Bases de Gröbner parciales y optimización combinatoria multiobjetivo

SEMINARIO DE GEOMETRÍA TÓRICA IV

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Multiobjective Integer Programming

$$\begin{array}{lrl} \min & \{c^1 x, \ldots, c^k x\} & = C x \\ s.a. & & \\ & A x & = b \\ & x & \in \mathbb{Z}^n_+ \end{array} \qquad (MIP_{A,C})$$

where $A \in \mathbb{Z}^{m \times n}$, $b \in Z^m$ and $C \in \mathbb{Z}^{k \times n}_+$

Partial order induced by C

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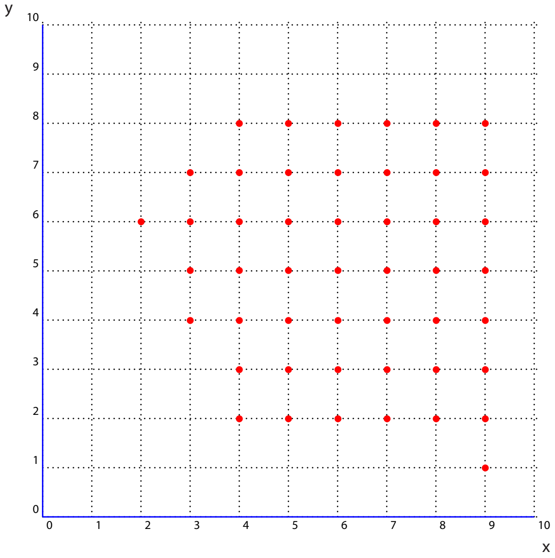
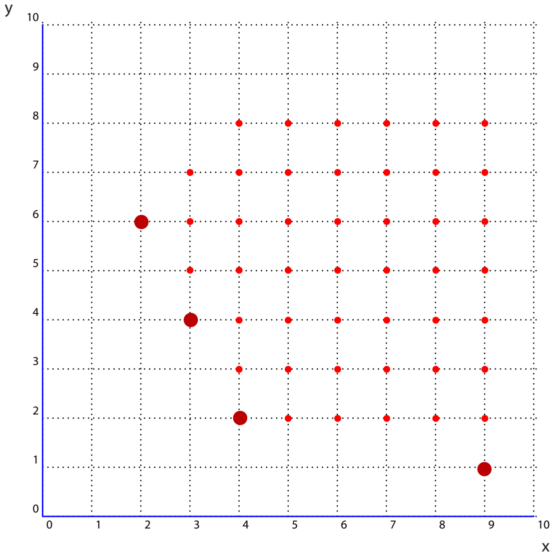


Partial order induced by C



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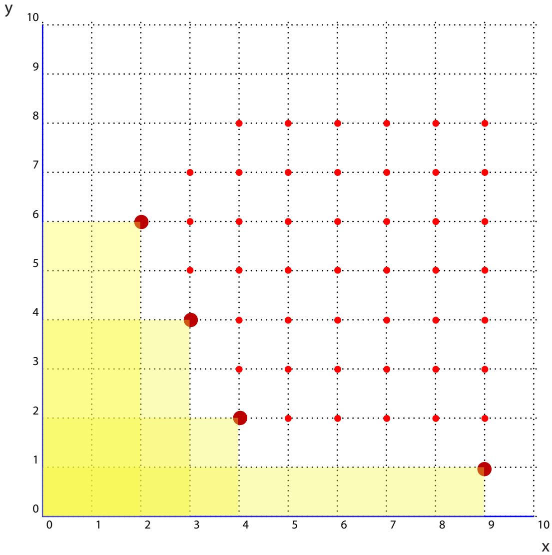

Partial order induced by C

Linear partial order over $\mathbb{Z}_+^n$:

$$x \prec_C y :\Longleftrightarrow Cx \lneqq Cy$$

Solution Notion for MOILP

Definition

A feasible vector $\widehat{x} \in \mathbb{R}^n$ is said to be a nondominated solution for $MIP_{A,C}(b)$ if there is no other feasible vector y such that

$$c_j\, y \leq c_j\, \widehat{x} \qquad \forall j = 1, \ldots, k$$

with at least one strict inequality for some j.

