Binomial arithmetical rank of toric ideals associated to graphs

Anargyros Katsabekis

12 November 2010

Binomial ideals

- Consider the polynomial ring $K[x_1, \ldots, x_m]$ over an algebraically closed field K.
- $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ for $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$.
- Binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is a difference of two monomials.
- Binomial ideal is an ideal generated by binomials.

Definition of a toric ideal of a graph

- G finite, connected and undirected graph having no loops and no multiple edges with vertices $\{v_1, \ldots, v_n\}$ and edges $\{e_1, \ldots, e_m\}$.
- To $e = \{v_i, v_j\}$ we associate a monomial $\mathbf{t}^e = t_i t_j \in K[t_1, \ldots, t_n]$.
- The toric ideal I_G is the kernel of the K-algebra homomorphism $\phi : K[x_1, \ldots, x_m] \longrightarrow K[t_1, \ldots, t_n]$ given by

$$\phi(x_i) = \mathbf{t}^{e_i}, \text{ for every } i = 1, \ldots, m.$$

Toric ideals are binomial ideals

- To $e = \{v_i, v_j\}$ we associate a vector $\mathbf{a}_e = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$.
- The toric ideal I_G is generated by all the binomials

$$x_1^{u_1} \cdots x_m^{u_m} - x_1^{v_1} \cdots x_m^{v_m}$$

such that

$$u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m = v_1\mathbf{a}_1 + \cdots + v_m\mathbf{a}_m.$$

Example

- G graph with 8 vertices and edges $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, $e_3 = \{3,4\}$, $e_4 = \{4,5\}$, $e_5 = \{5,6\}$, $e_6 = \{4,6\}$, $e_7 = \{4,7\}$, $e_8 = \{7,8\}$, $e_9 = \{3,8\}$, $e_{10} = \{1,3\}$.
- I_G is minimally generated by

$$x_3x_8 - x_7x_9, x_1x_3^2x_5 - x_2x_4x_6x_{10}.$$

- $\Gamma = (e_{i_1}, \ldots, e_{i_{2q}})$ even closed walk of G.

$$f_\Gamma = \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}} \in I_G.$$

- I_G is generated by all the binomials f_Γ.

Arithmetical rank

- The arithmetical rank $\mathrm{ara}(I_G)$ of I_G is the smallest integer s for which there exists polynomials $F_1, \ldots, F_s$ in I_G such that $\sqrt{I_G} = \sqrt{(F_1, \ldots, F_s)}$.
- G graph with 5 vertices and edges $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, $e_3 = \{3,4\}$, $e_4 = \{4,5\}$, $e_5 = \{1,5\}$, $e_6 = \{1,3\}$, $e_7 = \{2,5\}$, $e_8 = \{3,5\}$.
- I_G is minimally generated by

$$x_1x_8 - x_6x_7, x_2x_5 - x_6x_7, x_3x_5 - x_4x_6, x_2x_4 - x_3x_7.$$

Binomial arithmetical rank

1. The binomial arithmetical rank $\mathrm{bar}(I_G)$ of I_G is the smallest integer s for which there exists binomials $B_1, \ldots, B_s$ in I_G such that

$$\sqrt{I_G} = \sqrt{(B_1, \ldots, B_s)}.$$

What is the exact value of $\mathrm{bar}(I_G)$?

- If I_G is complete intersection, then $\mathrm{bar}(I_G) = \mathrm{ht}(I_G)$.
- For the ideal $J = (x_1x_6 - x_3x_4, x_1x_5 - x_2x_4, x_3x_5 - x_2x_6)$ it is known that $\mathrm{ara}(J) = 3$, so $\mathrm{bar}(J) = 3$.
- $J = I_G$ for G the bipartite graph with 5 vertices and edges $e_1 = \{1, 4\}, e_2 = \{2, 4\}, e_3 = \{3, 4\}, e_4 = \{1, 5\}, e_5 = \{2, 5\}, e_6 = \{3, 5\}$.
- A bipartite graph G is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one vertex in X and one vertex in Y.

