0n ideals of Castelnuovo-Mumford regularity equal 2

Marcel Morales

Institut Fourier

University Grenoble I

and University Lyon I

France

Introduction

- We consider $S := K[V]$ a (standard graded) polynomial ring on a finite set of variables V, $\mathcal{Q} \subset S$ an (homogeneous) ideal.

Introduction

- We consider $S := K[V]$ a (standard graded) polynomial ring on a finite set of variables V, $\mathcal{Q} \subset S$ an (homogeneous) ideal.

- $\mathcal{Q}$ has a $p-$ linear resolution $\Leftrightarrow$ $\mathcal{Q}$ is generated by elements in degree p, and the minimal free resolution of $\mathcal{Q}$ is linear, i.e.

Introduction

• We consider $S := K[V]$ a (standard graded) polynomial ring on a finite set of variables V, $\mathcal{Q} \subset S$ an (homogeneous) ideal.

• $\mathcal{Q}$ has a $p-$ linear resolution $\Leftrightarrow$ $\mathcal{Q}$ is generated by elements in degree p, and the minimal free resolution of $\mathcal{Q}$ is linear, i.e.

• For $i \geq 2$ the matrices M_i in a minimal free resolution of $\mathcal{Q}$ have linear entries.

$$0 \to F_s \stackrel{M_s}{\to} F_{s-1} \to .... \to F_1 \stackrel{M_1}{\to} \mathcal{Q} \to 0$$

the number s is the projective dimension of $S/\mathcal{Q}$.

Introduction

• We consider $S := K[V]$ a (standard graded) polynomial ring on a finite set of variables V, $\mathcal{Q} \subset S$ an (homogeneous) ideal.

• $\mathcal{Q}$ has a $p-$ linear resolution $\Leftrightarrow$ $\mathcal{Q}$ is generated by elements in degree p, and the minimal free resolution of $\mathcal{Q}$ is linear, i.e.

• For $i \geq 2$ the matrices M_i in a minimal free resolution of $\mathcal{Q}$ have linear entries.

$$0 \to F_s \stackrel{M_s}{\to} F_{s-1} \to .... \to F_1 \stackrel{M_1}{\to} \mathcal{Q} \to 0$$

the number s is the projective dimension of $S/\mathcal{Q}$.
Study of ideals having a $2-$ linear resolution was initiated by Castelnuovo, Del Pezzo and Bertini.

Introduction

In fact Castelnuovo, Del Pezzo, Bertini have showed that for all projective algebraic varieties $X \subset \mathbb{P}^n$ (irreducible)

$$\deg(X) \geq \operatorname{codim}(X) + 1$$

they have classified the projective algebraic varieties that satisfies the equality. They are known as varieties of minimal degree:
1) Quadratic hypersurfaces 2) Rational normal scrolls, that is varities such that its defining ideal is generated by the 2×2 minors of the following scroll matrix :

$$\mathcal{B} = \left(\begin{array}{ccc|c|ccc} L_{1,0} & \dots & L_{1,c_1} & \dots & L_{r,0} & \dots & L_{r,c_r} \\ L_{1,1} & \dots & L_{1,c_1+1} & \dots & L_{r,1} & \dots & L_{r,c_r+1} \end{array}\right)$$

where $L_{1,0}, L_{1,2}, \dots, L_{1,c_1+1}, \dots, L_{r,1}, L_{r,2}, \dots, L_{r,c_r+1}$ are all linearly independent forms. 3)The veronese surface in $\mathbb{P}^5$, which defining ideal $\mathcal{I}$ is generated by the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Introduction

• Xambo classified all projective algebraic sets $X \subset \mathbb{P}^n$ (reduced non irreducible) that satisfies the equality

$$\deg(X) = \operatorname{codim}(X) + 1$$

with the additional hypothesis that X is connected in codimension 1. That is : if $Y \subset X$ of codimension ≥ 2, then $X \setminus Y$ is connected.