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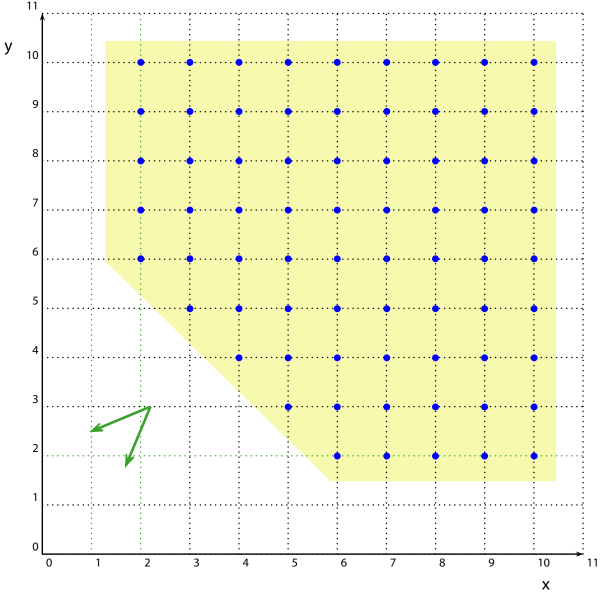
$X_E := \{$nondominated solutions$\}$.

$Y_E := \{$efficients solutions$\}$.

Example



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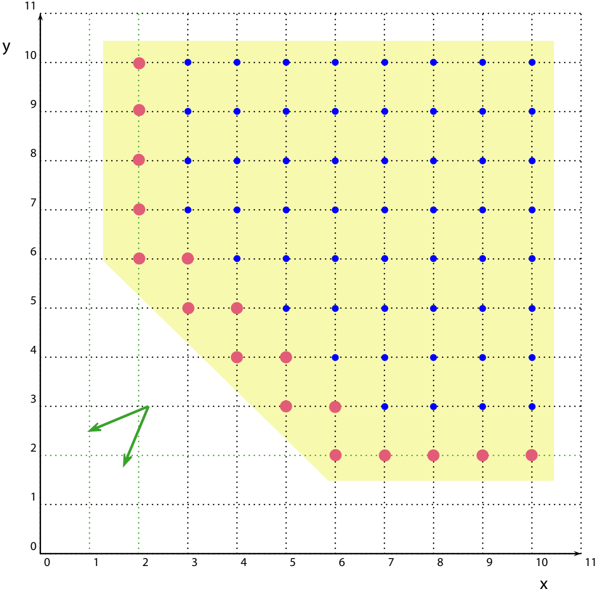

Standard Methods:

- Dynamic Programming (Li-Haimes, 1990).
- Implicit Enumeration (Fukuda-Matsui, 1992).
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Non-Standard Methods:

- Partial Gröbner Bases: Test Families.
- Barvinok Functions: Augmentation Algorithms, Digging, Binary Search.
- ...

Outline

1. **Test Families for MOILP**
2. **p-Gröbner bases**
3. **Solving MOILP problems**
4. **Computational Results**

Test Family

Definition (Test Family)

A finite collection $\mathcal{G}_C^1, \ldots, \mathcal{G}_C^r$ of sets in $\mathbb{Z}^n \times \mathbb{Z}_+^n$ is a test family for $MIP_{A,C}$ if and only if:

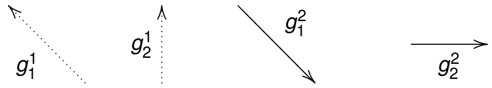
1. $\mathcal{G}_C^j$ is totally ordered by the second component with respect to $\prec_C$, for $j = 1, \ldots, r$.
2. For all $(g, h) \in \mathcal{G}_C^j$, $j = 1, \ldots, r$, $A\,(h - g) = A\,h$, $h, h - g \geq 0$.
3. If $x \in \mathbb{N}^n$ is dominated, there is some $\mathcal{G}_C^j$ in the collection and $(g, h) \in \mathcal{G}_C^j$, such that $x - g \prec_C x$.
4. If $x \in \mathbb{N}^n$ is a nondominated solution in a $MIP_{A,C}$ then for all $(g, h) \in \mathcal{G}_C^j$ and for all $j = 1, \ldots, n$ either $x - g$ is negative or $x - g$ does not compare with x.

