \otimes T^2)$ be the vector-valued n-form corresponding to a 2-contravariant tensor T^2.
   (a) If g is a covariant metric, and $T(D) := T^2 \cdot D$ is the endomorphism associated to T^2, prove that for any vector field D we have $\Pi_n \cdot D = i_{T(D)}\mathrm{d}X$, and $\mathrm{d}(\Pi_n \cdot D) = \big(\mathrm{div}\, T(D)\big)\mathrm{d}X$.
   (b) If a symmetric linear connection ∇ on X preserves the volume form, $\nabla(\mathrm{d}X) = 0$, show that $\mathrm{d}\Pi_n = \mathrm{d}X \otimes (\mathrm{div}_\nabla T^2)$. (*Hint*: At any point $p \in X$, consider a local base of vector fields $D_0, \ldots, D_n$ such that $(\nabla D_i)_p = 0$).

8. Let ∇ be a linear connection on a smooth manifold X. If $\rho \in \mathcal{C}^\infty(X)$ and U is a vector field, show that $\mathrm{div}_\nabla(\rho U \otimes U) = (U\rho + \rho\mathrm{div}_\nabla U)U + \rho U^\nabla U$. In the case of a perfect fluid of mass density ρ, mean velocity U and pressure p, show that the condition $\mathrm{div}_\nabla(\rho U \otimes U + ph) = 0$ is equivalent to the pair of equations
$$U\rho + \rho\mathrm{div}\, U = 0 \qquad \text{(Mass conservation law)}$$
$$\rho U^\nabla U = -\mathrm{grad}\, p \qquad \text{(Law of motion)}$$

9. Let R_2 be the Ricci tensor of a riemannian 3-manifold (X, g). Prove that $R_2 = 0$ implies that $R = 0$, and that $R_2 = fg$ implies that X has constant curvature.

10. The **Einstein** tensor G_2 of a riemannian (or Lorentz) metric g is defined to be $G_2 = R_2 - \frac{1}{2}\kappa g$, where $\kappa = C_{12}R_2$ is the **scalar curvature** (contraction of the two indices of R_2 with g). Check that $C_1^1(\nabla G_2) = 0$, and that $G_2 = 0$ if and only if $R_2 = 0$.

11. *Weyl's geometry*: Let X be a smooth manifold endowed with a riemannian (or Lorentz) metric g well-defined up to a constant factor at any point (i.e., up to a positive smooth function, so that we don't assume the existence of a length or time unit universally defined) and a linear connection ∇ preserving it: $\mathrm{d}g = \nabla g = \omega_1 \otimes g$ for some 1-form ω_1 (obviously depending on the representant g, i.e., on the fixed units). Prove the following statements:
   (a) $\mathrm{d}^2 g = \omega_2 \otimes g$, where ω_2 is a closed 2-form not depending on the fixed representant g.
   (b) $\omega_2 = 0$ if and only if locally there exists a representant g such that $\nabla g = 0$ (so that we have in fact a riemannian geometry).
   (c) The Einstein tensor $R_2 - \frac{1}{2}\kappa g$ does not depend on the representant g, where R_2 is the Ricci tensor of ∇, and $\kappa = C_{12}R_2$ is the contraction of the two indices of R_2 with g.

12. In case that you know what an holomorphic vector bundle $E \to X$ over an analytic manifold X is, prove that if we fix a smooth hermitian metric on E, then there is a unique linear connection ∇ on the sheaf of smooth sections satisfying the following equivalent conditions:

(a) $D(e\cdot v) = (D^\nabla e)\cdot v + e\cdot (D^\nabla v)$, for any smooth tangent vector field D and any pair e, v of smooth sections.

(b) $D(e\cdot v) = (D^\nabla e)\cdot v + e\cdot (\bar{D}^\nabla v)$, for any smooth complex vector field D and any pair e, v of smooth sections.

(c) $\mathrm{d}(e\cdot v) = (\mathrm{d}e)\cdot v + e\cdot(\mathrm{d}v)$, for any pair e, v of smooth sections.

(d) $\frac{\partial}{\partial \bar{z}_i}^\nabla e = 0$, for any holomorphic section e and any holomorphic local coordinates $(z_1,\dots,z_n)$.

(e) In a local basis $(e_1,\dots,e_r)$ of holomorphic sections, the structure 1-forms ω_{ij} are of (1,0) type (recall that $\mathrm{d}e_i = \sum_j \omega_{ij}e_j$).

(*Hint*: Put $h_{ij} = e_i\cdot e_j$. Then $\partial h_{ij} + \bar\partial h_{ij} = \mathrm{d}h_{ij} = \sum_k \omega_{ik}h_{kj} + \sum_k h_{ik}\bar\omega_{jk}$, so that if $\omega = (\omega_{ij})$ is of (1,0) type, it is fully determined, $\omega = (\partial h)h^{-1}$).

13. If $\pi\colon E\to X$ is a smooth vector bundle, show that $\bar\pi\colon J^1E\to X$ also has a natural structure of smooth vector bundle.

14. If $\pi\colon Y = \mathbb{R}\times X\to\mathbb{R}$ is the natural projection, show that $J^1Y = \mathbb{R}\times(TX)$, where TX is the tangent bundle of the smooth manifold X.

    If $\pi\colon Y = X\times\mathbb{R}\to X$ is the natural projection, show that $J^1Y = \mathbb{R}\times(T^*X)$, where T^*X is the cotangent bundle of the smooth manifold X.

15. The **Takens elementary complex** of a real vector space V is $S^\bullet\otimes\Lambda^\bullet V^*$, with the grading induced from $\Lambda^\bullet V^*$ and the differential d defined by the product with the identity $\mathrm{Id}\in V\otimes V^*$,

$$\cdots \xrightarrow{\mathrm{d}} S^\bullet V\otimes\Lambda^p V^* \xrightarrow{\mathrm{d}} S^\bullet V\otimes\Lambda^{p+1}V^* \xrightarrow{\mathrm{d}} \cdots$$

    If we consider the contraction of indices $c\colon S^\bullet V\otimes\Lambda^p V^*\to S^\bullet V\otimes\Lambda^{p-1}V^*$, show that

$$(c\mathrm{d} + \mathrm{d}c)(x) = (r+n-p)x,\ n = \dim V,$$

    whenever $x\in S^rV\otimes\Lambda^pV^*$, and conclude that the cohomology groups of the Takens elementary complex are zero except for $H^n(S^\bullet V\otimes\Lambda^\bullet V^*) = \Lambda^n V^*$.

