Algebraic analysis of system reliability: A combinatorial approach

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- System: A set of connected components forming a complex whole. Appear in nature, industry, also as processes.
- Reliability: Probability that a system will perform its intended function during a specified period of time under stated conditions.

Coherent Systems

- System: a set $\mathcal{S}$ of n components, with increasing efficiency levels $\{0, 1, \ldots\}$.
- Outcome is a nonnegative integer vector of length n describing the state of $\mathcal{S}$.
- Failure outcome is an outcome that leads to failure of $\mathcal{S}$.
- A system is coherent if we cannot move from nonfailure into a failure state by improving any of the components and vice-versa.

Coherent Systems: Example 1

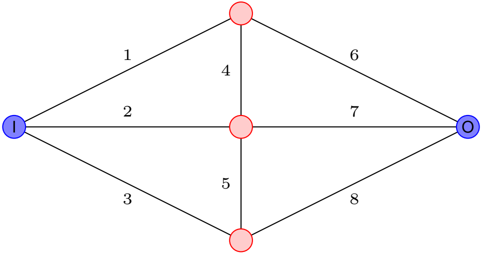

Figure: An example of coherent system: a network

- Each connection can either fail (0) or work (1), $\mathfrak{D} = \{0,1\}^8$
- A (minimal) failure state: $(0,0,0,1,1,1,1,1)$
- A (minimal) nonfailure state: $(1,0,0,0,0,1,0,0)$

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Denoting $\bar{\mathfrak{F}}$ to the nonfailure set and $\bar{\mathfrak{F}}^*$ the set of minties.

$$\mathcal{R} = Prob(\bar{\mathfrak{F}}) = Prob(\bigcup_{\alpha \in \bar{\mathfrak{F}}^*} Q_\alpha)$$

"probability of the union of all events that include at least one mintie".

Reliability Computation.

Variety of methods for Reliability evaluation and bounds.

- Parallel and series reductions.
- Pivotal decompositions.
- Inclusion-exclusion methods.
- Sum of disjoint products.
- Markov chain imbeddable structures
- Delta-Star and Star-Delta transformations
- ...

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Inclusion-exclusion.

The classical approach makes use of the inclusion-exclusion identity:

$$Prob(\bar{\mathfrak{F}}) = \sum_{\alpha\in\bar{\mathfrak{F}}^*} Prob(Q_\alpha) - \sum_{\alpha,\alpha'\in\bar{\mathfrak{F}}^*} Prob(Q_\alpha \cap Q'_\alpha) + \cdots + (-1)^{|\bar{\mathfrak{F}}^*|+1} Prob(\bigcap_{\alpha\in\bar{\mathfrak{F}}^*} Q_\alpha)$$

Truncations give upper and lower bounds and are known as Bonferroni inequalities:

$$Prob(\bar{\mathfrak{F}}) \leq \sum_{\substack{I\in\mathcal{P}(\bar{\mathfrak{F}}^*)\\ |I|\leq r}} (-1)^{|I|+1} Prob(\bigcap_{\alpha\in I} Q_\alpha) \quad \text{(r odd)}$$

$$Prob(\bar{\mathfrak{F}}) \geq \sum_{\substack{I\in\mathcal{P}(\bar{\mathfrak{F}}^*)\\ |I|\leq r}} (-1)^{|I|+1} Prob(\bigcap_{\alpha\in I} Q_\alpha) \quad \text{(r even)}$$

Abstract tubes.

Inclusion-Exclusion identities and bounds are very redundant in many situations.
A geometric-algebraic method to obtain improved Bonferroni inequalities is that of abstract tubes [Naiman, Wynn].

- A simplicial complex and a collection of subcomplexes with certain contractibility properties is associated to the system.
- To each subcomplex we associate a chain complex.
- The ranks of the modules in this chain complex provide improved Bonferroni inequalities for the reliability of the system.

