Applications of Discrete Optimization to Commutative Algebra.

Toric Geometry Seminar 2010

Jarandilla de la Vera, November 2010

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Outline

1. **Solving IP using CA**
   - Toric Gröbner Bases
   - Graver Bases
   - Lexicographic Gröbner Bases
   - Short generating functions
   - Sum of Squares (SOS)

2. **Discrete Optimization in CA**
   - Computing the omega invariant
   - Decomposing into m-irreducibles

Integer Programming

$$
\begin{array}{lll}
\min & c\,x & \\
s.t. & A\,x & = b \\
 & x & \in \mathbb{Z}_+^n
\end{array}
\qquad (\mathrm{IP}_{A,c}(b))
$$

$A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$.

Gröbner Bases of Toric Ideals and IP

- P. Conti and C. Traverso: Buchberger algorithm and integer programming, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes* (New Orleans, LA, 1991), 130–139, Lecture Notes in Comput. Sci., 539, Springer, Berlin, 1991.
- S. Hoşten and R. Thomas: *Gröbner bases and integer programming*, Gröbner bases and applications (Linz, 1998), 144–158, London Math. Soc. Lecture Note Ser., 251, Cambridge Univ. Press, Cambridge, 1998.
- S. Hoşten and B. Sturmfels: *GRIN: an implementation of Gröbner bases for integer programming*. Integer programming and combinatorial optimization (Copenhagen, 1995), 267–276, Lecture Notes in Comput. Sci., 920, Springer, Berlin, 1995.

How to solve IP using GB

$I_A = \langle x^u - x^v : Av = Au \rangle$

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$\mathcal{G}$: A Gröbner basis for I_A with respect to $\prec_c$:

$$u \prec_c v :\Leftrightarrow \begin{cases} cu < cv \quad \text{or} \\ cu = cv \text{ and } u \prec_{lex} v \end{cases}$$

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u_0: A feasible solution for $\mathrm{IP}_{A,c}(b)$.

$nf(x^{u_0}, \mathcal{G}) = x^{u^\star}$

$u^\star$ OPTIMAL SOLUTION for $\mathrm{IP}_{A,c}(b)$.

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Task: How to compute a System of generators for I_A?: Big-M method, GRIN method

Geometric GB

R. Thomas, A geometric Buchberger algorithm for integer programming. Math. Oper. Res. **20** (1995) 864–884.

Definition (Test Set)

A finite set $G = \{g_1, \dots, g_t\} \subseteq \mathbb{Z}^n$ is a test set for $IP_{A,c}$ if and only if:

1. *For all $g \in G$, $Ag = 0$.*
2. *If $x \in \mathbb{N}^n$ is a non optimal solution for $IP_{A,c}(b)$, with $b \in \mathbb{Z}^n_+$, there is some $g \in G$, such that $x - g \prec_c x$.*
3. *If $x \in \mathbb{N}^n$ is the optimal solution for $IP_{A,c}(b)$, with $b \in \mathbb{Z}^n_+$, then for all $g \in G$, $x - g$ is infeasible.*

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Theorem (Thomas 1995)

Let P *be the set of non-optimal solutions of* $\mathrm{IP}_{A,c}$. *Then, there exist* $\alpha_1, \ldots, \alpha_t \in P$ *such that:*

$$P = \bigcup_{i=1}^{t} (\alpha_i + \mathbb{N}^n)$$

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1. *For all $g \in G$, $A g = 0$.*
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Definition (Geometric GB)

For each $i = 1, \ldots, t$, let β_i the optimal solution of $\mathrm{IP}_{A,c}(A\alpha_i)$:

$$\mathcal{G}^G = \{g_i = \beta_i - \alpha_i : i = 1, \ldots, t\}$$

$\mathcal{G}^G$ is a minimal test set

Let α a non-optimal solution of $\mathrm{IP}_{A,c}(b)$. By the theorem above, there exists at least one α_i such that $\alpha \geq \alpha_i$, then:

$$\alpha - \alpha_i \geq 0$$

and then,

$$\alpha - \alpha_i + \beta_i = \alpha - g_i \geq 0$$

But also:

- α and $\alpha - g_i$ are both feasible solutions of $\mathrm{IP}_{A,c}(b)$: $A(\alpha - g_i) = A(\alpha - \alpha_i + \beta_i) = A\alpha$.
- $\alpha - g_i$ improves α: $c(\alpha - g_i) = c\alpha - c\alpha_i + c\beta_i \leq c\alpha$. ($\beta_i$ is optimal for $\mathrm{IP}_{A,c}(A\alpha_i)$)

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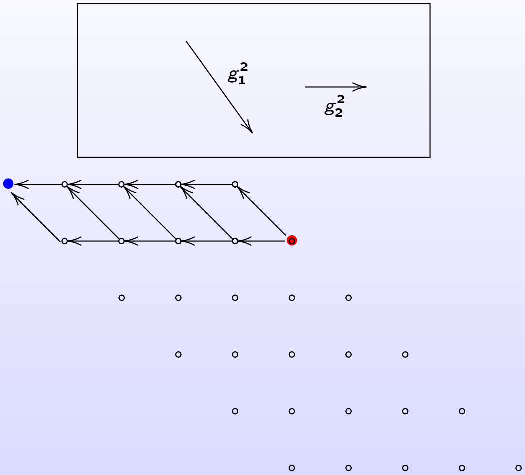
- $\alpha - g_i$ improves α: $c(\alpha - g_i) = c\alpha - c\alpha_i + c\beta_i \leq c\alpha$. ($\beta_i$ is optimal for $\mathrm{IP}_{A,c}(A\alpha_i)$)

$$\mathcal{G}^G = \{u - v : x^u - x^v \in \mathcal{G}\}$$


g_1^2
g_2^2

Applications and Extensions

- S. Hoşten and B. Sturmfels: *Computing the integer programming gap*, Combinatorica **23** (2007), 367–382.
- R.R. Thomas and R. Weismantel. *Truncated Gröbner bases for integer programming*. Applicable Algebra in Engineering, Communication and Computing **8** (4) 1997, 241–256
- B. Sturmfels and R. Thomas, *Variation of cost functions in integer programming*, Mathematical Programming **77** (1997) 357–387.
- V. Blanco and J. Puerto. (2009) *Partial Gröbner Bases for Multiobjective Integer Linear Optimization*. SIAM Journal on Discrete Mathematics 23 (2), 571–595.
- Tayur, S.R., Thomas, R.R., and Natraj, N.R. (1995). *An algebraic geometry algorithm for scheduling in presence of setups and correlated demands*. Mathematical Programming A, 69(3):369–401, 1995.
- Castro, F., Gago, J., Hartillo, I., Puerto, J., and Ucha, J.M. (2010). *An algebraic approach to Integer Portfolio problems* . arXiv:1004.0905

Graver bases and IP

- R. Hemmecke, J. De Loera, S. Onn, R. Weismantel, *N-fold Integer Programming*, Discrete Optimization 5 (2), 2008, 231–241.
- R. Hemmecke, J. De Loera, S. Onn, U.G. Rothblum, R. Weismantel, *Convex Integer Maximization via Graver Bases*, Journal of Pure and Applied Algebra 213 (8). 2008, 1569–1577
- R. Hemmecke and S. Onn, *Multicommodity flow in polynomial time*. Arxiv: arXiv:0906.5106 (June 2009)
- De Loera, J., Haws, D., Lee, J. and O'Hair, A. (2009) *Computation in Multicriteria Matroid Optimization*, To appear, Journal of Experimental Algorithmics, 2009.