16. Let $S_{2n+1}\subset\mathbb{C}^{n+1}$ be the sphere of unitary vectors (with the usual hermitian metric). Show that the natural map $S_{2n+1}\to\mathbb{P}_{n,\mathbb{C}}$ is a smooth principal bundle of group $S_1 = \{z\in\mathbb{C}\colon |z| = 1\}$, and that it is not a trivial bundle.

17. Let $L\to X$ be a smooth line bundle, and let L^* be the complement of the zero section. Show that $L^*\to X$ is a smooth principal bundle of group $K^* = K - \{0\}$ (where $K = \mathbb{R}$ or $\mathbb{C}$), and that so we obtain any smooth principal bundle of group K^*.

15. Algebraic Geometry II

1. Decompose $\mathbb{Q}/\mathbb{Z}$ as a direct sum of injective hulls $E(\mathbb{Z}/p\mathbb{Z})$.

2. If $(\mathcal{O},\mathfrak{m})$ is a finite local k-algebra, prove that the $\mathcal{O}$-module $\mathcal{O}^* = \mathrm{Hom}_k(\mathcal{O}, k)$ is the injective hull $E(\mathcal{O}/\mathfrak{m})$ of the residue field. (*Hint*: $(\mathcal{O}/\mathfrak{m})^*\simeq\mathcal{O}/\mathfrak{m}$).

3. Let x be a point of a noetherian scheme X, and let $i\colon \mathrm{Spec}\,\mathcal{O}_{X,x}\to X$ be the natural morphism. If M is a $\mathcal{O}_{X,x}$-module, show that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{N}, i_*\widetilde{M}) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{N}_x, M)$ for any $\mathcal{O}_X$-module $\mathcal{N}$.

    If I is the injective hull of the residue field $\kappa(x)$ of the local ring $\mathcal{O}_{X,x}$, prove that the quasi-coherent sheaf $i_*\widetilde{I}$ is an injective $\mathcal{O}_X$-module.

4. Let I be an injective module over a noetherian ring A. Prove that I_x is an injective A_x-module at any point $x\in\mathrm{Spec}\,A$.

5. Let $A = \mathbb{C}[\partial_1,\dots,\partial_n]$ be the ring of linear differential operators with constant coefficients, and let M be an A-module. Given operators $p_i\in\mathbb{C}[\partial_1,\dots,\partial_n]$ and elements $m_i\in M$, $i = 1,\dots,r$, we consider the system of differential equations

$$\begin{cases} p_1(\partial_1,\dots,\partial_n)u = m_1 \\ \dots\dots\dots\dots\dots\dots \\ p_r(\partial_1,\dots,\partial_n)u = m_r \end{cases}$$

Put $I=(p_1,\dots,p_r)$ and fix a free resolution $A^s \xrightarrow{(q_{ij})} A^r \xrightarrow{(p_i)} A \to A/I \to 0$.

(a) If $\operatorname{Ext}^1_A(A/I,M)=0$, prove that the system admits some solution $u\in M$ if and only the integrability conditions $\sum_i q_{ij}m_i = 0$, $i=1,\dots,r$, hold.

(b) Show that $M=\mathbb{C}[x_1,\dots,x_n]$ is an injective A-module. In fact, it is the injective hull of the residue field of A at the origin.

Moreover, if A/I is a finite $\mathbb{C}$-algebra, then $\operatorname{Hom}_A(A/I,M)=(A/I)^*$ and the dimension of the space of solutions of the homogeneous system $p_i(\partial_1,\dots,\partial_n)u=0$ is the degree of A/I.

(c) When M is the ring of complex smooth functions on $\mathbb{C}^n$ and A/I is a finite $\mathbb{C}$-algebra, prove that $\operatorname{Ext}^p_A(A/I,M)=0$, $p>0$. (*Hint*: When I is a maximal ideal, take the Koszul resolution of $A/I=\mathbb{C}$. In general, proceed by induction on the degree of A/I).

(d) Show that the ring of formal series $\mathbb{C}[[x_1,\dots,x_n]]$ is an injective A-module. (*Hint*: If A is a k-algebra, then A^* is an injective A-module because it represents the exact functor $N \rightsquigarrow N^*$, and $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[\partial_1,\dots,\partial_n],\mathbb{C})=\mathbb{C}[[x_1,\dots,x_n]]$).

6. Let $P_1,\dots,P_r$ be a regular sequence of homogeneous polynomials in $k[x_0,\dots,x_d]$, $r<d$, and let Y be the closed subscheme of $\mathbb{P}_d$ defined by the ideal $(P_1,\dots,P_r)$. If we put $d_i=\deg P_i$, prove the following statements (we agree that $\binom{n}{m}=0$ when $n<m$):

(a) $H^p(Y,\mathcal{O}_Y(n))=0$, when $p\neq 0, d-r$.

(b) $\dim H^{d-r}(Y,\mathcal{O}_Y(d_1+\ldots+d_r+n))=\binom{-n-1}{d}$, when $n>-d-1-\inf(d_1,\dots,d_r)$.

(c) $\dim H^0(Y,\mathcal{O}_Y(n))=\binom{n+d}{d}$, when $n<\inf(d_1,\dots,d_r)$.

7. Let C be a curve in $\mathbb{P}_r$ defined by a regular sequence $P_1,\dots,P_{r-1}$ of homogeneous polynomials. If we put $d_i=\deg P_i$, prove that

$$\chi(C,\mathcal{O}_C(n)) = d_1\dots d_{r-1}(n+1) - \sum_{j=1}^{r-1} d_1\dots d_{r-1}\tfrac{d_j-1}{2}\cdot$$

8. Prove that a ring A is regular if and only if so is the ring $A[x]$.

9. Let $\mathcal{O}$ be a noetherian local ring. If the annihilator of $f\in\mathcal{O}$ is a prime ideal $\mathfrak{p}$, prove that

$$\operatorname{depth}\mathcal{O}\le \dim(\mathcal{O}/\mathfrak{p}).$$

10. If $\mathcal{O}$ is a local Cohen-Macaulay ring, prove that $f\in\mathcal{O}$ is an algebraic zero divisor if and only if $(f)_0$ is a topological zero divisor (it has non empty interior).

