When is a simplicial toric ideal a complete intersection?

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Abstract

Let k be an arbitrary field and $k[x] = k[x_1, \ldots, x_n]$ and $k[t] = k[t_1, \ldots, t_m]$ two polynomial rings over k. Denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. A *binomial* f in $k[x]$ is a difference of two monomials, i.e. , $f = x^\alpha - x^\beta$ for some $\alpha, \beta \in \mathbb{N}^n$.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a set of nonzero vectors in $\mathbb{N}^m$. The kernel of the homomorphism of k-algebras $\phi\colon k[x] \to k[t];\ x_i \longmapsto t^{a_i}$ is called *a toric ideal* and will be denoted by $I_\mathcal{A}$. By [6, Corollary 4.3], it is an $\mathcal{A}$*-homogeneous binomial* ideal, i.e., if one sets the $\mathcal{A}$-degree of a monomial $x^\alpha \in k[x]$ as $\alpha_1 a_1 + \cdots + \alpha_n a_n \in \mathbb{N}^m$, and says that $f \in k[x]$ is $\mathcal{A}$-homogeneous if its monomials have the same $\mathcal{A}$-degree, then $I_\mathcal{A}$ is generated by $\mathcal{A}$-homogeneous binomials. According to [6, Lemma 4.2], the height of $I_\mathcal{A}$ is equal to $n - \mathrm{rk}(\mathbb{Z}\mathcal{A})$, where $\mathrm{rk}(\mathbb{Z}\mathcal{A})$ denotes the rank of the subgroup of $\mathbb{Z}^m$ generated by $\mathcal{A}$.

The ideal $I_\mathcal{A}$ is a *complete intersection* (CI) if there exists a system of $\mathcal{A}$-homogeneous binomials $g_1, \ldots, g_s$ such that $I_\mathcal{A} = (g_1, \ldots, g_s)$, where $s = \mathrm{ht}(I_\mathcal{A})$. The problem of determining complete intersection toric ideals has a long history; see the introduction of [5].

If $n > m$ and $\mathcal{A} = \{d_1 e_1, \ldots, d_m e_m, a_{m+1}, \ldots, a_n\}$, where $\{e_1, \ldots, e_m\}$ is the canonical basis of $\mathbb{Z}^m$, $I_\mathcal{A}$ is said to be a *simplicial toric ideal*. If $d_1 = \cdots = d_m = \sum_{j=1}^m a_{ij}$ for all $i \in \{m+1, \ldots, n\}$, $I_\mathcal{A}$ is called a *simplicial projective toric ideal* and $V(I_\mathcal{A}) \subset \mathbb{P}_k^{n-1}$ a *simplicial projective toric variety*. A theoretical characterization of the property of being a complete intersection in simplicial toric ideals via the *semigroup gluing* can be found in [4].

The purpose of this work is to provide and implement an algorithm for determining whether a simplicial toric ideal $I_\mathcal{A}$ is a complete intersection without computing a minimal system of generators of $I_\mathcal{A}$. The detailed proofs of the results here included are in [1].

Now we will present two results that can be applied to general toric ideals. The first one is based on the following idea, we associate to $\mathcal{A}$ another set $\mathcal{A}_{red} \subset \mathbb{N}^m$ with this property:

Theorem 0.1. *$I_\mathcal{A}$ is a* CI $\Longleftrightarrow$ *$\mathcal{A}_{red} = \emptyset$ or $I_{\mathcal{A}_{red}}$ is a* CI.

Whenever $\mathcal{A}_{red}$ is not empty, the ideal $I_{\mathcal{A}_{red}}$ is simpler that $I_\mathcal{A}$ in several senses.

Let us present the second result. For certain $i \in \{1, \ldots, n\}$ one can define

$$m_i := \min \left\{ b \in \mathbb{Z}^+ \mid b a_i \in \sum_{\substack{j \in \{1,\ldots,n\} \\ j \neq i}} \mathbb{N} a_j \right\}$$

If there exist $i, j : 1 \leq i < j \leq n$ such that $m_i a_i = m_j a_j$, we set $\mathcal{A}' := (\mathcal{A} \backslash \{a_i, a_j\}) \cup \{a_i'\}$ where $a_i' := (a_{i1}', \ldots, a_{im}') \in \mathbb{N}^m$ with $a_{ik}' = 0$ if $a_{ik} = 0$ and $a_{ik}' = \gcd\{a_{ik}, a_{jk}\}$ otherwise.

Theorem 0.2. *If $I_{\mathcal{A}}$ is a* CI*, then $I_{\mathcal{A}'}$ also is.*

Now, me study simplicial toric ideals and get the following:

Theorem 0.3. *Let $I_{\mathcal{A}}$ be a simplicial toric ideal. If $I_{\mathcal{A}}$ is a* CI*, then either $\mathcal{A}_{red} = \emptyset$ or there exist $i, j : 1 \leq i < j \leq n$ such that $m_i a_i = m_j a_j$.*

These results lead to an algorithm that determines whether a simplicial toric ideal is a complete intersection receiving as input the set $\mathcal{A}$. The algorithm answers TRUE if $I_{\mathcal{A}}$ is a complete intersection or FALSE otherwise. In case the answer is positive, the algorithm gives without any extra effort a minimal set of generators of the simplicial toric ideal.

Meanwhile, for simplicial projective toric ideals, we get the following result which provides a simpler algorithm for this case.

Theorem 0.4. *Let $I_{\mathcal{A}}$ be a simplicial projective toric ideal. $I_{\mathcal{A}}$ is a* CI $\Leftrightarrow \mathcal{A}_{red} = \emptyset$.

We have implemented both algorithms in C++ and in SINGULAR [3]. They will be soon included as a new library `cisimplicial.lib` [2]. Computational experiments show that our implementation is able to solve large-size instances.

As non-trivial consequences of Theorem 0.4, when k is an algebraically closed field, we completely classify which smooth simplicial projective toric varieties are ideal-theoretic complete intersection and which simplicial projective toric varieties having only one singular point are ideal-theoretic complete intersection.

References

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