Graver Bases

$$I_A = \langle x^u - x^v : Au = Av, u, v \in \mathbb{Z}_+ \rangle$$

Definition (Graver Bases)

$$\mathcal{G}r_A = \{x^u - x^v : \nexists x^w - x^z \in I_A \text{ such that } w \leq u \text{ and } z \leq v\}$$

$\mathcal{G}r_A^G = \{z \in ker_{\mathbb{Z}}(A) : z \text{ cannot be written as } z = u + v \text{ where } u, v \in \ker_{\mathbb{Z}}(A) \text{ and } u_i v_i \geq 0, \forall i\}$

$$x^u - x^v \in \mathcal{G}r \iff u - v \in \mathcal{G}r_A^G$$

Universal Gröbner bases

$UGB_A = \bigcup_c G_{A,c}$

$$\mathcal{G}r_A \subseteq UGB_A$$

... but in some special cases it is easier to compute than Gröbner bases: N-fold Integer Programs

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$\mathcal{G}r_A^G$ is a test set for IP_A

Polynomial IP and Systems of Polynomial Equations

- K. Hägglöf, P. Lindberg, L. Svensson, *Computing global minima to polynomial optimization problems using Gröbner bases*, Journal of Global Optimization 7 (2) (1995) 115–125.
- D. Bertsimas, D. Perakis, S. Tayur, *A new algebraic geometry algorithm for integer programming*, Management Sience 46 (7) (2000) 999–1008.
- V. Blanco and J. Puerto, *Some algebraic methods for solving multiobjective polynomial integer programs*, Journal of Symbolic Computation, 2010
- R. Datta. *Finding All Nash Equilibria of a Finite Game Using Polynomial Algebra*, Economic Theory 20. 2009.

$$\begin{array}{lrll} \min & f(x) & & \\ s.t. & & & \\ & g_j(x) & \leq 0 & = 1, \ldots, m \\ & h_r(x) & = 0 & r = 1, \ldots, s \\ & x & \in \mathbb{Z}^n & \end{array} \quad (1)$$

KKT necessary conditions for optimality [Karush, 1939] [Kuhn-Tucker, 1951]

Let x^* a feasible solution. Suppose that f and g_j, for $j = 1, \ldots, m$, are differentiable at x^*, that g_j, for $j \notin J$, is continuous at x^*,and that h_r, for $r = 1, \ldots, s$, is continuously differentiable at x^*. Further suppose that ∇g_j, for $j \in I$, and ∇h_r, for $r = 1, \ldots, s$, are linearly independent (regularity conditions). If x^* is a optimal solution, then there exist scalars λ_j, for $j = 1, \ldots, m$, and μ_r, for $r = 1, \ldots, s$, such that

$$
\begin{aligned}
\nabla f(x^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) + \sum_{r=1}^{s} \mu_r \nabla h_r(x^*) &= 0 & \\
\lambda_j \, g_j(x^*) &= 0 & \text{for } j = 1, \ldots, m \\
\lambda_j &\geq 0 & \text{for } j = 1, \ldots, m
\end{aligned}
\tag{KKT}
$$

NR

x^* is Non Regular for (1), if x^* is feasible and there exist λ_i, for $i = 1, \ldots, m$, and μ_j, for $j = 1, \ldots, s$ such that:

$$
\sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{s} \mu_j \nabla h_j(x^*) = 0
$$

$MOPIP_{\mathbf{f},\mathbf{g}}$

$$
\begin{array}{llll}
\min & (f_1(x), \ldots, f_k(x)) & & \\
s.t. & & & \\
& g_j(x) & \leq 0 & j = 1, \ldots, m \\
& h_r(x) & = 0 & r = 1, \ldots, s \\
& x & \in \mathbb{Z}_+^n &
\end{array} \tag{2}
$$

with $f_1, \ldots, f_k, g_1, \ldots, g_m, h_1, \ldots, h_s$ polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ and the constraints defining a bounded feasible region.

Chebishev Scalarization

Nondominance necessary conditions for the Chebyshev scalarization [Bowman 1976]

x^* is a nondominated solution if and only if there are positive real numbers $\omega_1, \ldots, \omega_k > 0$ so that x^* is an image unique solution of the following weighted Chebyshev approximation problem:

$$\begin{array}{lrll} \min\limits_{x} \max\limits_{i} & \omega_i \left(f_i(x) - \hat{y}_i\right) & & \\ s.t. & & & \\ & g_j(x) & \leq 0 & j = 1, \ldots, m \\ & h_r(x) & = 0 & r = 1, \ldots, s \\ & x_i(x_i - 1) & = 0 & i = 1, \ldots, n \\ & x \in \mathbb{R}^n & & \end{array} \qquad (P_\omega)$$

where $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_k) \in \mathbb{R}^k$ is a lower bound of $f = (f_1, \ldots, f_k)$, i.e., $\hat{y}_i \leq f_i(x)$ for all feasible solution x and $i = 1, \ldots, k$.

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$$
\begin{array}{lrll}
\min & \gamma & & \\
s.t. & & & \\
& \omega_i \left(f_i(x) - \hat{y}_i\right) - \gamma & \leq 0 & i = 1, \ldots, k \\
& g_j(x) & \leq 0 & j = 1, \ldots, m \\
& h_r(x) & = 0 & r = 1, \ldots, s \\
& x_i(x_i - 1) & = 0 & i = 1, \ldots, n \\
& \gamma \in \mathbb{R} & x \in \mathbb{R}^n &
\end{array} \qquad (P_\omega)
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Multiobjective FJ

FJ necessary conditions for non dominance [Zadeh, 1963; Cunha-Polack, 1967]

Let x^* a feasible solution. Suppose that f, g_j, for $j = 1, \dots, m$ and h_r, for $r = 1, \dots, s$, are continuously differentiable at x^*. If x^* is a nondominated solution, then there exist scalars ν_i, for $i = 1, \dots, k$, λ_j, for $j = 1, \dots, m$, and μ_r, for $r = 1, \dots, s$, such that

$$
\begin{aligned}
\sum_{i=1}^{k} \nu_i \nabla f_i(x^*) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) + \sum_{r=1}^{s} \mu_r \nabla h_r(x^*) &= 0 \\
\lambda_j g_j(x^*) &= 0 \quad \text{for } j = 1, \dots, m \\
\lambda_j &\geq 0 \quad \text{for } j = 1, \dots, m \\
\nu_i &\geq 0 \quad \text{for } i = 1, \dots, k \\
(\nu, \lambda, \mu) &\neq (\mathbf{0}, \mathbf{0}, \mathbf{0})
\end{aligned}
\tag{MO-FJ}
$$

Generating functions and IP

- Barvinok, A. *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Mathematics of Operations Research , 19 (1994), 769–779.
- Blanco, V and Puerto, J. *Short Rational Generating Functions For Multiobjective Linear Integer Programming*. Submitted. Available on Arxiv: 0712.4295. 2008.
- De Loera, J.A., Haws, D., Hemmecke, R., Huggins, P., and R. Yoshida *Three Kinds of Integer Programming Algorithms Based on Barvinok's Rational Functions*, Lecture Notes in Computer Science, Integer Programming and Combinatorial Optimization (2004) 3–9
- De Loera, J.A., Hemmecke, R., Köppe, M. (2008). *Pareto Optima of Multicriteria Integer Linear Programs*. INFORMS Journal on Computing, 2008.
- Woods, K. and Yoshida, R. (2005). *Short rational generating functions and their applications to integer programming* , SIAG/OPT Views and News, 16 , 15-19.

Generating functions of rational polytopes

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polytope in $\mathbb{R}^n$ with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$.

$$f(P; z) = \sum_{\alpha \in P \cap \mathbb{Z}^n} z^\alpha$$

where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, encodes the integer points inside P.
INTRACTABLE!

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INTRACTABLE!

$P = [0, N] \subset \mathbb{R}$:

$f(P, z) = \sum_{i=0}^{N} z^i = 1 + z + z^2 + \cdots + z^N$

Generating functions of rational polytopes

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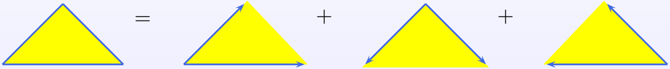
INTRACTABLE!