Algebraic approach

Proposition (Giglio and Wynn 04)

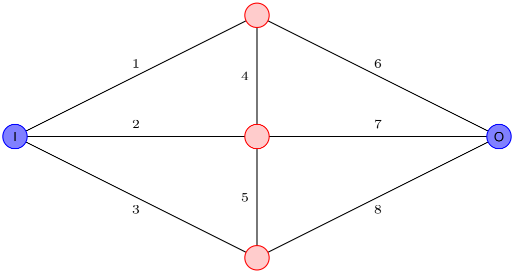
Given a system $\mathcal{S}$ of n components, its states can be seen as the exponent vectors of monomials in $R = \mathbf{k}[x_1, \ldots, x_n]$.

- *The points in $\bar{\mathfrak{F}}^*$, are the (exponents of the) minimal generators of a monomial ideal $I_{\mathcal{S}}$*
- *The points in $\bar{\mathfrak{F}}$ represent the (exponents of the) monomials belonging to the monomial ideal $I_{\mathcal{S}}$.*
- *The points in $\mathfrak{F}$ represent the (exponents of the) monomials belonging to the complement R/I.*

- The ideal property corresponds to coherency. Orthants correspond to divisibility.
- Computing the reliability of $\mathcal{S}$ amounts to count the monomials in $I_{\mathcal{S}}$.
- We have to compute the denominator of $H_{I_{\mathcal{S}}}(\mathbf{x})$.
- To provide bounds we need to express the multigraded Hilbert series of I in terms of some resolution of I.

$$H_{I_{\mathcal{S}}}(\mathbf{x}) = \frac{\sum (-1)^i \beta_{i,\mu} x^{\mu}}{\prod_i x_i}$$

- This method generalizes the clasical inclusion-exclusion approach (corresponding to Taylor resolution) and the abstract tubes approach (corresponding to Scarf resolution).



$$I_{\mathcal{S}} = \langle x_1x_6, x_1x_4x_7, x_2x_4x_6, x_1x_4x_5x_8, x_2x_7, x_3x_4x_5x_6, x_2x_5x_8, x_3x_5x_7, x_3x_8 \rangle$$

Method	Total	0	1	2	3	4	5	6	7	8
Taylor (inc-exc)	511	9	36	84	126	126	84	36	9	1
Scarf (abstr. tube)	103	9	27	37	24	6	0	0	0	0
Hilbert Series (min. res.)	87	9	25	31	18	4	0	0	0	0

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- Mayer-Vietoris trees
  - Fast, doesn't need to compute resolution.
  - Provides the form usable for bounds.
  - Can be used to analyse the structure of the ideal and its resolution.

Application to reliability:

- Non-structured systems:
  - General method.
  - Tight bounds.
  - Efficient computation.
- Structured systems:
  - K-out-of-N:
    - Explicit formulas for bounds and exact reliability.
  - Consecutive K-out-of-N:
    - Explicit formulas under i.i.d.
    - Recursion for bounds and reliability in general.
  - Series-parallel:
    - Recursive formulas for reliability and bounds.

K-out-of-N systems

A k-out-of-n system is one that fails if at least k out of a total of n components fail.
A k-out-of-n system can be modeled by the ideal

$$I_{k,n} = \langle x^\mu : x^\mu \text{ is a squarefree monomial of degree } k \text{ in } n \text{ variables}\rangle$$

It a squarefree stable ideal. The minimal free resolution is known [AHH]. Their Mayer-Vietoris tree is minimal.
$I_{k,n}$ has a minimal generating set which consists of $\binom{n}{k}$ monomials.

From the minimal free resolution, the Mayer-Vietoris tree or simplicial considerations we obtain.

$$\beta_i(I_{k,n}) = \binom{n}{k+i}\binom{i+k-1}{k-1} \qquad \forall 0 \leq i \leq n-k.$$

Multigraded Hilbert series of $I_{k,n}$:

$$\mathcal{H}(I_{k,n};x) = \frac{\sum_{i=0}^{n-k}(-1)^i\binom{i+k-1}{k-1}\left(\sum_{\alpha\in[n,k+i]} x^\alpha\right)}{\prod_i(1-x_i)},$$

where $[n, k+i]$ denotes the set of vectors with 1 in the indices of the $(k+i)$-subsets of $\{1,\dots,n\}$ and 0 in the other entries.