Introduction

• Xambo classified all projective algebraic sets $X \subset \mathbb{P}^n$ (reduced non irreducible) that satisfies the equality

$$\deg(X) = \operatorname{codim}(X) + 1$$

with the additional hypothesis that X is connected in codimension 1. That is : if $Y \subset X$ of codimension ≥ 2, then $X \setminus Y$ is connected.

• Xambo classified all projective algebraic sets $X \subset \mathbb{P}^n$ (reduced non irreducible) that satisfies the equality

$$\deg(X) = \operatorname{codim}(X) + 1$$

with the additional hypothesis that X is connected in codimension 1. That is : if $Y \subset X$ of codimension ≥ 2, then $X \setminus Y$ is connected.

Remark

• *By Barile-Morales the ideals $\mathcal{Q} \subset S$ defining algebraic sets of minimal degree are exactly ideals $\mathcal{Q}$ such that $S/\mathcal{Q}$ is Cohen-Macaulay and $\mathcal{Q}$ has a $2-$ linear resolution.*

Description of 2$-$ regular ideals follows from a series of works :

[B-M 1] Barile M., Morales M., *On the equations defining minimal varieties*, Comm. Alg., **28** (2000), 1223 – 1239.

[B-M 2] Barile M., Morales M., *On unions of scrolls along linear spaces*, Rend. Sem. Mat. Univ. Padova, **111** (2004), 161 – 178.

[E-G-H-P] Eisenbud D., Green M., Hulek K., Popescu S., *Restricting linear syzygies: algebra and geometry*, Compos. Math. **141** (2005), no. 6, 1460–1478.

[M] Morales, Marcel *Simplicial ideals, 2-linear ideals and arithmetical rank* Journal of Algebra 324 (2010), pp. 3431-3456.

Linearly joined ideals

$S = K[V]$. Let $\mathbf{V}$ the K-vector space generated by V. Remark that any ideal $\mathcal{J} \subset S$ can be written as $\mathcal{J} = (\mathcal{Q}, \mathcal{M})$, where $\mathcal{Q} \subset \mathbf{V}$ is a linear space, and $\mathcal{M}$ is an ideal generated in degree at least two. • A sequence of ideals $\mathcal{J}_1, ..., \mathcal{J}_l$, whith $\mathcal{J}_i = (\mathcal{Q}_i, \mathcal{M}_i)$ is *linearly joined* if for $k = 2, ..., l$:

$$(*) \qquad \mathcal{J}_k + \cap_{i=1}^{k-1} \mathcal{J}_i = (\mathcal{Q}_k) + (\cap_{i=1}^{k-1} \mathcal{Q}_i).$$

Main result

Theorem

TFAE:

1. the sequence of ideals $\mathcal{J}_1, ..., \mathcal{J}_l \subset S := K[\mathbf{V}]$ is linearly joined.
2. For all $i = 1, ..., l$, there exist sublinear spaces $\mathcal{Q}_i \subset \mathbf{V}$, and ideals $\mathcal{M}_i \subset K[\mathbf{V}]$ such that
   - a) for all $i = 1, ..., l$, $\mathcal{J}_i = (\mathcal{M}_i, \mathcal{Q}_i)$,
   - b) the sequence of ideals $\mathcal{Q}_1, ..., \mathcal{Q}_l \subset S := K[\mathbf{V}]$ is linearly joined,
   - c)$\mathcal{M}_i \subseteq (\mathcal{Q}_j)$ for all $i \neq j, i, j \in \{1, ..., l\}$.

Main result 2

Theorem

TFAE:

1. *the sequence of linear ideals* $\mathcal{Q}_1, ..., \mathcal{Q}_l \subset S := K[\mathbf{V}]$ *is linearly joined,*
2. *For all* $i = 1, ..., l$, *there exist sublinear spaces* $\mathcal{D}_i, \mathcal{P}_i \subset \mathbf{V}$, *with* $\mathcal{D}_l = 0, \mathcal{P}_1 = 0$, *such that*
   - *a)* $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i$
   - *b)* $\mathcal{D}_1 \supset \mathcal{D}_2 \supset ... \supset \mathcal{D}_l$ *(strictly decreasing).*
   - *c) Let* Δ_i *be a supplementary space of* $\mathcal{D}_i$ *in* $\mathcal{D}_{i-1}$. *Then* $\Delta_i \times \mathcal{P}_i \subset (\mathcal{P}_j)$ *for* $i < j$.