Algorithm

- If x^* is dominated:
  1. There is some j and $(g, h) \in \mathcal{G}_C^j$ such that $x^* - g$ is feasible and $x^* - g \prec_C x^*$. Discard x^* and add $x^* - g$ to the list.
  2. For the remaining chains there may exist some (g, h) such that $x^* - g$ is feasible but non-comparable with x^*. We keep tracks of all of them.
- If x^* is non-dominated:
  1. Keep it as an element in our solution set.
  2. Reducing x^* by the chains in the test family we can only obtain either non-comparable feasible solutions, that we maintain in our structure.

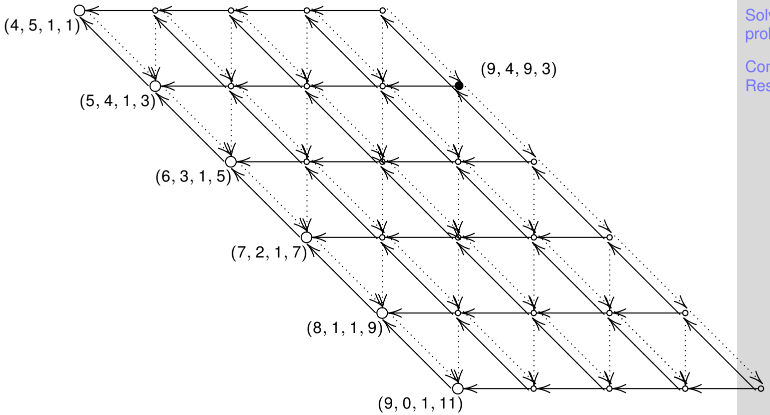
Example

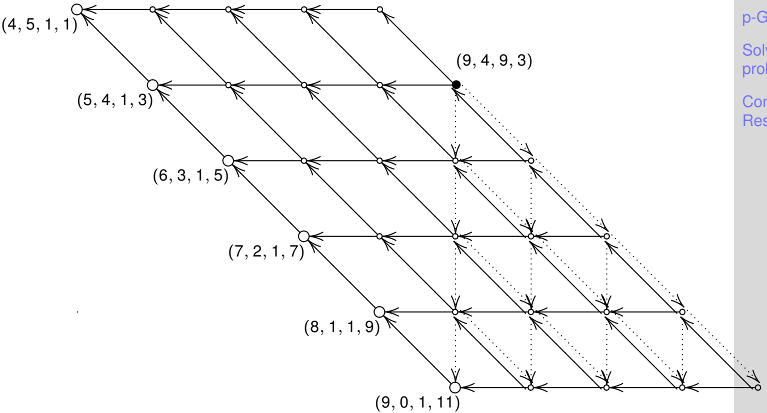
$$A = \begin{bmatrix} 2 & 2 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 10 & 1 & 0 & 0 \\ 1 & 10 & 0 & 0 \end{bmatrix}.$$



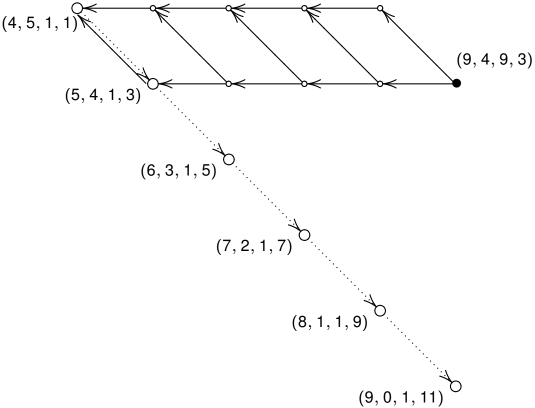
Elements in $\mathcal{G}_{\prec_C} = \{\mathcal{G}^1_{\prec_C}, \mathcal{G}^2_{\prec_C}\}$

$b = (17, 11)' \longrightarrow$ Feasible Solution: $(9, 4, 9, 3)$





(4, 5, 1, 1)
(9, 4, 9, 3)
(5, 4, 1, 3)
(6, 3, 1, 5)
(7, 2, 1, 7)
(8, 1, 1, 9)
(9, 0, 1, 11)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$$I_A = \langle x^u - x^v : u - v \in Ker(A), u, v \geq 0 \rangle = \langle x^{u_1} - x^{v_1}, \dots, x^{u_s} - x^{v_s} \rangle$$

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$x^u - x^v \equiv x^{h-g} - x^h$, $g \in Ker(A)$, $h, h - g \geq 0$, $h - g \in setlt(h - g, h)$.