Aims of the talk

Theorem

If G is bipartite, then $\mathrm{bar}(I_G) = \mu(I_G)$.

Theorem

If I_G is generated by quadratic binomials, then $\mathrm{bar}(I_G) = \mu(I_G)$.

Circuits of a toric ideal

- The support of a monomial $\mathbf{x}^{\mathbf{u}}$ is $\text{supp}(\mathbf{x}^{\mathbf{u}}) = \{i | x_i \text{ divides } \mathbf{x}^{\mathbf{u}}\}$.
- The support of a binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is $\text{supp}(B) = \text{supp}(\mathbf{x}^{\mathbf{u}}) \cup \text{supp}(\mathbf{x}^{\mathbf{v}})$.
- An irreducible binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_G$ is called a circuit of I_G if there is no binomial $B' \in I_G$ such that $supp(B') \subsetneqq supp(B)$.

Example

- G graph with 8 vertices and edges $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, $e_3 = \{3,4\}$, $e_4 = \{4,5\}$, $e_5 = \{5,6\}$, $e_6 = \{4,6\}$, $e_7 = \{4,7\}$, $e_8 = \{7,8\}$, $e_9 = \{3,8\}$, $e_{10} = \{1,3\}$.
- The circuits are

$$x_3x_8 - x_7x_9, x_1x_3^2x_5 - x_2x_4x_6x_{10},$$

$$x_1x_9^2x_7^2x_5 - x_2x_8^2x_4x_6x_{10}.$$

The graph Δ_G

- $\mathcal{C} = \{E \mid \text{supp}(\mathbf{x^u}) = E \text{ or } \text{supp}(\mathbf{x^v}) = E \text{ where } \mathbf{x^u} - \mathbf{x^v} \text{ is circuit}\}$.
- $\mathcal{C}_{\min}$ is the set of minimal elements of $\mathcal{C}$.
- The graph Δ_G has vertices the elements of $\mathcal{C}_{\min}$.
- $\{E, E'\}$ is an edge of Δ_G if and only if there exists a circuit $\mathbf{x^u} - \mathbf{x^v} \in I_G$ such that $\text{supp}(\mathbf{x^u}) = E$ and $\text{supp}(\mathbf{x^v}) = E'$.

Example

- G graph with 5 vertices and edges $e_1 = \{1,2\}$, $e_2 = \{1,3\}$, $e_3 = \{1,4\}$, $e_4 = \{2,3\}$, $e_5 = \{2,5\}$, $e_6 = \{3,4\}$, $e_7 = \{3,5\}$, $e_8 = \{4,5\}$.
- The circuits are

$$x_1x_7 - x_2x_5, x_1x_6 - x_3x_4, x_1x_8 - x_3x_5, x_4x_8 - x_5x_6,$$

$$x_3x_7 - x_2x_8, x_1x_6x_7 - x_2x_4x_8, x_2x_5x_6 - x_3x_4x_7.$$

Continue the example

- $\mathcal{C}_{\min} = \{E_1 = \{1,7\}, E_2 = \{2,5\}, E_3 = \{1,6\}, E_4 = \{3,4\}, E_5 = \{1,8\}, E_6 = \{3,5\}, E_7 = \{4,8\}, E_8 = \{5,6\}, E_9 = \{3,7\}, E_{10} = \{2,8\}\}$.
- The graph Δ_G has 5 connected components, namely

$$\{E_1, E_2\},\ \{E_3, E_4\},\ \{E_5, E_6\},\ \{E_7, E_8\},\ \{E_9, E_{10}\}.$$

Definitions

- A subset $\mathcal{M}$ of the edges of a graph G is called a matching in G if there are no two edges in $\mathcal{M}$ which are incident with a common vertex.
- $\mathcal{M}$ is a maximum matching if it has the maximum possible cardinality among all matchings.
- The cardinality $\tau(G)$ of a maximum matching in G is commonly known as its matching number.

