ALGEBRA SCHEMES AND THEIR REPRESENTATIONS

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Introduction

The equivalence (*Cartier duality*) between the category of topologically flat formal k-groups and the category of flat bialgebras has been treated as a duality of continuous vector spaces (of functions) [G, Exposé VII_B by P. Gabriel, 2.2.1]. This is owing to the fact that the reflexivity of vector spaces of infinite dimension does not hold if one does not provide them with a certain topology and does not consider the continuous dual. In this paper we obtain this duality without providing the vector spaces of functions with a topology.

Let R be a commutative ring with unit. It is natural to consider R-modules as $\mathcal{R}$-module functors in the following way: if M is an R-module, let $\mathcal{M}$ be the $\mathcal{R}$-module functor defined by $\mathcal{M}(S) := M \otimes_R S$ for every R-algebra S which belongs to the category $\mathcal{C}_R$ of R-algebras. Now, if $\mathbb{M}$ is a functor of $\mathcal{R}$-modules, its dual $\mathbb{M}^*$ can be defined in a natural way as the functor of $\mathcal{R}$-modules defined $\mathbb{M}^*(S) := \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathcal{S})$. In this work we will prove that the functor defined by an R-module is reflexive: $\mathcal{M} \tilde{\to} \mathcal{M}^{**}$, even in the case of R being a ring.

We call the functors $\mathcal{M}^*$ $\mathcal{R}$-module schemes and if they are $\mathcal{R}$-algebra functors too, we will say they are $\mathcal{R}$-algebra schemes. In section 2 we study and characterize the vector space schemes (2.3, 2.17) and we characterize when the module scheme closure of an $\mathcal{R}$-module functor $\mathbb{M}$ is equal to $\mathbb{M}^{**}$ (2.8, 2.9).

P. Gabriel [G, Exposé VII_B, 1.3.5] proved that the category of topologically flat formal R-varieties is equivalent to to the category of flat cocommutative R-coalgebras, where R is a pseudocompact ring. We prove (4.2) that the category of $\mathcal{R}$-algebra schemes is equivalent to the category of R-coalgebras, where R is a ring.

From this perspective, on the theory of algebraic groups and their representations $\mathcal{R}$-module schemes appear in a necessary way, as also do $\mathcal{R}$-algebra schemes as linear envelopes of groups. Let $G = \mathrm{Spec}\, A$ be an R-group and let $G^{\cdot}$ be the functor of points of G, i.e., $G^{\cdot}(S) = \mathrm{Hom}_{R-\mathrm{sch}}(\mathrm{Spec}\, S, G)$ for all $S \in \mathcal{C}_R$, and let $\mathcal{R}G^{\cdot}$ be the "linear envelope of $G^{\cdot}$" (see section 3). We prove that the $\mathcal{R}$-algebra scheme closure of $\mathcal{R}G^{\cdot}$ is the $\mathcal{R}$-algebra scheme $\mathcal{A}^*$ (3.3, 5.4) and the category of G-modules is equal to the category of $\mathcal{A}^*$-modules (5.5). So, the theory of linear representations of a group $G = \mathrm{Spec}\, A$ is a particular case of the theory of $\mathcal{A}^*$-modules (5.7, 5.8, 6.4, etc). Moreover, there is a bijective correspondence between the R-rational points of $\mathcal{A}^*$ and the multiplicative characters of G (5.6). When R is an algebraically closed field and G is smooth we prove that the completion of $\mathcal{R}G^{\cdot}$ by its ideal functors of finite codimension is also $\mathcal{A}^*$ (3.5, 5.9).

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Finally we prove that every $\mathcal{R}$-algebra scheme $\mathcal{A}^*$ is an inverse limit of finite $\mathcal{R}$-algebra schemes (4.12). We characterize the separable algebra schemes (7.4) and we prove the theorem of Wedderburn-Malcev (8.8) in the context of algebra schemes.

This paper is essentially self-contained.

1. $\mathcal{R}$-module schemes. Reflexivity theorem.

Let R be a commutative ring with unit, let $\mathcal{C}_R$ be the category of commutative R-algebras and let $\mathcal{R} : \mathcal{C}_R \to \mathcal{C}_R$ be the algebra functor that assigns the R-algebra $\mathcal{R}(S) := S$ to S. Let $\mathcal{C}_{Ab}$ be the category of commutative groups.

Definition 1.1. *A functor $\mathbb{M} : \mathcal{C}_R \to \mathcal{C}_{Ab}$ with a morphism of functors $\mathcal{R} \times \mathbb{M} \overset{\theta}{\to} \mathbb{M}$ is said to be an $\mathcal{R}$-module functor if $\mathbb{M}(S)$ with the morphism $S \times \mathbb{M}(S) \overset{\theta(S)}{\to} \mathbb{M}(S)$ is an S-module for each $S \in \mathcal{C}_R$.*

Given an R-module M, the functor $\mathcal{M}$ defined by $\mathcal{M}(S) := M \otimes_R S$ is an $\mathcal{R}$-module functor.

Unless otherwise stated, we assume that all functors considered in this article are functors from the category $\mathcal{C}_R$ to another one.

Definition 1.2. *Given a pair of $\mathcal{R}$-module functors $\mathbb{M}$ and $\mathbb{M}'$, we will denote by $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ the functor of all $\mathcal{R}$-linear morphisms from $\mathbb{M}$ to $\mathbb{M}'$, i.e.,*

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')(S) = \operatorname{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$$

where $\mathbb{M}_{|S}$ denotes the functor $\mathbb{M}$ restricted to the category of commutative S-algebras $\mathcal{C}_S$. An element of $\operatorname{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$ consists of assigning a morphism of T-modules $\mathbb{M}(T) \to \mathbb{M}'(T)$ to each S-algebra T.

We denote by $\mathbb{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ the dual functor of $\mathbb{M}$[1].

Proposition 1.3. *For every $\mathcal{R}$-module functor $\mathbb{M}$ and every R-module M, it holds that*

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}) = \operatorname{Hom}_R(M, \mathbb{M}(R))$$

Proof. Given an $\mathcal{R}$-linear morphism $f : \mathcal{M} \to \mathbb{M}$, we have for every R-algebra S a morphism of S-modules $f_S : M \otimes_R S \to \mathbb{M}(S)$ and a commutative diagram

$$\begin{array}{ccc} M \otimes_R S & \xrightarrow{f_S} & \mathbb{M}(S) \\ \uparrow & & \uparrow \\ M & \xrightarrow{f_R} & \mathbb{M}(R) \end{array}$$

Hence, the morphism of S-modules f_S is determined by f_R. □

Lemma 1.4. *Let S be an R-algebra, let M be an R-module and let $\mathbb{M}$, $\mathbb{M}'$ be $\mathcal{R}$-module functors. Then*

1. *$\mathcal{M}_{|S}$ is the functor associated to $M \otimes_R S$ on $\mathcal{C}_S$.*
2. *$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')_{|S} = \mathbb{H}om_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$.*

[1] If $\mathbb{M} = \mathcal{M}$ or $\mathbb{M} = \mathcal{M}^*$ then $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')(S)$ is a set (see 1.3, 1.6 and Yoneda's lemma) and $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}')$ is a functor. When we write $\mathbb{M}^*$ or $\mathbb{M}^{**}$ we will suppose that they are well-defined functors. However, for any $\mathbb{M}$ and $\mathbb{M}'$, in order for $\operatorname{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S})$ to be a set (it will be necessary in 2.2, 2.3 and 8.1), instead of taking into account the category of commutative algebras, we consider an infinite set X and the category of commutative algebras whose cardinal is less than or equal to $\operatorname{card}(X^{\mathbb{N}})$. See [D, General conventions].

Definition 1.5. *Given a commutative R-algebra A, we define the functor* $(\operatorname{Spec} A)^{\cdot}$ *to be* $(\operatorname{Spec} A)^{\cdot}(S) = \operatorname{Hom}_{\mathrm{R-alg}}(A, S)$ *for each commutative R-algebra S. This functor will be called the functor of points of* $\operatorname{Spec} A$.

By Yoneda's lemma (see [E, Appendix A5.3]*),* $\operatorname{Hom}_{\mathrm{func}}((\operatorname{Spec} A)^{\cdot}, \mathbb{M}) = \mathbb{M}(A)$.

Given an R-module M, we will denote by $S^{\cdot}_R M$ the symmetric algebra of M. Let us recall the next well-known lemma (see [D, II, §1, 2.1] or [G, Exposé VII_B, 1.2.4]).

Lemma 1.6. *If M is an R-module, then $\mathcal{M}^* = (\operatorname{Spec} S^{\cdot}_R M)^{\cdot}$ as $\mathcal{R}$-module functors.*

Proof. For every R-algebra S, it holds that

$$\mathcal{M}^*(S) = \operatorname{Hom}_{\mathcal{S}}(\mathcal{M}_{|S}, \mathcal{R}_{|S}) \overset{1.4}{=} \operatorname{Hom}_{\mathcal{S}}(\mathcal{M} \otimes_{\mathcal{R}} \mathcal{S}, \mathcal{S}) \overset{1.3}{=} \operatorname{Hom}_S(M \otimes_R S, S) =$$
$$= \operatorname{Hom}_R(M, S) = \operatorname{Hom}_{\mathrm{R-alg}}(S^{\cdot}_R M, S) = (\operatorname{Spec} S^{\cdot}_R M)^{\cdot}(S)$$

□

Definition 1.7. *The tensorial product of two functors $\mathbb{M}$, $\mathbb{N}$ in the category of $\mathcal{R}$-module functors is defined to be* $(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{N})(S) = \mathbb{M}(S) \otimes_S \mathbb{N}(S)$.

Proposition 1.8. *Let M, M' be R-modules. Then*

$$\mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = \mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}'$$

Proof. We know that $\mathcal{M}^*$ is represented by $\operatorname{Spec} S^{\cdot}_R M$, therefore

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') \subseteq \operatorname{Hom}_{\mathrm{func}}(\mathcal{M}^*, \mathcal{M}') = \mathcal{M}'(S^{\cdot}_R M) = S^{\cdot}_R M \otimes_R M'$$

However, in order for $w \in S^{\cdot}_R M \otimes_R M'$ to be a linear application, it must be $w \in M \otimes_R M'$. Hence, $\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = M \otimes_R M'$.

For every R-algebra S, we have that

$$\mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}')(S) = \operatorname{Hom}_S(\mathcal{M}^*{}_{|S}, \mathcal{M}'{}_{|S}) = \operatorname{Hom}_S((\mathcal{M} \otimes_{\mathcal{R}} \mathcal{S})^*, \mathcal{M}' \otimes_{\mathcal{R}} \mathcal{S})$$
$$= (M \otimes_R S) \otimes_S (M' \otimes_R S) = (\mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}')(S)$$

□

Remark 1.9. *If $\mathcal{C}_R$ is the category of commutative R-algebras whose cardinal is less than or equal to* $\operatorname{card}(X^{\mathbb{N}}) = m$, *then we have to suppose that $S^{\cdot}_R M \in \mathcal{C}_R$, i.e., that* $\operatorname{card} S^{\cdot}_R M \leq m$. *If M is a free R-module of whichever cardinal, we obtain the proposition again: Let $\{M_i\}$ be the set of quotients of M whose cardinal is less than or equal to m. It is easily seen that $\mathcal{M}^* = \varinjlim_i \mathcal{M}_i^*$. Then*

$$\mathbb{H}om_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}') = \mathbb{H}om_{\mathcal{R}}(\varinjlim_i \mathcal{M}_i^*, \mathcal{M}') = \varprojlim_i (\mathcal{M}_i \otimes_{\mathcal{R}} \mathcal{M}') \overset{*}{=} \mathcal{M} \otimes_{\mathcal{R}} \mathcal{M}'$$

where $\overset{*}{=}$ is a consequence of the equality $M \otimes_R M' \otimes_R S = \varprojlim_i (M_i \otimes_R M' \otimes_R S)$ for every R-algebra $S \in \mathcal{C}_R$. Even more, we can assume M is a projective R-module, i.e., a direct sum of a free R-module.

As a corollary we obtain the following

Theorem 1.10. *Let M be an R-module. Then*

$$\mathcal{M}^{**} = \mathcal{M}$$

Definition 1.11. *Quasi-coherent $\mathcal{R}$-modules are defined to be $\mathcal{R}$-module functors of the type $\mathcal{M}$, where M is any R-module. We shall say that $\mathcal{M}$ is a coherent $\mathcal{R}$-module if M is a finitely generated R-module.*

$\mathcal{R}$-module schemes are defined to be $\mathcal{R}$-module functors of the type $\mathcal{M}^*$.

If M is a free finitely generated R-module then $\mathcal{M}$ is a quasi-coherent $\mathcal{R}$-module and an $\mathcal{R}$-module scheme.

Theorem 1.12. *The category of quasi-coherent modules over $\mathcal{R}$ is equivalent to the category of R-modules. The category of quasi-coherent modules over $\mathcal{R}$ is anti-equivalent to the category of $\mathcal{R}$-module schemes (the correspondence is established by taking the dual functor).*

In [G, Exposé VII_B, 1.2.3], the anti-equivalence between the category of flat R-modules and the category of projective pseudocompact R-modules is established, where R is a (commutative) pseudocompact ring.

Proposition 1.13. *The $\mathcal{R}$-linear morphism $\mathcal{M} \to \mathcal{M}'$ is surjective, in the category of $\mathcal{R}$-module functors, if and only if the morphism $\mathcal{M}'^* \to \mathcal{M}^*$ is injective, in the category of $\mathcal{R}$-module functors.*

Proof. It follows immediately that if the morphism $\mathcal{M} \to \mathcal{M}'$ is surjective, then the morphism $\mathcal{M}'^* \to \mathcal{M}^*$ is injective. Inversely, let us suppose the morphism $\mathcal{M}'^* \to \mathcal{M}^*$ is injective. If V is the cokernel of the morphism $M \to M'$, we obtain $\mathcal{V}^* = 0$. Hence $\mathcal{V} = \mathcal{V}^{**} = 0$ and the morphism $\mathcal{M} \to \mathcal{M}'$ is surjective. □

If a morphism $\mathcal{M}'^* \to \mathcal{M}^*$ is surjective then the associated morphism $M \to M'$ is injective and it has a retraction. Let us consider the R-algebra $S := R \oplus M$, where $e_1 \cdot e_2 = 0$ for all $e_1, e_2 \in M$. Let $w \in \mathcal{M}^*(S) = \mathrm{Hom}_R(M, S)$ be defined by $w(e) := e$. Then, there exists a $w' \in \mathrm{Hom}_R(M', S)$ such that $w'(e) = e$ for all $e \in M$. If $\pi : S \to M$ is the natural projection, then $\pi \circ w'$ is a retraction of the morphism $M \to M'$.

Let us recall the Formula of adjoint functors.

Definition 1.14. *Let us consider the inclusion of categories*

$$\mathcal{C}_R = \{\textit{commutative } R\textit{-algebras}\} \stackrel{i}{\supset} \mathcal{C}_S = \{\textit{commutative } S\textit{-algebras}\}$$

where S is an R-algebra. Given a functor $\mathbb{N}$ on $\mathcal{C}_S$ we define $(i_*\mathbb{N})(R') := \mathbb{N}(S \otimes_R R')$ for each object R' of $\mathcal{C}_R$. Given a functor $\mathbb{M}$ on $\mathcal{C}_R$ we define $(i^*\mathbb{M})(S') := \mathbb{M}(S')$ for each object S' of $\mathcal{C}_S$.

Let us give a direct proof of the following theorem, although it can be obtained from [B, 8.4,8.5] after many precisions and complicated technical terms.

Theorem 1.15 (Formula of adjoint functors)**.** *Let $\mathbb{M}$ be an $\mathcal{R}$-module functor and let $\mathbb{N}$ be an $\mathcal{S}$-module functor. Then it holds that*

$$\mathrm{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N}) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$$

Proof. Given a $w \in \mathrm{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$, we have morphisms $w_{S\otimes R'} : \mathbb{M}(S \otimes R') \to \mathbb{N}(S \otimes R')$ for each R-algebra R'. By composition with the morphisms $\mathbb{M}(R') \to \mathbb{M}(S \otimes R')$, we have the morphisms $\phi_{R'} : \mathbb{M}(R') \to \mathbb{N}(S \otimes R') = i_*\mathbb{N}(R')$, which in their turn define $\phi \in \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$.

Given a $\phi \in \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$, we have morphisms $\phi_{S'} : \mathbb{M}(S') \to i_*\mathbb{N}(S') = \mathbb{N}(S \otimes S')$ for each S-algebra S'. By composition with the morphisms $\mathbb{N}(S \otimes S') \to$

$\mathbb{N}(S')$, we have the morphisms $w_{S'}: \mathbb{M}(S') \to \mathbb{N}(S')$, which in their turn define $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$.

Now we shall show that $w \mapsto \phi$ and $\phi \mapsto w$ are mutually inverse. Given $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$ we have $\phi \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$. Let us prove that the latter defines w again. We have the following diagram, where S' is an S-algebra,

$$\begin{array}{ccccc} \mathbb{M}(S') & \xrightarrow{i} & \mathbb{M}(S \otimes S') & \xrightarrow{w_{S\otimes S'}} & \mathbb{N}(S \otimes S') \\ & \searrow\!\!\searrow & \downarrow & & \downarrow p \\ & & \mathbb{M}(S') & \xrightarrow{w_{S'}} & \mathbb{N}(S') \end{array}$$

The composite morphism $p \circ w_{S\otimes S'} \circ i = p \circ \phi_{S'}$ is that assigned to ϕ, and coincides with $w_{S'}$ since the whole diagram is commutative.

