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Some problems in finite geometries where algebraic curves and surfaces appear

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Let $\mathrm{AG}(n, q)$ denote the n-dimensional affine space over $\mathbb{F}_q$, the finite field with q elements, and let $\mathrm{PG}(n, q)$ denote the projective space.

The finite field version of Kakeya's conjecture says that a set of points in $\mathrm{AG}(n, q)$, which contains a line of every direction has cardinality at least cq^n, where the constant c depends only on n. Proposed by Wolff in 1999 this conjecture was proven recently by Dvir (2008), who proved the lower bound $\binom{q+n-1}{n}$. For $n = 2$ and q odd Blokhuis and Mazzocca (2008) improved this bound to $\binom{q+1}{2} + \frac{q-1}{2}$ and classified all sets meeting the bound as more or less the external points to a conic C. For $n = 2$ and q even Dvir's bound is tight and for $n \geq 3$ no general good examples are known.

The Blokhuis-Mazzocca proof uses a method introduced by Segre (1956) who proved that an arc of size $q + 1$, a set of $q + 1$ points in $\mathrm{PG}(2, q)$ with the property that at most two points are collinear, is a conic, if q is odd. In fact Segre proved much more. He proved that a set of points dual to the tangent lines of an arc A is contained in an algebraic curve of degree at most $t = q + 2 - |A|$ if q is even and $2t$ if q is odd. Blokhuis, Bruen and Thas (1990) extended this result to arcs in $\mathrm{PG}(n, q)$, proving that there is an algebraic hypersurface which contains all the points dual to the hyperplanes which are incident with $n - 1$ points of an arc. The main aim is to show that there is no arc of size $q + 2$ unless q is even and $n = 2$ or $q - 2$. This would imply that the longest linear maximum distance separable codes are no longer than the Reed-Solomon codes, an open problem dating back to the fifties.

The proof of the bound in the planar Kakeya problem also provides a proof that the number of collinear triples in the graph of a permutation is at least $\frac{q-1}{2}$, a bound conjectured by Cooper and Solymosi (2005) and proved by Li (2008).

Functions over a finite field which determine few directions occur in the factorisations of abelian groups, permutation polynomials and finite geometry and we shall consider some of the more important results in this direction. We shall consider functions for which the number of idrections not determined is at least $\frac{q-1}{t} + t - 2$ for some integer t. In the case that q is prime it is conjectured by Ball and Gacs (2008) that the graph of f is contained in an algebraic curve of degree t. This has been proved for $t = 2$ by Lovász and Schrijver (1981) and for $t = 3$ by Gács (2003). We shall consider a new method proposed in joint work with Gacs (2008) which provides proofs for the cases $t = 2$ and 3 and partial results for $t \geq 4$.