$P = [0, N] \subset \mathbb{R}$:

$$f(P, z) = \sum_{i=0}^{N} z^i = 1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}$$

Useful results on SGF

SGF of rational polytopes can be computed with the SGF of their supported cones (Brion, 1984)



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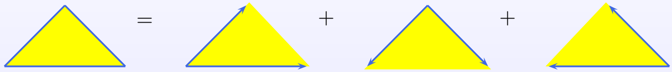

Theorem (Barvinok, 1994)

Assume n, the dimension, is fixed. Given a rational polyhedron $P \subset \mathbb{R}^n$, the generating function $f(P;z)$ can be computed in polynomial time in the form

$$f(P;z) = \sum_{i \in I} \varepsilon_i \frac{z^{u_i}}{\prod_{j=1}^{n} (1 - z^{v_{ij}})}$$

where I is a polynomial-size indexing set, and where $\varepsilon \in \{1, -1\}$ and $u_i, v_{ij} \in \mathbb{Z}^n$ for all i and j.

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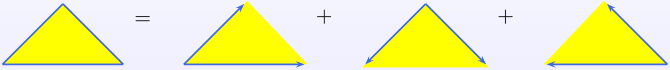

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Software: `LattE` and `barvinok`

$\max c\,x$ s.t. $A\,x \leq b$, $x \in \mathbb{Z}_+^n$

Theorem (De Loera et al., 2004)

Assume that the number of variables, n, is fixed. There is a polynomial-time algorithm for computing the optimal solution of a (single-objective) integer program using generating functions.

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Theorem (De Loera-Hemmecke-Köppe, 2008)

Assume that the number of variables, n and the number of objective of a multiobjective linear integer program are fixed. Then, the set of nondominated solutions can be encode in a short generating function.

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Theorem (B.-Puerto 2009)

Assume that ONLY the number of variables, n, is fixed. Then, we can encode, in polynomial time, the entire set of nondominated solutions for $MIP_{A,C}(b)$ *in a short sum of rational functions.*

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Theorem (B.-Puerto 2009)

Assume n is a constant. There is a polynomial-delay procedure to enumerate the entire set of nondominated solutions of $MIP_{A,C}(b)$.

Consequences...

- Blanco, V and Puerto, J. *Some complexity results on fuzzy integer programming*. Submitted. Available on Arxiv: 0712.4295. 2008.
- Köppe, M, Ryan, C.T., and Queyranne, M (2008). *Rational Generating Functions and Integer Programming Games*, Submitted. Available on Arxiv: 0809.0689.
- Köppe, M., Queyranne, M., and Ryan, C.T. (2009) *Parametric Integer Programming Algorithm for Bilevel Mixed Integer Programs.* Journal of Optimization Theory and Applications Volume 146, Number 1, 137-150
- Blanco,V., García-Sánchez, P.A., and Puerto, J. (2010) *Counting Numerical Semigroups with Short Generating Functions.* To Appear in the Int. J. of Algebra and Computation.

Sums of squares and the moment problem

- Nie, J., Demmel, J., and Sturmfels, B. 2006. *Minimizing Polynomials via Sum of Squares over the Gradient Ideal*. Math. Program. 106, 3 (May. 2006), 587–606.
- Nie, J., Powers, V., and Demmel, J. Re*presentations of positive polynomials on noncompact semialgebraic sets via KKT ideals*, Journal of pure and applied algebra 209 (1), 2007 , 189–200.
- Lasserre, J.B. *Global Optimization with Polynomials and the Problem of Moments*, SIAM Journal on Optimization, v.11 n.3, p.796–817, 2000.
- Henrion D., Lasserre J.B., and Lofberg J. (2009) *Gloptipoly 3: moments, optimization and semidefinite programming*. Optim. Methods and Softwares 24, pp. 761–779.
- Lasserre J.B. (2009) *Moments and sums of squares for polynomial optimization and related problems*. J. Global Optim. 45, pp. 39–61.

Numerical Semigroups

Definition

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$\{n_1, \ldots, n_p\}$ is a system of generators of S if

$S = \{\sum_{i=1}^{p} n_i x_i : x_i \in \mathbb{N}, i = 1, \ldots, p\}$. We denote $S = \langle n_1, \ldots, n_p \rangle$.

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Example: $S = \langle 2, 4, 5, 7, 10 \rangle = \langle 2, 5 \rangle = \{0, 2, 4, \rightarrow\} = \mathbb{N}\backslash\{1, 3\}$

Numerical Semigroups and Discrete Optimization

- *Multiplicity*: $m(S) = \min\{s \in S\backslash\{0\}\}$.
- *Frobenius Number*: $F(S) = \max\{n \in \mathbb{Z}\backslash S\}$.
- Kunz's polyhedron (Rosales et. al, 2002).
- Arithmetic invariants.
- Irreducibility: irreducible (Rosales-Branco, 2003) and m-irreducible (B.-Rosales, 2010) numerical semigroups.

The omega invariant: definition

Definition (Geroldinger, 1997)

Let $S = \langle n_1, \ldots, n_p \rangle$ *be a numerical semigroup. For* $s \in S$*, let* $\omega(S, s)$ *denote the smallest* $N \in \mathbb{N}_0 \cup \{\infty\}$ *with the following property:*

For all $n \in \mathbb{N}$ *and* $s_1, \ldots, s_n \in S$*, if* $\sum_{i=1}^{p} s_i - s \in S$*, then there exists a subset* $\Omega \subset \{1, \ldots, n\}$ *such that* $|\Omega| \leq N$ *and*

$$\sum_{j \in \Omega} s_j - s \in S\,.$$

Furthermore, we set

$$\omega(S) = \max\{\omega(S, n_i) : i = 1, \ldots, p\} \in \mathbb{N}.$$

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References: Geroldinger-Hassler (2008a-b), Geroldinger-Kainrath (2010), B.-GarcíaSánchez-Geroldinger (2010), Omidali (2010), Anderson-Chapman-Kaplan-Torkornoo (2010)...

The omega invariant: characterization

Theorem (B.-GarcíaSanchez-Geroldinger, 2010)

Let $S = \langle n_1, \ldots, n_p \rangle$ *be a numerical semigroup.*

1. *For every* $s \in S$,

   $$\omega(S, n) = \max\{\sum_{i=1}^{p} x_i : x \in \mathrm{Minimals}\,\mathsf{Z}(n + S)\},$$

2. $\omega(S) = \max\{\sum_{i=1}^{p} x_i : x \in \mathrm{Minimals}\big(\mathsf{Z}(n_i + S)\big)$ *for some* $i = 1, \ldots, p\}$.

$$\mathsf{Z}(n) = \{(x_1, \ldots, x_p) \in \mathbb{N}^p : n = \sum_{i=1}^{p} x_i n_i\}$$

$$\mathsf{Z}(n + S) = \{(x_1, \ldots, x_p) \in \mathbb{N}^p : n + s = \sum_{i=1}^{p} x_i n_i, \text{ para algún } s \in S\}$$

Optimization over and integer efficient set

$$
\begin{array}{ll}
\max \ c(x) & \\
s.t. & x \text{ is a non-dominated solution of} \\
v-\min \ C(x) & = (C_1(x), \ldots, C_k(x)) \\
s.t. & \\
 & Ax = b \\
 & x \in \mathbb{Z}_+^n
\end{array}
\tag{OES}
$$

($x \in \mathbb{Z}_+^n$ feasible, is a *non dominated* solution if there is no other feasible solution $y \in \mathbb{Z}_+^n$ such that $C(y) \leq C(x)$ y $C(x) \neq C(y)$)

