Applications of Discrete Optimization to Commutative Algebra

Víctor Blanco

Departamento de Álgebra, Universidad de Granada

vblanco@ugr.es

Applying discrete optimization usually means solving real-world problems in industry, transportation, location, etc, but there are many problems arising in abstract mathematics, specially in commutative algebra, that can be formulated as integer programming problems and that have centered the attention of many researchers. In particular, problems in classical number theory are specially interesting and only a few papers appear in the literature where operational researchers deal with these type of problems.

In the recent years, there have appeared several methods based on algebraic tools to solve (single and multi-objective) discrete optimization problems as Gröbner bases or short generating functions, amongst others. Here, we intend to give an inverse viewpoint, that is, how optimization tools can help to solve algebraic problems and then getting closer both mathematical fields, algebra and operational research.

We study a specific problem in a basic algebraic structure: numerical semigroups. It is a simple framework where developing optimization tools to make computations that are usually done by brute force.

A numerical semigroup is a set of non-negative integers, closed under addition, containing zero and such that its complement in $\mathbb{N}$ is finite (see [4] for further details).

New arithmetic invariants for commutative semigroups, and then, in particular for numerical semigroups that have recently appeared in the literature are the tame degree, the catenary degree and the ω invariant (see [1], [2], [3] for a complete algebraic description of these indices). These arithmetic invariants come from the field of commutative algebra and factorization theory, and their definitions are not intuitive from a combinatorial viewpoint. In particular, the ω invariant allows to derive crucial finiteness properties in the theory of non-unique factorizations for semigroups (tameness).

In this talk we present a new method for computing the ω invariant of a numerical semigroup by optimizing a linear function over the non-dominated solutions of a multiobjective integer programming problem. In contrast to usual integer programming problems, in multiobjective problems there are several objective functions to be optimized.

We recall in this talk the applications of commutative algebra to discrete otimization as well as new research lines where discrete optimization may help solving problems in semigroup theory.

References

[1] Anderson, D.F., Chapman, S.T., Kaplan, N., and Torkornoo, D. (2010). An algorithm to compute ω-primality in a numerical monoid, Semigroup Forum.

[2] Blanco, V., García-Sánchez, P.A., and Geroldinger, A. (2010). *Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids*. Submitted. arxiv:1006.4222[math.AC].

[3] Chapman, S.T., García-Sánchez, P.A., and Llena, D. (2009). *The catenary and tame degree of numerical monoids*, Forum Math. **21**, 117 – 129.

[4] Rosales, J.C. and García-Sánchez, P.A. (2009), *Numerical Semigroups*, Springer.