For example,

$$I_{3,5} = \langle xyz, xyu, xyv, xzu, xzv, xuv, yzu, yzv, yuv, zuv \rangle$$

$$\mathcal{H}(I_{3,5}; \mathbf{x}) = \frac{xyz + xyu + xyv + xzu + xzv + xuv + yzu + yzv + yuv + zuv}{(1-x)(1-y)(1-z)(1-u)(1-v)}$$

$$-\frac{3(xyzu + xyzv + xyuv + xzuv + yzuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)} + \frac{6(xyzuv)}{(1-x)(1-y)(1-z)(1-u)(1-v)},$$

the Betti numbers of $I_{3,5}$ are then: $\beta_0 = 10$, $\beta_1 = 15$ and $\beta_2 = 6$.

Consecutive K-out-of-N

A *k-out-of-n* system works (fails) if at least k consecutive components work (fail).
It is an important system in different applications [KZ05]

- Microwave stations of telecom network
- Oil pipeline system
- Vacuum system in electron accelerator
- Photography of nuclear accelerator
- Scan statistics in gene expression
- Pattern detection in DNA sequences

Evaluation of the reliability of consecutive k-out-of-n systems:

- Assuming i.i.d. components:
  - Combinatorial recursive and closed form formulas for exact reliability (Chiang and Niu, Bolloinger and Salvia, Derman et al.)
- Systems with independent components
  - Recursive formula (Chiang and Niu, Shantikumar, Papastavridis et al.)
  - Imbedded Markov chain approach: recursive tables (Hwang and Wright)
- Bounds
  - Only under i.i.d or independence assumptions (Chiang and Nu, Zuo, Papastavridis et al.)

Algebraic approach:
The ideal corresponding to a consecutive k-out-of-n system is of the form:

$$I = \langle x_1 x_2 \cdots x_k, x_2 x_3 \cdots x_{k+1}, \ldots, x_{n-k+1} x_{n-k+2} \cdots x_n \rangle$$

It has $n-k+1$ generators in n variables. The minimal resolution of these ideals is provided by Mayer-Vietoris trees, and therefore the bounds obtained in this way are tightest among those produced with our approach [SW08].

The structure of Mayer-Vietoris trees of $C(k,n)$ ideals provide recursive formulas for their (graded,multigraded) Betti numbers.
Example: Total Betti numbers
For $n \leq 2k$ we have

$$\begin{aligned} \beta_{0,k,n} &= n-k+1 \\ \beta_{1,k,n} &= n-k \\ \beta_{i,k,n} &= 0, \text{ for } i \geq 2 \end{aligned}$$

For $n \geq 2k+1$ we have

$$\begin{aligned} \beta_{0,k,n} &= n-k+1 \\ \beta_{1,k,n} &= n-2k+1+\beta_{1,k,n-1} \\ \beta_{i,k,n} &= \beta_{i-2,k,n-k-1}+\beta_{i-1,k,n-k-1}+\beta_{i,k,n-1}, \text{ for } i \geq 2 \end{aligned}$$

Using standard methods we obtain a generating function for the Betti numbers of the $C(k,n)$ ideal:

$$G_k(x,y) = \sum_{i=0}^{\infty} \sum_{n=k}^{\infty} \beta_{i,k,n} x^i y^n = \frac{y^k(1+xy)}{(1-y)(1-x^2y^{k+1}-xy^{k+1}-y)}.$$

- Bounds for reliability.
- Asymptotic behaviour of the system can be analyzed (e.g. application in scan statistics for gene expression).