Given $\phi \in \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, i_*\mathbb{N})$ we have $w \in \operatorname{Hom}_{\mathcal{S}}(i^*\mathbb{M}, \mathbb{N})$. Let us see that the latter defines ϕ. We have the following diagram, where R' is an R-algebra,

$$\begin{array}{ccccc} \mathbb{M}(R') & \xrightarrow{r} & \mathbb{M}(S \otimes R') & \xrightarrow{w_{S\otimes R'}} & \mathbb{N}(S \otimes R') \\ \downarrow \phi_{R'} & & \downarrow \phi_{S\otimes R'} & & \uparrow p \\ (i_*\mathbb{N})(R') & \xrightarrow{j} & (i_*\mathbb{N})(S \otimes R') & = & \mathbb{N}(S \otimes S \otimes R') \end{array}$$

The composite morphism $w_{S\otimes R'} \circ r$ assigned to w agrees with $\phi_{R'}$, since $p \circ j = Id$ and the whole diagram is commutative.

□

For simplicity of notation, given a functor $\mathbb{M}$ we will sometimes write $w \in \mathbb{M}$ instead of $w \in \mathbb{M}(S)$.

Proposition 1.16. *Let $\mathbb{M}_i$ be K-vector space functors and let M be a K-vector space. It holds that*

$$\mathbb{H}om_{\mathcal{K}}(\prod_i \mathbb{M}_i, \mathcal{M}) = \bigoplus_i \mathbb{H}om_{\mathcal{K}}(\mathbb{M}_i, \mathcal{M})$$

Proof. From the injective morphism $\bigoplus_i \mathbb{M}_i \xrightarrow{j} \prod_i \mathbb{M}_i$ one obtains the morphism

$$j^*: \mathbb{H}om_{\mathcal{K}}(\prod_i \mathbb{M}_i, \mathcal{M}) \to \mathbb{H}om_{\mathcal{K}}(\bigoplus_i \mathbb{M}_i, \mathcal{M}) = \prod_i \mathbb{H}om_{\mathcal{K}}(\mathbb{M}_i, \mathcal{M})$$

The aim is to prove that this morphism is injective and its image is $\bigoplus_i \mathbb{H}om_{\mathcal{K}}(\mathbb{M}_i, \mathcal{M})$.

For the first question, let $w \in \mathbb{H}om_{\mathcal{K}}(\prod_i \mathbb{M}_i, \mathcal{M})$ be a linear form such that $w \neq 0$ but $w_{|\bigoplus_i \mathbb{M}_i} = 0$. Then there exists a K-algebra S and elements $f_i \in \mathbb{M}_i(S)$ such that $w((f_i)_i) \neq 0$, and composing with the morphisms $\phi: \prod_i \mathcal{S} \to \prod_i \mathbb{M}_{i_{|S}}$, $\phi((s_i)_i) = (s_i f_i)_i$ we get a linear form $w \circ \phi \in \operatorname{Hom}_{\mathcal{S}}(\prod_i \mathcal{S}, \mathcal{M} \otimes \mathcal{S})$ that is not null but is null on $\bigoplus_i \mathcal{S}$, which is impossible since $\mathbb{H}om_{\mathcal{S}}(\prod_i \mathcal{S}, \mathcal{M} \otimes \mathcal{S}) = \mathbb{H}om_{\mathcal{S}}((\bigoplus_i \mathcal{S})^*, \mathcal{M} \otimes \mathcal{S}) \overset{1.8}{=} (\bigoplus_i \mathcal{S}) \otimes_{\mathcal{S}} (\mathcal{M} \otimes \mathcal{S}) = \bigoplus_i \mathcal{M} \otimes \mathcal{S}$.

To prove that $\mathrm{Im} j^* = \oplus_i \mathbb{H}om_{\mathcal{K}}(\mathbb{M}_i, \mathcal{M})$ it is enough to prove that $\mathrm{Hom}_{\mathcal{K}}(\prod_i \mathbb{M}_i, \mathcal{M}) = \oplus_i \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}_i, \mathcal{M})$, because in that case we will have

$$\mathrm{Hom}_{\mathcal{S}}(\prod_i \mathbb{M}_{i_{|S}}, \mathcal{M}_{|S}) \overset{1.15}{=} \mathrm{Hom}_{\mathcal{K}}(\prod_i \mathbb{M}_i, \mathcal{M} \otimes \mathcal{S}) = \oplus_i \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}_i, \mathcal{M} \otimes \mathcal{S})$$
$$\overset{1.15}{=} \oplus_i \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{i_{|S}}, \mathcal{M}_{|S})$$

Given a linear form $w \in \mathrm{Hom}_{\mathcal{K}}(\prod_i \mathbb{M}_i, \mathcal{M})$ we have to prove that there exists at most a finite subset of indices i such that $w_{|\mathbb{M}_i} \neq 0$. Let us suppose that this is not true, i.e., that there exists a set of indices i_n, where $n \in \mathbb{N}$, and K-algebras S_n such that $w(m_{i_n}) \neq 0$ for some $m_{i_n} \in \mathbb{M}_{i_n}(S_n)$. Let $S = \otimes_n S_n$ and denote by $h_r : S_r \hookrightarrow \otimes_n S_n$ the natural injections and by $\tilde{m_r}$ the image of m_{i_r} by the induced morphism $\mathbb{M}_{i_r}(h_r) : \mathbb{M}_{i_r}(S_r) \to \mathbb{M}_{i_r}(\otimes_n S_n)$. It is easy to see that $w(\tilde{m_r}) = h_r(w(m_{i_r})) \neq 0$. Therefore, we get a linear form $\bar{w} : \prod_n \mathcal{S} \to \mathcal{M} \otimes \mathcal{S}$, $\bar{w}((s_n)_n) := w((s_n \tilde{m}_n)_n)$, which is not null on any factor $\mathcal{S} \subset \prod_n \mathcal{S}$. Again this contradicts the fact that $\mathbb{H}om_{\mathcal{S}}(\prod_n \mathcal{S}, \mathcal{M} \otimes \mathcal{S}) = \oplus_n \mathcal{M} \otimes \mathcal{S}$. □

2. Characterizations of vector space schemes.

Let R be a commutative ring with unit and let K be a commutative field.

Definition 2.1. *An $\mathcal{R}$-module functor $\mathbb{M}$ is said to be reflexive if $\mathbb{M} = \mathbb{M}^{**}$.*

Theorem 2.2. *If $\mathbb{M}$ is a reflexive functor of $\mathcal{K}$-vector spaces such that $\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, -)$ commutes with direct sums, i.e.,*

$$\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \oplus_{i\in I} \mathbb{M}_i) = \oplus_{i\in I} \mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \mathbb{M}_i)$$

for all $\mathcal{K}$-vector space functors $\mathbb{M}_i$, then $\mathbb{M}$ is a $\mathcal{K}$-vector space scheme.

Proof. From the adjoint functor formula, given a commutative K-algebra S, we have that

$$\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{S}) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, i_* i^* \mathcal{K}) = \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathcal{S}) = \mathbb{M}^*(S)$$

However, $\mathcal{S} = \oplus_{i\in I} \mathcal{K}$ and the property that $\mathbb{M}$ satisfies by hypothesis means that $\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{S}) = \mathbb{M}^*(K) \otimes S$. Hence, $\mathbb{M}^*(S) = \mathbb{M}^*(K) \otimes_K S$ and $\mathbb{M}^* = \mathcal{M}$, where $M = \mathbb{M}^*(K)$, and therefore $\mathbb{M} = \mathbb{M}^{**} = \mathcal{M}^*$. □

We can now rephrase this result in terms of direct limits. The definition that we work with is taken from [E, Appendix 6].

Theorem 2.3. *Let $\mathbb{M}$ be a reflexive $\mathcal{K}$-vector space functor. The functor on the category of quasi-coherent $\mathcal{K}$-vector spaces, $\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, -)$, commutes with direct limits if and only if $\mathbb{M}$ is a $\mathcal{K}$-vector space scheme.*

Proof. The necessary condition is a consequence of the previous theorem, since it was only necessary that $\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, -)$ commuted with direct sums of quasi-coherent vector spaces for $\mathbb{M}$ to be a $\mathcal{K}$-vector space scheme.

The sufficient condition is obtained as an immediate consequence of Proposition 1.8, since the functor $\lim\limits_{\overrightarrow{i\in I}} \mathcal{M}_i$ is again a quasi-coherent $\mathcal{K}$-vector space and

$$\mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \lim_{\overrightarrow{i\in I}} \mathcal{M}_i) \overset{1.8}{=} \mathbb{M}^* \otimes (\lim_{\overrightarrow{i\in I}} \mathcal{M}_i) = \lim_{\overrightarrow{i\in I}} (\mathbb{M}^* \otimes \mathcal{M}_i) \overset{1.8}{=} \lim_{\overrightarrow{i\in I}} \mathbb{H}om_{\mathcal{K}}(\mathbb{M}, \mathcal{M}_i)$$

□

Definition 2.4. *Given an $\mathcal{R}$-module functor $\mathbb{M}$, we shall say that $\bar{\mathbb{M}}$ is the $\mathcal{R}$-module scheme closure of $\mathbb{M}$ if $\bar{\mathbb{M}}$ is an $\mathcal{R}$-module scheme and*

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{M}^*) = \mathrm{Hom}_{\mathcal{R}}(\bar{\mathbb{M}}, \mathcal{M}^*)$$

for every R-module M.

As $\bar{\mathbb{M}}$ is defined to be the representant on the category of $\mathcal{R}$-module schemes of the functor $\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, -)$ it is unique up to isomorphisms, and there exists a canonical morphism $\mathbb{M} \to \bar{\mathbb{M}}$ corresponding to the identity morphism $\bar{\mathbb{M}} \to \bar{\mathbb{M}}$.

Notation 2.5. *We mean by $\mathbb{M}^*(\mathcal{R})$ the quasi-coherent $\mathcal{R}$-module corresponding to the R-module $\mathbb{M}^*(R)$, i.e., $\mathbb{M}^*(\mathcal{R})(S) = \mathbb{M}^*(R) \otimes_R S$.*

Lemma 2.6. *Let $\mathbb{M}$, $\mathbb{N}$ be functors of $\mathcal{R}$-modules. Then*

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N}^*) = \mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*)$$

Proof.

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N}^*) = \mathbb{H}om_{\mathcal{R}}(\mathbb{M} \otimes_{\mathcal{R}} \mathbb{N}, \mathcal{R}) = \mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*)$$

□

Proposition 2.7. *Let $\mathbb{M}$ be an $\mathcal{R}$-module functor. It holds that $\bar{\mathbb{M}} = \mathbb{M}^*(\mathcal{R})^*$.*

Proof.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{M}^*) &\overset{2.6}{=} \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}^*) \overset{1.3}{=} \mathrm{Hom}_{R}(M, \mathbb{M}^*(R)) \\ &\overset{1.3}{=} \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}^*(\mathcal{R})) \overset{2.6}{=} \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}^*(\mathcal{R})^*, \mathcal{M}^*)\end{aligned}$$

□

Unfortunately, the $\mathcal{R}$-module scheme closure of an $\mathcal{R}$-module functor $\mathbb{M}$ is not stable under base change.

Proposition 2.8. *Let $\mathbb{M}$ be an $\mathcal{R}$-module functor. If $\mathbb{M}^*$ is a quasi-coherent $\mathcal{R}$-module then $\bar{\mathbb{M}} = \mathbb{M}^{**}$ and $\bar{\mathbb{M}}^* = \mathbb{M}^*$.*

Proof. If $\mathbb{M}^*$ is a quasi-coherent $\mathcal{R}$-module then

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{M}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{M}^*) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{M}^{**}, \mathcal{M}^*)$$

Therefore $\bar{\mathbb{M}} = \mathbb{M}^{**}$. Moreover, $\bar{\mathbb{M}}^* = (\mathbb{M}^{**})^* = (\mathbb{M}^*)^{**} = \mathbb{M}^*$. □

Proposition 2.9. *The $\mathcal{R}$-module scheme closure of an $\mathcal{R}$-module functor $\mathbb{M}$ is stable under base change if and only if $\mathbb{M}^*$ is a quasi-coherent $\mathcal{R}$-module.*

Proof. If $\bar{\mathbb{M}}_{|S} = \overline{\mathbb{M}_{|S}}$, then taking $\mathrm{Hom}_S(-, \mathcal{S})$ we obtain that $\mathbb{M}^*(S) = \mathbb{M}^*(R) \otimes_R S$. Inversely, if $\mathbb{M}^*$ is quasi-coherent then $\bar{\mathbb{M}}_{|S} = {\mathbb{M}^{**}}_{|S} = {\mathbb{M}_{|S}}^{**} = \overline{\mathbb{M}_{|S}}$. □

Example 2.10. *If $\mathbb{M}_1, \ldots, \mathbb{M}_n$ are $\mathcal{R}$-module functors whose duals are quasi-coherent $\mathcal{R}$-modules, then $(\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_n)^* = \mathbb{M}_1^* \otimes \ldots \otimes \mathbb{M}_n^*$, which in particular is a quasi-coherent $\mathcal{R}$-module:*

$$\begin{aligned} \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_n, \mathcal{R}) &= \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_{n-1}, \mathbb{M}_n^*) \\ &= \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_{n-2}, \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_{n-1}, \mathbb{M}_n^*)) \\ &= \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_{n-2}, \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_n^{**}, \mathbb{M}_{n-1}^*)) \\ &\overset{1.8}{=} \mathbb{H}om_{\mathcal{R}}(\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_{n-2}, \mathbb{M}_{n-1}^* \otimes \mathbb{M}_n^*) \\ &= \ldots = \mathbb{M}_1^* \otimes \ldots \otimes \mathbb{M}_n^* \end{aligned}$$

Hence, $\overline{\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_n} = (\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_n)^{**} = (\mathbb{M}_1^* \otimes \ldots \otimes \mathbb{M}_n^*)^*$ and $\overline{\mathbb{M}_1 \otimes \ldots \otimes \mathbb{M}_n} = \overline{\bar{\mathbb{M}}_1 \otimes \ldots \otimes \bar{\mathbb{M}}_n}$.

If we denote by $\bar{\otimes}$ the tensorial product in the category of $\mathcal{R}$-module schemes then $\mathcal{M}_1^* \bar{\otimes} \mathcal{M}_2^* = \overline{\mathcal{M}_1^* \otimes \mathcal{M}_2^*} = (\mathcal{M}_1 \otimes \mathcal{M}_2)^*$. Moreover, $\bar{\otimes}$ commutes with inverse limits:

$$\begin{aligned} (\lim_{\overleftarrow{i}} \mathcal{M}_i^*) \bar{\otimes} \mathcal{M}^* &= (\lim_{\overrightarrow{i}} \mathcal{M}_i)^* \bar{\otimes} \mathcal{M}^* = ((\lim_{\overrightarrow{i}} \mathcal{M}_i) \otimes \mathcal{M})^* = (\lim_{\overrightarrow{i}} (\mathcal{M}_i \otimes \mathcal{M}))^* \\ &= \lim_{\overleftarrow{i}} (\mathcal{M}_i \otimes \mathcal{M})^* = \lim_{\overleftarrow{i}} (\mathcal{M}_i^* \bar{\otimes} \mathcal{M}^*) \end{aligned}$$

Henceforward, we shall only work with functors of K-vector spaces.

Proposition 2.11. *The morphism $\mathbb{M} \to \mathbb{M}^{**}$ is injective if and only if the morphism $\mathbb{M} \to \bar{\mathbb{M}}$ is injective.*

Proof. Let us prove the necessary condition. Given $s \in \mathbb{M}(S)$ such that $s = 0$ in $\bar{\mathbb{M}}(S) = \mathbb{M}^*(\mathcal{K})^*(S) = \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}^*(\mathcal{K})_{|S}, \mathcal{S}) \overset{1.15}{=} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}^*(\mathcal{K}), \mathcal{S}) \overset{1.3}{=} \mathrm{Hom}_K(\mathbb{M}^*(K), S)$, then $s(w) := w(s) = 0$ for all $w \in \mathbb{M}^*(K)$.

Given a commutative S-algebra T, if one writes $T = \oplus K \cdot e_i$, one notices that

$$\mathbb{M}^*(T) = \mathrm{Hom}_{\mathcal{T}}(\mathbb{M}_{|T}, \mathcal{T}) \overset{1.15}{=} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{T}) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \oplus \mathcal{K}) \subset \prod \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{K})$$

which assigns to every $w_T \in \mathbb{M}^*(T)$ a $(w_i) \in \prod \mathbb{M}^*(K)$. Explicitly, given $t \in \mathbb{M}(T)$, then $w_T(t) = \sum_i w_i(t) \cdot e_i$. Therefore $w_T(s) = 0$ for all $w_T \in \mathbb{M}^*(T)$. Since the morphism $\mathbb{M} \hookrightarrow \mathbb{M}^{**}$ is injective, this means that $s = 0$, i.e., the morphism $\mathbb{M} \to \bar{\mathbb{M}}$ is injective.