11. Let $I\subseteq\mathfrak{m}$ be an ideal of a noetherian local ring $(\mathcal{O},\mathfrak{m})$. Let us consider the cone $C=\operatorname{Spec}G_I\mathcal{O}$ and the closed point x of the vertex $V=\operatorname{Spec}(\mathcal{O}/I)\subseteq C$.

(a) If I is $\mathfrak{m}$-primary, show that $\dim_x C=\dim\mathcal{O}$.

(b) In general, if y is the generic point of an irreducible component of the vertex V, show that the local ring of C at y is just the local ring of the cone $\operatorname{Spec}(G_{I_y}\mathcal{O}_y)$ at the closed point y, and conclude that $\dim_y C=\dim\mathcal{O}_y$.

(c) If $\mathcal{O}$ is Cohen-Macaulay and the cone C is Cohen-Macaulay at x, prove that $\dim_x C=\dim\mathcal{O}$.

12. Let $A\to B$ be a finite faithfully flat morphism. Prove that A is Cohen-Macaulay if and only if so is B. Conclude that an affine algebraic variety $X=\operatorname{Spec}A$ over a field is Cohen-Macaulay if and only if it admits a finite flat morphism $X\to\mathbb{A}_n$ onto the affine space.

13. Let E be a k-vector space. We say that a **coproduct** linear map $\mu\colon E\to E\otimes_k E$ defines a structure of (commutative) **coalgebra** on E, with **counity** a linear map $u\colon E\to k$, if the diagrams

$$\begin{CD} E @>{\mu}>> E\otimes_k E \\ @V{\mu}VV @VV{\mu\otimes \mathrm{Id}}V \\ E\otimes_k E @>{\mathrm{Id}\otimes\mu}>> E\otimes_k E\otimes_k E \end{CD} \qquad \begin{array}{ccc} E & \xrightarrow{\ \mu\ } & E\otimes_k E \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle s} \\ & & E\otimes_k E \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\ \mu\ } & E\otimes_k E \\ & {\scriptstyle \mathrm{Id}}\searrow & \downarrow{\scriptstyle u\otimes\mathrm{Id}} \\ & & k\otimes_k E \end{array}$$

commute, where $s(e\otimes v)=v\otimes e$. Prove that (E,μ,u) is a coalgebra if and only if $\mu^*\colon E^*\otimes_k E^*\to E^*$ and $u^*\colon k\to E^*$ define a structure of k-algebra on E^*, the unity being $u\in E^*$.

With the notations of p. 432, show that any injective morphism of $\mathcal{O}\otimes_k\mathcal{O}$-modules $\delta\colon I\hookrightarrow I\otimes_k I$ defines a structure of coalgebra on I. Moreover, $I\otimes_k I$ is an injective hull of the residue field of $\mathcal{O}\otimes_k\mathcal{O}$, and the counity $r\colon I\to k$ is the unique linear map such that $r(k)=k$ and the diagram

$$\begin{array}{ccc} I & \xrightarrow{\ \mu\ } & I\otimes_k I \\ & {\scriptstyle r}\searrow & \downarrow{\scriptstyle r\otimes r} \\ & & k \end{array}$$

commutes. Conclude that the local residue Res_x coincides, up to a factor $0\neq\lambda\in k$, with the counity of the coproduct $\delta_x\colon H^1_x(\Omega_C)\to H^1(\Omega_C)\otimes_k H^1(\Omega_C)$.

14. Let S be a scheme. Show that the functor of points $X^\bullet=\mathrm{Hom}_S(-,X)$ of a S-scheme X is always a sheaf, in the sense that $U\rightsquigarrow X^\bullet(U)$ is a sheaf on any S-scheme.

Let $F\colon$ *S*-**schemes** $\rightsquigarrow$ **Sets** be a contravariant functor. Given a S-scheme $R\to S$, any R-scheme is obviously a S-scheme, so that F induces a contravariant functor $F_R\colon$ *R*-**schemes** $\rightsquigarrow$ **Sets**. If F is a sheaf, and it is locally representable, in the sense that F_R is representable for some open cover $R=\oplus_i U_i\to S$, prove that F is representable.

(*Hint*: Take $X'\to R$ representing F_R with certain $\xi'\in F_R(X')=F(X')$. Since F_R comes from F, it is endowed with an isomorphism $\phi\colon \pi_1^*F_R\xrightarrow{\sim}\pi_2^*F_R$, where $\pi_i\colon\ \oplus_{i,j}U_i\cap U_j=R\times_S R\rightrightarrows R$ are the canonical projections, with obvious transitivity conditions on $R\times_S R\times_S R=\oplus_{i,j,k}U_i\cap U_j\cap U_k$. Hence we have an isomorphism $\varphi\colon \pi_1^*X'\xrightarrow{\sim}\pi_2^*X'$, $\phi(\pi_1^*\xi')=\varphi^*(\pi_2^*\xi')$, with the same transitivy conditions. Prove that there is a S-scheme X and an isomorphism $X'=X\times_S R$ such that φ corresponds to the canonical isomorphism $\pi_1^*X_R=\pi_2^*X_R$, and we have $\pi_1^*(\xi_R)=\pi_2^*(\xi_R)$, where $\xi_R\in F(X_R)$ corresponds to $\xi'\in F(X')$. Since F is a sheaf, show that ξ_R comes from a unique $\xi\in F(X)$. Conclude that $\xi\colon X^\bullet\to F$ is an isomorphism, since both are sheaves).

(a) Show that fibred products $X\times_S Y$ exist in the category of schemes.

(b) Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Show the existence of a S-scheme $\mathrm{Spec}\,\mathcal{A}$ representing the functor $F(T)=\mathrm{Hom}_{\mathcal{O}_T\text{-alg}}(\mathcal{A}\otimes_{\mathcal{O}_S}\mathcal{O}_T,\mathcal{O}_T)$.