Main result 2

Theorem

TFAE:

1. *the sequence of linear ideals* $\mathcal{Q}_1, ..., \mathcal{Q}_l \subset S := K[\mathbf{V}]$ *is linearly joined,*
2. *For all* $i = 1, ..., l$*, there exist sublinear spaces* $\mathcal{D}_i, \mathcal{P}_i \subset \mathbf{V}$*, with* $\mathcal{D}_l = 0, \mathcal{P}_1 = 0$*, such that*
   - *a)* $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i$
   - *b)* $\mathcal{D}_1 \supset \mathcal{D}_2 \supset ... \supset \mathcal{D}_l$ *(strictly decreasing).*
   - *c) Let* Δ_i *be a supplementary space of* $\mathcal{D}_i$ *in* $\mathcal{D}_{i-1}$*. Then* $\Delta_i \times \mathcal{P}_i \subset (\mathcal{P}_j)$ *for* $i < j$*.*

- Generators of linearly joined ideals:

$$\mathcal{J} = (\bigcup_{i=2}^{l} \Delta_i \times \mathcal{P}_i, \mathcal{M}_1, ..., \mathcal{M}_l).$$

Primary decomposition of reduced 2−regular ideals

Theorem

TFAE

1. *The reduced ideal* $\mathcal{J}$ *is* 2−*regular.*
2. *There exists a linearly joined sequence of prime ideals* $\mathcal{J}_1, ..., \mathcal{J}_l \subset S := K[\mathbf{V}]$ *such that* $\operatorname{reg}(\mathcal{J}_i) \leq 2$ *for any* i, *and*

$$\mathcal{J} = \bigcap_{i=1}^{l} \mathcal{J}_i.$$

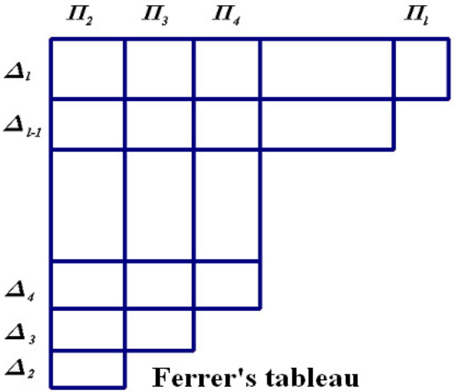
The difficult part is the implication " $\Rightarrow$ " and is due to [E-G-H-P].

Theorem

Let $\mathcal{Q}_i \subset \mathbf{V}$ *be linear ideals for* $i = 1, ..., l$. $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l$ *be a linearly joined ideal. Then*

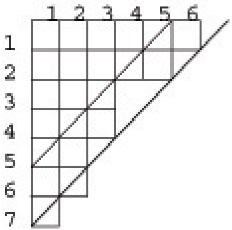
1. $\operatorname{projdim}(S/\mathcal{Q}) = \text{máx}_{2 \leq i \leq l}\{\operatorname{card}(\mathcal{P}_i) + \operatorname{card}(D_{i-1}) - 1\}$.
2. *The connectedness dimension* $c(S/\mathcal{Q}) = \operatorname{depth}(S/\mathcal{Q}) - 1$.
3. *The arithmetical rank* $\operatorname{ara}(\mathcal{Q}) = \operatorname{projdim}(S/\mathcal{Q})$.
4. *The cohomological dimension:* $\operatorname{cd}(\mathcal{Q}) = \operatorname{projdim}(S/\mathcal{Q})$.

Ferrer's diagrams, Ferrer's ideals



Ferrer's ideals are linearly joined and have 2-linear resolution.