Optimization over and integer efficient set: Omega

Let $S = \langle n_1, \ldots, n_p \rangle$ be a numerical semigroup. Then, for each $j \in \{1, \ldots, p\}$, $\omega(S, n_j)$ is the solution of the following OES problem:

$$\begin{aligned}
&\max \sum_{i=1}^{n} x_i \\
&s.t. \\
&\qquad x \in v - \min \ (x_1, \ldots, x_p) \\
&\qquad s.t. \\
&\qquad \sum_{i=1}^{p} x_i n_i - \sum_{i=1}^{p} y_i n_i = n_j \\
&\qquad x_i \leq ub_i = \max_k \mathrm{UB}_{ik} \\
&\qquad x_j = 0 \\
&\qquad x, y \in \mathbb{Z}_+^p
\end{aligned} \tag{OES_j}$$

where $\mathrm{UB}_{ik} = \min\{x_i : x_i n_i - \sum_{j=1}^{p} y_j n_j = n_k, y_k = 0, x_i \in \mathbb{Z}_+, y \in \mathbb{Z}_+^p\}$

($x_j = 0$: e_j is a non-dominated solution, but non optimal since $\omega(S, n_j) > 1$ (B.-GarcíaSánchez-Geroldinger, 2010))

Solving the problem of optimizing over and integer efficient set: general scheme (Jorge, 2009)

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Solving the problem of optimizing over and integer efficient set: general scheme (Jorge, 2009)

- Solve a relaxed (single objective) problem (feasible solution).
- Obtain a non-dominated solution dominating the feasible solution (Ecker-Kouada, 1975).
- Check if the solution is optimal (Nemhauser-Wolsey, 1988), otherwise, move to another feasible solution.

Computing the omega invarian: initial solution

$$\max \sum_{i=1}^{n} x_i$$

$$s.t.$$

$$\sum_{i=1}^{p} x_i n_i - \sum_{i=1}^{p} y_i n_i = n_j \qquad (\mathrm{R}_j)$$

$$x_i \leq ub_i$$

$$x_j = 0$$

$$x, y \in \mathbb{Z}_+^p.$$

Lemma

Problem (R_j) is feasible. Furthermore, the optimal solutions of (R_j) are not non-dominated solution of (SMIP_j).

Computing the omega invariant: generating efficient solutions

Let x^* be an optimal solution of (R_j) and $(\overline{s}, \overline{x}, \overline{y})$ an optimal solution of the following problem

$$\begin{aligned}
&\max \sum_{i=1}^{n} s_i \\
&s.t. \\
&\qquad x_i + s_i = x_i^* && i = 1, \ldots, p \\
&\qquad \sum_{i=1}^{p} x_i n_i - \sum_{i=1}^{p} y_i n_i = n_j \\
&\qquad x_i \leq ub_i && i = 1, \ldots, p \\
&\qquad x_j = 0 \\
&\qquad x, y \in \mathbb{Z}_+^p.
\end{aligned} \qquad (\mathrm{EK}_j(x^*))$$

Then, $\overline{x}$ is a non-dominated solution of (SMIP_j) that dominates x^*.

Computing the omega invariant: generating new feasible solutions

Let $\overline{x}^1, \ldots, \overline{x}^s$ be non-dominated solutions of $(\mathrm{SMIP_j})$ and $(\hat{x}, \hat{y})$ an optimal solution of the following problem

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n} x_i & \\
s.t. & & \\
& x_i \leq z_i^k(\overline{x}_i^k - 1) - M_i(z_i^k - 1) & i = 1, \ldots, p, k = 1, \ldots, s \\
& \displaystyle\sum_{i=1}^{p} x_i n_i - \sum_{i=1}^{p} y_i n_i = n_j & \\
& x_i \leq ub_i & i = 1, \ldots, p \\
& x_j = 0 & \\
& \displaystyle\sum_{i=1}^{p} z_i^k \geq 1 & k = 1, \ldots, s \\
& x, y \in \mathbb{Z}_+^p, z^k \in \{0,1\}^p & k = 1, \ldots, s
\end{array}
\qquad (\mathrm{NW}_j(\overline{x}^1, \ldots, \overline{x}^s))
$$

where $M_j = \max\{x_j : n_j x_j = \sum_{i \neq j}^{p} n_i\, y_i, x_i \leq ub_i, i = 1, \ldots, p, x, y \in \mathbb{Z}_+^p\}$.

Then, $\hat{x}$ is a feasible solution of $(\mathrm{SMIP_j})$ that is not dominated by

Improvements...

1. Better bounds (B.-GarcíaSánchez-Geroldinger, 2010): $\omega(S) \leq n_p$.
2. $\sum_{i=1}^{p} n_i\, y_i \leq \max \bigcup_{i=1}^{p} \mathrm{Ap}(S, n_i)$ y $\sum_{i=1}^{p} n_i\, y_i \geq \min \bigcup_{i=1}^{p} \mathrm{Ap}(S, n_i) = \min\{n_1, \ldots, n_p\}$.
   Apéry set: $\mathrm{Ap}(S, a) = \{s \in S \mid s - a \notin S\}$, $a \in S$.
3. Controlling the bounds at each iteration.

Experiments

1. 250 instances. Embedding dimension $\in \{5, 10, 15, 20\}$ (`RandomListOfNS`) with $n_i \in [2, 1000]$
2. Implemented in Xpress-Mosel 7.0 in a Intel Core 2 Quad 2x 2.50 Ghz and 4 GB of RAM.
3. Compared to GAP package on numerical semigroups (brute force).
4. Limit: 2h.

Experiments

S	n_j	$\omega(S, n_j)$	min	it	$time_j$	GAPtime	tottime	avtime	#min
	20	4	[0,0,0,0,4]	9	0.54	6.03			12
	354	60	[60,0,0,0,0]	12	0.95	11.35			14
S5(1)	402	63	[63,0,0,0,0]	16	1.439	12.54	5.921	1.184	17
	417	60	[60,0,0,0,0]	15	1.43	12.68			16
	429	60	[60,0,0,0,0]	17	1.55	12.43			20
	7	3	[0,3,0,0,0]	10	0.55	12.48			11
	292	93	[93,0,0,0,0]	9	0.37	23.72			11
S5(2)	359	93	[93,0,0,0,0]	11	0.43	27.33	2.84	0.56	13
	645	200	[200,0,0,0,0]	14	0.67	45.92			15
	755	200	[200,0,0,0,0]	18	0.81	75.59			19
	5	2	[0,0,0,2,0]	8	0.285	1.201			11
	86	37	[37,0,0,0,0]	11	0.294	2.527			12
S5(3)	99	37	[37,0,0,0,0]	11	0.34	2.82	1.69	0.33	12
	148	60	[60,0,0,0,0]	12	0.37	4.1			13
	152	60	[60,0,0,0,0]	12	0.39	2.29			13
	41	14	[0,14,0,0,0]	12	0.893	5.64			14
	65	22	[22,0,0,0,0]	13	0.988	6.02			14
S5(4)	155	24	[24,0,0,0,0]	16	1.1	8.22	7.39	1.47	18
	317	22	[21,0,1,0,0]	22	2.916	13.96			28
	377	31	[31,0,0,0,0]	18	1.49	18.7			35
	28	10	[0,10,0,0,0]	11	0.5	10.71			12
	55	25	[25,0,0,0,0]	8	0.381	11.45			12
S5(5)	125	27	[27,0,0,0,0]	13	0.71	20.18	4.719	0.94	15
	233	26	[26,0,0,0,0]	13	0.732	42.37			17
	590	30	[24,5,0,1,0]	23	2.38	109.38			48