(c) If $\mathcal{E}$ is a locally free $\mathcal{O}_S$-module (of finite rank), show the existence of a S-scheme $E=\mathrm{Spec}\,(S^\bullet\mathcal{E}^*)$ representing the functor $F(T)=\Gamma(T,\mathcal{E}\otimes_{\mathcal{O}_S}\mathcal{O}_T)$.

15. Let S be a scheme and let $\mathcal{A}^\bullet=\mathcal{A}_0\oplus\mathcal{A}_1\oplus\ldots\oplus\mathcal{A}_n\oplus\ldots$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. If the natural morphism $S^\bullet\mathcal{A}_1\to\mathcal{A}^\bullet$ is surjective, prove the existence of a S-scheme $\mathrm{Proj}\,\mathcal{A}^\bullet$ such that for any affine open set $U\to S$ we have a natural isomorphism $(\mathrm{Proj}\,\mathcal{A}^\bullet)\times_S U=\mathrm{Proj}\,\Gamma(U,\mathcal{A}^\bullet)$.

16. Let U be an open subset of a ringed space $(X,\mathcal{O}_X)$. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, prove that $\mathcal{I}|_U$ is an injective $(\mathcal{O}|_U)$-module.

17. Let X be a noetherian scheme. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, prove that any stalk $\mathcal{I}_x$ is an injective $\mathcal{O}_{X,x}$-module.

18. If X is a noetherian scheme, prove that any direct sum of injective $\mathcal{O}_X$-modules is injective.

19. Let A be a noetherian ring, and put $X=\mathrm{Spec}\,A$. If I is an injective A-module, prove that $\widetilde{I}$ is an injective $\mathcal{O}_X$-module.

If M is a finitely generated A-module, conclude that $\underline{\mathrm{Ext}}^n_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})=\mathrm{Ext}^n_A(M,N)^\sim$.

20. Let X be a noetherian scheme. Prove that any quasi-coherent $\mathcal{O}_X$-module is the inductive limit of its coherent submodules. Moreover, if $\mathcal{I}$ is an injective quasi-coherent sheaf, prove that $\mathcal{I}|_U$ is an injective quasi-coherent $\mathcal{O}_U$-module for any open set $U\subset X$. (*Hint*: Deligne's formula).

21. Let Y be a closed subscheme of a noetherian scheme X. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module, prove that
$$H^p_Y(X,\mathcal{M}) = \varinjlim \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{p}^n_Y, \mathcal{M}).$$

22. Show that $K'(\mathbb{Z}) = K(\mathbb{Z}) = \mathbb{Z}$.

23. If $\mathcal{O}$ is a local ring, show that $\operatorname{rk}: K'(\operatorname{Spec}\mathcal{O}) \to \mathbb{Z}$ is an isomorphism.
If $\mathcal{O}$ is a regular local ring, show that $\operatorname{rk}: K(\operatorname{Spec}\mathcal{O}) \to \mathbb{Z}$ is an isomorphism.

24. Let E be a locally free $\mathcal{O}_X$-module of rank $r+1$. Prove that $K(\mathbb{P}(E)) = \bigoplus_{i=0}^r K(X)y^i_E$, where we put $y_E = 1 - \mathcal{O}_{\mathbb{P}(E)}(1)$. (*Hint*: $y_E = -\mathcal{O}(1)x_E$ and $\mathcal{O}(1)$ is invertible).

25. Let $\mathfrak{m}$ be the maximal ideal of a rational point x of a smooth surface X, and let
$$\pi\colon \bar{X} = \operatorname{Proj}(\mathcal{O}_X \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \ldots) \longrightarrow X$$
be the blow-up of X at x. Let E be the exceptional fibre, defined by the ideal $\mathfrak{p}_E = \mathfrak{m}\mathcal{O}_{\bar{X}}$.
Show that $\mathfrak{p}_E = \mathcal{O}_{\bar{X}}(1)$, and $\mathfrak{p}^n_E = \mathcal{O}_{\bar{X}}(n)$ for all $n \geq 1$.
Show that $\mathcal{O}_E \cdot \mathcal{O}_{\bar{X}}(1) = \mathcal{O}_E(1)$ in the ring $K(\bar{X})$, and $(E \cap E) = -1$.

26. Let $X \to \mathbb{P}_2$ be the blow-up of a projective plane at a rational point x, and let E be the exceptional fibre. Prove that
   (a) $GK^1(X) = \mathbb{Z}H \oplus \mathbb{Z}E$, where H is a line not incident to x.
   (b) $GK^2(X) = \mathbb{Z}p$, where p is any rational point.
   (c) $H \cdot H = p$, $E \cdot E = -p$ and $H \cdot E = 0$.

27. Let X be a smooth variety. If $\mathfrak{p}$ is the ideal of a smooth closed subvariety $Y \to X$, prove that
$$Y \cdot Y = \textstyle\sum_i (-1)^i \Lambda^i(\mathfrak{p}/\mathfrak{p}^2) \in GK^\bullet(X).$$

28. If E is a locally free $\mathcal{O}_X$-module of rank r and L is a line sheaf, prove that $c_r(E-L) = c_r(E \otimes_{\mathcal{O}_X} L^*)$ in $GK^\bullet(X)$.

29. If $c(E) = (1+\alpha_1)\ldots(1+\alpha_r) \in GK^\bullet(X)$, show that $c(-E) = \prod_{i=1}^{r}(1 - \alpha_i + \alpha_i^2 - \alpha_i^3 + \ldots)$.

30. Let X be a smooth variety. If N is the normal bundle of a smooth closed subvariety $i\colon Y \to X$ of codimension d, prove that $i^*i_*(y) = y \cdot c_d(N) \in GK^\bullet(Y)$, where $y \in GK^\bullet(Y)$.