1) Consider the partition:
$6 \geq 5 \geq 3 \geq 3 \geq 2 \geq 2 \geq 1$,



the corresponding Ferrer's ideal $\mathcal{I}$ is generated by

$$\mathcal{I} = (x_1y_1, x_1y_2, ..., x_1y_6, x_2y_1, ..., x_2y_5, x_3y_1, ..., x_3y_3,$$
$$x_4y_1, ..., x_4y_3, x_5y_1, x_5y_2, x_6y_1, x_6y_2, x_7y_1),$$

and its minimal primary decomposition is

$$\mathcal{I} = (x_1, ..., x_7) \cap (x_1, ..., x_6, y_1) \cap (x_1, ..., x_4, y_1, y_2)$$
$$\cap (x_1, x_2, y_1, y_2, y_3) \cap (x_1, y_1, ..., y_5) \cap (y_1, ..., y_6),$$

2) Matroidal ideals generated in degree 2. Let $V = \bigcup S_i$ be a partition of V. The ideal $\mathcal{Q}$ generated by $\bigcup S_i \times S_j$ has a 2−linear resolution.

Examples

2) Matroidal ideals generated in degree 2. Let $V = \bigcup S_i$ be a partition of V. The ideal $\mathcal{Q}$ generated by $\bigcup S_i \times S_j$ has a $2-$linear resolution.

3)Let $S = K[a, b, c, x, y, z, u, v, w]$ and $\mathcal{J}_1 = (a, b, c, x)$;

$\mathcal{J}_2 = (x(x-u)-c^2, a, b, y, z)$; $\mathcal{J}_3 = (b, x, z-u, c)$; $\mathcal{J}_4 = (x, y-u, a, c)$.

$\mathcal{J} := \bigcap_{i=1}^{4} \mathcal{J}_i$ has a 2-linear resolution, is generated by $x(x-u)-c^2$ and the elements :

$$
\begin{array}{ccccccc}
 & & & bc & & & \\
 & & bx & & ac & & \\
 & ba & & ax & & cy & \\
b(y-u) & & a(z-u) & & xy & & cz \\
 & & & & xz, & &
\end{array}
$$

$\mathrm{projdim}\,(S/\bigcap_{i=1}^{4} \mathcal{J}_i) = 5$, $\mathrm{cd}\mathcal{J} = \mathrm{ara}\,\mathcal{J} = 5$, and
$\mathcal{J} = \mathrm{rad}\,(x(x-u)-c^2, bx, ab+ax, b(y-u)+a(z-u)+xy, xz)$.

We give now one simple open case.

Example

Consider $S = K[a,b,c,d,e,f]$ the ring of polynomials, $\mathcal{B} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $F = ad - bc$, then the ideal $\mathcal{J} = \bigcap_{i=1}^{3} \mathcal{J}_i$, where

$$\mathcal{J}_1 = (F,e,f); \mathcal{J}_2 = (b,d,f); \mathcal{J}_3 = (a,c,e);$$

has a 2-linear resolution, $\mathrm{cd}(\mathcal{J}) = \mathrm{projdim}\,(S/\mathcal{J}) = 3$. In this example we have that
$\mathcal{J} = (F, fe, eb, fa, ed, fc) = \sqrt{(F, fe, eb + fa, ed + fc)}$ so
$3 \leq \mathrm{ara}\,(\mathcal{J}) \leq 4$, we guess that $\mathrm{ara}\,(\mathcal{J}) = 3$.

Theorem

Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_l$ *be a linearly joined sequence of linear ideals,* $\mathcal{Q} := \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_l$. *Thus* $\mathcal{Q} = (\bigcup_{2 \leq i \leq l} \Delta_i \times \mathcal{P}_i)$. *There exists an ordered subset* $x_1, \ldots, x_n$ *of* $\bigcup_{2 \leq i \leq l} \Delta_i \cup \mathcal{P}_i$ *such that we can range the generators of* $\mathcal{Q}_1$ *into a triangle having* $\operatorname{projdim} S/\mathcal{Q}$ *lines, as follows:*

