

Finiteness theorems in étale cohomology

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Theorem 1 [1]. *Let ℓ be a prime and $f : X \rightarrow Y$ a morphism of finite type between quasi-excellent noetherian $\mathbb{Z}[1/\ell]$ -schemes and F a constructible sheaf of $\mathbb{Z}/\ell^r$ -modules on $X_{\text{ét}}$. Then the sheaves $R^i f_* F$ on $Y_{\text{ét}}$ are constructible and vanish except for finitely many i 's.*

There are analogues of theorem 1 for non-abelian coefficients. For direct images of sheaves of sets the quasi-excellence condition is not needed. The conclusion of theorem 1 also holds for morphisms between S -schemes of finite type where S is a noetherian $\mathbb{Z}[1/\ell]$ -scheme of dimension ≤ 1 (this is reduced to results of [3]). If X is normal excellent and $j : U \rightarrow X$ an open immersion with $\text{cod}(X - U, X) \geq 2$ one can show by another method (which uses ultraproducts) that the sheaves $R^1 j_* G$ are constructible for every finite group G .

Let S be a noetherian scheme and $\mathcal{C}$ the category of reduced S -schemes of finite type $X \xrightarrow{f} S$ such that f sends maximal points to maximal points and is generically finite. The alteration topology is the Grothendieck topology on $\mathcal{C}$ defined by the pretopology generated by Zariski covers and proper surjective maps. A family $\{X_i \rightarrow X\}$ in $\mathcal{C}$ is covering for the alteration topology iff it is covering for the Voevodsky h -topology. For a prime ℓ , the ℓ' -alteration topology on $\mathcal{C}$ is the Grothendieck topology generated by

- (i) families of étale morphisms $\{X_i \rightarrow X\}$ such that every $x \in X$ is the image of some point $y \in X_i$ with $[\kappa(y) : \kappa(x)]$ prime to ℓ ,
- (ii) proper surjective maps $X' \rightarrow X$ satisfying the same condition for maximal points of X .

Theorem 2. *Let X be a reduced quasi-excellent noetherian scheme and $Z \subset X$ a nowhere dense closed subset. Then there is a covering family $\{X_i \rightarrow X\}$ for the alteration topology such that each X_i is regular and the inverse image of Z in X_i is the support of a normal crossings divisor.*

The conclusion of theorem 2 can be reformulated in terms of points of the alteration topology, which correspond to valuations. There is a refined form with the ℓ' -alteration topology when a prime ℓ is invertible. With further work one can add the condition that the $X_i \rightarrow X$ are generically étale.

One proof of a part of theorem 1 (constructibility for each i) uses theorem 2 and cohomological descent for oriented product toposes, and the proof of the full result uses the refined form of theorem 2. We discuss the use of reduction to the complete case (Artin-Popescu approximation) and Epp's theorem in the proof of theorem 2, and a modification theorem (theorem 3) which is used in the proof of the refined form.

Let X be a locally noetherian scheme and $Z \subset X$ a closed subset. We say that (X, Z) is log-regular if X is log-regular when equipped with some fs (étale)

log-structure with locus of triviality $X - Z$. The log-structure is then given by the subsheaf of $\mathcal{O}_X$ of functions invertible outside Z .

Theorem 3. *Let (X, Z) be a noetherian log-regular scheme equipped with an action of a finite group G which is generically free, admissible (so that $p : X \rightarrow X/G$ exists) and such that for every $x \in X$ the order of the inertia group G_x is invertible in $\kappa(x)$. Then there is a projective birational map $q : Y' \rightarrow X/G$ such that (Y', Z') is log-regular where $Z' = q^{-1}p(Z \cup (\text{locus of non-trivial inertia}))$.*

This requires some form of canonical desingularization in characteristic zero.

As a consequence of theorem 3 (version with log-smoothness over a base) and de Jong's results one has

Theorem 4. *Let ℓ be a prime and R an excellent Dedekind ring over $\mathbb{Z}[1/\ell]$ and X a separated integral flat R -scheme of finite type. Then there is an alteration $q : X' \rightarrow X$ of degree prime to ℓ (generically étale if R is a perfect field), with X' regular integral quasi-projective over R , such that X' has an fs log-structure such that there is a log-smooth R -morphism $X' \rightarrow \text{Spec}(R')$ where R' is the normalization of R in a finite extension of $\text{Frac}(R)$ and the log-structure of $\text{Spec}(R')$ is defined by a finite set of codimension one points.*

Theorem 4 is used in a new proof of absolute cohomological purity in the mixed characteristic case.

We mention cohomological dimension results

- (i) Affine Lefschetz: If R is an excellent strictly henselian noetherian local ring with $1/\ell \in R$, the ℓ -cohomological dimension of affine open subschemes of $\text{Spec}(R)$ is $\leq \dim(R)$.
- (ii) For R as above a domain, $cd_\ell(\text{Frac}(R)) = \dim(R)$ holds without the excellence assumption.
- (iii) Kato's conjecture on p -cohomological dimension [2].

One can prove the existence of dualizing complexes adapted to dimension functions [4].

REFERENCES

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