31. Let $j\colon Y \to X$ be a smooth closed hypersurface of a smooth variety X. Prove the existence of an exact sequence $0 \to j^*L_{-Y} \to j^*\Omega_X \to \Omega_Y \to 0$, and conclude that in $GK^\bullet(X)$ we have
$$j_*c_i(Y) = \sum_{m=0}^{j}(-1)^m c_m(X) \cdot Y^{j+1-m}.$$

32. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of locally free $\mathcal{O}_X$-modules, then the morphisms $\Lambda^i E' \otimes \Lambda^{n-i}E \to \Lambda^n E$ define a filtration of $\Lambda^n E$, the graded module being $\oplus_i(\Lambda^i E' \otimes \Lambda^{n-i}E'')$. Hence $\Lambda^n E = \sum_i (\Lambda^i E')(\Lambda^{n-i}E'')$ in $K'(X)$. We put $\lambda^i(E) = \Lambda^i E \in K'(X)$. The function $\lambda_t(E) = \sum_i \lambda^i(E)t^i$, valued in the multiplicative group of formal series with coefficients in $K'(X)$ and constant term 1, is additive, and we have a well-defined series $\lambda_t(x) = \sum_i \lambda^i(x)t^i$ for any $x \in K'(X)$.
If L is a line sheaf, show that $\lambda_t(-L) = (1+Lt)^{-1} = \sum_i (-1)^i L^i t^i$ and $\lambda^i(-L) = (-1)^i L^i$.

33. If a locally free $\mathcal{O}_X$-module E of rank r has a section without zeros, show that $0 = \sum_i (-1)^i \Lambda^i E$ in $K'(X)$. (*Hint*: $E - \mathcal{O}_X$ is the class of a locally free sheaf of rank $r-1$, so that $0 = \lambda^r(E-1)$).

34. If E is a locally free $\mathcal{O}_X$-module of rank r, prove that $0 = \sum_i (-1)^{r-i}\lambda^i(E^*)\mathcal{O}_{\mathbb{P}(E)}(r-i)$ in $K'(\mathbb{P}(E))$. (*Hint*: The exact sequence $0 \to \bar{E} \to \pi^*E^* \to \mathcal{O}(1) \to 0$ shows that $\lambda^r\big(E^* - \mathcal{O}_{\mathbb{P}(E)}(1)\big) = 0$).

35. If X is a smooth variety, we define the γ-operations in $K(X)$ as
$$\gamma^n(x) = \lambda^n(x+n-1) = \sum_{i=0}^{n}(-1)^{n-i}\lambda^i(x+n), \ n \geq 0.$$
If we put $\gamma_t(x) = \sum_n \gamma^n(x)t^n$, prove that

(a) $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x)$,
$\lambda_t(x) = \gamma_{\frac{t}{1+t}}(x)$.

(b) $\gamma^0(x) = 1$,
$\gamma^1(x) = x$,
$\gamma^n(x+y) = \sum\limits_{i+j=n} \gamma^i(x)\gamma^j(y)$.

(c) If E is a locally free sheaf of rank r, then $\gamma^{r+1}(E-r) = 0$.

(d) If L is a line sheaf, then $\gamma^n(L) = L$ and $\gamma^n(1-L) = (1-L)^n$.

(e) Let E be a locally free sheaf of rank r and let $\xi = \mathcal{O}_{\mathbb{P}(E)}(-1)$ be the tautological line sheaf of the projective bundle $\pi\colon \mathbb{P}(E) \to X$. If we put $x = 1 - \xi \in K(\mathbb{P}(E))$, prove that

$$\sum_{i=0}^{r} \pi^!\big[\gamma^i(E-r)\big](1-\xi)^{r-i} = 0$$

and conclude that $c_i^K(E) = \gamma^i(E-r) \in K(X)$ and that $c_i(E) = [\gamma^i(E-r)] \in GK^i(X)$.

36. If E is a locally free sheaf on a smooth variety X, we put $\sigma^i(E) = S^iE \in K(X)$. Check that $\sigma_t(E) = \sum_i \sigma^i(E)t^i$ is an additive function, valued in the multiplicative group of formal series with coefficients in $K(X)$ and constant term 1, and we have a well-defined series $\sigma_t(x) = \sum_i \sigma^i(x)t^i$ for any $x \in K(X)$.

If L is a line sheaf on X, show that $\sigma^i(-L) = 0$, $i \geq 2$, and $\sigma^i(1-L) = 1-L$, $i \geq 1$.

Finally, prove that $\sigma_t(x)\lambda_{-t}(x) = 1$. (*Hint*: Reduce to the case of a line sheaf).

37. Let D be a tangent vector field to a smooth variety X of dimension n. If D has a finite number of zeros $x_1, \ldots, x_r$, show that $c_n(X) = \mu_1 x_1 + \ldots + \mu_r x_r \in GK^n(X)$, where $\mu_i = l\big(\mathcal{O}_{X,x_i}/(f_1, \ldots, f_n)\big)$ and $D = f_1\partial_1 + \ldots + f_n\partial_n$ in a neighborhood of x_i.

38. Let X, Y be smooth algebraic k-varieties. Prove that any closed embedding $Y \to X$ is regular. (*Hint*: The ideal $\mathfrak{p}_y$ of Y at a point $y \in Y$ contains a sequence of parameters of length $\operatorname{codim}_y Y$, generating $\mathfrak{p}_y$, according to the exact sequence $\mathfrak{p}_y/\mathfrak{m}_{X,y}\mathfrak{p}_y \to \mathfrak{m}_{X,y}/\mathfrak{m}_{X,y}^2 \to \mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2 \to 0$).

39. Let $E \to X$ be a vector bundle, and let U be the complement of the incidence relation in $\mathbb{P}(E)\times_X\mathbb{P}(E^*)$. Show that both $\pi_1\colon U \to \mathbb{P}(E)$ and $\pi_2\colon U \to \mathbb{P}(E^*)$ are affine bundles, and that the natural map $\pi_1^*\mathcal{O}_{\mathbb{P}(E)}(-1) \otimes_{\mathcal{O}_U} \pi_2^*\mathcal{O}_{\mathbb{P}(E^*)}(-1) \to \mathcal{O}_U$ is an isomorphism, so that $\pi_1^*\mathcal{O}_{\mathbb{P}(E)}(-1) = \pi_2^*\mathcal{O}_{\mathbb{P}(E^*)}(1)$.