$$x_1 x_n \tag{1}$$

$$x_2 x_n, \quad x_1 x_{1,1} \tag{2}$$

$$x_3 x_n, \quad x_2 x_{2,1}, \quad x_1 x_{1,2} \tag{3}$$

$$\ldots$$

$$x_{j-1} x_n, \qquad \ldots \qquad x_1 x_{1,j-2} \tag{$j-1$}$$

$$x_j x_n, \quad x_{j-1} x_{j-1,1}, \quad \ldots \qquad x_1 x_{1,j-1} \tag{j}$$

$$\ldots$$

Betti numbers

Let consider a 2-regular linear ideal

$$\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \ldots \cap \mathcal{Q}_l,$$

the codimension of $\mathcal{Q}$ is $c = \text{mín}\{\dim \mathcal{Q}_i\}$, the multiplicity is the number $e = \operatorname{card}\{i / \dim \mathcal{Q}_i = c\}$, it is well known that $e = \deg(S/\mathcal{Q})$.

Theorem

Let consider a 2-regular linear ideal $\mathcal{Q}$ with its tableau. Let $d = \dim S/\mathcal{Q}$, $s_{d-1} = \operatorname{card} \mathcal{D}_{c+1}, s_{d-2} = \operatorname{card} \mathcal{D}_{c+2}, s_{d-i} = \operatorname{card} \mathcal{D}_{c+i}$, where $\mathcal{D}_k$ is the k line of the above triangle. Then :

1. *c the codimension of $\mathcal{Q}$ is equal to the number of complete lines and $e = c + 1 - s_{d-1}$.*
2. $$\beta_j(S/\mathcal{Q}) = j\binom{c+1}{j+1} + \sum_{i=0}^{d-1} s_i \binom{n-i-1}{j-1}$$

Example

. $S = K[a, b, c, x, y, z, u]$, and

$$\mathcal{J}_1 = (a, b, c); \mathcal{J}_2 = (a, b, y, z); \mathcal{J}_3 = (a, x, z - u, c);$$

$$\mathcal{J}_4 = (x - u, y - u, b, c).$$

$\mathcal{J} := \bigcap_{i=1}^{4} \mathcal{J}_i$. The minimal free resolution of $S/\mathcal{J}$ will be:

$$0 \to S^3 \to S^{12} \to S^{17} \to S^9 \to S \to S/\mathcal{J} \to 0.$$

Corollary

Two ideals with 2-linear resolutions have the same Betti numbers if and only if they have the same numbers $c, s_{d-1}, \ldots, s_0$. In each class of ideals with 2-linear resolutions, having the same Betti numbers there is a unique Ferrer ideal.

Theorem:
Let $\mathcal{Q} := \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_l \subset K[V]$ be a 2-regular square free monomial ideal, that is $\mathcal{Q}_i$ is generated by a set of variables $Q_i \subset V$. We have a decomposition $Q_i = D_i \cup P_i$ satisfying the properties of Theorem 2.
Then $H^j_{\mathcal{Q}}(S) \neq 0$ if and only if either

1. $j = \operatorname{card}(Q_k)$ for some $k = 1, \ldots, l$, and in this case $\dim_K(H^j_{\mathcal{Q}}(S))_{-\alpha(Q_k)} = 1$.
2. $j + 1 = \operatorname{card}(D_{k-1} \cup P_k)$ for some $k = 2, \ldots, l$, and in this case $\dim_K(H^j_{\mathcal{Q}}(S))_{-\alpha(D_{k-1} \cup P_k)}$ is the number of occurrences of the set $D_{k-1} \cup P_k$ among the sets $D_1 \cup P_2, \ldots, D_{l-1} \cup P_l$.