For the sufficient condition, we consider the morphism $\mathbb{M}^*(\mathcal{K}) \to \mathbb{M}^*$, which, by taking duals, becomes $\mathbb{M}^{**} \to \mathbb{M}^*(\mathcal{K})^*$. Since the composite morphism $\mathbb{M} \to \mathbb{M}^{**} \to \bar{\mathbb{M}} = \mathbb{M}^*(\mathcal{K})^*$ is injective, so is the morphism $\mathbb{M} \to \mathbb{M}^{**}$. □

Lastly, we will show another characterization of $\mathcal{K}$-vector space schemes by means of complete reflexive functors. First, we need some technical results before Definition 2.14.

Proposition 2.12.

(1) *The morphism $\mathbb{M}^*(K) \to \mathbb{M}(K)^*$ is injective if and only if the morphism $\mathbb{M}^* \to \mathbb{M}(\mathcal{K})^*$ is injective.*
(2) *The morphism $\mathbb{M}^*(K) \to \mathbb{M}(K)^*$ is injective if and only if for every quasi-coherent vector space $\mathcal{V}$ the image of any $\mathcal{K}$-linear morphism $\mathbb{M} \to \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$.*

Proof.

(1) If the morphism $\mathbb{M}^* \to \mathbb{M}(\mathcal{K})^*$ is injective then taking sections on K the morphism $\mathbb{M}^*(\mathcal{K}) \to \mathbb{M}(\mathcal{K})^*$ is injective. Inversely, from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \oplus\mathcal{K}) & \to & \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}(\mathcal{K}), \oplus\mathcal{K}) \\ \cap & & \cap \\ \prod \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{K}) & \hookrightarrow & \prod \mathrm{Hom}_K(\mathbb{M}(K), K) \end{array}$$

one has that $\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \oplus\mathcal{K}) \subset \mathrm{Hom}_K(\mathbb{M}(K), \oplus K)$. Since $S = \oplus K$, then

$$\begin{aligned} \mathbb{M}^*(S) &= \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathcal{S}) \overset{1.15}{=} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \oplus\mathcal{K}) \subset \mathrm{Hom}_K(\mathbb{M}(K), \oplus K) \\ &= \mathrm{Hom}_S(\mathbb{M}(K) \otimes_K S, S) \overset{1.3}{=} \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}(\mathcal{K}) \otimes_{\mathcal{K}} \mathcal{S}, \mathcal{S}) \\ &\overset{1.4}{=} \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}(\mathcal{K})_{|S}, \mathcal{S}) = \mathbb{H}om_{\mathcal{K}}(\mathbb{M}(\mathcal{K}), \mathcal{K})(S) = \mathbb{M}(\mathcal{K})^*(S) \end{aligned}$$

i.e., the morphism $\mathbb{M}^* \to \mathbb{M}(\mathcal{K})^*$ is injective.

(2) Let us suppose that the image of any morphism $\mathbb{M} \to \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$. Given $w \in \mathbb{M}^*(K)$, i.e., a morphism $w : \mathbb{M} \to \mathcal{K}$, $\mathrm{Im}\, w$ is equal to the quasi-coherent vector space associated to $w(\mathbb{M}(K))$. Hence if $w(\mathbb{M}(K)) = 0$ then $w = 0$.

Inversely, let V' be the image of the morphism $\mathbb{M}(K) \to V$ and consider $\mathbb{M} \to \mathcal{W} := \mathcal{V}/\mathcal{V}'$. The morphism $\mathcal{W}^* \to \mathbb{M}(\mathcal{K})^*$ is null. Hence the morphism $\mathcal{W}^* \to \mathbb{M}^*$ is null, and the composite morphism $\mathbb{M} \to \mathbb{M}^{**} \to \mathcal{W}^{**} = \mathcal{W}$ is null. Therefore, the image of the morphism $\mathbb{M} \to \mathcal{V}$ is $\mathcal{V}'$.

□

Corollary 2.13. *Let $\mathbb{M}$ be a reflexive functor and let V be a K-vector space. Then the image of any $\mathcal{K}$-linear morphism $\mathbb{M} \to \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$.*

Proof. If $\mathbb{M} = \mathbb{M}^{**}$, by Proposition 2.11 the morphism $\mathbb{M}^* \to \overline{\mathbb{M}^*} = \mathbb{M}(\mathcal{K})^*$ is injective. Then by Proposition 2.12 the proof is complete. □

Definition 2.14. *Given a $\mathcal{K}$-vector space functor $\mathbb{M}$ such that the image of any $\mathcal{K}$-linear morphism $\mathbb{M} \to \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$, let us consider the $\mathcal{K}$-vector space subfunctors $\mathbb{M}_i \subset \mathbb{M}$ such that $\mathbb{M}/\mathbb{M}_i$ are coherent $\mathcal{K}$-vector spaces. Then we define $\hat{\mathbb{M}} := \lim\limits_{\overleftarrow{i}} \mathbb{M}/\mathbb{M}_i$.*

The direct limit of quasi-coherent vector spaces, in the category of $\mathcal{K}$-vector space functors, is a quasi-coherent vector space. Therefore, the inverse limit of $\mathcal{K}$-vector space schemes is a $\mathcal{K}$-vector space scheme. Hence $\hat{\mathbb{M}}$ is a $\mathcal{K}$-vector space scheme, namely, $\hat{\mathbb{M}} := (\lim\limits_{\overrightarrow{i}} (\mathbb{M}/\mathbb{M}_i)^*)^*$.

Proposition 2.15. *Let V be a K-vector space. Then, $\mathcal{V}^*$ is complete and separate, i.e., $\widehat{\mathcal{V}^*} = \mathcal{V}^*$.*

Proof. By the reflexivity theorem, the coherent cokernels of $\mathcal{V}^*$ correspond to the subspaces $V' \subset V$ of finite dimension. Hence,

$$\widehat{\mathcal{V}^*} = \lim_{\overleftarrow{\dim_K V' < \infty}} (\mathcal{V}')^* = (\lim_{\overrightarrow{\dim_K V' < \infty}} \mathcal{V}')^* = \mathcal{V}^*$$

□

Proposition 2.16. *Let $\mathbb{M}$ be a $\mathcal{K}$-vector space functor such that the image of any $\mathcal{K}$-linear morphism $\mathbb{M} \to \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$. Then the vector space closure of $\mathbb{M}$ is equal to the completion of $\mathbb{M}$, i.e., $\hat{\mathbb{M}} = \bar{\mathbb{M}}$.*

In particular, $\hat{\mathbb{M}} = \mathcal{V}^*$, where $\mathcal{V} = \mathbb{M}^*(\mathcal{K})$, and $\hat{\mathbb{M}}$ is complete, separate, and reflexive.

Proof. First, let us suppose that V is a finite-dimensional space. Observe that the dual of an inverse limit of K-vector space schemes is equal to the direct limit of the quasi-coherent dual vector spaces, $(\lim\limits_{\leftarrow i} \mathcal{V}_i^*)^* = (\lim\limits_{\rightarrow i} \mathcal{V}_i)^{**} = \lim\limits_{\rightarrow i} \mathcal{V}_i$, then

$$\mathrm{Hom}_{\mathcal{K}}(\hat{\mathbb{M}}, \mathcal{V}^*) = \mathrm{Hom}_{\mathcal{K}}(\lim_{\leftarrow i} \mathbb{M}/\mathbb{M}_i, \mathcal{V}^*) = \lim_{\rightarrow i} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}/\mathbb{M}_i, \mathcal{V}^*) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{V}^*)$$

In general, $\mathcal{V}^* = \lim\limits_{\leftarrow i} \mathcal{V}_i^*$, where $\dim_K V_i < \infty$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}(\hat{\mathbb{M}}, \mathcal{V}^*) &= \mathrm{Hom}_{\mathcal{K}}(\hat{\mathbb{M}}, \lim_{\leftarrow i} \mathcal{V}_i^*) = \lim_{\leftarrow i} \mathrm{Hom}_{\mathcal{K}}(\hat{\mathbb{M}}, \mathcal{V}_i^*) = \lim_{\leftarrow i} \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{V}_i^*) \\ &= \mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathcal{V}^*)\end{aligned}$$

Therefore, $\hat{\mathbb{M}} = \bar{\mathbb{M}}$. □

Theorem 2.17. *Let $\mathbb{M}$ be a reflexive K-vector space functor. Then $\mathbb{M}$ is a $\mathcal{K}$-vector space scheme if and only if $\mathbb{M}$ is complete and separate.*

3. Linearizations of varieties. Closure of linearizations.

Definition 3.1. *Let $X = \mathrm{Spec}\, A$ be an affine $\mathcal{R}$-scheme and let us denote by $X^{\cdot}$ the functor of points of X, i.e., $X^{\cdot}(S) = \mathrm{Hom}_{\mathrm{R-alg}}(A, S)$. Let $\mathcal{R}X^{\cdot}$ be the $\mathcal{R}$-module functor defined by $\mathcal{R}X^{\cdot}(S) := \underset{X^{\cdot}(S)}{\oplus} S = \{$ the formal finite S-linear combinations of points of X in S $\}$.*

It is clear that for every $\mathcal{R}$-module functor $\mathbb{M}$ it holds that

$$\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}X^{\cdot}, \mathbb{M}) = \mathrm{Hom}_{\mathrm{functors}}(X^{\cdot}, \mathbb{M})$$

Since every morphism of R-algebras $A \to S$ is in particular R-linear, we have a morphism of functors $\phi : X^{\cdot} \hookrightarrow \mathcal{A}^*$, where the morphism between schemes is given by the natural epimorphism of R-algebras $S^{\cdot}_R A \to A$. Then we have a morphism $\mathcal{R}X^{\cdot} \to \mathcal{A}^*$.

Notation 3.2. *It is usual the notation $X_S = \mathrm{Spec}\, A \times_R \mathrm{Spec}\, S = \mathrm{Spec}(A \otimes_R S)$ and $A_S = A \otimes_R S$.*

Theorem 3.3. *Let $X = \mathrm{Spec}\, A$ be an affine R-scheme. It holds that*

(1) $\mathcal{R}X^{\cdot *} = \mathcal{A}$
(2) $\overline{\mathcal{R}X^{\cdot}} = \mathcal{R}X^{\cdot **} = \mathcal{A}^*$

Proof.

$$\mathcal{R}X^{\cdot *}(R) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}X^{\cdot}, \mathcal{R}) = \mathrm{Hom}_{\mathrm{func}}(X^{\cdot}, \mathcal{R}) = A$$

and likewise

$$\mathcal{R}X^{\cdot *}(S) = \mathrm{Hom}_{\mathcal{S}}(\mathcal{R}X^{\cdot}_{|S}, \mathcal{S}) = \mathrm{Hom}_{\mathcal{S}}(\mathcal{S}X^{\cdot}_S, \mathcal{S}) = A_S = \mathcal{A}(S)$$

Hence, $\mathcal{R}X^{\cdot *} = \mathcal{A}$ and taking duals $\mathcal{A}^* = \mathcal{R}X^{\cdot **} \overset{2.8}{=} \overline{\mathcal{R}X^{\cdot}}$. □

Theorem 3.4. *If $X = \operatorname{Spec} A$ is an R-scheme and $\mathbb{M}$ is a reflexive $\mathcal{R}$-module functor, then the morphism*

$$\begin{array}{rcl} \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) & \to & \operatorname{Hom}_{\text{functors}}(X^{\cdot}, \mathbb{M}) \\ \mathcal{A}^* \to \mathbb{M} & \mapsto & X^{\cdot} \overset{\phi}{\hookrightarrow} \mathcal{A}^* \to \mathbb{M} \end{array}$$

is an isomorphism.

Moreover, if A is a free R-module such linear applications of functors are determined each by its value on global sections, i.e.,

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) \subset \operatorname{Hom}_R(A^*, \mathbb{M}(R))$$

Proof. Firstly, we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) &\overset{2.6}{=} \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}^*, \mathcal{A}) \overset{3.3}{=} \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}^*, \mathcal{R}X^{\cdot *}) \\ &\overset{2.6}{=} \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}X^{\cdot}, \mathbb{M}) = \operatorname{Hom}_{\text{func}}(X^{\cdot}, \mathbb{M}) \end{aligned}$$

which is the isomorphism to compose with ϕ.

Secondly, since $A = \oplus R \subseteq \prod R$ we get

$$\begin{aligned} \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) &\overset{2.6}{=} \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}^*, \mathcal{A}) \subseteq \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}^*, \prod \mathcal{R}) \\ &\overset{2.6}{=} \operatorname{Hom}_{\mathcal{R}}(\oplus \mathcal{R}, \mathbb{M}) \overset{1.3}{=} \operatorname{Hom}_R(\oplus R, \mathbb{M}(R)) \end{aligned}$$

Since the injective morphism $\operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) \hookrightarrow \operatorname{Hom}_R(\oplus R, \mathbb{M}(R))$ factors through $\operatorname{Hom}_R(A^*, \mathbb{M}(R))$, the morphism $\operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) \to \operatorname{Hom}_R(A^*, \mathbb{M}(R))$ is injective. □

Theorem 3.5. *Let us suppose that the only function $a \in A$ of the K-scheme $X = \operatorname{Spec} A$ that is null on every K-rational point is the zero function $a = 0$. Then, it holds that $\widehat{\mathcal{K}X^{\cdot}} = \mathcal{A}^*$.*

Proof. By hypothesis the morphism $\mathcal{K}X^{\cdot *}(K) = A \hookrightarrow (\mathcal{K}X^{\cdot})(K)^*$ is injective, hence we are under the hypothesis of Definition 2.14 and Proposition 2.12. Therefore, by Proposition 2.16 $\widehat{\mathcal{K}X^{\cdot}} = \overline{\mathcal{K}X^{\cdot}} \overset{3.3}{=} \mathcal{A}^*$. □

Maybe it is more natural the definition

$$\mathcal{R}X^{\cdot\prime}(S) := < \operatorname{Hom}_{\text{R-alg}}(A, S) >_S \subset \operatorname{Hom}_R(A, S)$$

i.e., $\mathcal{R}X^{\cdot\prime}$ is the image of $\mathcal{R}X^{\cdot}$ in $\mathcal{A}^*$.

Proposition 3.6. *It holds*

(1) $\operatorname{Hom}_{\mathcal{R}}(\mathcal{R}X^{\cdot\prime}, \mathbb{M}) = \operatorname{Hom}_{\text{func}}(X^{\cdot}, \mathbb{M})$ *for every reflexive functor* $\mathbb{M}$.
(2) $\mathcal{R}X^{\cdot\prime *} = \mathcal{A}$.
(3) $\overline{\mathcal{R}X^{\cdot\prime}} = \mathcal{A}^*$.
(4) *The minimum reflexive subfunctor of $\mathcal{A}^*$ that contains $\mathcal{R}X^{\cdot\prime}$ is $\mathcal{A}^*$.*

Proof.

(1) It is a consequence of the equalities

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}) \overset{3.4}{=} \operatorname{Hom}_{\text{func}}(X^{\cdot}, \mathbb{M}) = \operatorname{Hom}_{\mathcal{R}}(\mathcal{R}X^{\cdot}, \mathbb{M})$$

(2),(3) They are consequences of (1).

(4) Let us suppose we have morphisms $\mathcal{R}X^{\cdot\prime} \hookrightarrow \mathbb{M} \hookrightarrow \mathcal{A}^*$, where $\mathbb{M}$ is a reflexive functor. Taking double duals, we obtain that the composite morphism $\mathcal{A}^* \to \mathbb{M} \to \mathcal{A}^*$ is the identity morphism. Therefore, the morphism $\mathbb{M} \to \mathcal{A}^*$ is surjective and (4) follows.

□

4. Algebra schemes.

Definition 4.1. *We call an $\mathcal{R}$-module scheme $\mathcal{A}^*$ an $\mathcal{R}$-algebra scheme if it is also an $\mathcal{R}$-algebra functor (i.e., $\mathcal{A}^*(S)$ is a S-algebra and the morphisms $\mathcal{A}^*(S) \to \mathcal{A}^*(S')$ are morphisms of S-algebras for every morphism $S \to S'$ of R-algebras).*

Proposition 4.2. *The category of coalgebras with counit, $\mathcal{C}_{coalg}$, is anti-equivalent to the category of algebra schemes, $\mathcal{C}_{\mathbf{alg}}$. The functors which give the equivalence are $\mathcal{C}_{coalg} \to \mathcal{C}_{\mathbf{alg}}$, $B \rightsquigarrow \mathcal{B}^*$ and $\mathcal{C}_{\mathbf{alg}} \to \mathcal{C}_{coalg}$, $\mathcal{A}^* \rightsquigarrow A$.*

Proof. Observe that $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}_1^* \otimes \ldots \otimes \mathcal{M}_n^*, \mathcal{N}^*) \overset{2.6}{=} \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, (\mathcal{M}_1^* \otimes \ldots \otimes \mathcal{M}_n^*)^*) \overset{2.10}{=} \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{M}_1 \otimes \ldots \otimes \mathcal{M}_n)$.