Conclude that in any cohomology theory $A(\mathbb{P}(E)) = A(X) \oplus A(X)x_E \oplus \ldots A(X)x_E^{r-1}$, where we put $x_E = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ and $r = \operatorname{rk} E$. Moreover, $x_E^r - c_1(E^*)x_E^{r-1} + \ldots + (-1)^r c_r(E^*) = 0$.

40. Let $F(t) = \sum_i a_i t^i$ be a formal series with coefficients in $A(\operatorname{Spec} k)$. If E is a locally free sheaf, by definition $F_+(E) = (\operatorname{rk} E)a_0 + S(c_1(E), \ldots, c_n(E), \ldots)$ for a well-defined power series S. Show that the additive extension $F_+\colon K(X) \to A(X)$ is just $F_+(x) = (\operatorname{rk} x)a_0 + S(c_1(x), \ldots, c_n(x), \ldots)$.

Prove an analogous statement for the multiplicative extension $F_\times\colon K(X) \to A(X)$.

41. Given a cohomology theory (A, f_*) on the smooth quasi-projective k-varieties, let us consider a new direct image f_*^{new} so that (A, f_*^{new}) also is a cohomology theory. Prove the existence of an invertible formal series $F(t) = a_0 + a_1 t + \ldots$ with coefficients in $A(\operatorname{Spec} k)$ such that $f_*^{\text{new}}(a) = f_*(F_\times(-T_f)a)$. (*Hint*: Show that $c_1^{\text{new}}(\mathcal{O}_{\mathbb{P}_1}(1)) = a_0 c_1(\mathcal{O}_{\mathbb{P}_1}(1))$, where $a_0 \in A(\operatorname{Spec} k)$ is invertible. Hence $c_1^{\text{new}}(\mathcal{O}_{\mathbb{P}_d}(1)) = c_1(\mathcal{O}_{\mathbb{P}_d}(1)) \cdot F[c_1(\mathcal{O}_{\mathbb{P}_d}(1))]$ for some invertible series $F(t)$, and conclude by Panin's lemma).

42. In the case of a smooth projective variety of dimension g with trivial tangent bundle, show that for any divisor D we have $\chi(L_D) = \frac{1}{g!}\deg D^g$.

43. Let E be a locally free sheaf of rank r on the projective space $\mathbb{P}_d$. If $x = c_1(\mathcal{O}(1)) \in GK^\bullet(\mathbb{P}_d)$, then

$$c(E) = 1 + m_1 x + \ldots + m_r x^r = (1 + \alpha_1 x)\ldots(1 + \alpha_r x)$$

for some integer numbers $m_1, \ldots, m_r$ and some complex numbers $\alpha_1, \ldots, \alpha_r$. Prove that

$$\chi(\mathbb{P}_d, E(n)) = \operatorname{Res}\left(\sum_{i=1}^{r} e^{(\alpha_i+n)t}\frac{\mathrm{d}t}{(1-e^{-t})^{d+1}}, t=0\right) = \sum_{i=1}^{r}\binom{n+d+\alpha_i}{d}.$$

(*Hint*: Use the variable change $u = 1 - e^{-t}$ to calculate the residue). Conclude that

(a) When $r = 2$, $d = 3$, the number $m_1 m_2$ is even. (*Hint*: $\binom{d+\alpha_1}{d} + \binom{d+\alpha_2}{d}$ is integer).

(b) When $r = 2$, $d = 4$, we have $m_2(1 + m_2 - 3m_1 - 2m_1^2) \equiv 0 \pmod{12}$.

(c) There is no vector bundle E on $\mathbb{P}_3$ such that $E|_{\mathbb{P}_2}$ is the tangent bundle of $\mathbb{P}_2$.

44. Assuming that $GK^\bullet \otimes \mathbb{Q}$ is a cohomology theory, prove that $\mathrm{ch}\colon K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to GK^\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism for any smooth projective variety X.

45. If A is a cohomology theory, prove that any functorial group morphism $\phi\colon K(X) \to A(X)$ is the additive extension of a formal power series $F(t)$ with coefficients in $A(\operatorname{Spec} k)$.

If A follows the law $x + y$, then ϕ is a ring morphism when $F(x + y) = F(x)F(y)$ and $F(0) = 1$.

If A follows the law $x + y - xy$, then ϕ is a ring morphism when $F(x + y - xy) = F(x)F(y)$ and $F(0) = 1$. For example $F(t) = (1 - t)^k$, $k \in \mathbb{Z}$.

46. Define the **Adams operations** $\Psi^k\colon K(X) \to K(X)$ so that $\Psi^k(L_1 + \ldots + L_n) = L_1^k + \ldots + L_n^k$ when $L_1, \ldots, L_n$ are line sheaves, $k \in \mathbb{Z}$. Prove that Ψ^k is in fact a ring morphism.

State and prove a Riemann-Roch type theorem for Ψ^k.

47. Let $\mathcal{D}$ be a bounded complex of injective quasi-coherent sheaves over a noetherian scheme X. If we put $D(\mathcal{M}) = \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{M}, \mathcal{D})$, prove that the following conditions are equivalent:

(a) The natural morphism $\mathcal{M}^\bullet \to DD(\mathcal{M}^\bullet)$ is a quasi-isomorphism for any bounded complex $\mathcal{M}^\bullet$ of quasi-coherent sheaves, with coherent cohomology sheaves $\mathcal{H}^p(\mathcal{M}^\bullet)$.

(b) The natural morphism $\mathcal{O}_X \to DD(\mathcal{O}_X)$ is a quasi-isomorphism.

(c) The natural morphism $\kappa(x) \to DD(\kappa(x))$ is a quasi-isomorphism at any closed point x.

48. Let $\mathcal{O}$ be a Cohen-Macaulay local ring of dimension d with residue field $k = \mathcal{O}/\mathfrak{m}$, and let ω be a biduality module (a biduality complex with a unique nonzero term). Prove that

(a) $\mathrm{Ext}^d_{\mathcal{O}}(k, \omega) = k$ and $\mathrm{Ext}^p_{\mathcal{O}}(k, \omega) = 0$, $p \neq d$.
(*Hint*: $\mathbf{R}\mathrm{Hom}_{\mathcal{O}}(k, \mathcal{O}) = \mathbf{R}\mathrm{Hom}_{\mathcal{O}}(k, \mathbf{R}\mathrm{Hom}_{\mathcal{O}}(\omega, \omega)) = \mathbf{R}\mathrm{Hom}_{\mathcal{O}}(k \underset{=}{\otimes} \omega, \omega)$).