Experiments

S	n_j	$\omega(S, n_j)$	min	it	$time_j$	GAPtime	tottime	avtime	#min
	43	5	[0,0,5,0,0,0,0,0,0,0]	49	3.36	5.41			58
	63	8	[8,0,0,0,0,0,0,0,0,0]	48	2.7	8.61			65
	68	8	[8,0,0,0,0,0,0,0,0,0]	49	3.16	13.18			69
	108	7	[5,0,2,0,0,0,0,0,0,0]	52	4.15	18.26			81
	120	8	[8,0,0,0,0,0,0,0,0,0]	57	4.24	12.65			94
S10(1)	135	9	[9,0,0,0,0,0,0,0,0,0]	68	5.95	15.5	67.89	6.78	108
	142	9	[9,0,0,0,0,0,0,0,0,0]	88	9.24	16.75			125
	150	7	[4,2,1,0,0,0,0,0,0,0]	66	8.07	19.85			116
	177	9	[7,2,0,0,0,0,0,0,0,0]	70	6.88	49.26			149
	224	9	[7,0,2,0,0,0,0,0,0,0]	113	20.1	65.16			246
	15	3	[0,0,3,0,0,0,0,0,0,0]	36	1.801	6.64			45
	46	9	[9,0,0,0,0,0,0,0,0,0]	39	1.66	10.32			48
	58	10	[10,0,0,0,0,0,0,0,0,0]	38	1.69	12.23			50
	89	9	[7,0,1,0,0,0,0,1,0,0]	47	2.681	17.33			68
	108	15	[15,0,0,0,0,0,0,0,0,0]	63	3.278	24			83
S10(2)	114	16	[16,0,0,0,0,0,0,0,0,0]	57	3.07	28.81	35.87	3.58	78
	117	15	[15,0,0,0,0,0,0,0,0,0]	63	4.316	21.65			88
	126	16	[16,0,0,0,0,0,0,0,0,0]	73	4.243	22.48			99
	130	22	[22,0,0,0,0,0,0,0,0,0]	64	3.399	38.59			98
	173	23	[23,0,0,0,0,0,0,0,0,0]	107	9.73	80.49			161
	20	4	[0,0,0,4,0,0,0,0,0,0]	39	1.48	5.1			43
	22	5	[5,0,0,0,0,0,0,0,0,0]	43	1.59	5.41			45
	24	5	[5,0,0,0,0,0,0,0,0,0]	36	1.3	5.41			45
	26	5	[3,2,0,0,0,0,0,0,0,0]	33	1.64	3.49			44
S10(3)	54	6	[0,6,0,0,0,0,0,0,0,0]	52	2.77	14.05	99.49	9.94	88
	77	9	[7,0,2,0,0,0,0,0,0,0]	93	13.27	26.72			176
	83	9	[6,0,2,1,0,0,0,0,0,0]	109	19.41	33.83			198
	89	10	[10,0,0,0,0,0,0,0,0,0]	100	13.85	41.18			219
	93	10	[10,0,0,0,0,0,0,0,0,0]	109	21.75	46.17			254
	95	10	[10,0,0,0,0,0,0,0,0,0]	114	22.4	52.46			251
	131	7	[5,0,0,0,2,0,0,0,0,0]	63	8.34	61.44			102
	136	6	[3,1,0,0,2,0,0,0,0,0]	47	7.23	54.38			88
	171	6	[2,2,0,0,2,0,0,0,0,0]	65	11.18	56.92			102
	173	7	[3,1,3,0,0,0,0,0,0,0]	60	9.87	116.22			118
S10(4)	239	8	[5,2,0,0,0,0,0,1,0,0]	83	16.81	104.66	225.55	22.55	155
	278	10	[10,0,0,0,0,0,0,0,0,0]	80	14.93	129.1			208
	287	10	[10,0,0,0,0,0,0,0,0,0]	62	11.628	128.1			178
	364	10	[7,3,0,0,0,0,0,0,0,0]	128	34.053	227.12			260
	483	11	[9,1,0,0,0,0,1,0,0,0]	204	105.146	497			427

Experiments

S	n_j	ω	min	it	$time_j$	GAPtime	tottime	avtime	#min
	146	8	[0,6,2,0,0,0,0,0,0,0]	42	8.048	100.82			70
	173	10	[10,0,0,0,0,0,0,0,0,0,0]	71	15.43	115.39			99
	207	10	[10,0,0,0,0,0,0,0,0,0,0]	60	11.77	138.87			82
	359	12	[7,5,0,0,0,0,0,0,0,0]	60	14.69	198.246			152
S10(5)	426	12	[12,0,0,0,0,0,0,0,0,0,0]	77	16.23	290.08	315.14	31.51	130
	548	12	[0,12,0,0,0,0,0,0,0,0,0]	105	38.525	470.76			209
	604	15	[15,0,0,0,0,0,0,0,0,0,0]	124	43.81	499.9			244
	606	13	[13,0,0,0,0,0,0,0,0,0,0]	98	28.4	422.96			243
	657	12	[0,8,4,0,0,0,0,0,0,0]	105	65.01	558.71			244
	702	14	[14,0,0,0,0,0,0,0,0,0,0]	159	73.19	718.58			362
	47	6	[6,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	114	8.386	13.78			129
	65	5	[4,0,0,1,0,0,0,0,0,0,0,0,0,0,0]	112	10.358	35.64			159
	79	5	[3,1,0,0,1,0,0,0,0,0,0,0,0,0,0]	105	9.392	7.21			165
	82	6	[0,6,0,0,0,0,0,0,0,0,0,0,0,0,0]	141	13.664	17.93			184
	84	6	[4,1,0,0,1,0,0,0,0,0,0,0,0,0,0]	112	11.156	28.24			192
	91	7	[7,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	101	6.863	9.34			173
S15(1)	96	7	[4,3,0,0,0,0,0,0,0,0,0,0,0,0,0]	104	11.98	52.225	612.099	40.8066	250
	100	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	187	26.425	29.725			251
	109	7	[7,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	129	12.659	35.725			245
	121	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	154	19.271	48.225			307
	124	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	214	37.796	203.8			364
	134	7	[5,1,1,0,0,0,0,0,0,0,0,0,0,0,0]	168	29.652	241.9			383
	139	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	183	30.122	199.05			394
	169	8	[5,2,0,0,0,1,0,0,0,0,0,0,0,0,0]	285	38.405	164.625			680
	46	5	[0,1,3,0,1,0,0,0,0,0,0,0,0,0,0]	83	8.383	79.4			98
	115	6	[2,0,3,0,1,0,0,0,0,0,0,0,0,0,0]	94	10.454	112.65			109
	155	17	[17,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	123	14.728	139.425			151
	286	15	[12,0,3,0,0,0,0,0,0,0,0,0,0,0,0]	137	20.015	291.65			206
	289	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	109	14.545	293.575			190
	341	17	[17,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	174	65.975	401			252
	342	15	[14,0,0,0,1,0,0,0,0,0,0,0,0,0,0]	192	32.986	406.775			265
S15(2)	348	15	[13,0,2,0,0,0,0,0,0,0,0,0,0,0,0]	193	113.383	427.3	1683.63	112.242	291
	393	20	[20,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	228	135.869	550.35			320
	436	25	[25,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	273	74.036	736.575			413
	445	24	[24,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	311	96.198	784.625			434
	449	19	[19,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	294	82.945	795.45			425
	504	22	[22,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	354	177.154	1161.45			594
	527	20	[20,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	367	166.69	1345.275			610
	584	26	[26,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	438	670.269	1988.875			737