Giving an $\mathcal{R}$-algebra functor structure on a scheme $\mathcal{A}^*$ is equivalent to giving the morphism of multiplication $\mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ and the unit $\mathcal{R} \hookrightarrow \mathcal{A}^*$, so that the diagrams that state distributive, associative and the like properties are commutative. This is equivalent to giving morphisms $A \to A \otimes A$, $A \to R$ which endow A with a coalgebra structure with counit. □

Notation 4.3. *From now on, in this and next sections, $\mathcal{A}^*$ denotes an $\mathcal{R}$-algebra scheme.*

Definition 4.4. *Let $\mathbb{M}$ be an $\mathcal{R}$-module functor and let $\mathbb{A}$ be an $\mathcal{R}$-algebra functor. We say that $\mathbb{M}$ is an $\mathbb{A}$-module if there exists a morphism of $\mathcal{R}$-algebra functors $\mathbb{A} \to \mathbb{E}nd_{\mathcal{R}}(\mathbb{M})$.*

We will say that an R-module M is an $\mathbb{A}$-module if $\mathcal{M}$ is an $\mathbb{A}$-module.

Giving a structure of $\mathcal{A}^*$-module on M is equivalent to the existence of a morphism $\mathcal{A}^* \otimes \mathcal{M} \to \mathcal{M}$ verifying the obvious properties, which is equivalent to the existence of a morphism $\mathcal{M} \to \mathcal{A} \otimes \mathcal{M}$ verifying the obvious properties, since

$$\begin{aligned}\mathrm{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{H}om_{\mathcal{R}}(\mathcal{M}, \mathcal{M})) &= \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}^* \otimes \mathcal{M}, \mathcal{M}) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathbb{H}om_{\mathcal{R}}(\mathcal{A}^*, \mathcal{M})) \\ &= \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{A} \otimes \mathcal{M}) = \mathrm{Hom}_R(M, A \otimes M)\end{aligned}$$

By these equivalences, if we have the morphism $M \to A \otimes M$, $m \mapsto \sum_i a_i \otimes m_i$, then $w \cdot m = \sum_i w(a_i) m_i$ given $w \in \mathcal{A}^*$. If w is the general linear form, i.e., $w = \mathrm{Id} \in \mathcal{A}^*(A) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{A})$, then $w \cdot m = \sum_i a_i \otimes m_i$.

If $\mathcal{A}^*$ is an algebra scheme, then A is in a natural way a right and left $\mathcal{A}^*$-module as it follows:

$$(w \cdot a)(w') := a(w' \cdot w)$$
$$(a \cdot w)(w') := a(w \cdot w')$$

where $a \in \mathcal{A}$, $w, w' \in \mathcal{A}^*$. We shall say that A is the regular $\mathcal{A}^*$-module.

Given an R-submodule $M' \subset M$ we will say that $\mathcal{M}' \to \mathcal{M}$ is a cuasicoherent submodule.

Lemma 4.5. *Let $M_1,\dots,M_n$ be projective R-modules and let M_0 be an R-module. The $\mathcal{R}$-linear morphism $T:\mathcal{M}_1^*\otimes_{\mathcal{R}}\dots\otimes_{\mathcal{R}}\mathcal{M}_n^*\to\mathcal{M}_0$ factors via an epimorphism onto a coherent submodule of $\mathcal{M}_0$.*

Proof. As $M_1,\dots,M_n$ are projective R-modules, they are direct summands of free modules $L_1,\dots,L_n$. Then, $\mathcal{M}_i^*$ is a direct summand of $\mathcal{L}_i^*$ and we can assume that $M_i=L_i$ are free modules.

By Proposition 1.8, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}_1^*\otimes_{\mathcal{R}}\dots\otimes_{\mathcal{R}}\mathcal{M}_n^*,\mathcal{M}_0)=M_1\otimes_R\dots\otimes_R M_n\otimes M_0$. Let $\{e_{ij}\}_i$ be a basis for M_j, for every j. Then for every $T\in\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}_1^*\otimes_{\mathcal{R}}\dots\otimes_{\mathcal{R}}\mathcal{M}_n^*,\mathcal{M}_0)$ we can write

$$T=\sum_{i_1,\dots,i_n} e_{i_1 1}\otimes\dots\otimes e_{i_n n}\otimes e_{i_1\dots i_n}$$

where only a finite number of the elements $e_{i_1\dots i_n}\in M_0$ are not null. It is easy to check that T factors via an epimorphism onto the image of the coherent $\mathcal{R}$-module associated to $M=<e_{i_1\dots i_n}>_{i_1\dots i_n}$. □

Proposition 4.6. *Let $\mathcal{A}^*$ be an $\mathcal{R}$-algebra scheme, let M be an $\mathcal{A}^*$-module and let $M'\subset M$ be an R-submodule. Let us suppose that A is a projective R-module. Then, M' is an $\mathcal{A}^*$-submodule of M if and only if M' is an A^*-submodule of M.*

Proof. Obviously, if M' is an $\mathcal{A}^*$-submodule of M then M' is an A^*-submodule of M. Inversely, let us suppose M' is an A^*-submodule of M and let us consider the natural morphism of multiplication $\mathcal{A}^*\otimes\mathcal{M}'\to\mathcal{M}$. By the previous lemma the morphisms $\mathcal{A}^*\to\mathcal{M}$, $w\mapsto w\cdot m'$, for each $m'\in M'$, factors via $\mathcal{M}'$, then $\mathcal{A}^*\otimes\mathcal{M}'=\mathcal{A}^*\otimes M'\to\mathcal{M}$ factors via $\mathcal{M}'$, which proves that M' is an $\mathcal{A}^*$-submodule of M. □

Proposition 4.7. *Let $\mathcal{A}^*$ be an $\mathcal{R}$-algebra scheme and let M be an $\mathcal{A}^*$-module (respectively a right and left $\mathcal{A}^*$-module). Let us suppose A is a projective R-module. Every finitely generated R-submodule of M is included in an $\mathcal{A}^*$-submodule of M (respectively a right and left $\mathcal{A}^*$-module) that is a finitely generated R-submodule.*

Proof. Given a finitely generated R-module $M'\subset M$ then $A^*\cdot M'$ (respectively $A^*\cdot M'\cdot A^*$), the obvious image of the morphism $A^*\otimes M'\to M$ (respectively $A^*\otimes M'\otimes A^*\to M$), is an $\mathcal{A}^*$-submodule (respectively a right and left $\mathcal{A}^*$-submodule) of M that is a finitely generated R-module. □

Remark 4.8. *In particular, an $\mathcal{A}^*$-module M is a K-vector space of finite dimension if and only if is a finitely-generated $\mathcal{A}^*$-module, i.e., there exists an epimorphism of $\mathcal{A}^*$-modules $\mathcal{A}^*\oplus\overset{n}{\dots}\oplus\mathcal{A}^*\to\mathcal{M}$.*

Definition 4.9. *Let $\mathcal{A}^*$ be an $\mathcal{R}$-algebra scheme. We will say that a submodule scheme $\mathcal{I}^*\subseteq\mathcal{A}^*$ is an ideal scheme if it is an ideal subfunctor. We will say that $\mathcal{I}^*\subseteq\mathcal{A}^*$ is a bilateral ideal scheme if it is a bilateral ideal subfunctor.*

The kernel of a morphism of algebra schemes is a bilateral ideal scheme.

Definition 4.10. *Given a finite R-algebra B, we will say that $\mathcal{B}$ is a coherent $\mathcal{R}$-algebra.*

Remark 4.11. *Owing to the categorial equivalence between the category of R-modules and the category of quasi-coherent $\mathcal{R}$-modules, there is an obvious categorial equivalence between finite R-algebras and coherent $\mathcal{R}$-algebras.*

Proposition 4.12. *Let $\mathcal{A}^*$ be an $\mathcal{R}$-algebra scheme. Let us suppose A is a projective R-module. Then $\mathcal{A}^*$ is an inverse limit of quotients $\mathcal{B}_i$, which are coherent $\mathcal{R}$-algebras.*

Proof. A is a direct limit of its finitely generated R-submodules $M_i \subset A$. Then by Proposition 4.7 it is a direct limit of its right and left $\mathcal{A}^*$-submodules N_i that are finitely generated R-modules.

The kernels of the morphisms $\mathcal{A}^* \to \mathcal{N}_i^*$ are bilateral ideal schemes $\mathcal{I}_i^* = (\mathcal{A}/\mathcal{N}_i)^*$ of $\mathcal{A}^*$. Let $R^n \to N_i$ an epimorphism of R-modules. The composite morphism $\mathcal{A}^* \to \mathcal{N}_i^* \hookrightarrow (\mathcal{R}^n)^* = \mathcal{R}^n$ factors via the epimorphism $\mathcal{A}^* \to \mathcal{B}_i$, where $B_i = A^*/\mathcal{I}_i^*$ and it is a finite R-algebra, by Lemma 4.5. Dually, we obtain the morphisms $\mathcal{N}_i \to \mathcal{B}_i^* \hookrightarrow \mathcal{A}$. Taking direct limit we obtain the sequence $\mathcal{A} \to \varinjlim_i \mathcal{B}_i^* \hookrightarrow \mathcal{A}$. Hence, $\varinjlim_i \mathcal{B}_i^* = \mathcal{A}$. Dually, $\varprojlim_i \mathcal{B}_i = \mathcal{A}^*$. □

5. Closure of an algebra functor.

Definition 5.1. *Let $\mathbb{M}$ be an $\mathcal{R}$-algebra functor. We define $\tilde{\mathbb{M}}$ to be the representant on the category of $\mathcal{R}$-algebra schemes, if it exists, of the functor* $\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathbb{M}, -)$. *I.e.,*

$$\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathbb{M}, \mathcal{A}^*) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\tilde{\mathbb{M}}, \mathcal{A}^*)$$

Notation 5.2. *We will denote by $\mathcal{A}^* \tilde{\otimes} \mathcal{B}^*$ the representant, on the category of $\mathcal{R}$-algebra schemes, of the functor* $\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathcal{A}^* \otimes \mathcal{B}^*, -)$.

Then, we have that

$$\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\widetilde{\mathcal{A}^* \otimes \mathcal{B}^*}, \mathcal{C}^*) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathcal{A}^* \otimes \mathcal{B}^*, \mathcal{C}^*) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathcal{A}^* \tilde{\otimes} \mathcal{B}^*, \mathcal{C}^*)$$

Therefore, $\widetilde{\mathcal{A}^* \otimes \mathcal{B}^*} = \mathcal{A}^* \tilde{\otimes} \mathcal{B}^*$.

Proposition 5.3. *If $\mathbb{M}$ is an $\mathcal{R}$-algebra functor such that $\mathbb{M}^*$ is a quasi-coherent $\mathcal{R}$-module, then $\tilde{\mathbb{M}} = \bar{\mathbb{M}} \overset{2.8}{=} \mathbb{M}^{**}$. Moreover, if $\mathbb{E}$ is an $\mathcal{R}$-module functor such that $\mathbb{N} := \mathbb{E}^*$ is an $\mathcal{R}$-algebra functor, then*

$$\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathbb{M}, \mathbb{N}) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\tilde{\mathbb{M}}, \mathbb{N})$$

Proof. By Lemma 2.6, Example 2.10 and Proposition 2.8 it holds for every $\mathcal{R}$-module functor $\mathbb{N}_1 := \mathbb{N}_2^*$ that

$$\mathrm{Hom}_{\mathcal{R}}(\mathbb{M} \otimes \ldots \otimes \mathbb{M}, \mathbb{N}_1) = \mathrm{Hom}_{\mathcal{R}}(\mathbb{N}_2, \mathbb{M}^* \otimes \ldots \otimes \mathbb{M}^*) = \mathrm{Hom}_{\mathcal{R}}(\bar{\mathbb{M}} \otimes \ldots \otimes \bar{\mathbb{M}}, \mathbb{N}_1)$$

If we consider $\mathbb{N}_1 = \bar{\mathbb{M}}$, it follows easily that the structure of algebra of $\mathbb{M}$ define a structure of algebra on $\bar{\mathbb{M}}$. Finally, if we consider $\mathbb{N}_1 = \mathbb{N}$, we see at once that $\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathbb{M}, \mathbb{N}) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\tilde{\mathbb{M}}, \mathbb{N})$. □

Remark 5.4. *In particular,*

1. *If $G = \mathrm{Spec}\, A$ is an R-group, then $\widetilde{\mathcal{R}G^{\cdot}} = \overline{\mathcal{R}G^{\cdot}}$.*
2. *If $\mathcal{A}^*$ and $\mathcal{B}^*$ are $\mathcal{R}$-algebra schemes, then $\mathcal{A}^* \tilde{\otimes} \mathcal{B}^* = \mathcal{A}^* \bar{\otimes} \mathcal{B}^*$.*

Theorem 5.5. *Let $G = \mathrm{Spec}\, A$ be an R-group scheme. The category of G-modules is equal to the category of $\mathcal{A}^*$-modules.*

Proof. Let M be an R-module. Let us observe that $\mathbb{E}nd_{\mathcal{R}}(\mathcal{M}) = (\mathcal{M}^* \otimes \mathcal{M})^*$. Therefore, by Proposition 5.3 and Theorem 3.3, (2), $\mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathcal{R}G^{\cdot}, \mathbb{E}nd_{\mathcal{R}}(\mathcal{M})) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathcal{A}^*, \mathbb{E}nd_{\mathcal{R}}(\mathcal{M}))$. In conclusion, endowing M with a structure of G-module is equivalent to endowing M with structure of $\mathcal{A}^*$-module.

Defining a morphism $\mathcal{R}G^{\cdot} \otimes \mathcal{M} \to \mathcal{M}$ is equivalent to defining a morphism $\mathcal{A}^* \otimes \mathcal{M} \to \mathcal{M}$, because $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}G^{\cdot} \otimes \mathcal{M}, \mathcal{M}) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}^* \otimes \mathcal{M}, \mathcal{M})$ by Lemma 2.6, since $(\mathcal{R}G^{\cdot} \otimes \mathcal{M})^* = (\mathcal{A}^* \otimes \mathcal{M})^*$. Now it is easy to check that $\mathrm{Hom}_{G-\mathrm{mod}}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_{\mathcal{A}^*}(\mathcal{M}, \mathcal{M}')$. □

Proposition 5.6. *Let $G = \mathrm{Spec}\ A$ be an R-group and let $G_m = \mathbb{A}ut_{\mathcal{R}}(\mathcal{R}) \subset \mathbb{E}nd_{\mathcal{R}}(\mathcal{R})$. It holds that*

$$\mathrm{Hom}_{R-\mathrm{grp}}(G, G_m) = \mathrm{Hom}_{\mathcal{R}-\mathrm{alg}}(\mathcal{A}^*, \mathcal{R})$$

Proposition 5.7. [W, 3.3] *Let V be a G-module. Every vector subspace of V of finite dimension is included in a G-submodule of V of finite dimension.*

Proof. It is a consequence of Proposition 4.7 □

Proposition 5.8. [W, 3.4] *If $G = \mathrm{Spec}\, A$ is an algebraic group then it is a subgroup of a linear group Gl_n.*

Proof. Let us consider the natural inclusion $G^{\cdot} \hookrightarrow \mathcal{A}^*$. By Proposition 4.12 we know that $\mathcal{A}^* = \lim\limits_{\leftarrow i} \mathcal{A}_i^*$ is an inverse limit of finite quotient K-algebras. By the noetherianity of G, there exists an index i such that the morphism $G^{\cdot} \to \mathcal{A}_i^*$ is injective. However, we have the natural injection $\mathcal{A}_i^* \hookrightarrow \mathbb{E}nd_{\mathcal{K}}(\mathcal{A}_i)$, then an injection $G^{\cdot} \hookrightarrow \mathbb{E}nd_{\mathcal{K}}(\mathcal{A}_i)$. □

In this section, from now on, $\mathbb{A}$ will be an algebra functor such that the image of any $\mathcal{K}$-linear morphism $\mathbb{A} \to \mathcal{V}$ is a quasi-coherent subspace of $\mathcal{V}$, for example, if $\mathbb{A}$ is a reflexive $\mathcal{K}$-vector space functor.

Theorem 5.9. $\tilde{\mathbb{A}} = \lim\limits_{\leftarrow i} \mathbb{A}/\mathbb{I}_i$, *where $\{\mathbb{I}_i\}_i$ is the set of bilateral ideal subfunctors of $\mathbb{A}$ such that $\mathbb{A}/\mathbb{I}_i$ is a coherent $\mathcal{K}$-vector space.*

Proof. Let us denote $\mathbb{A}' = \lim\limits_{\leftarrow i} \mathbb{A}/\mathbb{I}_i$. We must proof the functorial expression

$$\mathrm{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A}, \mathcal{B}^*) = \mathrm{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A}', \mathcal{B}^*)$$

First let us suppose that $\mathcal{B}^*$ is a finite $\mathcal{K}$-algebra scheme. Every morphism of $\mathcal{K}$-algebra functors $\mathbb{A} \to \mathcal{B}^*$ has as kernel an $\mathbb{I}_i$, then it factors through $\mathbb{A}/\mathbb{I}_i$, then through $\mathbb{A}'$. Inversely, let us see that every morphism $\mathbb{A}' \to \mathcal{B}^*$ factors through $\mathbb{A}/\mathbb{I}_i$:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}}(\mathbb{A}', \mathcal{B}^*) &= \mathrm{Hom}_{\mathcal{K}}(\lim_{\substack{\leftarrow \\ i}} \mathbb{A}/\mathbb{I}_i, \mathcal{B}^*) = \mathrm{Hom}_{\mathcal{K}}(\mathcal{B}, \lim_{\substack{\rightarrow \\ i}} (\mathbb{A}/\mathbb{I}_i)^*) \\ &\overset{*}{=} \lim_{\substack{\rightarrow \\ i}} \mathrm{Hom}_{\mathcal{K}}(\mathcal{B}, (\mathbb{A}/\mathbb{I}_i)^*) = \lim_{\substack{\rightarrow \\ i}} \mathrm{Hom}_{\mathcal{K}}(\mathbb{A}/\mathbb{I}_i, \mathcal{B}^*) \end{aligned}$$

where $\overset{*}{=}$ holds because B is a finite-dimensional K-vector space.