Partial Reduction

The reduction of $(g, h) \in \mathbb{Z}^n \times \mathbb{Z}^n_+$ by an ordered set $\mathcal{G} \subseteq Ker(A) \times \mathbb{Z}^n_+$, consists of:

Algorithm 1: Partial Reduction Algorithm

input : $R = \{(g, h)\}$, $S = \{(g, h)\}$, $\mathcal{G} = \{g_1, \ldots, g_t\}$
For each $(\tilde{g}, \tilde{h}) \in S$:
for $i = 1, \ldots, t$ **do**
  repeat
    if $\tilde{h} - g_i$ *and* $\tilde{h} - \tilde{g}$ *are comparable by* $\prec_C$ **then**
      $R_o = \{(\tilde{g} - g_i, \max_{\prec_C}\{\tilde{h} - \tilde{g}_i, \tilde{h} - \tilde{g}\})\}$
    else
      $R_o = \{(\tilde{g} - g_i, \tilde{h} - g_i), (\tilde{g} - g_i, \tilde{h} - \tilde{g})\}$
    end
    For each $r \in R_o$ and $s \in R$:
    if $r \prec_C s$ **then**
      $R := R \setminus \{s\}$;
    end
    $S := R_o$
    $R := R \cup R_o$;
  until $\{i : \tilde{h} - h_i \geq 0\} = \emptyset$;
end
output: R, the partial reduction set of (g, h) by $\mathcal{G}_C$

Extension for a finite collection of ordered sets of pairs in $\mathbb{Z}^n \times \mathbb{Z}^n_+$:

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Theorem

Let $\mathcal{G}$ be a set in $\mathbb{Z}^n \times \mathbb{Z}^n_+$, whose maximal chains are $\mathcal{G}_1, \ldots, \mathcal{G}_t$, and σ, σ' two sequences of the indices $(1,\ldots,t)$. Then,

$$pRem((g,h),\mathcal{G})_\sigma = pRem((g,h),\mathcal{G})_{\sigma'} \quad (g,h) \in \mathbb{Z}^n \times \mathbb{Z}^n_+$$

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$pRem((g,h),(\mathcal{G}_i)) = pRem((g,h),(\mathcal{G}_i))_\sigma$ for any σ.

Partial Gröbner Basis

Definition (Partial Gröbner basis)

A family $\mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_n\} \subseteq Ker(A) \times \mathbb{Z}^n_+$ is a partial Gröbner basis (p-Gröbner basis) for the family of problems $MIP_{A,C}$, if $\mathcal{G}_1, \ldots, \mathcal{G}_n$ are the maximal chains for the partially ordered set $\bigcup_i \mathcal{G}_i$ and for any $(g, h) \in \mathbb{Z}^n \times \mathbb{Z}^n_+$, with $h - g \geq 0$:

$$g \in Ker(A) \Longleftrightarrow pRem((g, h), \mathcal{G}) = \{0\}.$$

A p-Gröbner basis is said to be *reduced* if every element at each maximal chain cannot be obtained by reducing any other element of the same chain.

Theorem

The reduced p-Gröbner basis for $MIP_{A,C}$ is the unique minimal test family for $MIP_{A,C}$

S-polynomials

$$S^1((g,h),(g',h')) =$$

$$\begin{cases} (g - g' - 2(h - h'), \gamma + g - 2h) & \text{if } \gamma + g - 2h \prec_C \gamma + g' - 2h' \\ (g' - g - 2(h' - h), \gamma + g' - 2h') & \text{if } \gamma + g' - 2h' \prec_C \gamma + g - 2h \\ (g - g' - 2(h - h'), \gamma + g - 2h) & \text{if } \gamma + g' - 2h' \nsim \gamma + g - 2h \end{cases}$$

$$S^2((g,h),(g',h')) =$$

$$\begin{cases} (g - g' - 2(h - h'), \gamma + g - 2h) & \text{if } \gamma + g - 2h \prec_C \gamma + g' - 2h' \\ (g' - g - 2(h' - h), \gamma + g' - 2h') & \text{if } \gamma + g' - 2h' \prec_C \gamma + g - 2h \\ (g' - g - 2(h' - h), \gamma + g' - 2h') & \text{if } \gamma + g' - 2h' \nsim \gamma + g - 2h \end{cases}$$

where $\gamma \in \mathbb{N}^n$ whose components are $\gamma_i = \max\{h_i, h_i'\}$, $i = 1, \ldots, n$.