We also obtain the following recurrence relationship for the graded Betti numbers, $\beta_{i,j,k,n}$, where j is the degree of the term corresponding monomial.

$$\begin{aligned}
\beta_{0,k,n,k} &= n-k+1 \\
\beta_{1,k,n,k+1} &= n-k \ \text{ for } k \geq \tfrac{n}{2} \\
\beta_{1,k,n,k+1} &= 1+\beta_{1,k,n-1,k+1} \ \text{ for } k < \tfrac{n}{2} \\
\beta_{1,k,n,2k} &= n-2k+\beta_{1,k,n-1,2k} \ \text{ for } k < \tfrac{n}{2} \\
\beta_{i,k,n,j} &= \beta_{i-2,k,n-k-1,j-k-1}+\beta_{i-1,k,n-k-1,j-k}+\beta_{i,k,n-1,j}, \ \text{ for } i \geq 2
\end{aligned}$$

Example: 4 out of n, i.i.d. components with p=0.9

n	Exact	Lou, Fu (lower)	Lou, Fu (upper)	MVT L1	MVT U1	MVT L2	MVT U2
50	0.9958	0.9950	0.9958	0.9953	0.9958	0.9958	0.9958
100	0.9913	0.9908	0.9913	0.9913	0.9913	0.9913	0.9913
1000	0.9141	0.9048	0.9142	0.9003	0.9151	0.9140	0.9141

Example: 4 out of 11, independent components with
$p_i = 0.7 + 0.02(i-1) \quad 1 \leq i \leq 11$

Computation of $\mathcal{U}$ using Embedded Markov chain

I	j=0	j=1	j=2	j=3	j=4
0	1.000000	0.000000	0.000000	0.000000	0.000000
1	0.700000	0.300000	0.000000	0.000000	0.000000
2	0.720000	0.196000	0.084000	0.000000	0.000000
3	0.740000	0.187200	0.050960	0.021840	0.000000
4	0.760000	0.177600	0.044928	0.012230	0.005242
5	0.775912	0.167200	0.039072	0.009884	0.007932
6	0.793654	0.155182	0.033440	0.007814	0.009909
7	0.811875	0.142858	0.027933	0.006019	0.011316
8	0.830495	0.129900	0.022857	0.004469	0.012279
9	0.849440	0.116269	0.018186	0.003200	0.012904
10	0.868644	0.101933	0.013952	0.002182	0.013288
11	0.888040	0.086868	0.010193	0.001395	0.013507

Computation of MVT(C(4,11)) takes 0.00 seconds, the Hilbert series numerator contains 33 elements of depth up to 4. Substituting the variables x_i for the corresponding p_i we obtain the bounds. The fourth of them is the actual reliability:

r	bound on $\mathcal{U}$	s_r
1	0.016558	8
2	0.013503	21
3	0.013507	30
4	0.013507	33

Weighted k-out-of-n

System with n components, each with its own positive integer weight such that the system is failed if and only if the total weight of failed components is at least k.
Let $\{w_1, \ldots, w_n\}$ be the weights of the components of $W_{k,n}$
Consider the set of products of the variables with their weights:

$$J_{W_{k,n}} = \{\prod_{i \in \sigma} x_i^{w_i} | \sigma \subseteq \{1, \ldots, n\}\}$$

then the ideal of the system is given by

$$I_{W_{k,n}} = \langle x^\mu \in J_{W_{k,n}} | deg(x^\mu) \geq k \rangle$$

To compute the reliability of the system, we consider the multigraded Hilbert series of $I_{W_{k,n}}$ putting p_i, the probability of component i being in a working state in place of $x_i^{w_i}$.

Example:
Weighted 5-out-of-3 system with weights $2, 6, 4$

$$I_{W_{5,3}} = \langle x^2 z^4, y^6 \rangle$$

.
The numerator of the Hilbert series of $I_{W_{k,n}}$ is $x^2 z^4 + y^6 - x^2 y^6 z^4$
Reliabilty of the system is $R(W_{5,3}) = p_1 p_3 + p_2 - p_1 p_2 p_3$.

