On the distribution of the roots of sparse systems of polynomial equations

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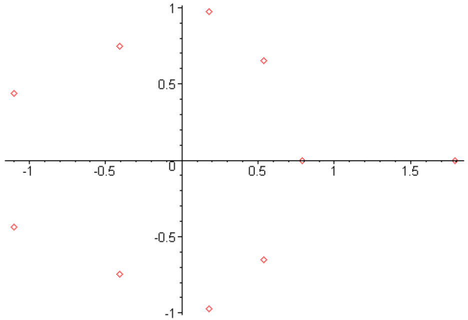
Toric Geometry Seminar 2010 – Jarandilla de la Vera

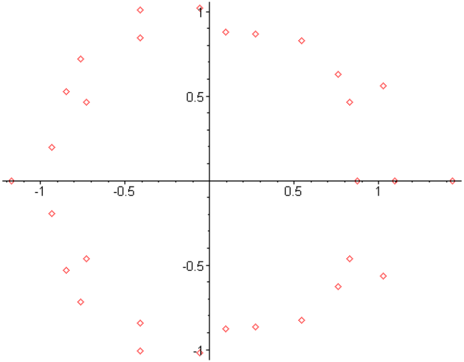
Activity

Let f be a polynomial of degree $d \gg 0$ with coefficients ± 1 or 0. I will plot all complex solutions of $f = 0$, then we will see what it happens...

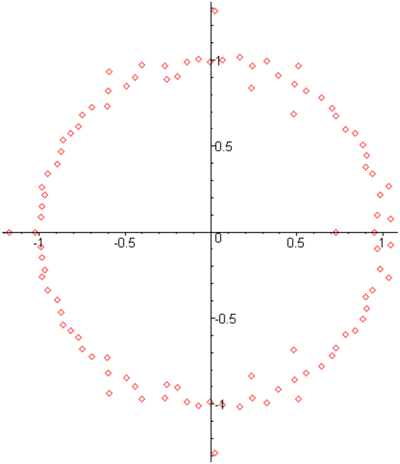
For instance, let $d = 10$ and
$f = -x^{10} + x^9 + x^8 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$



$d = 30$ and
$f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdot$



$d = 100$ and
$f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$

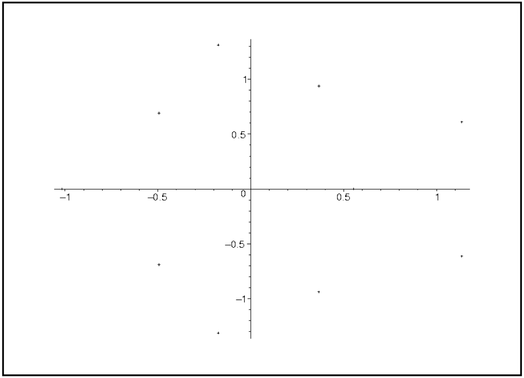


Conclusion???

A more ambitious activity

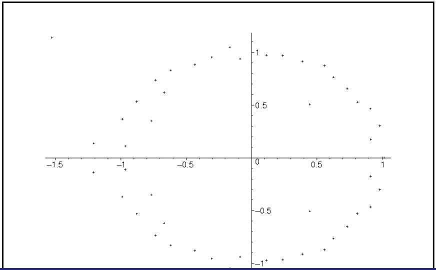
Let us say now that f has degree $d \gg 0$ with coefficients between $-d$ and d. What happens now?

$d = 10$ and
$f = -6 + 8x - x^2 + 10x^3 - 3x^4 + 8x^5 + 4x^6 - 9x^7 + 9x^8 - 6x^9 + 5x^{10}$