S-polynomial Criterion

Theorem (Extended Buchberger Criterion)

Let $\mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_t\}$ with $\mathcal{G}_i \subseteq I_A$ for all $i = 1, \ldots, t$, be the maximal chains for the partially ordered set $\{g_i : g_i \in \mathcal{G}_i, \text{ for some } i = 1, \ldots, t\}$. Then the following statements are equivalent:

1. *$\mathcal{G}$ is a p-Gröbner basis for the family $MIP_{A,C}$.*
2. *For each $i, j = 1, \ldots, t$ and $(g, h) \in \mathcal{G}_i$, $(g', h') \in \mathcal{G}_j$, $pRem(S^k((g,h),(g',h')), \mathcal{G}) = \{0\}$, for $k =, 1, 2$.*

Algorithm for computing p-Gröbner basis

Algorithm 2: Partial Buchberger Algorithm

input : A generating set for $I_A \equiv \langle (g,h) : g \in Ker(A), h, h-g \in \mathbb{Z}^+ \rangle$: $\mathcal{G}$
repeat
  Compute, $\mathcal{G}_1, \ldots, \mathcal{G}_t$, the maximal chains for $\mathcal{G}$.
  for *$i, j \in \{1, \ldots, t\}$, $i \neq j$, and each pair $(g,h) \in \mathcal{G}_i$, $(g', h') \in \mathcal{G}_j$* **do**
    Compute $R^k = pRem(S^k((g,h),(g^{'},h^{'})),\mathcal{G})$, $k = 1, 2$.
    if $R^k = \{0\}$ **then**
      Continue with other pair.
    else
      Add $\phi(\mathcal{F}(r))$ to $\mathcal{G}$, for each $r \in R^k$.
    end
  end
until $R^k = \{0\}$ *for every pairs* ;
output: $\mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_Q\}$
p-Gröbner basis for $MIP_{A,C}$.

Computing the nondominated solutions

Algorithm 3: Nondominated solutions computation for $MIP_{A,C}(b)$

input : $MIP_{A,C}(b)$

STEP 1. Compute a generating set for I_A.
([Hosten-Sturmfels, 1995], [Di Biase-Urbanke, 1996])

STEP 2. Compute the partial reduced Gröbner basis for $MIP_{A,C}$, $\mathcal{G}_C = \{\mathcal{G}_1, \ldots, \mathcal{G}_t\}$.

STEP 3. Compute an initial feasible solution, α_o, for $MIP_{A,C}(b)$: A solution for the diophantine system of equations $Ax = b$, $x \in \mathbb{Z}^n$.

STEP 4. Calculate the set of partial remainders:
$R := pRem((\alpha_o, \alpha_o), \mathcal{G}_C)$.

output: Nondominated Solutions : R.

Computational Experiments

Multiobjective Knapsack

Problem	sog	pgröbner	pos	total	act_pGB
knap4_2	0.063	249.369	1.265	250.697	164.920
knap4_3	0.063	1002.689	2.012	1004.704	772.772
knap4_4	0.063	1148.574	2.374	1151.011	763.686
knap5_2	0.125	1608.892	0.875	609.892	1187.201
knap5_3	0.125	3500.831	2.035	3503.963	2204.123
knap5_4	0.125	3956.534	2.114	3958.773	3044.157
knap6_2	0.185	2780.856	2.124	2783.165	2241.091
knap6_3	0.185	3869.156	2.018	3871.359	2790.822
knap6_4	0.185	4598.258	3.006	4601.449	3096.466

Multiobjective Transportation Problems

Problem	sog	pgröbner	pos	total	act_pGB
tranp3x2_2	0.015	11.813	0.000	11.828	7.547
tranp3x2_3	0.015	7.218	13.108	30.341	6.207
tranp3x2_4	0.015	6.708	15.791	21.931	4.561
tranp3x3_2	0.047	1545.916	1.718	1547.681	928.222
tranp3x3_3	0.047	3194.333	11.235	3205.615	2172.146
tranp3x3_4	0.047	3724.657	7.823	3732.527	2112.287
tranp4x2_2	0.046	675.138	2.122	677.306	398.093
tranp4x2_3	0.046	1499.294	6.288	1505.628	119.519
tranp4x2_4	0.046	2285.365	7.025	2292.436	1654.048