In the general case,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A},\mathcal{B}^*) &\overset{4.12}{=} \operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A},\lim_{\substack{\leftarrow\\ i}}\mathcal{B}_i^*) = \lim_{\substack{\leftarrow\\ i}}\operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A},\mathcal{B}_i^*)\\ &= \lim_{\substack{\leftarrow\\ i}}\operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A}',\mathcal{B}_i^*) = \operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A}',\lim_{\substack{\leftarrow\\ i}}\mathcal{B}_i^*)\\ &= \operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A}',\mathcal{B}^*)\end{aligned}$$

□

Proposition 5.10. *Let $\mathbb{A}$ be a $\mathcal{K}$-algebra functor. Then,*

(1) *The category of $\mathcal{K}$-coherent $\mathbb{A}$-modules is the same as the category of $\mathcal{K}$-coherent $\tilde{\mathbb{A}}$-modules.*
(2) *The natural morphism $\bar{\mathbb{A}} \to \tilde{\mathbb{A}}$ is surjective.*

Proof.

(1) If $\mathbb{I}_i \hookrightarrow \mathbb{A}$ is a bilateral ideal functor such that $\mathbb{A}/\mathbb{I}_i$ is a coherent $\mathcal{K}$-vector space, then the epimorphism $\mathbb{A} \to \mathbb{A}/\mathbb{I}_i$ factors through $\tilde{\mathbb{A}}$ and hence the morphism $\tilde{\mathbb{A}} \to \mathbb{A}/\mathbb{I}_i$ is surjective.

If $\tilde{\mathbb{I}}_i \hookrightarrow \tilde{\mathbb{A}}$ is a bilateral ideal functor such that $\tilde{\mathbb{A}}/\tilde{\mathbb{I}}_i$ is a coherent $\mathcal{K}$-vector space, then the image of the morphism $\mathbb{A} \to \tilde{\mathbb{A}}/\tilde{\mathbb{I}}_i$ is an algebra scheme, therefore the induced morphism $\tilde{\mathbb{A}} \to \tilde{\mathbb{A}}/\tilde{\mathbb{I}}_i$ values on that image. In conclusion, the morphism $\mathbb{A} \to \tilde{\mathbb{A}}/\tilde{\mathbb{I}}_i$ is surjective.

Now (1) follows easily.

(2) By the last argument, the composite morphism $\bar{\mathbb{A}} \to \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}/\tilde{\mathbb{I}}_i$ is surjective. The inverse limit of such surjections is surjective, because dually the direct limit of injections of quasi-coherent vector spaces is an injection. Then the morphism $\bar{\mathbb{A}} \to \tilde{\mathbb{A}}$ is surjective.

□

Theorem 5.11. *Let $\mathbb{A}$ be a $\mathcal{K}$-algebra functor such that $\bar{\mathbb{A}}$ is a $\mathcal{K}$-algebra functor and $\mathbb{A} \to \bar{\mathbb{A}}$ is a morphism of $\mathcal{K}$-algebra functors. Then $\bar{\mathbb{A}} = \tilde{\mathbb{A}}$.*

Proof. The morphism of $\mathcal{K}$-algebra functors $\mathbb{A} \to \bar{\mathbb{A}}$ factors through a morphism $i : \tilde{\mathbb{A}} \to \bar{\mathbb{A}}$. The morphism of $\mathcal{K}$-algebra functors $\mathbb{A} \to \tilde{\mathbb{A}}$ is a $\mathcal{K}$-linear morphism, then it factors through a morphism $j : \bar{\mathbb{A}} \to \tilde{\mathbb{A}}$. As $i \circ j : \bar{\mathbb{A}} \to \bar{\mathbb{A}}$ is the identity morphism on $\mathbb{A}$, $i \circ j = \mathrm{Id}$. Then the morphism j is injective and, since it is surjective by the previous proposition, this proves that $\bar{\mathbb{A}} = \tilde{\mathbb{A}}$. □

Definition 5.12. *Let $\mathbb{A}$ be a $\mathcal{K}$-algebra functor. We call the K-vector space of distributions of finite support of $\mathbb{A}$, and we denote it by D, the vector subspace $D \subseteq \mathbb{A}^*(K)$ consisting of linear 1-forms of $\mathbb{A}$ that are null on some bilateral ideal of $\mathbb{A}$ whose cokernel is a coherent K-vector space.*

By Theorem 5.9, $\tilde{\mathbb{A}}^* = \mathcal{D}$, then $\tilde{\mathbb{A}} = \mathcal{D}^*$. It holds that

$$\operatorname{Hom}_{\mathrm{coalg}}(B,D) = \operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathcal{D}^*,\mathcal{B}^*) = \operatorname{Hom}_{\mathcal{K}-\mathrm{alg}}(\mathbb{A},\mathcal{B}^*)$$

for every coalgebra B.

Given a commutative K-algebra A and a closed point $x \in \operatorname{Spec} A$, if we consider it as an ideal of A we will write $\mathfrak{m}_x$ for it.

Proposition 5.13. *Let A be a commutative K-algebra of finite type. It holds that*

(1) $\tilde{\mathcal{A}} = \prod_{x \in \mathrm{Spec}_{max} A} \hat{\mathcal{A}}_x$, *where* $\hat{\mathcal{A}}_x := \lim\limits_{\overleftarrow{n}} \mathcal{A}/\mathfrak{m}_x^n$.

(2) *The natural morphism* $\bar{\mathcal{D}} \to \mathcal{A}^*$ *is surjective, where* D *is the* K*-vector space of the distributions of finite support of* $\mathcal{A}$.

Proof.

(1) If A/I is a finite K-algebra, then $\mathrm{Spec}(A/I)$ correspond to a finite number of closed points of $\mathrm{Spec}\, A$, $\{x_1, \dots, x_n\}$, and there exists an $m \in \mathbb{N}$ such that $(\mathfrak{m}_{x_1} \cdot \ldots \cdot \mathfrak{m}_{x_n})^m \subset I$. Therefore,

$$\begin{aligned} \tilde{\mathcal{A}} &= \lim_{\overleftarrow{x_1,\dots,x_n,m}} \mathcal{A}/(\mathfrak{m}_{x_1} \cdot \ldots \cdot \mathfrak{m}_{x_n})^m \\ &= \lim_{\overleftarrow{x_1,\dots,x_n,m}} \mathcal{A}/\mathfrak{m}_{x_1}^m \times \ldots \times \mathcal{A}/\mathfrak{m}_{x_n}^m = \prod_{x \in \mathrm{Spec}_{max} A} \hat{\mathcal{A}}_x \end{aligned}$$

(2) The morphism $\bar{\mathcal{D}} \to \mathcal{A}^*$ is surjective if and only if the morphism $A \to \bar{\mathcal{D}}^*(K) = D^*$ is injective. By (1) this morphism is obviously injective.

□

Lemma 5.14. *Let* $\phi : \mathbb{M}_1 \to \mathbb{M}_2$ *be a morphism of vector space functors and let* $\bar{\phi} : \bar{\mathbb{M}}_1 \to \bar{\mathbb{M}}_2$ *be the induced morphism on the vector space scheme closure. It holds that* $\overline{\mathrm{Coker}\, \phi} = \mathrm{Coker}\, \bar{\phi}$ *and* $\bar{\phi}(\bar{\mathbb{M}}_1)$ *is the vector space scheme closure of the image of* $\mathbb{M}_1$ *in* $\bar{\mathbb{M}}_2$.

Proof. Obviously, $\bar{\phi}(\bar{\mathbb{M}}_1)$ is the same as the minimum vector space subscheme in $\bar{\mathbb{M}}_2$ that contains the image of $\mathbb{M}_1$. It follows immediately from the functorial definition of Coker and the vector space scheme closure that $\overline{\mathrm{Coker}\, \phi} = \mathrm{Coker}\, \bar{\phi}$. □

Notation 5.15. *In the next proposition, given* $\mathbb{M} \subset \mathcal{M}^*$*, we will denote by* $\mathbb{M}'$ *the module scheme closure of* $\mathbb{M}$ *in* $\mathcal{M}^*$.

Proposition 5.16. *Let* $\mathcal{I}_1^*, \dots, \mathcal{I}_n^* \subseteq \mathcal{A}^*$ *be bilateral ideal schemes and let* $\mathcal{M}$ *be an* $\mathcal{A}^*$*-module. It holds that*

(1) $\mathcal{I}_1^* \cdot \mathcal{M}$ *is a quasi-coherent submodule of* $\mathcal{M}$.
(2) $\mathcal{I}_1^* \cdot \mathcal{I}_2^* \cdot \mathcal{M} = (\mathcal{I}_1^* \cdot \mathcal{I}_2^*)' \cdot \mathcal{M}$.
(3) $\{e \in \mathcal{M} : \mathcal{I}_1^* \cdot e = 0\}$ *is a quasi-coherent submodule of* $\mathcal{M}$.
(4) $(\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*)'$ *is an* $\mathcal{A}^*$*-submodule of* $\mathcal{M}^*$ *and to take the module scheme closure is stable under base change, i.e., given a morphism of rings* $K \to B$*, then* $(\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*)'_{|B} = (\mathcal{M}^*{}_{|B} \cdot \mathcal{I}_{1|B}^* \cdot \ldots \cdot \mathcal{I}_{n|B}^*)'$*. Therefore,* $(\mathcal{M}^*/(\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*)')^* = (\mathcal{M}^*/(\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*))^*$.
(5) $((\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_r^*)' \cdot (\mathcal{I}_{r+1}^* \cdot \ldots \cdot \mathcal{I}_n^*)')' = (\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*)'$.

Proof.

(1) The image of the morphism of $\mathcal{A}^*$-modules $\mathcal{I}_1^* \otimes_{\mathcal{K}} \mathcal{M} \to \mathcal{M}$ is a quasi-coherent $\mathcal{A}^*$-submodule and it coincides with $\mathcal{I}_1^* \cdot \mathcal{M}$.

(2) It is enough to prove $\mathcal{I}_1^* \cdot \mathcal{I}_2^* \cdot e = (\mathcal{I}_1^* \cdot \mathcal{I}_2^*)' \cdot e$. Let us consider the commutative diagram

$$\begin{array}{ccc} \mathcal{I}_1^* \cdot \mathcal{I}_2^* & \xrightarrow{\cdot e} & \mathcal{M} \\ \big\downarrow & & \big\| \\ \mathcal{A}^* & \xrightarrow{\cdot e} & \mathcal{M} \end{array}$$

The image of the top horizontal morphism is a vector space of finite dimension, then it is closed and it coincides with $\mathcal{I}_1^* \cdot \mathcal{I}_2^* \cdot e$. The counterimage of $\mathcal{I}_1^* \cdot \mathcal{I}_2^* \cdot e$ by the bottom horizontal morphism must contain $(\mathcal{I}^* \cdot \mathcal{I}_2^*)'$. Hence, $(\mathcal{I}^* \cdot \mathcal{I}_2^*)' \cdot e \subseteq \mathcal{I}^* \cdot \mathcal{I}_2^* \cdot e$ and the equality follows.

(3) Let us consider the exact sequence $\mathcal{I}_1^* \otimes \mathcal{M}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{M}^* \cdot \mathcal{I}_1^* \to 0$. Then the kernel of the morphism

$$\mathcal{M} \to \mathcal{I}_1 \otimes \mathcal{M}$$

is $\{e \in \mathcal{M} : \mathcal{I}_1^* \cdot e = 0\}$.

(4) The image of the morphism of $\mathcal{A}^*$-modules $\overline{\mathcal{M}^* \otimes \mathcal{I}_1^* \otimes \ldots \otimes \mathcal{I}_n^*} = (\mathcal{M} \otimes \mathcal{I}_1 \otimes \ldots \otimes \mathcal{I}_n)^* \to \mathcal{M}^*$ is the same as $(\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*)'$. Therefore, $(\mathcal{M}^* \cdot \mathcal{I}_1^* \cdot \ldots \cdot \mathcal{I}_n^*)'$ is an $\mathcal{A}^*$-module and the module scheme closure is stable under base change.

(5) It follows from $\overline{\mathcal{M}^* \otimes \mathcal{I}_1^* \otimes \ldots \otimes \mathcal{I}_n^*} = \overline{\overline{\mathcal{M}^* \otimes \mathcal{I}_1^* \otimes \ldots \otimes \mathcal{I}_r^*} \otimes \overline{\mathcal{I}_{r+1}^* \otimes \ldots \otimes \mathcal{I}_n^*}}$. □

Notation 5.17. *From now on, when we are in the context of algebra schemes and vector spaces, given a bilateral ideal scheme $\mathcal{I}^* \subset \mathcal{A}^*$ and a right $\mathcal{A}^*$-module $\mathcal{M}^*$ we will understand by $\mathcal{M}^* \cdot \mathcal{I}^*$ its module scheme closure $(\mathcal{M}^* \cdot \mathcal{I}^*)'$ in the vector space scheme $\mathcal{M}^*$.*

6. Maximal quotient semisimple algebra scheme.

Definition 6.1. *We say a $\mathcal{K}$-algebra scheme $\mathcal{A}^*$ is simple if it does not contain any proper bilateral ideal. We say that an $\mathcal{A}^*$-module $V \neq 0$ is simple if it does not contain any proper $\mathcal{A}^*$-submodule. We say that an $\mathcal{A}^*$-module V is semisimple if it is a sum of simple $\mathcal{A}^*$-modules.*

An $\mathcal{A}^*$-module V is semisimple if and only if it is a direct sum of simple $\mathcal{A}^*$-modules.

By Proposition 4.7, the simple $\mathcal{A}^*$-modules are K-vector spaces of finite dimension.

Theorem 6.2. *$\mathcal{A}^*$ is simple if and only if it is isomorphic to the endomorphism ring of a finite-dimensional vector space over a non-commutative field of finite degree.*

Proof. If $\mathcal{A}^*$ is simple, by Proposition 4.12 $\mathcal{A}^*$ is a finite $\mathcal{K}$-algebra scheme. Now, this theorem is a consequence of Wedderburn Theorem ([P, 3.5]). □

Theorem 6.3. *Every simple $\mathcal{A}^*$-module is an $\mathcal{A}^*$-submodule of the regular module. Every $\mathcal{A}^*$-module is a submodule of a direct sum of regular modules.*

Proof. If V is a simple left $\mathcal{A}^*$-module, therefore of finite dimension, then $\mathcal{V}^*$ is a simple right $\mathcal{A}^*$-module. Hence, for every $w \in V^*$ not null, $\mathcal{V}^* = w \cdot \mathcal{A}^*$. I.e., $\mathcal{V}^*$ is a quotient of $\mathcal{A}^*$, as a right $\mathcal{A}^*$-module. Therefore, V is a submodule of A, as left modules. Let us suppose now that V is not simple. The morphism of multiplication $\mathcal{V}^* \otimes \mathcal{A}^* \to \mathcal{V}^*$ is obviously surjective and it is of right $\mathcal{A}^*$-modules, where $\mathcal{A}^*$ acts on $\mathcal{V}^* \otimes \mathcal{A}^*$ by the second factor (on the right). Taking duals we have the desired injection $V \hookrightarrow A \otimes V = \oplus A$. □

Corollary 6.4. [W, 3.5] *Every simple G-module is a G-submodule of the regular G-module. Every G-module is a G-submodule of a direct sum of regular modules.*

Definition 6.5. *We say that a $\mathcal{K}$-algebra scheme $\mathcal{A}^*$ is a semisimple $\mathcal{K}$-algebra scheme if every quasi-coherent $\mathcal{A}^*$-module is semisimple.*

Proposition 6.6. *$\mathcal{A}^*$ is a semisimple algebra scheme if and only if A is a semisimple $\mathcal{A}^*$-module.*

Proof. If $\mathcal{A}^*$ is a semisimple algebra scheme then in particular A is a semisimple $\mathcal{A}^*$-module. Inversely, if A is a semisimple $\mathcal{A}^*$-module, as by Proposition 6.3 every $\mathcal{A}^*$-module V is a submodule of a direct sum of A's, that is semisimple, we have that V is semisimple. Then $\mathcal{A}^*$ is a semisimple algebra scheme. □

Definition 6.7. *A bilateral ideal scheme $\mathcal{I}^* \subset \mathcal{A}^*$ is said to be a maximal bilateral ideal scheme if $\mathcal{A}^*/\mathcal{I}^*$ is simple. We shall call maximal spectrum of $\mathcal{A}^*$ the set of its maximal bilateral ideal schemes, which we will denote by $\mathrm{Spec}_{max}\mathcal{A}^*$.*

If $\mathcal{A}^* = \mathcal{A}_1^* \times \mathcal{A}_2^*$, then

$$\mathrm{Spec}_{max}\mathcal{A}^* = \mathrm{Spec}_{max}\mathcal{A}_1^* \sqcup \mathrm{Spec}_{max}\mathcal{A}_2^*$$

because every bilateral ideal scheme $\mathcal{I}^* \subseteq \mathcal{A}^*$ is $\mathcal{I}^* = \mathcal{I}_1^* \times \mathcal{I}_2^*$, where $\mathcal{I}_i^*$ is a bilateral ideal scheme of $\mathcal{A}_i^*$. Therefore, every epimorphism from a product of two $\mathcal{K}$-algebra schemes to a simple $\mathcal{K}$-algebra scheme factors through the projection on one of the two factors. If $\mathcal{A}_i^*$, $\mathcal{B}_j^*$ are simple $\mathcal{K}$-algebras and $\phi : \mathcal{A}_1^* \times \ldots \times \mathcal{A}_r^* \to \mathcal{B}_1^* \times \ldots \times \mathcal{B}_s^*$ is an epimorphism, then there exist isomorphisms $\phi_j : \mathcal{A}_{i_j}^* \to \mathcal{B}_j^*$ ($i_j \neq i_k$, if $j \neq k$) such that $\phi(a_1, \ldots, a_r) = (\phi_1(a_{i_1}), \ldots, \phi_s(a_{i_s}))$.