$d = 50$ and

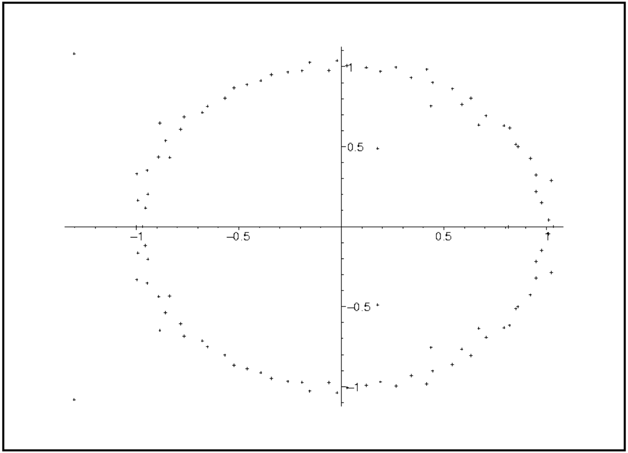
$$f = -24 + 12x - 44x^{48} - 48x^{49} - 42x^{28} + 15x^{29} + 34x^{26} + 22x^{27} - 24x^{24} + 29x^{25} + 14x^{2} - 40x^{3} - 48x^{4} + 35x^{5} + 24x^{6} + 27x^{7} - 3x^{8} - 15x^{9} - 21x^{10} + 12x^{14} - 15x^{50} - 14x^{33} + 38x^{34} + 10x^{35} - 23x^{36} + 48x^{37} + 30x^{38} - 23x^{39} - 31x^{40} + 2x^{41} + 24x^{42} + 9x^{43} - 15x^{44} - 29x^{45} + 45x^{46} + 40x^{47} + 40x^{31} - 40x^{32} + 38x^{11} + 8x^{12} - 16x^{13} - 39x^{15} + 2x^{16} - 38x^{17} - x^{18} + 16x^{19} - 44x^{20} - 20x^{21} + 22x^{22} + 28x^{23} + 32x^{30}$$



$d = 100$ and

$f=$

$$30-45x-91x^{74}-33x^{75}+4x^{73}-59x^{79}+35x^{92}-57x^{48}+49x^{49}+2x^{93}-87x^{28}-16x^{29}-78x^{26}-31x^{27}+19x^{50}-73x^{24}-63x^{25}+98x^{2}+29x^{3}-97x^{4}+47x^{5}+46x^{6}-88x^{7}-74x^{8}-60x^{9}-62x^{10}-27x^{81}-82x^{80}-92x^{78}-50x^{77}-41x^{76}-21x^{95}+8x^{66}-7x^{67}+75x^{64}-19x^{94}-48x^{63}+92x^{65}-18x^{60}+53x^{61}+84x^{59}-15x^{57}-13x^{58}-64x^{91}+84x^{90}-54x^{89}+67x^{55}-81x^{56}-27x^{54}-61x^{88}+43x^{87}+49x^{86}+51x^{84}-12x^{85}-64x^{83}+52x^{82}+43x^{70}-91x^{71}-97x^{72}+76x^{68}+14x^{69}+73x^{99}-56x^{97}+41x^{98}+73x^{96}+44x^{100}+2x^{51}-79x^{52}+87x^{53}-43x^{14}+39x^{62}+50x^{33}+53x^{34}+64x^{35}+57x^{36}-57x^{37}-31x^{38}+85x^{39}+30x^{40}-49x^{41}+6x^{42}-82x^{43}+34x^{44}+59x^{45}+7x^{46}+91x^{47}+59x^{31}+58x^{32}-4x^{11}-71x^{12}-68x^{13}+74x^{15}+60x^{16}-3x^{17}+23x^{18}-55x^{19}+80x^{20}-32x^{21}+17x^{22}-14x^{23}-69x^{30}$$


1
0.5
–1
–0.5
0.5
1
–0.5
–1

Conclusion???

The Erdös-Turán theorem

Let $f(x) = a_d x^d + \cdots + a_0 = a_d(x - \rho_1 \, e^{i\theta_1}) \cdots (x - \rho_d \, e^{i\theta_d})$

Definition

The *angle discrepancy* of f is

$$\Delta_\theta(f) := \sup_{0 \le \alpha < \beta < 2\pi} \left| \frac{\#\{k : \alpha \le \theta_k < \beta\}}{d} - \frac{\beta - \alpha}{2\pi} \right|$$

The ε*-radius discrepancy* of f is

$$\Delta_{\mathrm{r}}(f;\varepsilon) := \frac{1}{d} \, \#\left\{k : 1 - \varepsilon < \rho_k < \frac{1}{1-\varepsilon}\right\}$$

Also set $||f|| := \sup_{|z|=1} |f(z)|$

Theorem [Erdös-Turán 1948], [Hughes-Nikeghbali 2008]

$$\Delta_\theta(f) \leq c\sqrt{\frac{1}{d}\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)} \quad , \quad 1-\Delta_{\mathrm{r}}(f;\varepsilon) \leq \frac{2}{\varepsilon d}\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)$$