Experiments

S	n_j	ω	min	it	$time_j$	GAPtime	tottime	avtime	#min
	40	3	[0,1,0,0,1,0,0,0,0,1,0,0,0,0,0]	89	7.53	24.85			113
	84	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	119	11.648	39.95			139
	126	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	114	13.082	67.375			195
	130	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	117	12.165	68.425			188
	132	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	126	12.849	69.1			186
	135	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	149	23.85	72.4			212
	142	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	156	20.35	78.35			216
S15(3)	152	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	176	22.83	90.65	1407.684	93.8456	267
	165	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	183	31.199	109.15			305
	183	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	189	38.537	134.025			324
	217	11	[9,1,0,0,1,0,0,0,0,0,0,0,0,0,0]	244	77.582	219.075			446
	221	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	250	70.445	229.625			454
	229	13	[13,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	274	89.679	253.775			478
	273	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	353	133.473	475.725			701
	323	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	578	842.465	3487.55			1076
	75	5	[0,4,0,1,0,0,0,0,0,0,0,0,0,0,0]	92	8.229	42.65			123
	104	6	[4,0,0,1,1,0,0,0,0,0,0,0,0,0,0]	96	10.708	52.2			140
	114	7	[5,0,0,2,0,0,0,0,0,0,0,0,0,0,0]	86	8.044	54.875			144
	128	7	[5,0,2,0,0,0,0,0,0,0,0,0,0,0,0]	104	10.528	61.075			147
	216	8	[3,0,5,0,0,0,0,0,0,0,0,0,0,0,0]	151	23.513	126.85			254
	219	9	[8,1,0,0,0,0,0,0,0,0,0,0,0,0,0]	145	19.017	126.95			251
	241	9	[9,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	139	22.934	156.65			302
S15(4)	271	8	[4,3,1,0,0,0,0,0,0,0,0,0,0,0,0]	188	52.956	202.2	892.038	59.4692	357
	309	9	[6,1,1,1,0,0,0,0,0,0,0,0,0,0,0]	211	62.039	284.25			439
	310	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	188	34.511	281.525			429
	321	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	269	67.673	320.65			523
	327	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	185	43.396	341.225			502
	340	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	212	42.852	374.325			509
	352	10	[7,3,0,0,0,0,0,0,0,0,0,0,0,0,0]	262	300.513	2434.3			622
	371	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	315	185.125	505.925			661
	29	5	[0,5,0,0,0,0,0,0,0,0,0,0,0,0,0]	97	3.899	258.275			106
	50	5	[5,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	99	4.911	533.775			116
	95	6	[3,2,0,0,0,1,0,0,0,0,0,0,0,0,0]	115	9.017	73.725			154
	96	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	120	7.652	24.55			171
	99	9	[9,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	127	9.08	25.7			170
	109	9	[9,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	138	11.984	30.225			187
	110	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	124	7.249	31.025			192
S15(5)	119	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	127	9.411	36.375	685.921	45.72807	211
	131	10	[9,1,0,0,0,0,0,0,0,0,0,0,0,0,0]	121	10.939	45.3			252
	134	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	174	17.188	48.075			258
	135	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	173	20.678	225.525			288
	152	10	[8,2,0,0,0,0,0,0,0,0,0,0,0,0,0]	180	28.469	144.375			338

Experiments ($n = 20$)

n_j	ω	min	it	$time_j$	GAPtime	tottime	avtime	#min
131	8	[0,8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	188	35.321	321.325			264
145	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	191	34.252	332.3			265
249	9	[6,2,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	233	54.57	550.725			340
257	9	[6,2,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	233	53.913	569.15			352
260	8	[8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	197	47.428	573.775			355
319	9	[1,7,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	244	90.809	785.925			451
354	9	[4,5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	256	97.12	938.925			500
459	10	[0,10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	398	182.085	1700.525			787
465	9	[9,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	356	363.725	1747.575			752
469	11	[3,8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	317	239.548	1802.75	19603.67	980.1835	796
487	9	[9,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	384	408.842	2011.8			865
572	12	[6,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	399	826.33	3290.4			1160
575	10	[0,10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	542	3123.4	3414.425			1235
587	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	487	1356.94	3657.525			1273
606	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	507	2100.84	4119.5			1389
607	10	[5,5,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0]	787	2894.21	4181.7			1387
652	11	[9,0,0,0,0,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	683	2013.25	5538.2			1673
674	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	821	1999.747	6288.875			1732
694	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	726	2991.08	7185.075			1851
762	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	118	690.26	11142.2			2375
57	4	[0,2,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0]	146	15.003	108.725			186
105	7	[7,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	169	20.95	158.525			211
182	9	[6,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	195	31.558	311.125			304
186	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	272	61.55	328.45			364
201	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	319	300.1	374.2			409
204	9	[9,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	284	63.721	394.55			427
254	9	[8,0,0,0,1,0,0,0,0,0,0,0,0,0,0]	378	153.266	635.725			599
259	10	[7,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	295	72.789	653.9			530
263	10	[6,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	294	69.81	695.8			612
274	9	[4,5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	336	172.839	751.05	11703.9	585.1952	587
275	11	[8,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	414	217.261	798.075			702
294	8	[5,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	359	396.194	915.575			612
295	11	[8,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	488	2342.7	975.65			802
298	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	463	174.03	990.675			773
307	10	[6,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	391	187.431	1084.4			808
338	13	[13,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	484	432.837	1546.65			1001
367	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	608	2000.37	2113.475			1248
393	12	[3,9,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	658	2531.045	2889.325			1502
417	11	[5,0,2,0,0,0,0,0,0,2,3,0,0,0,0,0,0,0,0,0]	563	1093.34	3737.325			1686
431	14	[6,8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	781	1367.11	*			*

Experiments ($n = 20$)

n_j	ω	min	it	$time_j$	GAPtime	tottime	avtime	#min
85	4	[0,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	147	25.646	341.175			230
298	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	316	99.513	864.5			455
333	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	322	91.273	1007.75			465
342	16	[16,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	326	99.809	1026.075			466
349	16	[16,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	307	76.393	1075.375			512
358	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	324	86.003	1092.975			480
401	12	[10,0,0,0,0,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	394	154.683	1372.975			631
415	16	[16,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	448	293.327	1474			687
462	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	361	135.922	1827.75			691
480	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	527	261.581	1982.85	8912.075	445.6038	786
556	16	[16,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	610	592.666	2975.05			1028
569	18	[18,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	695	2453.98	3284.75			1158
583	19	[19,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	711	1496.19	3440.55			1164
609	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	57	11.671	4037.25			1290
619	13	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	912	990.061	4518.725			1386
708	18	[18,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	582	569.397	*			*
710	15	[11,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	733	673.064	*			*
752	18	[18,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	777	722.174	*			*
821	21	[18,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0]	108	35.333	*			*
853	21	[19,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	108	43.389	*			*
81	5	[0,5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	140	17.706	219.175			214
107	6	[6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	176	27.955	264.375			251
168	9	[7,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	185	26.037	416.725			291
194	9	[6,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	239	57.189	527.75			427
230	8	[4,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	186	39	707.575			471
236	9	[7,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	198	49.312	735.875			474
274	9	[8,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	284	524.051	1027.225			590
277	9	[7,2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	276	113.266	1079.7			679
286	9	[6,2,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	321	676.315	1143.675			698
290	8	[1,7,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	312	1775.88	1177.15	11215.3	560.7652	630
305	10	[7,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	256	188.763	1345.125			683
310	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	297	85.818	1407.6			704
348	10	[7,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	403	392.226	2039.675			953
351	11	[11,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	432	193.712	2079.4			949
366	10	[9,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	383	295.208	2346.175			912
379	10	[3,7,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	296	1007.8	2735.2			1116
396	11	[10,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	560	3095.66	3283.85			1222
416	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	541	693.549	*			*
521	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	611	955.771	*			*
583	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	982	1000.085	*			*

Experiments ($n = 20$)

n_j	ω	min	it	$time_j$	GAPtime	tottime	avtime	#min
101	8	[0,8,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	57	8.393	488.75			246
141	7	[7,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	146	33.759	567.975			260
279	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	99	23.478	1170.65			502
314	10	[10,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	199	68.421	1328.55			457
329	11	[7,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	85	17.706	1428.15			461
369	11	[5,5,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	106	25.178	1747.875			493
399	11	[7,4,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	156	90.957	2166.55			711
425	11	[5,6,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	65	26.208	2477.4			718
438	13	[13,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	115	37.799	2648.8			732
447	15	[15,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	60	11.342	2771.675	10007.05	500.3524	808
477	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	138	51.901	3357.7			929
501	16	[16,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	40	7.425	3884.325			1026
534	12	[12,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	180	145.075	4557.05			983
536	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	83	23.15	4752.25			1090
555	13	[9,3,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	166	63.804	5404.225			1190
574	13	[13,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	345	777.094	*			*
620	14	[14,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	721	654.063	*			*
727	18	[18,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	882	2140.435	*			*
786	17	[17,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	734	3000.11	*			*
871	17	[17,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]	813	2800.75	*			*

- $av\ \frac{\mathtt{time}_j}{\mathtt{GAPtime}} = 0.23.$
- $av\ \frac{\mathtt{it}}{\#\mathtt{min}} = 0.59.$
- GAP was not able to solve 14 problem in 2h. ($*$).