Two-stage weighted k-out-of-n systems

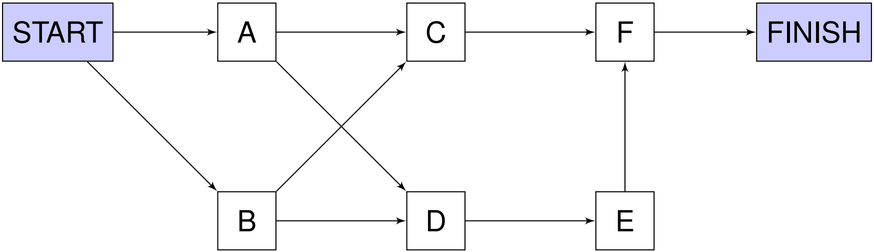
Such a system consists of a number of subsystems each of which has a weighted-k-out-of-n structure (which is called the second-level structure).
The relation among the subsystems is given by a certain structure, which is called the first-level structure.
Series weighted k-out-of-n systems are used to model project management, while parallel weighted k-out-of-n systems can be used to model shortest path problems [Chen & Yang 05].

In the algebraic translation:

- The second level structure is given by the ideal corresponding to each weighted k-out-of-n system.
- The first level structure corresponds to operations among the ideals involved.
    - Series structures correspond to union of ideals.
    - Parallel structures correspond to intersection of ideals.

Consider an assembly project with the following activities:

Activity	**Estimated Duration**	**Possible delays**
A=train workers	6 days	1 day
B=purchase raw materials	9 days	3 days
C=produce product 1	8 days	3 days
D=produce product 2	7 days	2 days
E=test product 2	10 days	2 days
F=assemble products 1 and 2	12 days	4 days



- Each activity is subject to delays with a probability of $1-p$.
- $x_i = 0$ means activity i is delayed, $x_i = 1$ means activity i is not delayed.
- The project fails to be completed within deadline if at least one of the paths finishes in more than 40 days.
- Paths ACF and BCF cannot fail, we consider just paths $ADEF$ and $BDEF$.
- $I_{ADEF} = \langle d^2 f^4, e^2 f^4 \rangle$ and $I_{BDEF} = \langle b^3, f^4, d^2 e^2 \rangle$ they are respectively a 6-out-of-4 system with weights $1, 2, 2, 4$ and a 3-out-of-4 system with weights $3, 2, 2, 4$.
- $I = I_{ADEF} + I_{BDEF} = \langle b^3, f^4, d^2 e^2 \rangle$ (series composition since we need both to work).
- Using Hilbert series we obtain $R(I) = 2p^3 - p^4$

Mincut ideals of two-terminal networks

Definition

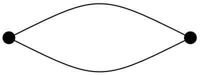
Let N be a two-terminal network and I_N its mincut ideal. Let V_N the set of nodes of the network and E_N the set of its connections. We say that a *path* in N is a sequence of nodes $n_1, \cdots, n_k$ such that (n_i, n_{i+1}) is an edge, all n_i are distinct and $n_1 = s$, $n_k = t$. A *shortest path* in N is a path in N whose length is minimal among all pathes in N. We denote $lsp(N)$ the length of a shortest path in N.

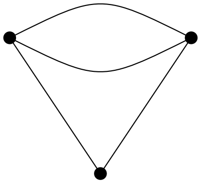
Proposition

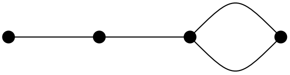
Let N be a two-terminal network and I_N its mincut ideal. Then $dim(I) := dim(R/I) = n - lsp(N)$.

Definition

An edge p joining two nodes s and t is a basic series-parallel network. N is a parallel-series network if it is a basic series-parallel network, or if $N = N_1 + N_2$ or $N = N_1 \times N_2$, where N_1, N_2 are series-parallel networks, $+$ denotes series composition and $\times$ denotes parallel composition .