(b) If ω_1, ω_2 are two biduality $\mathcal{O}$-modules, then they are (non-canonically) isomorphic. (*Hint*: If $t \in \mathfrak{m}$ is regular, by induction on d show that $\omega_i/t^n\omega_i$ is a biduality $(\mathcal{O}/t^n\mathcal{O})$-module, so that $\mathrm{Hom}_{\mathcal{O}}(\omega_1/t^n\omega_1, \omega_2/t^n\omega_2) = \mathcal{O}/t^n\mathcal{O}$ and we have compatible isomorphisms $\omega_1/t^n\omega_1 \simeq \omega_2/t^n\omega_2$ defining an isomorphism $\widehat{\omega}_1 \simeq \widehat{\omega}_2$. Conclude that $\mathrm{Hom}_{\mathcal{O}}(\omega_1, \omega_2) \simeq \mathcal{O}$, any generator defining an isomorphism $\omega_1 \simeq \omega_2$).

(c) $H^d_x(\operatorname{Spec} \mathcal{O}, \omega)$ is an injective hull of k, where x is the closed point of $\operatorname{Spec} \mathcal{O}$.

(d) $H^p_x(\operatorname{Spec} \mathcal{O}, M)^* = \mathrm{Ext}^{d-p}(M, \omega)^{\widehat{}}$ for any finitely generated $\mathcal{O}$-module M.

49. Prove that a sheaf morphism $\mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\mathcal{F}(X) \to \mathcal{G}(X)$ is injective for any object X, and for any section $t \in \mathcal{G}(X)$ there is a cover $\{U_i \to X\}$ where the sections $t|_{U_i}$ come from some sections $s_i \in \mathcal{F}(U_i)$.

50. Show that the fpqc topology on the category of S-schemes is defined by the families $\{V_\alpha \to X\}$ refined by some family $\{\operatorname{Spec} B_{ij} \to \operatorname{Spec} A_i \hookrightarrow X\}$, where $X = \bigcup_i \operatorname{Spec} A_i$ is an affine open cover and $\{\operatorname{Spec} B_{ij} \to \operatorname{Spec} A_i\}$ is a finite surjective family of flat morphisms. (*Hint*: These families $\{V_\alpha \to X\}$ define a pretopology because any cover $\{V_\alpha \to U\}$ of an affine scheme U is refined, U being compact, by a finite surjective family of flat morphisms $\{\operatorname{Spec} B_i \to U\}$.)

Conclude that, for any morphism $T \to S$, the induced topology on $\mathbf{Sch}|_T$ is just the fpqc topology of $\mathbf{Sch}|_T$.

51. Show that a sequence $\mathcal{F}' \overset{f_1, f_2}{\rightrightarrows} \mathcal{F} \to \mathcal{F}''$ of sheaves on a situs is exact if and only if

(a) Two sections $s, \bar{s} \in \mathcal{F}(X)$ coincide in $\mathcal{F}''(X)$ if and only if there is a cover $\{U_i \to X\}$ sch that for any index i we have that $s|_{U_i}$ and $\bar{s}|_{U_i}$ coincide in the cokernel of $\mathcal{F}'(U_i) \rightrightarrows \mathcal{F}(U_i)$.

(b) Any section $s'' \in \mathcal{F}''(X)$ comes, on some cover of X, from sections of $\mathcal{F}$.

(*Hint*: The sheafification Coker of the presheaf $U \rightsquigarrow \mathrm{Coker}(\mathcal{F}'(U) \rightrightarrows \mathcal{F}(U))$ is the cokernel of f_1, f_2, and it fulfills both conditions. Conversely, condition 2 shows that $\mathcal{F} \to \mathcal{F}''$ factors through a morphism $\phi\colon \mathrm{Coker} \to \mathcal{F}''$, and if two sections $s_1, s_2 \in \mathrm{Coker}(X)$ coincide in $\mathcal{F}''(X)$, then condition 1 states that they coincide on a cover of X, so that $s_1 = s_2$. Now condition 2 states that locally any section $s'' \in \mathcal{F}''(X)$ is defined by sections of $\mathcal{F}$, hence of Coker, so that ϕ is an isomorphism.)

52. If M is an A-module and $A \to B$ is a faithfully flat morphism, show that we have an exact sequence

$$M \to M \otimes_A B \xrightarrow{\mathrm{d}} M \otimes_A B \otimes_A B \xrightarrow{\mathrm{d}} M \otimes_A B \otimes_A B \otimes_A B \ldots$$

where $\mathrm{d}(m \otimes b_1 \otimes \ldots \otimes b_p) = \sum_{k=0}^{p} m \otimes b_1 \otimes \ldots b_k \otimes 1 \otimes \ldots \otimes b_p$. (*Hint*: After a faithfully flat base change, we may assume that $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ admits a section).

Now, if we consider the functor of points of the additive group, $\mathbb{G}_a(\mathrm{Spec}\, B) = B$, conclude that $\check{H}^p(B/A, \mathbb{G}_a) = 0$, $p \geq 1$.

In the case of a Galois extension of cyclic group $G = (\sigma)$, obtain the additive **Hilbert's theorem 90**: Any element $\beta \in K$ of null trace is $\beta = \sigma(\alpha) - \alpha$ for some $\alpha \in K$.

53. Let k be a field. For any k-algebra A, let $\aleph_A$ be the supremum of the cardinals of the subalgebras of A which are fields, so that $\aleph_A \leq \aleph_B$ whenever there is a morphism of k-algebras $f\colon A \to B$. Let $\mathcal{P}(A)$ be the first well-order of cardinal $\aleph_A$ and let $\mathcal{P}(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$ be the unique isomorphism of $\mathcal{P}(A)$ onto an initial ray of $\mathcal{P}(B)$, so that $\mathcal{P}(f) = \mathcal{P}(h)$ for any other morphism $h\colon A \to B$. Show that $\mathcal{P}$ is a separated presheaf of sets on $(\mathbf{Aff}|_k)_{\mathrm{fp}} = (\mathrm{k}\text{-}\mathbf{alg})^{\mathbf{op}}_{fp}$ such that the "sheafification" $L\mathcal{P}$ is not a sheaf of sets.