Lower bound for bar

Definition

For the graph Δ_G we define $\delta(\Delta_G) = \#\mathrm{Vertices}(\Delta_G) - \tau(\Delta_G)$.

Theorem

For a toric ideal I_G we have that

$$\mathrm{bar}(I_G) \geq \delta(\Delta_G).$$

Indispensable monomials and binomials

- A binomial $B \in I_G$ is called indispensable if every system of binomial generators of I_G contains B or $-B$, while a monomial M is called indispensable if every system of binomial generators of I_G contains a binomial B such that the M is a monomial of B.

Bipartite graphs

Theorem

Let G be a bipartite graph, then $\mathrm{bar}(I_G) = \mu(I_G)$.

Sketch of the proof.

- $\mathcal{C}_{\min} = \{\mathsf{supp}(M_1), \ldots, \mathsf{supp}(M_r)\}$, where $M_1, \ldots, M_r$ are the indispensable monomials.
- $\{E, E'\}$ is an edge of Δ_G if and only if there is an indispensable binomial $M_i - M_j \in I_G$ with $E = \mathrm{supp}(M_i)$ and $E' = \mathrm{supp}(M_j)$.
- Every connected component of Δ_G is an edge.

Complete graph on 4 vertices

- $\mathcal{K}_4$ complete graph on 4 vertices with edges $e_1 = \{1, 2\}$, $e_2 = \{2, 3\}$, $e_3 = \{3, 4\}$, $e_4 = \{1, 4\}$, $e_5 = \{1, 3\}$, $e_6 = \{2, 4\}$.
- $I_{\mathcal{K}_4}$ is minimally generated by

$$B_1 = x_1x_3 - x_2x_4, B_2 = x_1x_3 - x_5x_6.$$

- The circuits are:

$$B_1, B_2, B_3 = x_2x_4 - x_5x_6.$$

Toric ideals generated by quadratic binomials

Theorem

If I_G is generated by quadratic binomials, then $\mathrm{bar}(I_G) = \mu(I_G)$.

Sketch of the proof.

- $\mathcal{C}_{\min} = \{\mathrm{supp}(M_1), \ldots, \mathrm{supp}(M_r)\}$, where $M_1, \ldots, M_r$ are the indispensable monomials.
- Δ_G has g connected components which are edges, where g is the number of indispensable binomials in I_G.
- Δ_G has $\frac{t-g}{2}$ connected components which are triangles, where $t = \mu(I_G)$.

Compute $\mathrm{bar}(I_{\mathcal{K}_n})$

- $\mathcal{K}_n$, $n \geq 4$, is the complete graph on the vertex set $\{v_1, \ldots, v_n\}$.
- Every connected component of $\Delta_{\mathcal{K}_n}$ is a triangle.
- $\Delta_{\mathcal{K}_n}$ has $3\binom{n}{4}$ vertices and $\binom{n}{4}$ triangles, corresponding to all complete subgraphs of $\mathcal{K}_n$ of order 4.
- $\mathrm{bar}(I_{\mathcal{K}_n}) = 2\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{12}$.
- $\binom{n}{2} - n = \frac{n(n-3)}{2} \leq \mathrm{ara}(I_{\mathcal{K}_n}) \leq \binom{n}{2} = \frac{n(n-1)}{2}$.

G-homogeneous arithmetical rank

- $\deg_G(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$.
- $\deg_G(\mathbf{x}^{\mathbf{u}}) = u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m$.
- The G-homogeneous arithmetical rank $\mathrm{ara}_G(I_G)$ of I_G is the smallest integer s for which there exists G-homogeneous polynomials $F_1, \ldots, F_s$ in I_G such that

$$\sqrt{I_G} = \sqrt{(F_1, \ldots, F_s)}.$$

- The equality $\mathrm{ara}_G(I_G) = \mu(I_G)$ holds for the two cases studied in this talk.