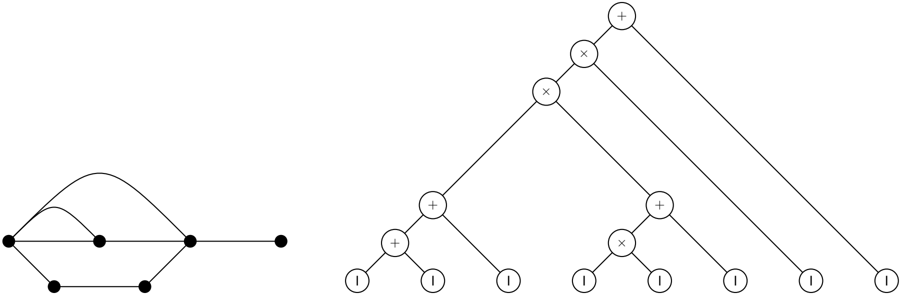


Figure: A series-parallel network and its corresponding SP-tree

Proposition (SW09)

Let N_1 and N_2 be two networks the edges of which are labelled $x_1, \ldots, x_{n_1}$ and $x_{n_1+1}, \ldots, x_{n_1+n_2}$. Let $I_{N_1} \subset \mathbf{k}[x_1, \ldots, x_{n_1}]$ and $I_{N_2} \subset \mathbf{k}[x_{n_1+1}, \ldots, x_{n_1+n_2}]$ be their corresponding mincut ideals. Then the mincut ideals of their series and parallel compositions are given by

$$I_{N_1+N_2} = I_{N_1} + I_{N_2} \qquad I_{N_1 \times N_2} = I_{N_1} \cap I_{N_2}$$

where $I_{N_1+N_2}$ and $I_{N_1 \times N_2}$ are ideals in $\mathbf{k}[x_1, \ldots, x_{n_1+n_2}]$

Theorem

Let N be a series-parallel network and I_N its mincut ideal. The minimal free resolution of I_N is obtained as an iterated mapping cone (Mayer Vietoris tree).

Proposition

Let A and B be two series-parallel networks. We denote by $S(A, B)$ the network obtained by the series combination of A and B, and by $P(A, B)$ the parallel combination of A and B. For any network N, we denote by $lsp(N)$ the length of the shortest path of the network N, by $numgens(N)$ the number of minimal generators of I_N, by $pdim(N)$ the projective dimension and by $reg(N)$ the Castelnuovo-Mumford regularity of I_N. Then, we have:

1. $lsp(S(A, B)) = lsp(A) + lsp(B)$;
   $lsp(P(A, B)) = min\{lsp(A), lsp(B)\}$
2. $numgens(S(A, B)) = numgens(A) + numgens(B)$;
   $numgens(P(A, B)) = numgens(A) * numgens(B)$
3. $pdim(P(A, B)) = pdim(A) + pdim(B)$;
   $pdim(S(A, B)) = pdim(A) + pdim(B) + 1$
4. $reg(P(A, B)) = reg(A) + reg(B)$;
   $reg(S(A, B)) = reg(A) + reg(B) - 1$

Proposition

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Proposition

A series-parallel network is Cohen-Macaulay if and only if it is a series combination of pure parallel networks.

Conclusions

The Algebraic approach provides a general treatment of the reliability of coherent systems

- Exact reliability, bounds, recursive relations, asymptotic behaviour
- Binary and multivalued systems
- Allows avoiding assumptions on the components' probability distributions
- Structured and nonstructured problems

Conclusions

Computer algebra is a neccessary condition for the use of this approach in reasonable systems

- Minimal resolution, unfeasible
- Mayer-Vietoris trees provides fast algorithms

Conclusions

In the Application to reliability analysis we have:

- Best performance in non-structured problems, or under few assumptions
- Competitive in structured systems (k-out-of-n, series parallel,...)

Future work

- Other structured systems (in progress)
- Interaction with existing methods (in particular those that subdivide the systems)
- Produce software for algebraic reliabilty evaluation (in progress)
- The role of distributions
- Multistate case
- Applications (structural reliability, scan statistics, etc...)
- ...