Irreducible and m-irreducible numerical semigroups

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- *A numerical semigroup is* ***irreducible*** *if it cannot be expressed as an intersection of two numerical semigroups containing it properly.*

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Problem: Decompose (minimally) into m-irreducibles

How are those *m*-irreducibles?

Proposition (B.-Rosales, 2010)

S is m-irreducible if $\mathrm{m}(S) = m$ and it is maximal (w.r.t $\subseteq$) among the set of numerical semigroup with Frobenius number $\mathrm{F}(S)$ and multipliity m.

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Corollary (B.-Rosales, 2010)

A numerical semigroup, S, with multiplicity m is m-irreducible if and only if one of the following conditions holds:

1. $S = \{0, m, \rightarrow\}$.
2. $S = \{0, m, \rightarrow\} \backslash \{f\}$ *with* $f \in \{m+1, \ldots, 2m-1\}$.
3. S *is an irreducible numerical semigroup.*

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Corollary (B.-Rosales, 2010)

Let S be a numerical semigroup with multiplicity m. Then, S is m-irreducible if and only if $\mathrm{g}(S) \in \left\{ m-1, m, \left\lceil \frac{\mathrm{F}(S)+1}{2} \right\rceil \right\}$.

Where to look for those m-irreducible n.s. in the decomposition?

Definition (Oversemigroups)

Let S be a numerical semigroup with multiplicity m. The set of oversemigroups of S is

$$\mathcal{O}(S) = \{S' \text{ numerical semigroup} : S \subseteq S\}$$

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Definition

Let S be a numerical semigroup. The special gaps of S is the following set:

$$\mathrm{SG}(S) = \{z \in \mathrm{G}(S) : S \cup \{z\} \text{ is a numerical semigroup}\}$$

where $\mathrm{G}(S)$ is the set of gaps of S.

$\mathrm{SG}_m(S) = \{z \in \mathrm{SG}(S) : z > m\}$. $\#\mathrm{SG}_m(S) \leq m - 1$

Where to look for those m-irreducible n.s. in the decomposition?

Lemma

Let $S \in \mathcal{S}(m)$ and $S_1, \ldots, S_n \in \mathcal{O}_m(S)$. Then, $S = S_1 \cap \cdots \cap S_n$ if and only if for all $h \in \{x \in \mathrm{SG}(S) : x > m\}$ there exists $i \in \{1, \ldots, n\}$ such that $h \notin S_i$.

Proposition

Assume that $\mathrm{Minimals}_{\subseteq}\mathcal{I}_m(S) = \{S_1, \ldots, S_n\}$. Then, $S = S_{i_1} \cap \cdots \cap S_{i_r}$ if and only if $\mathrm{SG}_m(S) \cap (\mathrm{G}(S_{i_1}) \cup \cdots \cup \mathrm{G}(S_{i_r})) = \mathrm{SG}_m(S)$, where $\{S_{i_1}, \ldots, S_{i_r}\} \subseteq \{S_1, \ldots, S_n\}$.

Translating the problem: Kunz coordinates

Definition

Let S be a numerical semigroup with multiplicity m. If $\mathrm{Ap}(S,m) = \{w_0 = 0, w_1, \ldots, w_{m-1}\}$, the ***Kunz coordinates*** *of S is the vector $x \in \mathbb{Z}_+^{m-1}$ with components $x_i = \frac{w_i - i}{m}$ for $i = 1, \ldots, m-1$.*

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Theorem (Rosales et. al, 2002)

Each numerical semigroup is one-to-one identified with its Kunz coordinates.
Furthermore, the set of Kunz coordinates of the numerical semigroups with multiplicity m is the set of solutions of the following system of diophantine inequalities:

$$\begin{array}{rcl} x_i \geqslant & 1 & \text{for all } i\in\{1,\ldots,m-1\},\\ x_i+x_j-x_{i+j}\geqslant & 0 & \text{for all } 1\leqslant i\leqslant j\leqslant m-1,\ i+j\leqslant m-1,\\ x_i+x_j-x_{i+j-m}\geqslant & -1 & \text{for all } 1\leqslant i\leqslant j\leqslant m-1,\ i+j>m \end{array}$$

Translating the problem: Kunz coordinates

- $\mathrm{m}(x) = \mathrm{m}(S) = m$ (Multiplicity of x.)
- $\mathrm{F}(x) = \mathrm{F}(S) = \max_i\{mx_i + i\} - m$ (Frobenius number)
- $\mathrm{G}(x) = \mathrm{G}(S) = \{n \in \mathbb{Z} : mx_{n \pmod m} + n \pmod m > n\}$ (Gaps of x.)
- $\mathrm{g}(x) = \mathrm{g}(S) = \displaystyle\sum_{i=1}^{m-1} x_i$.(Genus of x.)
- $\mathrm{SG}_m(x) = \mathrm{SG}_m(S)$.(Special Gaps greater than m of x.)
- $\mathcal{U}_m(x) = \mathcal{O}_m(S)$. (Undercoordinates of x: $S \subseteq S' \iff x \geq x'$)

Algorithm 1: Computing the special gaps greater than the multiplicity of a Kunz coordinate.

Input : A Kunz coordinate $x \in \mathbb{Z}_+^{m-1}$.
Compute $M_1 = \{m(x_i - 1) + i : x_i + x_j > x_{i+j},$ for all j with $i + j < m\}$ and $M_2 = \{m(x_i - 1) + i : x_i + x_j > x_{i+j-m} - 1,$ for all j with $i + j > m\}$.

Output: $\mathrm{SG}_m(x) = \{z \in M_1 \cap M_2 : z > m$ and $2z \geq m\, x_{2z \pmod m} + 2z \pmod m)\}$.

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Algorithm 2: Computing the special gaps greater than the multiplicity of a Kunz coordinate.

Input : A Kunz coordinate $x \in \mathbb{Z}_+^{m-1}$.
Compute $M_1 = \{m(x_i - 1) + i : x_i + x_j > x_{i+j},$ for all j with $i + j < m\}$ and
$M_2 = \{m(x_i - 1) + i : x_i + x_j > x_{i+j-m} - 1,$ for all j with $i + j > m\}$.