Here $\sqrt{2} \leq c \leq 2{,}5619$ [Amoroso-Mignotte 1996]

Corollary: the equidistribution

Let $f_d(x)$ of degree d such that $\log\left(\frac{||f_d||}{\sqrt{|a_{d,0}a_{d,d}|}}\right) = o(d)$, then

$$\lim_{d\to\infty}\frac{1}{d}\#\left\{k : \alpha \leq \theta_{dk} < \beta\right\} = \frac{\beta-\alpha}{2\pi}$$

$$\lim_{d\to\infty}\frac{1}{d}\#\left\{k : 1-\varepsilon < \rho_{dk} < \frac{1}{1-\varepsilon}\right\} = 1$$

Some consequences

1. The number of real roots of f is $\leq 51\sqrt{d\log\left(\frac{||f||}{\sqrt{|a_0 a_d|}}\right)}$ [Erhardt-Schur-Szego]

2. If $g(z) = 1 + b_1 z + b_2 z^2 + \ldots$ converges on the unit disk, then the zeros of its d-partial sums distribute uniformely on the unit circle as $d \to \infty$ [Jentzsch-Szego]

Equidistribution in several variables

For $f_1,\ldots,f_n \in \mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ consider

$$V(f_1,\ldots,f_n) = \{\xi \in (\mathbb{C}^\times)^n : f_1(\xi) = \cdots = f_n(\xi) = 0\} \subset (\mathbb{C}^\times)^n$$

and V_0 the subset of isolated points Set $Q_i := \mathrm{N}(f_i) \subset \mathbb{R}^n$ the

Newton polytope, then

$$\# V_0 \leq \mathrm{MV}_n(Q_1,\ldots,Q_n) =: D \quad \text{[BKK]}$$

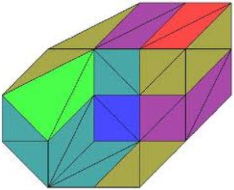
From now on, we will assume $\# V_0 = D$, in particular $V(\boldsymbol{f}) = V_0$.

Problem

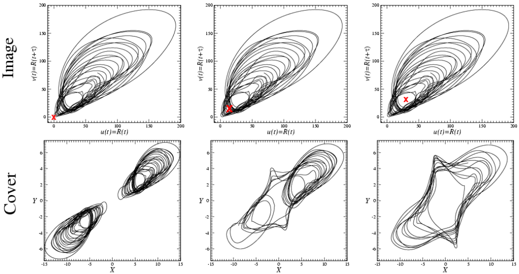
Estimate $\Delta_\theta(\boldsymbol{f})$ and $\Delta_{\mathrm{r}}(\boldsymbol{f},\varepsilon)$

A toric variety in the background...

$\#V_0 = D$ is equivalent to the fact that the system $f_1 = 0, \ldots, f_n = 0$ does not have solutions in the toric variety associated to the polytope $Q_1 + Q_2 + \ldots + Q_n$ [Bernstein 1975], [Huber-Sturmfels 1995]



Some Evidence



(a) $u_0 = 0$ (b) $u_0 = 15$ (c) $u_0 = 30$

Singularities of families of algebraic plane curves with "controlled" coefficients tend to the equidistribution
[Diaconis-Galligó]

More Evidence: equidistribution of algebraic points

A sequence of algebraic points $\{p_k\}_{k\in\mathbb{N}} \subset (\mathbb{C}^*)^n$ such that $\deg(p_k) = k$ and $\lim_{k\to\infty} h(p_k) = 0$ "equidistributes" in
$S^1 \times S^1 \times \ldots \times S^1$
[Bilú 1997]

What do we mean by "Equidistribution"?

- $\mu_k := \frac{1}{k} \sum_{f_k(z)=0} \delta_z$
- $\lim_{k\to\infty} \int_{\mathbb{C}} g\, d\mu_k = \int_{S^1} g\, d\mu$
  $\forall g \in C_0(\mathbb{C})$

$d\mu$ is the Haar measure on S^1

Our approach (jointly with Galligó and Sombra)

For $v \in \mathbb{R}^n \setminus \{0\}$ let $\pi_v : \mathbb{R}^n \to v^\perp$ the orthogonal projection and

$$\gamma(\boldsymbol{f}) := \frac{1}{D} \sup_{v \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \sum_{j=1}^{n} \mathrm{MV}_{n-1}\left(\pi_v(Q_k) : k \neq j\right) \log ||f_j||$$