Theorem 6.8. *$\mathcal{A}^*$ is a semisimple $\mathcal{K}$-algebra scheme if and only if it is a direct product of simple $\mathcal{K}$-algebras.*

Proof. Let us suppose that $\mathcal{A}^*$ is a semisimple algebra scheme. We know that $\mathcal{A}^*$ is an inverse limit of quotients $\mathcal{A}_i^*$ which are finite $\mathcal{K}$-algebras. Obviously, the $\mathcal{A}_i^*$-modules are $\mathcal{A}^*$-modules , then $\mathcal{A}_i^*$ is a semisimple algebra. By the theory of semisimple rings $\mathcal{A}_i^*$ is a direct product of simple finite $\mathcal{K}$-algebras, therefore $\mathcal{A}^*$ is a direct product of simple finite $\mathcal{K}$-algebras. □

Proposition 6.9. *Every $\mathcal{A}^*$-module $V \neq 0$ contains an only maximal semisimple $\mathcal{A}^*$-submodule not null.*

Proof. The maximal semisimple submodule is the sum of every semisimple submodule. As well there exist simple submodules, since given $0 \neq e \in V$, this e is contained in a finite-dimensional $\mathcal{A}^*$-module, which contains simple $\mathcal{A}^*$-submodules. □

Proposition 6.10. *The dual of the maximal semisimple submodule of A is the maximal semisimple quotient algebra scheme of $\mathcal{A}^*$, i.e., any other semisimple quotient $\mathcal{K}$-algebra scheme of $\mathcal{A}^*$ is a quotient of this one.*

Proof. Let $A_M \subset A$ be the maximal semisimple submodule. Let us see that it is bilateral. We must prove that it is a right $\mathcal{A}^*$-module. Given $w \in A^*$, it is clear that $A_M \cdot w$ is a left A^*-submodule of A. Then it is a left $\mathcal{A}^*$-submodule of A. It is also clear that it is semisimple, then $A_M \cdot w \subseteq A_M$. Hence A_M is a right A^*-submodule of A, then it is a right $\mathcal{A}^*$-submodule of A.

Moreover, the counit $w : A \to K$ (i.e., the unit of A^*) is not null on the whole A_M: if $m(w) = w(m) = 0$ for every $m \in A_M$, then $0 = (m \cdot w')(w) = m(w' \cdot w) = m(w')$ for every $w' \in A^*$ and $m = 0$, then $A_M = 0$, a contradiction.

Therefore, $\mathcal{A}^*_M$ is a $\mathcal{K}$-algebra scheme. A_M as an $\mathcal{A}^*_M$-module is semisimple because as an $\mathcal{A}^*$-module it is semisimple. Hence, by Proposition 6.6 $\mathcal{A}^*_M$ is a semisimple $\mathcal{K}$-algebra scheme. If $\mathcal{B}^*$ is a semisimple quotient of $\mathcal{A}^*$ then B is a $\mathcal{B}^*$-semisimple module, then it is a semisimple $\mathcal{A}^*$-submodule of A. Therefore $B \subset A_M$ and $\mathcal{B}^*$ is a quotient of $\mathcal{A}^*_M$. □

Notation 6.11. *We will denote by $\mathcal{A}^*_M$ the maximal semisimple quotient algebra scheme of $\mathcal{A}^*$.*

If V is a simple $\mathcal{A}^*$-module then the image of the natural morphism $\mathcal{A}^* \to \mathbb{E}nd_{\mathcal{K}}(\mathcal{V})$ is a simple $\mathcal{K}$-algebra scheme, then it is a quotient of $\mathcal{A}^*_M$. Then V is an $\mathcal{A}^*_M$-module. If V is a semisimple $\mathcal{A}^*$-module then it is a semisimple $\mathcal{A}^*_M$-module. Obviously,

$$\mathrm{Spec}_{max}\mathcal{A}^* = \mathrm{Spec}_{max}\mathcal{A}^*_M = \{\text{Set of isomorphism classes of simple } \mathcal{A}^*\text{-modules}\}$$

Definition 6.12. *We call the radical (ideal) of a $\mathcal{K}$-algebra scheme the kernel of the quotient morphism from the algebra scheme to its maximal semisimple quotient algebra scheme.*

Let V be an $\mathcal{A}^*$-module and let $\mathcal{I}^*$ be the radical of $\mathcal{A}^*$. V is semisimple if and only if it is an $\mathcal{A}^*_M$-module, i.e., if it is cancelled by $\mathcal{I}^*$. If $0 \neq V_1 \subseteq V$ is the maximal semisimple $\mathcal{A}^*$-submodule of V, then

$$V_1 = \{e \in V : \mathcal{I}^* \cdot e = 0\}$$

or equivalently, $V_1 = \{e \in V : \mathcal{I}^*(K) \cdot e = 0\}$.

Proposition 6.13. *Let V be an $\mathcal{A}^*$-module and let $\mathcal{I}^*$ be the radical of $\mathcal{A}^*$. Let V_1 be the maximal semisimple submodule of V, then*

$$\mathcal{V}_1 = (\mathcal{V}^* \otimes_{\mathcal{A}^*} \mathcal{A}^*_M)^* = (\mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^*)^*$$

Proof. By base change it is enough to prove that $V_1 = \mathrm{Hom}_{\mathcal{K}}(\mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^*, \mathcal{K})$. However, $\mathrm{Hom}_{\mathcal{K}}(\mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^*, \mathcal{K})$ identifies with the vectors $e \in \mathrm{Hom}_{\mathcal{K}}(\mathcal{V}^*, \mathcal{K}) = V$ such that $e(\mathcal{V}^* \cdot \mathcal{I}^*) = 0$. As $e(w \cdot i) = w(i \cdot e)$ for every $w \in \mathcal{V}^*$ and $i \in \mathcal{I}^*$, it follows that $e \in V$ holds that $e(\mathcal{V}^* \cdot \mathcal{I}^*) = 0$ if and only if $e \in V_1$. □

The functor $F(V) := V_1$ from the category of $\mathcal{A}^*$-modules to the category of $\mathcal{A}^*_M$-modules is a left exact functor represented by $\mathcal{A}^*_M$, because

$$F(V) = V_1 = \mathrm{Hom}_{\mathcal{A}^*}(\mathcal{A}^*_M, \mathcal{V})$$

Let us consider the quotient $V' = V/V_1$ and V'_1 the maximal semisimple $\mathcal{A}^*$-submodule of V'. Let $V_2 := \pi^{-1}(V'_1)$, where $\pi : V \to V'$ is the quotient morphism. Then $V_1 \subset V_2$ and $V_2/V_1 = V'_1$. So on we construct a canonical chain $V_1 \subset V_2 \subset V_3 \subset \ldots$, such that every quotient V_i/V_{i+1} is a semisimple $\mathcal{A}^*$-module and $V_i/V_{i+1} = \{\bar{e} \in V/V_{i-1} : \mathcal{I}^* \cdot \bar{e} = 0\}$. Inductively we deduce that

$$V_i = \{e \in V : \mathcal{I}^{*i} \cdot e = 0\}$$

Again, as in Proposition 6.13, we obtain that

$$\mathcal{V}_i = (\mathcal{V}^* \otimes_{\mathcal{A}^*} \mathcal{A}^*/\mathcal{I}^{*i})^* = (\mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^{*i})^*$$

Notation 6.14. *Given an $\mathcal{A}^*$-module V, we will denote by $V_1 \subset V_2 \subset \ldots$ the canonical chain of $\mathcal{A}^*$-submodules of V we have just constructed. We will denote $G(V) := \bigoplus_{i=1}^{\infty} V_i/V_{i-1}$, where $V_0 = 0$ and $G_{\mathcal{I}^*}\mathcal{V}^* := \prod_{i=1}^{\infty} (\mathcal{V}^* \cdot \mathcal{I}^{*i-1}/\mathcal{V}^* \cdot \mathcal{I}^{*i})$.*

Proposition 6.15. *Let V be an $\mathcal{A}^*$-module. Then*

$$(G_{\mathcal{I}^*}\mathcal{V}^*)^* = G(\mathcal{V})$$

In case of the regular $\mathcal{A}^*$-module A, the canonical chain of semisimple factors is $A_1 \subset A_2 \subset \ldots \subset A$ where $\mathcal{A}_i = (\mathcal{A}^*/\mathcal{I}^{*i})^*$.

Lemma 6.16. *Let V be a finitely-generated $\mathcal{A}^*$-module and let $\mathcal{I}^*$ be the radical of $\mathcal{A}^*$. There exists an $n >> 0$ such that $\mathcal{I}^{*n} \cdot \mathcal{V} = 0$.*

Proof. In the natural chain $V_1 \subseteq V_2 \subseteq \ldots$ of V, an inclusion $V_n \subseteq V_{n+1}$ is an equality when $V_n = V$ by Proposition 6.9. Because V is of finite dimension the equality $V = V_n$ must be true for some $n \in \mathbb{N}$. Therefore $\mathcal{I}^{*n} \cdot \mathcal{V} = 0$. □

Theorem 6.17. *Let V be an $\mathcal{A}^*$-module and let $\mathcal{I}^*$ be the radical of $\mathcal{A}^*$. It holds that*

(1) $V = \lim\limits_{\overrightarrow{i}} V_i$

(2) $\mathcal{V}^* = \lim\limits_{\overleftarrow{n}} \mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^{*n}$. *In particular,* $\bigcap\limits_{n=0}^{\infty} \mathcal{V}^* \cdot \mathcal{I}^{*n} = 0$.

Proof.

(1) Every $e \in V$ is included in a finite-dimensional $\mathcal{A}^*$-submodule V' of V. Therefore, there exists a $n \in \mathbb{N}$ such that $\mathcal{I}^{*n} e = 0$. Then $V = \lim\limits_{\overrightarrow{i}} V_i$.

(2) As $V = \lim\limits_{\overrightarrow{i}} V_i$, taking duals and remembering that $\mathcal{V}_i = (\mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^{*i})^*$, it follows that $\mathcal{V}^* = \lim\limits_{\overleftarrow{n}} \mathcal{V}^*/\mathcal{V}^* \cdot \mathcal{I}^{*i}$.

□

Proposition 6.18. (Nakayama) *Let V, V' be $\mathcal{A}^*$-modules and let $\mathcal{A}^*_M$ be the maximal semisimple quotient algebra scheme of $\mathcal{A}^*$.*

(1) $\mathcal{V}^* = 0 \iff \mathcal{V}^* \otimes_{\mathcal{A}^*} \mathcal{A}^*_M = 0$.

(2) *A morphism of $\mathcal{A}^*$-modules $\mathcal{V}^* \to \mathcal{V}'^*$ is surjective $\iff$ the morphism $\mathcal{V}^* \otimes_{\mathcal{A}^*} \mathcal{A}^*_M \to \mathcal{V}'^* \otimes_{\mathcal{A}^*} \mathcal{A}^*_M$ is surjective.*

(3) *The morphism $\mathcal{V}^* \to \mathcal{V}'^*$ is an isomorphism $\Leftrightarrow$ the morphism $G_{\mathcal{I}^*}\mathcal{V}^* \to G_{\mathcal{I}^*}\mathcal{V}'^*$ is an isomorphism.*

Proof.

(1) If $\mathcal{V}^* \otimes_{\mathcal{A}^*} \mathcal{A}^*_M = 0$ then $\mathcal{V}_1 = (\mathcal{V}^* \otimes_{\mathcal{A}^*} \mathcal{A}^*_M)^* = 0$, then $V = 0$ and $\mathcal{V}^* = 0$.

(2) (1) must be applied to the cokernels.

(3) If the morphism $G_{\mathcal{I}^*}\mathcal{V}^* \to G_{\mathcal{I}^*}\mathcal{V}'^*$ is an isomorphism, then so is the morphism between the completions, which coincide with the vector space schemes themselves by Theorem 6.17 (2).

□

If V is an $\mathcal{A}^*$-module of finite dimension we define the character associated to V, $\chi_V : \mathcal{A}^* \to \mathcal{K}$, to be

$$\chi_V(w) := \operatorname{tr} h_w$$

where h_w is the homotethy on V of factor $w \in \mathcal{A}^*$ and $\operatorname{tr} h_w$ is the trace of such linear endomorphism. But the trace of $h_w : V \to V$ is the same as the trace of the induced endomorphism $h_w : G(V) := \underset{i}{\oplus} V_i/V_{i-1} \to \underset{i}{\oplus} V_i/V_{i-1} =: G(V)$. So we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\chi_V} & \mathcal{K} \\ & \searrow \quad \nearrow_{\chi_{\underset{i}{\oplus} V_i/V_{i-1}}} & \\ & \mathcal{A}^*_M & \end{array}$$

Proposition 6.19. *Let us suppose that* $\operatorname{car} K = 0$. *Then* $\chi_V = \chi_{V'}$ *if and only if* $G(V)$ *and* $G(V')$ *are isomorphic* $\mathcal{A}^*_M$*-modules.*

Proposition 6.20. *Let* K *be an algebraically closed field. The characters associated to the simple* $\mathcal{A}^*$*-modules are linearly independent.*

Notation 6.21. *Let* $\mathcal{V}^*$ *be an* $\mathcal{R}$*-module scheme (respectively* $\mathcal{R}$*-algebra scheme) and let* $R \to S$ *be an extension of commutative rings. We will denote by* $\mathcal{V}^*_{\mathcal{S}}$ *the* S*-module scheme (respectively* S*-algebra scheme)* $\mathcal{V}^*{}_{|S} = (\mathcal{V} \otimes_{\mathcal{R}} \mathcal{S})^*$ *associated to the* S*-module* $V \otimes_R S$. *We will say that* $\mathcal{V}^*$ *is* $\mathcal{V}^*_{\mathcal{S}}$ *under the base change* $R \to S$.

Proposition 6.22. *Let* $\mathcal{A}^*$ *be an algebra scheme, let* $\mathcal{A}^*_M$ *be its maximal semisimple quotient algebra scheme and let* $K \to K'$ *be an extension of commutative fields. The maximal semisimple quotient* $\mathcal{K}'$*-algebra scheme of* $\mathcal{A}^*_{\mathcal{K}'}$ *is a quotient of* $(\mathcal{A}^*_M)_{\mathcal{K}'}$. *If* K *is algebraically closed, then the maximal semisimple quotient* $\mathcal{K}'$*-algebra scheme of* $\mathcal{A}^*_{\mathcal{K}'}$ *is the same as* $(\mathcal{A}^*_M)_{\mathcal{K}'}$.

Proof. Let $0 \subset A_1 \subset A_2 \subset \dots$ be the canonical filtration of $\mathcal{A}^*$-modules of A. Let us consider the filtration $A_1 \otimes_K K' \subset A_2 \otimes_K K' \subset \dots$ of $A \otimes_K K'$. If V is a simple $\mathcal{A}^*_{\mathcal{K}'}$-module, then it injects into $A \otimes_K K'$ and for some i there exists an injection $V \hookrightarrow (A_i \otimes_K K')/(A_{i-1} \otimes_K K')$ of $\mathcal{A}^*_{\mathcal{K}'}$-modules. However, $(A_i \otimes_K K')/(A_{i-1} \otimes_K K')$ is an $(\mathcal{A}^*_M)_{\mathcal{K}'}$-module, then V is an $(\mathcal{A}^*_M)_{\mathcal{K}'}$-module. In conclusion, every morphism from $\mathcal{A}^*_{\mathcal{K}'}$ to a simple algebra factors through $(\mathcal{A}^*_M)_{\mathcal{K}'}$. Therefore, the maximal semisimple quotient K'-algebra scheme of $\mathcal{A}^*_{\mathcal{K}'}$ is a quotient of $(\mathcal{A}^*_M)_{\mathcal{K}'}$.

If K is algebraically closed then $\mathcal{A}^*_M = \prod_i \mathcal{M}_{n_i}(K)$, then $(\mathcal{A}^*_M)_{\mathcal{K}'} = \prod_i \mathcal{M}_{n_i}(\mathcal{K}')$ is semisimple and the maximal semisimple quotient K'-algebra scheme of $\mathcal{A}^*_{\mathcal{K}'}$ is isomorphic to $(\mathcal{A}^*_M)_{\mathcal{K}'}$. □

Proposition 6.23. *Let* $\mathcal{A}^*_M$ *be the maximal semisimple quotient algebra scheme of* $\mathcal{A}^*$ *and let* $\mathcal{B}^*_M$ *be the maximal semisimple quotient algebra scheme of* $\mathcal{B}^*$. *Then the maximal semisimple quotient algebra scheme of* $\mathcal{A}^* \bar{\otimes} \mathcal{B}^*$ *is a quotient of* $\mathcal{A}^*_M \bar{\otimes} \mathcal{A}_N{}^*$.