54. Let $f^{-1}\colon \mathbf{Aff} \hookrightarrow \mathbf{Sch}$ be the inclusion of the category of affine schemes into the category of schemes. If $\mathcal{P}$ is a presheaf of sets on $\mathbf{Aff}$, show that $f^*\mathcal{P}$ is a presheaf of sets. Conclude that the "categories" of sheaves on $\mathbf{Sch}_{\mathrm{fpqc}}$ and $\mathbf{Aff}_{\mathrm{fpqc}}$ are equivalent.

55. In an arbitrary category $\mathbf{C}$, a **topology** is defined by a family of sieves for any object X, named **covering sieves** of X, such that

    (a) (Stability by base change) If $\mathcal{R}$ is a covering sieve of X, then $R \times_{X^\bullet} Y^\bullet$ is a covering sieve of Y for any morphism $Y \to X$.

    (b) (Local character) Let $\mathcal{C}$ be a sieve of X. If $\mathcal{R}$ is a covering sieve of X and for any morphism $U \to X$ in $\mathcal{R}$ we have that $\mathcal{C} \times_{X^\bullet} U^\bullet$ covers U, then $\mathcal{C}$ covers X.

    (c) $X^\bullet$ is a covering sieve of X, for any object X.

    (and it is sensible to assume that moreover any covering sieve contains a set-generated covering sieve). Prove that any sieve $\mathcal{C}$ of X containing a covering sieve $\mathcal{R}$ also is a covering sieve. (*Hint*: For any morphism $U \to X$ in $\mathcal{R}$ we have that $C \times_{X^\bullet} U^\bullet = U^\bullet$ covers U. Moreover, the intersection $R \cap R'$ of two covering sieves of X also covers X, because for any morphism $U \to X$ in R we have that $(R \cap R') \times_{X^\bullet} U^\bullet = R' \times_{X^\bullet} U^\bullet$ covers U.)

56. Let $\mathbf{C}$ be a category with finite fibred products. Show that any pretopology defines a topology in the above sense. (*Hint*: To prove the second condition, consider a cover $\{U_i \to X\}$ generating a sieve contained in $\mathcal{R}$. For each index i there is a cover $\{V_{ij} \to U_i\}$ generating a sieve contained in $\mathcal{C} \times_{X^\bullet} U_i^\bullet$, so that the morphisms $V_{ij} \to X$ are in $\mathcal{C}$. Hence $\mathcal{C}$ contains the sieve of X generated by the morphisms $V_{ij} \to X$, defining a cover of X.)

    Given a topology on $\mathbf{C}$ show that the sets $\{U_i \to X\}$ generating a covering sieve define a pretopology on $\mathbf{C}$. (*Hint*: If $\{U_i \to X\}$ generates a sieve $\mathcal{R}$, then $\{U_i \times_X Y \to Y\}$ generates $\mathcal{R} \times_{X^\bullet} Y^\bullet$, for any morphism $Y \to X$. Moreover, if $\{U_i \to X\}$ generates a covering sieve $\mathcal{R}$, and for any index i we have a family $\{V_{ij} \to U_i\}$ generating a covering sieve of U_i, then the sieve $\mathcal{R}'$ of X generated by the morphisms $V_{ij} \to X$ fulfills that $\mathcal{R}' \times_{X^\bullet} U_i^\bullet$ covers U_i (it contains the morphisms $V_{ij} \to U_i$). Hence for any morphism $Y \to U_i \to X$ in $\mathcal{R}$ we have that $\mathcal{R}' \times_{X^\bullet} Y^\bullet = (R' \times_{X^\bullet} U_i^\bullet) \times_{U_i^\bullet} Y^\bullet$ covers Y, so that $\mathcal{R}'$ covers X.)

57. Show that the opposite category of fields, with ring morphisms, has not fibred products, and that the sieves containing a finite (resp. separable, algebraic, purely inseparable,...) extension define a topology.

58. Show that the category of smooth manifolds has not fibred products, and that the sieves containing a surjective regular projection define a topology, finer than the classical topology. (*Hint*: If $P \to X$ is a regular projection, then $P \times_X Y$ exists for any smooth map $Y \to X$ and the natural map $P \times_X Y \to Y$ is a regular projection).

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Colophon

As S. Lang pointed out in his acute review of EGA I, in Algebraic Geometry a theorem is not true anymore because one can draw a picture, it is true because it is functorial. And in fact natural statements often may be reduced to a very simple obvious case precisely because of their naturalness.

And what does it mean the term canonical, or natural that we so often use in these notes without a precise formal definition?

In our daily life we use canonical as authorized and recognized, specially in the catholic ecclesiastical law; and natural is the Latin translation of the Greek term $\varphi\upsilon\sigma\iota\zeta$, a central concept of the ancient Greek philosophy that nowadays we still use in the word Physics to point that this science studies Nature.

In Heidegger's words (*Einführing im die Metaphysic* I §3):

> *Die $\Phi\upsilon\sigma\iota\zeta$ ist das Sein selbst, kraft dessen das Seiende erst beobachtbar wird und bleibt... Als Gegenerscheinung tritt heraus, was die Griechen $\theta\epsilon\sigma\iota\zeta$, Setzung, Satzung nennen oder $\nu o\mu o\zeta$, Gesetz, Regel im Sinne des Sittlichen... dem Gegensatz zu $\tau\epsilon\chi\nu\eta$ – was weder Kunst noch Technik besagt, sondern ein Wissen...* [8]

... where we most clearly perceive His aroma.

[8] $\Phi\upsilon\sigma\iota\zeta$ means the force that prevails, brings-forth and remains regulated by itself... As an opposite manifestation the Greeks introduced what they called $\theta\epsilon\sigma\iota\zeta$, what you put, or $\nu o\mu o\zeta$, custom, human convention... or $\tau\epsilon\chi\nu\eta$, which means production from a knowledge...