For f_i dense of degree d_i we have $\gamma(\boldsymbol{f}) = \sqrt{n} \sum_j \frac{\log ||f_j||}{d_j}$

Theorem (D-Galligo-Sombra)

If $f_1, \ldots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and the system has the right number of zeroes, then

$$\Delta_\theta(f) \leq c(n)\gamma(\boldsymbol{f})^{\frac{1}{2(n+1)}} \quad , \quad 1 - \Delta_{\mathrm{r}}(f; \varepsilon) \leq \frac{2}{\varepsilon D}\gamma(\boldsymbol{f})$$

with $c(n) \leq 2^{3n} n^{\frac{n+1}{2}}$

A more general result

Theorem (D-Galligo-Sombra)

If $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and the system has the right number of zeroes, then there exist functions $c, d : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$\Delta_\theta(\boldsymbol{f}) \leq c(\delta)\gamma(\boldsymbol{f})^{\frac{1}{2(n+1)}} \quad , \quad 1 - \Delta_{\mathrm{r}}(\boldsymbol{f};\varepsilon) \leq \frac{2}{\varepsilon D} d(\delta)\,\gamma(\boldsymbol{f})$$

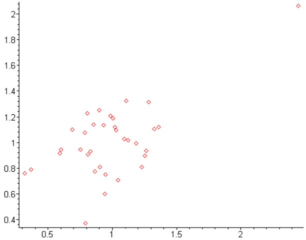
with $\delta := \mathrm{dist}(V_0, X_\infty)$

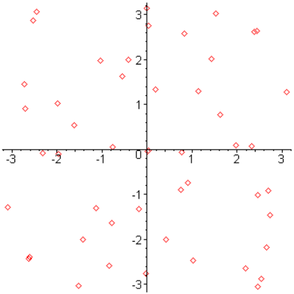
Example

$f = x^7 + x^6 y + x^5 y^2 - x^4 y^3 + x^3 y^4 + xy^6 - y^7 - x^6 + x^4 y^2 - x^3 * y^3 + x^2 * y^4 + x * y^5 + y^6 + \cdots$

$g = -x^7 - x^5 y^2 + x^4 y^3 + x^3 y^4 - x^2 y^5 - y^7 + x^5 y - xy^5 - y^6 + x^5 + x^4 y - x^2 y^3 - xy^4 + x^2 y^2 + \cdots$

The joint modulus and arguments of $f = g = 0$ plot as





A variant of the arithmetic Bezout theorem

For $a \in \mathbb{Z}^n \setminus \{0\}$ consider the monomial projection $\chi_a : (\mathbb{C}^\times)^n \to \mathbb{C}^\times,\ \xi \mapsto \xi^a = \xi_1^{a_1} \cdots \xi_n^{a_n}$ The associated *eliminant polynomial* is

$$E(\boldsymbol{f}, a)(z) := k \prod_{\xi \in V} (z - \chi_a(\xi))^{\mathrm{mult}(\xi)} \in \mathbb{C}[z]$$

It is a divisor of the *sparse resultant* associated to the system, where we take the variables in a monomial space ortoghonal to a

Theorem (D-Galligó-Sombra)

$$\log ||E(\boldsymbol{f}, a)|| \leq ||a|| \sum_{j=0}^{n} \mathrm{MV}_{n-1}\left(\pi_a(Q_k) : k \neq j\right) \log ||f_j||$$

The equidistribution via $E(\boldsymbol{f}, a)$

- For Δ_{r} we apply Erdos-Turán to $E(\boldsymbol{f}, e_i)$, with $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{Z}^n$
- For Δ_θ, we apply E-T to $E(\boldsymbol{f}, a)$ for all $a \in \mathbb{Z}^n$ to estimate the exponential sums on its roots, then recover V by tomography *via* Fourier analysis

What if we change the metric?

- $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$
- $N(f_i) = Q$
- $d :=$ the "degree" of Q
- $\langle f, g \rangle = \int_{S^{2n-1}} f \overline{g} d\mu$ with $d\mu$ the Haar measure
- $\mathcal{A}_Q := m^{-1}\left(\frac{Q}{d}\right) \subset (\mathbb{C}^*)^n$ with m the moment map

Theorem [Shiffman-Zelditch 2004]

The equidistribution happens in $\mathcal{A}_Q$

Moltes Gràcies!