Theorem

Let $\mathcal{Q}$ be a 2-regular square free monomial ideal. We set
$\mathcal{A}_{i,\mathcal{Q}} = \{(Q_j)/ \operatorname{card} Q_j = i, \}$,
$\mathcal{B}_{i,\mathcal{Q}} = \{(D_{k-1}, P_k), \operatorname{card} D_{k-1} + \operatorname{card} P_k = i+1\}$. *We say that an element* $(D_{k-1}, P_k) \in \mathcal{B}_{i,\mathcal{Q}}$ *belongs to the set* $\widetilde{\mathcal{B}_{i,\mathcal{Q}}}$ *if* $(D_{s-1}, P_s) = (Q_j) + (Q_k)$, *for some* j, k *such that* $\operatorname{card} Q_j, \operatorname{card} Q_k < i$. *With these notations, we have:*

$$Ass(H^i_{\mathcal{Q}}(S)) = \mathcal{A}_{i,\mathcal{Q}} \cup \widetilde{\mathcal{B}_{i,\mathcal{Q}}}.$$

$p-$joined ideals.

Definition

Let V be a finite set, $S = K[V]$. The sequence of ideals $\mathcal{J}_1, ..., \mathcal{J}_l$ is *p-joined* if

1. For all $m = 1, ..., l$, $\mathcal{J}_j = (\mathcal{D}_j, \mathcal{P}_j)$, where $\mathcal{D}_j$ is a linear ideal generated by a set $D_j \subset V$,
2. $V \supset D_1 \supset D_2 \supset ... \supset D_{l-1} \supset D_l = \emptyset$,
3. $\mathcal{P}_1 = 0$ and for all $k = 2, ..., l$ we have a minimal set of generators P_k of $\mathcal{P}_k$ such that every element in P_k does not contains any variable in D_{k-1},
4. for all $k = 2, ..., l$:

$$(*) \qquad \mathcal{J}_k + \cap_{j=1}^{k-1} \mathcal{J}_j = \mathcal{D}_{k-1} + \mathcal{P}_k.$$

In the above situation we will say that $\mathcal{J} := \cap_{j=1}^{l} \mathcal{J}_j$ is $p-$joined.

Lemma

Suppose that the sequence of ideals $\mathcal{J}_1, ..., \mathcal{J}_l$ is $p-$joined, Let Δ_i be the complementary set of D_i in D_{i+1}. Then for all $k = 2, ..., l$:

1. $\Delta_j \times P_j \subset P_k$ *for all* $j < k$,
2. $\bigcap_{j=1}^{k} \mathcal{J}_j = (D_k, \bigcup_{j=2}^{k} \Delta_j \times P_j)$.

Theorem

We assume that $\mathcal{J}$ is $p-$joined

1. *If (P_j) is a $p-$regular ideal for all j, then* $\operatorname{reg}(\mathcal{J}) = p + 1$.

Numerical invariants.

Theorem

We assume that $\mathcal{J}$ is $p-$joined

1. *If* (P_j) *is a* $p-$*regular ideal for all* j*, then* $\mathrm{reg}\,(\mathcal{J}) = p + 1$.
2. $\mathrm{projdim}\ S/\mathcal{J} = \text{máx}\{\ \mathrm{card}\, D_{k-1} + \ \mathrm{projdim}\ S/(P_k) \mid k = 2, ..., l\} - 1.$

Numerical invariants.

Theorem

We assume that $\mathcal{J}$ is $p-$joined

1. *If (P_j) is a $p-$regular ideal for all j, then* $\operatorname{reg}(\mathcal{J}) = p + 1$.
2. $\operatorname{projdim} S/\mathcal{J} = \text{máx}\{\operatorname{card} D_{k-1} + \operatorname{projdim} S/(P_k) \mid k = 2, \ldots, l\} - 1$.
3. *If $S/\mathcal{J}_1, \ldots, S/\mathcal{J}_l$ are Cohen–Macaulay rings of the same dimension d and for all $i = 2, \ldots, l$, $S/(D_{k-1}, P_k)$ is a Cohen–Macaulay ring of dimension $d - 1$ then $S/(\bigcap_{j=1}^{l} \mathcal{J}_j)$ is a Cohen–Macaulay ring of dimension d.*

Numerical invariants.