If k *is algebraically closed, then* $\mathcal{A}^*_M \bar{\otimes} \mathcal{B}^*_M$ *is the maximal semisimple quotient algebra scheme of* $\mathcal{A}^* \bar{\otimes} \mathcal{B}^*$ *and an* $\mathcal{A}^* \otimes \mathcal{B}^*$*-module is simple if and only if it is a tensorial product of a simple* $\mathcal{A}^*$*-module and a simple* $\mathcal{B}^*$*-module.*

Proof. Let V be a simple $\mathcal{A}^* \bar{\otimes} \mathcal{B}^*$-module. In particular $\dim_k V < \infty$. Let us consider the canonical chain $V_1 \subset V_2 \subset \dots \subset V_n = V$ of V as an $\mathcal{A}^*$-module. As $\mathcal{B}^*$ commutes with $\mathcal{A}^*$ in $\mathcal{A}^* \otimes \mathcal{B}^*$, then it has to leave stable the chain. Since V is simple, then $V = V_1$, i.e., V is a semisimple $\mathcal{A}^*$-module. Likewise, V is a semisimple $\mathcal{B}^*$-module. In conclusion, V is an $\mathcal{A}^*_M \otimes \mathcal{B}^*_M$-module, then it is an $\mathcal{A}^*_M \bar{\otimes} \mathcal{B}^*_M$-module. Therefore, every morphism from $\mathcal{A}^* \bar{\otimes} \mathcal{B}^*$ to a simple algebra scheme factors through $\mathcal{A}^*_M \bar{\otimes} \mathcal{B}^*_M$, then the maximal semisimple quotient algebra scheme of $\mathcal{A}^* \bar{\otimes} \mathcal{B}^*$ is a quotient of $\mathcal{A}^*_M \bar{\otimes} \mathcal{B}^*_M$.

Let K be an algebraically closed field. As $\mathrm{End}_K(V)\otimes_K \mathrm{End}_K(V') = \mathrm{End}_K(V\otimes_K V')$ ($\dim_K V, V' < \infty$), $\mathcal{A}^*_M = \prod\limits_i \mathbb{E}nd_{mathcalK}(\mathcal{V}_i)$ and $\mathcal{B}^*_M = \prod\limits_j \mathbb{E}nd_{\mathcal{K}}(\mathcal{V}'_j)$, then

$$\mathcal{A}^*_M \bar{\otimes} \mathcal{B}^*_M = \prod_{i,j} \mathbb{E}nd_{\mathcal{K}}(\mathcal{V}_i \otimes \mathcal{V}'_j)$$

□

Corollary 6.24. *Let K be an algebraically closed field. $\mathcal{A}^*$, $\mathcal{B}^*$ are semisimple $\mathcal{K}$-algebra schemes if and only if $\mathcal{A}^*\bar{\otimes}\mathcal{B}^*$ is semisimple.*

Let us give some final examples of the application of the representation theory of algebra schemes to the representation theory of algebraic groups.

Let $G = \mathrm{Spec}\, A$ be an algebraic group. It is easy to prove that G is unipotent if and only if $\mathcal{A}^*$ is local (i.e., it only contains one bilateral ideal scheme) and that G is triangulable if and only if $\mathcal{A}^*$ is basic (i.e., $\mathcal{A}^*_M = \prod \mathcal{K}$). It is also easy to prove that subschemes and quotients of a basic algebra scheme (respectively local and basic) are basic (respectively local and basic).

Corollary 6.25. *Subgroups, quotients, direct products of triangulable (respectively unipotent) groups are triangulable (respectively unipotent) groups.*

We shall say that $X \subset \mathcal{A}^*$ is a dense subset in $\mathcal{A}^*$ if the minimum vector space subscheme of $\mathcal{A}^*$ that contains X is $\mathcal{A}^*$. Dually, X is dense in $\mathcal{A}^*$ if the only $a \in A$ such that $a(x) = 0$ for all $x \in X$ is $a = 0$.

Proposition 6.26. *Let K be an algebraically closed field, let $\chi : \mathcal{A}^* \to \mathcal{K}$ be a morphism of functors of $\mathcal{K}$-algebras and let $X \subset \mathcal{A}^*$ be a dense subset in $\mathcal{A}^*$. $\mathcal{A}^*$ is local if and only if anyone of the following conditions holds:*

1. *For every $x \in X$, $x - \chi(x)$ belongs to the radical of $\mathcal{A}^*$.*
2. *For every $x \in X$ and every morphism of $\mathcal{K}$-algebra functors $\phi : \mathcal{A}^* \to \mathbb{E}nd_{\mathcal{K}}(\mathcal{V})$, where $\dim_K V < \infty$, $\phi(x - \chi(x))$ is nilpotent.*

Proof.

1. If $\mathcal{A}^*$ is local then $x - \chi(x)$ belongs to the kernel of χ, which is the radical of $\mathcal{A}^*$. Let us see the inverse. Let V be a simple $\mathcal{A}^*$-module and let $\phi : \mathcal{A}^* \to \mathbb{E}nd_{\mathcal{K}}(\mathcal{V})$ be the natural epimorphism. Given an $x \in X$, as $x - \chi(x)$ belongs to the radical of $\mathcal{A}^*$ it holds that $\phi(x - \chi(x))$ is nilpotent, then $\chi_V(x - \chi(x)) = 0$. Therefore $\chi_V(x) = n\chi(x)$, where $n = \dim_K V$. Then $\chi_V = n\chi$ because X is dense in $\mathcal{A}^*$. From here it follows that $V = K$ and $\chi_V = \chi$. In conclusion, $\mathcal{A}^*$ is local.
2. The proof above works here.

□

Let $\chi : \mathcal{A}^* \to \mathcal{K}$ be a morphism of $\mathcal{K}$-algebra schemes. An $\mathcal{A}^*$-module V is called χ-unipotent if there exists a filtration of $\mathcal{A}^*$-modules $0 \subset V'_1 \subset V'_2 \subset \cdots \subset V$ such that $\mathcal{A}^*$ operates on V'_i/V'_{i-1} via χ for all i (and $V = \underset{i}{\cup} V'_i$). Then $\mathcal{A}^*$ is local if and only if A is χ-unipotent. If $V' \to V$ is an epimorphism of $\mathcal{A}^*$-modules and V' is χ-unipotent then V is χ-unipotent. If V is χ-unipotent then $V \otimes_K \overset{n}{\cdots} \otimes_K V$ is χ-unipotent.

Corollary 6.27. *Let K be an algebraically closed field and let $G \subseteq Gl_n$ be an integral algebraic group. G is unipotent if and only if every closed point $g \in G$ is a unipotent matrix, i.e, $g - id$ is nilpotent.*

7. Separable algebra schemes.

See for instance [P, 10] for a study of separable algebras.

Definition 7.1. *We call a $\mathcal{K}$-algebra scheme $\mathcal{A}^*$ separable if and only if under every base change $K \hookrightarrow K'$, $\mathcal{A}^*_{\mathcal{K}'} := (\mathcal{A} \otimes_{\mathcal{K}} \mathcal{K}')^*$ is a semisimple $\mathcal{K}'$-algebra scheme.*

Definition 7.2. *Let $\mathbb{A}$ be a $\mathcal{K}$-algebra functor. We will call centre of $\mathbb{A}$ the $\mathcal{K}$-algebra subfunctor of $\mathbb{A}$, that we denote by $Z(\mathbb{A})$, defined by*

$$Z(\mathbb{A})(S) := \{a \in \mathbb{A}(S) \mid a\cdot = \cdot a\}$$

where $a\cdot : \mathbb{A}_{|S} \to \mathbb{A}_{|S}$, $b \mapsto a \cdot b$ and $\cdot a : \mathbb{A}_{|S} \to \mathbb{A}_{|S}$, $b \mapsto b \cdot a$.

It holds that $Z(\mathcal{A}^*)(S) = \{w \in \mathcal{A}^*(S) \mid \mathcal{A}^*_S \overset{w\cdot = \cdot w}{\to} \mathcal{A}^*_S\}$ coincides with the centre of the S-algebra $(A \otimes_K S)^* = \mathcal{A}^*(S)$.

$Z(\mathcal{A}^*)$ is a $\mathcal{K}$-algebra scheme: $Z(\mathcal{A}^*)$ is the kernel of the morphism $\phi : \mathcal{A}^* \to \mathbb{E}nd_{\mathcal{K}}(\mathcal{A})$, $w \mapsto w \cdot - \cdot w$, and $\mathbb{E}nd_{\mathcal{K}}(\mathcal{A})$ is included in the $\mathcal{K}$-vector space scheme $\mathbb{H}om_{\mathcal{K}}(\mathcal{A}, \bar{\mathcal{A}})$.

It holds that $Z(\mathcal{A}^* \bar{\otimes} \mathcal{B}^*) = Z(\mathcal{A}^*) \bar{\otimes} Z(\mathcal{B}^*)$: $\mathcal{B}^*$ is a $\mathcal{K}$-vector space scheme isomorphic to $\prod \mathcal{K}$, then $\mathcal{A}^* \bar{\otimes} \mathcal{B}^*$ is a right and left $\mathcal{A}^*$-module isomorphic to $\prod \mathcal{A}^*$. Now it is easily seen that $Z(\mathcal{A}^* \bar{\otimes} \mathcal{B}^*) \subseteq Z(\mathcal{A}^*) \bar{\otimes} \mathcal{B}^*$. Hence, $Z(\mathcal{A}^* \bar{\otimes} \mathcal{B}^*) \subseteq Z(\mathcal{A}^*) \bar{\otimes} Z(\mathcal{B}^*)$.

Notation 7.3. *Given a ring $(R, +, \cdot)$ we denote by $(R^\circ, +, *)$ the ring that is the same as R as a set, with the same addition and whose product $*$ is defined by $a * b := b \cdot a$.*

Theorem 7.4. *Let $\mathcal{A}^*$ be a $\mathcal{K}$-algebra scheme. The next conditions are equivalent:*

(1) *$\mathcal{A}^*$ is separable.*
(2) *$\mathcal{A}^*_{\bar{\mathcal{K}}}$ is a direct product of algebras of matrices, where $\bar{K}$ is an algebraically closed field.*
(3) *$\mathcal{A}^*$ is semisimple and its centre is a separable algebra scheme.*
(4) *$\mathcal{A}^* \bar{\otimes}_{\mathcal{K}} \mathcal{A}^{*\circ}$ is semisimple.*

Proof. 1) $\Rightarrow$ 2) It is obvious.

2) $\Rightarrow$ 1) If $\mathcal{A}^* \bar{\otimes}_{\mathcal{K}} \bar{\mathcal{K}}$ is a direct product of algebras of matrices then by base change it is also a direct product of algebras of matrices, whose radical is null. If $\mathcal{A}^*$, or any base change of it, has radical not null, then by base change it would not be null either.

1), 2) $\Rightarrow$ 3) $Z(\mathcal{A}^*)_{\bar{\mathcal{K}}} = Z(\mathcal{A}^*_{\bar{\mathcal{K}}}) = \prod \bar{\mathcal{K}}$, then $Z(\mathcal{A}^*)$ is separable. Obviously $\mathcal{A}^*$ is semisimple.

3) $\Rightarrow$ 2) $\mathcal{A}^*$ is a direct product of simple algebras. As the centre of a direct product is the direct product of the centres, we can assume that $\mathcal{A}^*$ is simple, that is a finite $\mathcal{K}$-algebra. We can write $A^* = \mathcal{A}^*$. In this case $Z(A^*)$ is a field, because $A^* = \mathrm{End}_{K'}(V)$ and $Z(A^*) = Z(K')$. Therefore, A^* is an Azumaya $Z(A^*)$-algebra and $A^* \otimes_K \bar{K} = A^* \otimes_{Z(A^*)} Z(A^*) \otimes_K \bar{K} = A^* \otimes_{Z(A^*)} \prod \bar{K}$ which is a direct product of algebras of matrices.

2) $\Rightarrow$ 4) It is enough to prove that $\mathcal{A}^* \bar{\otimes}_{\mathcal{K}} \mathcal{A}^{*\circ}$ under base change to the algebraic closure of K is semisimple. As the tensorial product of algebras of matrices is an algebra of matrices, (4) is proved.

4) $\Rightarrow$ 3) Because $Z(\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}) = Z(\mathcal{A}^*)\bar{\otimes}Z(\mathcal{A}^*)$ and since $\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}$ is a direct product of algebras of matrices (over algebras of division of finite degree), it follows that $Z(\mathcal{A}^*)\bar{\otimes}Z(\mathcal{A}^*)$ is a direct product of commutative fields (of finite degree) and, hence, $Z(\mathcal{A}^*)$ is a direct product of separable finite extensions of commutative fields of K, then it is separable. By Proposition 6.23 $\mathcal{A}^*$ is semisimple. □

Lemma 7.5. *If $\mathcal{A}^*$ is a semisimple $\mathcal{K}$-algebra scheme, every $\mathcal{A}^*$-module scheme is injective and projective (in the category of $\mathcal{A}^*$-module schemes).*

Proof. Dually, we must prove that every $\mathcal{A}^*$-module V is projective and injective. However, because $\mathcal{A}^*$ is semisimple every exact sequence of $\mathcal{A}^*$-modules is split, which implies that every $\mathcal{A}^*$-module V is projective and injective. □

Definition 7.6. *Let $\mathbb{A}$ be a $\mathcal{K}$-algebra functor. We shall say $D \in \mathrm{Hom}_{\mathcal{K}}(\mathbb{A}, \mathbb{M})$ is a derivation from $\mathbb{A}$ to an $\mathbb{A}\otimes\mathbb{A}^\circ$-module $\mathbb{M}$ if $D(ab) = (Da)b + a(Db)$, for every $a, b \in \mathbb{A}$. We will denote by $\mathrm{Der}_{\mathcal{K}}(\mathbb{A}, \mathbb{M})$ the set of all derivations from $\mathbb{A}$ to $\mathbb{M}$.*

Lemma 7.7. [P, 11.5] *Let $\Delta_{\mathbb{A}}$ be the kernel of the morphism $\mathbb{A}\otimes_{\mathcal{K}}\mathbb{A} \to \mathbb{A}$, $a\otimes b \mapsto ab$. It holds that*

$$\mathrm{Der}_{\mathcal{K}}(\mathbb{A}, \mathbb{M}) = \mathrm{Hom}_{\mathbb{A}\otimes\mathbb{A}^\circ}(\Delta_{\mathbb{A}}, \mathbb{M})$$

Notation 7.8. *Given a $\mathcal{K}$-algebra scheme $\mathcal{A}^*$ let us denote by $\bar{\Delta}_{\mathcal{A}^*}$ the kernel of the morphism $\mathcal{A}^*\bar{\otimes}_{\mathcal{K}}\mathcal{A}^* \to \mathcal{A}^*$, $a\otimes a' \mapsto aa'$. Let us observe that $\bar{\Delta}_{\mathcal{A}^*}$ is the $\mathcal{K}$-module scheme closure of $\Delta_{\mathcal{A}^*}$, since $\mathcal{A}^*\otimes_{\mathcal{K}}\mathcal{A}^* = \mathcal{A}^*\oplus\Delta_{\mathcal{A}^*}$.*

Proposition 7.9. *Let $\pi : \mathcal{B}^* \to \mathcal{A}^*$ be an epimorphism of $\mathcal{K}$-algebra schemes with kernel $\mathcal{I}^*$. Then the "sequence of differentials"*

$$0 \to \mathcal{I}^*/\mathcal{I}^{*2} \xrightarrow{d} \bar{\Delta}_{\mathcal{B}^*}\bar{\otimes}_{\mathcal{B}^*\bar{\otimes}\mathcal{B}^{*\circ}}(\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}) \to \bar{\Delta}_{\mathcal{A}^*} \to 0$$

is exact, where $d\bar{i} := \overline{i\otimes 1 - 1\otimes i}$ for all $i \in \mathcal{I}^*$.

Proof. If we apply $\mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(-, \mathcal{V}^*)$ to the sequence of differentials we obtain the exact sequence

$$0 \to \mathrm{Der}_{\mathcal{K}}(\mathcal{A}^*, \mathcal{V}^*) \to \mathrm{Der}_{\mathcal{K}}(\mathcal{B}^*, \mathcal{V}^*) \to \mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(\mathcal{I}^*, \mathcal{V}^*)$$

Therefore, there is only left to prove that d is injective. Let $s : \mathcal{A}^* \to \mathcal{B}^*$ be a section of $\mathcal{K}$-vector space schemes of the epimorphism $\pi : \mathcal{B}^* \to \mathcal{A}^*$. The map

$$\bar{\Delta}_{\mathcal{B}^*}\bar{\otimes}_{\mathcal{B}^*\bar{\otimes}\mathcal{B}^{*\circ}}(\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}) \to \mathcal{I}^*/\mathcal{I}^{*2},\ \overline{\sum_i b_i\otimes b'_i} \mapsto \overline{\sum_i (b_i - s(\pi(b_i)))\cdot b'_i}$$

is a retraction of d. □

Theorem 7.10. *$\mathcal{A}^*$ is a separable $\mathcal{K}$-algebra scheme if and only if $\mathcal{A}^*$ is a projective $\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}$-module.*

Proof. If $\mathcal{A}^*$ is a separable $\mathcal{K}$-algebra then $\mathcal{A}^*\bar{\otimes}\mathcal{A}^*$ is a semisimple algebra and every module scheme is projective.

