A. Melle-Hernández

On pencils of rational cuspidal curves on surfaces

(joint work with Daniel Daigle)

Let C be an irreducible projective plane curve in the complex projective space $\mathbb{P}^2$ with singular points $\{p_i\}_{i=1}^{\nu}$. From C one can extract the following information: its degree d, and the local embedded topological types T_i of the local singular germs $(C, p_i) \subset (\mathbb{P}^2, p_i)$. It is a very interesting, and still open problem, to characterize those collections of local embedded topological types $\{T_i\}_{i=1}^{\nu}$ (without fixing the positions of the points p_i) which can be realized by such a projective curve C of degree d.

For our propose of study we will be interested on rational cuspidal projective plane curves. In such a case the open surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic if and only if C is a rational cuspidal curve. One of the integers which help in the classification problem is the logarithmic Kodaira dimension $\bar{\kappa}$ of open surface $\mathbb{P}^2 \setminus C$. The classification of curves with $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ has been recently finished by Miyanishi and Sugie [5], Tsunoda [9]. and Tono [8].

This remarkable problem of classification is not only important for its own sake, but it is also connected with crucial properties, problems and conjectures in the theory of open surfaces, and in the classical algebraic geometry:

• **Coolidge and Nagata problem**, see [1, 6]. It predicts that every rational cuspidal curve can be transformed by a Cremona transformation into a line, (it is verified in all known cases).

• **Orevkov's conjecture** [7] which formulates an inequality involving the degree d and numerical invariants of local singularities. In a different formulation, this is equivalent with the positivity of the virtual dimension of the space of curves with fixed degree and certain local type of singularities which can be geometrically realized.

• **Rigidity conjecture** of Flenner and Zaidenberg, [3]. Fix one of 'minimal logarithmic compactifications' (V, D) of $\mathbb{P}^2 \setminus C$, that is V is a smooth projective surface with a normal crossing divisor D, such that $\mathbb{P}^2 \setminus C = V \setminus D$, and (V, D) is minimal with these properties. The *rigidity conjecture* asserts that every $\mathbb{Q}$-acyclic affine surfaces $\mathbb{P}^2 \setminus C$ with logarithmic Kodaira dimension $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$ is rigid and has unobstructed deformations.

In this talk we will be interested in the case of rational unicuspidal curves C of degree d. We will show that in such a case there always exists a pencil Λ_C and a net N_C of curves of degree d such that C is is a element of such linear systems. A curve C is called of non-negative type if the self-intersection of the strict transform of C in the minimal (not embbeded) resolution of C is non-negative. We will show that every rational unicuspidal curves C of non-negative type can be transform into a line by a Cremona transformation (Coolidge and Nagata problem). We will show that C is non-negative if and only if a general element of Λ_C is a rational curve (this is true in all known curves) We will show that C is of positive type if and only if a general member of the neat N_C is a rational curve.

References

[1] J.L. Coolidge, *A treatise of algebraic plane curves*, Oxford Univ. Press. Oxford, (1928).

[2] D. Daigle and A. Melle-Hernández, On pencils of rational cuspidal curves on surfaces, preprint 2009.

[3] H. Flenner and M. Zaidenberg, On a class of rational cuspidal plane curves. *Manuscripta Math.* **89** (1996) 439–459.

[4] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi, On rational cuspidal projective plane curves, *Proc. London Math. Soc.* (3) **92** (2006), no. 1, 99–138

[5] M. Miyanishi and T. Sugie, On a projective plane curve whose complement has logarithmic Kodaira dimension $-\infty$, *Osaka J. Math.* , **18** (1981), 1–11.

[6] M. Nagata, On rational surfaces I. Irreducible curves of arithmetic genus 0 and 1, *Memoirs of College of Science,* Univ. of Kyoto, Series A, Vol. **XXXII** (3) (1960), 351-370.

[7] S.Yu. Orevkov, On rational cuspidal curves, I. Sharp estimate for degree via multiplicity, *Math. Ann.* **324** (2002).

[8] K. Tono, On rational unicuspidal plane curves with $\bar{\kappa} = 1$, *RIMS-Kôkyûroku* **1233** (2001) 82–89.

[9] Sh. Tsunoda, The Structure of Open Algebraic Surfaces and Its Application to Plane Curves, *Proc. Japan Acad. Ser. A* **57** (1981).