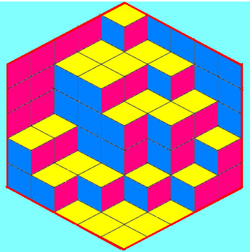
Theorem

We assume that $\mathcal{J}$ is $p-$joined

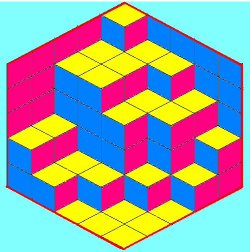
1. *If (P_j) is a $p-$regular ideal for all j, then* $\operatorname{reg}(\mathcal{J}) = p + 1$.
2. $\operatorname{projdim} S/\mathcal{J} = \text{máx}\{\operatorname{card} D_{k-1} + \operatorname{projdim} S/(P_k) \mid k = 2, \ldots, l\} - 1$.
3. *If $S/\mathcal{J}_1, \ldots, S/\mathcal{J}_l$ are Cohen–Macaulay rings of the same dimension d and for all $i = 2, \ldots, l$, $S/(D_{k-1}, P_k)$ is a Cohen–Macaulay ring of dimension $d-1$ then $S/(\bigcap_{j=1}^{l} \mathcal{J}_j)$ is a Cohen–Macaulay ring of dimension d.*
4. *Connectedness dimension, cohomological dimension and arithmetical rank can be computed under some assumptions.*

$p-$Ferrer'ideals are $p-1$-joined ideals

The following picture corresponds to the $3-$Ferrer diagram given by:



The following picture corresponds to the $3-$Ferrer diagram given by:



5	4	4	3	2
4	4	3	3	1
4	4	3	1	0
2	1	1	0	0
2	1	0	0	0

For a $p-$Ferrer's ideal $\mathcal{I}$ the projective dimension $\operatorname{projdim}(S/\mathcal{I})$ is equal to the number of diagonals in the above picture, and $\operatorname{ara}(\mathcal{I}) = \operatorname{projdim}(S/\mathcal{I})$, the Betti numbers "counts" the numbers of elements in each diagonal.

Matroidal ideals

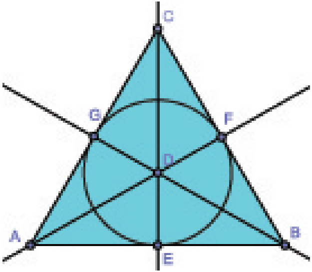
Definition

$S := K[V]$, V. A square free monomial ideal $\mathcal{I}$ (with set of generators $G(\mathcal{I})$ is a matroidal ideal if the following property holds: $(*)$ For any $u, v \in G(\mathcal{I})$, and any $x \in supp(u) \setminus supp(v)$ there exists $y \in supp(v) \setminus supp(u)$ such that $y\frac{u}{x} \in G(\mathcal{I})$.
All the generators of a matroidal ideal have the same degree d. We say that $\mathcal{I}$ is a matroidal ideal of rank d.

Matroidal ideals, examples

•Let $V_1 \cup ... \cup V_m = V$ be a partition of V. For any $1 \leq d \leq m$, the ideal generated by $\bigcup_{1 \leq i_1 < ... < i_d \leq m} V_{i_1} \times ... \times V_{i_d}$ is a matroidal ideal of rank d. We denote it by $\mathcal{M}_d(V_1, ..., V_m)$.

•The Fano Matroid has rank 3 and is simple.



the basis are all subsets of 3 elements of $\{A, B, C, D, E, F, G\}$ except the 7 drawn lines.

Matroidal ideals, examples

$\mathcal{I}_F$ the ideal associated to the Fano matroid, is generated by the following monomials:

								cfg			
					cef			afg	ecg		
				ace	acf	bef		dfg	aeg	bcg	
	abd		ade	bce	cdf	abf	aef	bfg	edg	abg	cdg
abc	acd	bcd	bde			bdf	def	beg	adg		

We have $\operatorname{projdim} S/\mathcal{I}_F = 5 = ara(\mathcal{I}_F) = \operatorname{cd}(\mathcal{I}_F)$.

Theorem

Any Matroidal ideal of degree d is $d-1$-joined.