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m-irreducible numerical semigroup $\Rightarrow$ ***m-irreducible Kunz coordinates***
$(\sum_{i=1}^{m} x_i \in \{m, m-1, \left\lceil \frac{\mathrm{F}(x)+1}{2} \right\rceil)$

Corollary

The set of Kunz coordinates in $\mathbb{Z}_+^{m-1}$ with genus g and Frobenius number F is the set of solutions of the following system of diophantine inequalities:

$$
\begin{aligned}
x_i &\geqslant 1 && \text{for all } i \in \{1, \dots, m-1\},\\
x_i + x_j - x_{i+j} &\geqslant 0 && \text{for all } 1 \leqslant i \leqslant j \leqslant m-1,\ i+j \leqslant m-1,\\
x_i + x_j - x_{i+j-m} &\geqslant -1 && \text{for all } 1 \leqslant i \leqslant j \leqslant m-1,\ i+j > m,\\
\sum_{i=1}^{m-1} x_i &= g && \\
F = \max_i\{m x_i + i\} - m && && ,\\
x_i &\in \mathbb{Z} && \text{for all } i \in \{1, \dots, m-1\},
\end{aligned}
$$

Translating the problem: Kunz coordinates

If x is a Kunz coordinate, the set of m-irreducible undercoordinates of x are those $x' \in \mathbb{Z}^{m-1}$ in the form $x' = x - y$ with $y \in \mathbb{Z}_+^{m-1}$, i.e., y verifying the following inequalities:

$$
\begin{array}{ll}
y_i \leqslant x_i - 1 & \text{for all } i \in \{1, \ldots, m-1\}, \\
y_i + y_j - y_{i+j} \leqslant x_i + x_j - x_{i+j} & \text{for all } 1 \leqslant i \leqslant j \leqslant m-1,\ i+j \leqslant m-1, \\
y_i + y_j - y_{i+j} \leqslant x_i + x_j - x_{i+j} + 1 & \text{for all } 1 \leqslant i \leqslant j \leqslant m-1,\ i+j > m \\
\displaystyle\sum_{i=1}^{m-1} y_i \in M(x,y) &
\end{array}
\tag{$\mathrm{P^m(x)}$}
$$

where

$$M(x,y) = \{\sum_{i=1}^{m-1} x_i - m, \sum_{i=1}^{m-1} x_i - m + 1, \sum_{i=1}^{m-1} x_i - \left\lceil \frac{\max_i\{m(x_i - y_i) + i\} - m + 1}{2} \right\rceil\}$$

$$
\begin{array}{ll}
y_i \leqslant x_i - 1 & i = 1, \ldots, m-1, \\
y_i + y_j - y_{i+j} \leqslant x_i + x_j - x_{i+j} & i + j \leqslant m-1, \\
y_i + y_j - y_{i+j} \leqslant x_i + x_j - x_{i+j} + 1 & i + j > m \\
m(x_k - y_k) + k \geq m(x_i - y_i) + i & \forall i \\
2\sum_{i=1}^{m-1} y_i - m y_k \geqslant 2\sum_{i=1}^{m-1} x_i - m x_k - k + m - 2 & \\
2\sum_{i=1}^{m-1} y_i - m y_k \leqslant 2\sum_{i=1}^{m-1} x_i - m x_k - k + m - 1 &
\end{array}
\qquad (\mathrm{P}_k^m(x))
$$

$k = 1, \ldots, m-1$, and

$$
\begin{array}{ll}
y_i \leqslant x_i - 1 & i = 1, \ldots, m-1, \\
y_i + y_j - y_{i+j} \leqslant x_i + x_j - x_{i+j} & i + j \leqslant m-1, \\
y_i + y_j - y_{i+j} \leqslant x_i + x_j - x_{i+j} + 1 & i + j > m \\
\sum_{i=1}^{m-1} y_i = \sum_{i=1}^{m-1} x_i - m &
\end{array}
\qquad (\mathrm{P}_m^m(x))
$$

Solving the above problems, we obtain a decomposition into m-irreducibles... but clearly, it is not minimal.

Denote by $\mathcal{I}_m(x)$ the maximal elements (w.r.t $\leq$) in the set $\mathcal{O}_m(x)$.

Theorem

Let $x \in \mathbb{Z}^{m-1}$ a Kunz coordinates. The elements $\mathcal{I}_m(x)$ are in the form $x - \hat{y}$ where $\hat{y}$ is a nondominated solution of the any of the following multiobjective linear integer programming problems.

$$\begin{aligned} & v - \min(y_1, \ldots, y_{m-1}) \\ & s.t. \\ & \qquad\qquad y \in \mathrm{P}_k(x) \end{aligned} \tag{$\mathrm{MIP}_k(x)$}$$

for $k = 1, \ldots, m-1, m$.

Theorem

Let x be a Kunz coordinate. Then, the elements in a minimal decomposition into m-irreducible Kunz coordinates can be found by solving the following problems:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m-1} y_i \\ s.t. \quad & \\ & y \in \mathrm{P}_k(x) \\ & m y_k \leq m x_k + k - h + 1 \end{aligned} \tag{$\mathrm{IP}_k(x,h)$}$$

where $k = h \pmod{m}$*.*

$$\begin{aligned} \min \quad & \sum_{i=1}^{m-1} y_i \\ s.t. \quad & \\ & y \in \mathrm{P}_m(x) \quad m y_k \leq m x_k + k - h + 1 \end{aligned} \tag{$\mathrm{IP}_m(x,h)$}$$

Corollary

For each $h \in \mathrm{SG}_m(x)$*, it is enough to solve* $\mathrm{IP}_{h \pmod m}(x, h)$ *if* $h > 2m$ *or* $\mathrm{IP}_m(x, h)$ *if* $h < 2m$*.*
Then, at most $\#\mathrm{SG}(S)$ *(*$\leq m - 1$*) problems must be solved.*

Discarding Solutions: Set Covering

Let x be a Kunz coordinates, $s = \#\mathrm{SG}(x)$, and $\{x^1, \dots, x^s\}$ a set of m-irreducible coordinates decompose in x (solutions of $\mathrm{IP}_k(x, h)$ for each $h \in \mathrm{SG}(x)$).
We consider s decision variables

$$z_i = \begin{cases} 1 & \text{if } x^i \text{ is selected for the minimal decomposition,} \\ 0 & \text{otherwise.} \end{cases}$$

We formulate the problem of selecting a minimal set of m-irreducible Kunz coordinates as

$$\begin{aligned} \min \quad & \sum_{i=1}^{s} z_i \\ s.t. \quad & \\ & \sum_{i/ m x^i_k + k \geq h+1} z_i \geq 1 \quad, \forall h \in \mathrm{SG}(S), k = h \pmod{m}, \end{aligned} \tag{SC(x)}$$

Algorithm 3: Decomposition into m-irreducible numerical semigroups.

Input : A numerical semigroup S with multiplicity m.

Compute the Kunz coordinates of S: $x \in \mathbb{Z}_+^{m-1}$. (Computing the Apéry set.)

$\mathrm{D} = \{\}$. $\mathrm{DmIRNS} = \{\}$

1. Compute $\mathrm{SG}_m(x)$.
2. **for** $h \in \mathrm{SG}_m(x)$ *with* $h = k \pmod{m}$ **do**
   - **if** $k = h - m$ **then**
     - Solve $\mathrm{P}_m(x, h)$: $\hat{y}$. Add $x - \hat{y}$ to D
   - **else**
     - Solve $\mathrm{P}_k(x, h)$: $\hat{y}$. Add $x - \hat{y}$ to D.
3. Select a minimal decomposition from $\mathrm{D}\}$: Solve $(\mathrm{SC}(x))$.
   $\mathrm{DmIR} = \{x' \in \mathrm{D} : z = 1\}$

for $x' \in \mathrm{DmIR}$ **do**
- $S' = \langle \{m\} \cup \{m x'_i + i : i = 1, \ldots, m-1\} \rangle$
- Add S' to DmIRNS.

Output: DmIRNS.

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Discarding: No solution are discarded because S is not 3-irreducible ($\mathrm{g}(S) = 14$ and $\mathrm{F}(S) = 23$).

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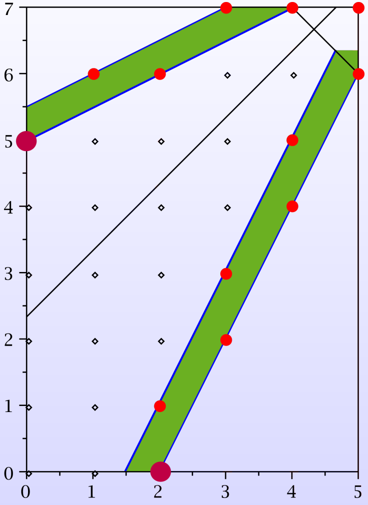
Minimal Decomposition into 3-irreducibles:
$\langle 3, 19, 26 \rangle = \langle 3, 11, 19 \rangle \cap \langle 3, 13 \rangle$



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