Inversely, let us suppose that $\mathcal{A}^*$ is a projective $\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}$-module. Therefore, the sequence

$$0 \to \bar{\Delta}_{\mathcal{A}^*} \to \mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ} \to \mathcal{A}^* \to 0 \tag{1}$$

is split. Let $\bar{K}$ be the algebraic closure of K. For simplicity of notation we write $\mathcal{A}^*$ instead of $\mathcal{A}^*_{\bar{\mathcal{K}}}$. Let $\mathcal{A}^*_M$ be the maximal semisimple quotient scheme of $\mathcal{A}^*$ and let $\mathcal{I}^*$ be the radical of $\mathcal{A}^*$. If we apply $-\otimes_{\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}}\mathcal{A}^*_M\bar{\otimes}\mathcal{A}^{*\circ}_M$ to (1) we obtain that $\bar{\Delta}_{\mathcal{A}^*}\otimes_{\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}}(\mathcal{A}^*_M\bar{\otimes}\mathcal{A}^{*\circ}_M) = \bar{\Delta}_{\mathcal{A}^*_M}$.

From the exact sequence $0 \to \mathcal{I}^* \to \mathcal{A}^* \to \mathcal{A}^*_M \to 0$ and the exact sequence of differentials from 7.9

$$0 \to \mathcal{I}^*/\mathcal{I}^{*2} \xrightarrow{d} \bar{\Delta}_{\mathcal{A}^*} \bar{\otimes}_{\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}} (\mathcal{A}^*_M \bar{\otimes} \mathcal{A}^{*\circ}_M) \to \bar{\Delta}_{\mathcal{A}^*_M} \to 0$$

we obtain that $\mathcal{I}^*/\mathcal{I}^{*2} = 0$. Therefore, $\mathcal{I}^* = \mathcal{I}^{*2}$ and by Theorem 6.17 $\mathcal{A}^* = \lim\limits_{\substack{\leftarrow \\ n}} \mathcal{A}^*/\mathcal{I}^{*n} = \mathcal{A}^*/\mathcal{I}^* = \mathcal{A}^*_M$ and $\mathcal{A}^*$ is separable. □

Remark 7.11. *In the proof we have seen that if (and only if) the sequence of* $\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}$*-modules* $0 \to \bar{\Delta}_{\mathcal{A}^*} \to \mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ} \to \mathcal{A}^* \to 0$ *is split, then* $\mathcal{A}^*$ *is a separable* $\mathcal{K}$*-algebra scheme.*

8. Extensions of algebra schemes.

In this section the cohomological arguments and descent theory are concisely used to give a proof of the Principal Theorem of Wedderburn-Malcev (see [P, 11.6] or [M, X, 3.2]) in the context of algebra schemes.

Proposition 8.1. *Let* $\mathbb{A}$ *be a* $\mathcal{K}$*-algebra functor, let* $\mathcal{C}_{\mathbb{A}-Mod}$ *be the category of* $\mathbb{A}$*-modules and let* $\mathcal{C}_{Vect}$ *be the category of* $\mathcal{K}$*-vector space functors. The functor "forget the structure of* $\mathbb{A}$*-module"* $\phi : \mathcal{C}_{\mathbb{A}-Mod} \to \mathcal{C}_{Vect}$, $\mathbb{M} \rightsquigarrow \mathbb{M}$ *has got an adjoint functor, which is* $\mathrm{Ad}(\phi) : \mathcal{C}_{Vect} \to \mathcal{C}_{\mathbb{A}-Mod}$, $\mathbb{M} \rightsquigarrow \mathbb{H}om_{\mathcal{K}}(\mathbb{A}, \mathbb{M})$*. I.e., if* $\mathbb{M}$ *is an* $\mathbb{A}$*-module and* $\mathbb{N}$ *is a* $\mathcal{K}$*-vector space functor it holds that*

$$\mathrm{Hom}_{\mathcal{K}}(\mathbb{M}, \mathbb{N}) = \mathrm{Hom}_{\mathbb{A}}(\mathbb{M}, \mathbb{H}om_{\mathcal{K}}(\mathbb{A}, \mathbb{N})) \tag{2}$$

Let us denote $R^0 := \mathrm{Ad}(\phi) \circ \phi$, i.e., $R^0(\mathbb{M}) = \mathbb{H}om_{\mathcal{K}}(\mathbb{A}, \mathbb{M})$. The morphism $\mathrm{Id} : \mathbb{M} \to \mathbb{M}$ defines a natural morphism $\mathbb{M} \to R^0(\mathbb{M}) = \mathbb{H}om_{\mathcal{K}}(\mathbb{A}, \mathbb{M})$ by the equation (2). If we apply R^0 to this morphism then we obtain a new morphism $R^0(\mathbb{M}) \to R^0(R^0(\mathbb{M})) = \mathbb{H}om_{\mathcal{K}}(\mathbb{A} \otimes \mathbb{A}, \mathbb{M})$ besides the natural one, and so on we will obtain the sequence

$$\mathbb{M} \to R^0(\mathbb{M}) \rightrightarrows R^0(R^0(\mathbb{M}))\dots$$

Let us denote by $\mathbb{M} \to R^{\cdot}(\mathbb{M})$ this complex, which is exact: The identity morphism $Id : \mathrm{Ad}(\phi)(\mathbb{N}) \to \mathrm{Ad}(\phi)(\mathbb{N})$ defines by adjointness a canonical morphism $(\phi \circ \mathrm{Ad}(\phi))(\mathbb{N}) \to \mathbb{N}$, then we have canonical morphisms $\phi(R^i(\mathbb{M})) \to \phi(R^{i-1}(\mathbb{M}))$ that turn out to be some operators of homotopy of the complex $\phi(\mathbb{M}) \to \phi(R^{\cdot}(\mathbb{M}))$. Therefore, this complex is homotopic to zero and $\mathbb{M} \to R^{\cdot}(\mathbb{M})$ is exact.

If we now consider $\mathbb{A} = \mathcal{A}^*$, then $R^0(\mathcal{V}) = \mathcal{A} \otimes \mathcal{V}$ and it turns out to be an injective quasi-coherent $\mathcal{A}^*$-module, because $\mathrm{Hom}_{\mathcal{A}^*}(-, \mathcal{A} \otimes \mathcal{V}) = \mathrm{Hom}_{\mathcal{K}}(-, \mathcal{V})$ is exact on the category of quasi-coherent $\mathcal{A}^*$-modules. Therefore $R^{\cdot}(\mathcal{V})$ is a resolution of $\mathcal{V}$ by injective quasi-coherent $\mathcal{A}^*$-modules.

Let E be an extension of quasi-coherent $\mathcal{A}^*$-modules of the quasi-coherent $\mathcal{A}^*$-module V by the quasi-coherent $\mathcal{A}^*$-module W (see [H, III, 1] or [K, 2.6] for the definition of extension of modules). The automorphisms of extensions of E identifies with $\mathrm{Hom}_{\mathcal{A}^*}(V, W)$. If W is an injective $\mathcal{A}^*$-module then $E = W \oplus V$. By the standard arguments from the descent theory we get the following

Theorem 8.2. *The extensions of* $\mathcal{A}^*$*-modules of* V *by* W*, modulo isomorphisms of extensions, are classified by the group* $\mathrm{Ext}^1_{\mathcal{A}^*}(V, W)$*.*

Given a morphism of $\mathcal{K}$-algebra functors $\chi : \mathcal{A}^* \to \mathcal{K}$ and a quasi-coherent $\mathcal{A}^*$-module V let us denote by $V^\chi := \{e \in V \mid w \cdot e = \chi(w) \cdot e \,\forall\, w \in \mathcal{A}^*\}$. We will denote by $H^{\cdot}(\chi, V)$ the derived functors of the functor $V \mapsto V^\chi$. Let us notice that

$$V^\chi = \mathrm{Hom}_{\mathcal{A}^*}(K, V)$$

hence $H^{\cdot}(\chi, V) = \mathrm{Ext}^{\cdot}_{\mathcal{A}^*}(K, V)$.

Let $G = \mathrm{Spec}\ A$ be a K-group. Given a G-module V let $V^G = \{e \in V \mid g \cdot e = e \,\forall\, g \in G^{\cdot}\}$. $H^{\cdot}(G, V)$ is defined as the derived functors of the functor $V \rightsquigarrow V^G$. The morphism $G \to \{1\}$, $g \mapsto 1$ defines a morphism $\chi : \mathcal{A}^* \to \mathcal{K}$ and $V^G = V^\chi$. By Theorem 5.5 next proposition holds.

Proposition 8.3. $H^{\cdot}(G, V) = H^{\cdot}(\chi, V) = \mathrm{Ext}^{\cdot}_{\mathcal{A}^*}(K, V)$

Corollary 8.4. [S, 8.6] *Let $G = \mathrm{Spec}\, A$ be an algebraic group. The extensions of G-modules of K by V are classified by $H^1(G, V)$.*

Proof. The extensions of G-modules of K by V are classified by $\mathrm{Ext}^1_{\mathcal{A}^*}(K, V) = H^1(G, V)$. □

Dually, let us consider the category of right $\mathcal{A}^*$-module schemes (i.e., left $\mathcal{A}^{*\circ}$-modules). Given a right $\mathcal{A}^*$-module $\mathcal{V}^*$, we will have the resolution by projective $\mathcal{A}^*$-module schemes of $\mathcal{V}^*$

$$\dots \mathcal{V}^* \bar{\otimes} \mathcal{A}^* \bar{\otimes} \mathcal{A}^* \rightrightarrows \mathcal{V}^* \bar{\otimes} \mathcal{A}^* \to \mathcal{V}^* \tag{3}$$

where the morphisms are $\mathcal{V}^* \bar{\otimes} \mathcal{A}^* \bar{\otimes} \overset{n}{\dots} \bar{\otimes} \mathcal{A}^* \to \mathcal{V}^* \bar{\otimes} \mathcal{A}^* \bar{\otimes} \overset{n-1}{\dots} \bar{\otimes} \mathcal{A}^*$, $a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto a_0 \otimes a_1 \otimes \overset{i}{\dots} \otimes a_i a_{i+1} \otimes \dots \otimes a_n$ for $0 \le i \le n$ and $a_0 \in V^*$. So we have that

$$\mathrm{Ext}^i_{\mathcal{A}^*}(W, V) = \mathrm{Ext}^i_{\mathcal{A}^{*\circ}}(\mathcal{V}^*, \mathcal{W}^*)$$

Let us suppose $\mathcal{V}^* = \mathcal{A}^*$, which is also a left $\mathcal{A}^*$-module, i.e., precisely it is an $\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}$-module. Then the equation (3) is a resolution of $\mathcal{A}^*$ by projective $\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}$-module schemes, which is split as a sequence of left $\mathcal{A}^*$-modules. Given a morphism of $\mathcal{K}$-algebras $\chi : \mathcal{A}^* \to \mathcal{K}$, every left $\mathcal{A}^*$-module $\mathcal{W}^*$ can be seen as an $\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}$-module, where $\mathcal{A}^*$ operates on the right by χ. It holds that

$$\mathrm{Ext}^i_{\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}}(\mathcal{A}^*, \mathcal{W}^*) = \mathrm{Ext}^i_{\mathcal{A}^*}(\mathcal{K}, \mathcal{W}^*)$$

Let us suppose V is a G-vector space of finite dimension, then

$$H^i(G, V) = \mathrm{Ext}^i_{\mathcal{A}^*}(K, V) = \mathrm{Ext}^i_{\mathcal{A}^* \bar{\otimes} \mathcal{A}^{*\circ}}(\mathcal{A}^*, \mathcal{V})$$

Let B be a (singular) extension of algebras of an algebra A by an $A \otimes_K A^\circ$-module M (see [M, X, 3] for the definition of extension of algebras). Giving an isomorphism of extensions of algebras from B to the trivial extension $T = A \oplus M$ ($m_i \cdot m_j = 0 \,\forall\, m_i, m_j \in M$) is equivalent to giving a K-derivation $D : B \to M$ such that D on M is the identity morphism. Let us suppose $M = \mathrm{Hom}_K(A \otimes_k A^\circ, N)$. If we apply $\mathrm{Hom}_{A \otimes A^\circ}(-, M) = \mathrm{Hom}_K(-, N)$ to the exact sequence of differentials

$$0 \to M \to \Delta_B \otimes_{B \otimes B^\circ} A \otimes A^\circ \to \Delta_A \to 0$$

associated to the exact sequence $0 \to M \to B \to A \to 0$, we obtain that the morphism $Der_K(B, M) \to \mathrm{Hom}_{A \otimes A^\circ}(M, M)$ is a surjection. Therefore there exist a derivation $D : B \to M$ such that on M is the identity. In conclusion, if $M = \mathrm{Hom}_K(A \otimes_k A^\circ, N)$ then B is isomorphic to the trivial extension.

Theorem 8.5. *The extensions of $\mathcal{K}$-algebra schemes of a $\mathcal{K}$-algebra scheme $\mathcal{A}^*$ by an $\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}$-module scheme $\mathcal{V}^*$ are classified by the group*

$$\mathrm{Ext}^2_{\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}}(\mathcal{A}^*,\mathcal{V}^*)=\mathrm{Ext}^2_{\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}}(\mathcal{V},\mathcal{A})$$

Proof. Let us follow the standard notation from of the descent theory (see [W, 17]). The extensions of algebra functors of $\mathcal{A}^*$ by $\mathcal{V}^*$ are classified by the group

$$\begin{aligned} H^1(R^\circ(\mathcal{V}^*)/\mathcal{V}^*, Aut_{alg.ext.}(\mathcal{A}^*\oplus\mathcal{V}^*)) &= H^1(R^\circ(\mathcal{V}^*)/\mathcal{V}^*, \mathrm{Der}_{\mathcal{K}}(\mathcal{A}^*,-)) \\ &= H^1(R^\circ(\mathcal{V}^*)/\mathcal{V}^*, \mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(\Delta_{\mathcal{A}^*},-)) \\ &= H^1(\mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(\Delta_{\mathcal{A}^*}, R^{\cdot}(\mathcal{V}^*)) \\ &\overset{*}{=} H^2(\mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(\mathcal{A}^*, R^{\cdot}(\mathcal{V}^*))) \end{aligned}$$

where $\overset{*}{=}$ follows from applying $\mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(-,R^{\cdot}(\mathcal{V}^*))$ to the $\mathcal{K}$-split exact sequence $0\to\Delta_{\mathcal{A}^*}\to\mathcal{A}^*\otimes\mathcal{A}^*\to\mathcal{A}^*\to 0$ and later taking cohomology.

Finally, we have

$$\begin{aligned} H^2(\mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(\mathcal{A}^*, R^{\cdot}(\mathcal{V}^*))) &= \dots \\ &= H^2(\mathrm{Hom}_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(V, R^{\cdot}(A))) = \mathrm{Ext}^2_{\mathcal{A}^*\otimes\mathcal{A}^{*\circ}}(V,A). \end{aligned}$$

□

Let $G=\mathrm{Spec}\,A$ be a K-group and let V be a G-vector space where $\dim_K V<\infty$.

Definition 8.6. [S, 8.2] *Let us denote by $1+V$ the algebraic group $\mathrm{Spec}\,S^{\cdot}V^*$, whose functor of points is $\mathcal{V}$. Given an exact sequence of affine K-groups*

$$1\to 1+V\to G'\overset{\pi}{\to} G\to 1$$

such that $g'\cdot(1+v)\cdot g'^{-1}=1+g(v)$ for all $g'\in G'$ and $v\in V$ (where $g=\pi(g')$) we shall say G' is an extension of groups of G by V.

The morphism $G\to 1$ induces the morphism of $\mathcal{K}$-algebra schemes $\chi:\mathcal{A}^*\to\mathcal{K}$.

Theorem 8.7. *The set of extensions of groups of G by V, modulo isomorphisms, is equal to the set of extensions of algebras of $\mathcal{A}^*$ by the $\mathcal{A}^*\otimes\mathcal{A}^{*\circ}$-module V, modulo isomorphisms (where $\mathcal{A}^*$ operates on V on the right by χ).*

Proof. By [H, VI, 10.3] or [S, 8.8], the set of extension of groups of G by V, modulo isomorphisms, is equal to $H^2(G,V)=\mathrm{Ext}^2_{\mathcal{A}^*}(K,V)=\mathrm{Ext}^2_{\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}}(\mathcal{A}^*,\mathcal{V})$, which coincides with the set of extensions of algebras of $\mathcal{A}^*$ by the $\mathcal{A}^*\bar{\otimes}\mathcal{A}^{*\circ}$-module V, modulo isomorphisms. □

Let us give explicitly the correspondence between the extensions of groups of $G=\mathrm{Spec}\,A$ by V and the extensions of algebras of $\mathcal{A}^*$ by V.

Let $\mathcal{B}^*$ be a (singular) extension of algebras of $\mathcal{A}^*$ by V, i.e., we have the exact sequence

$$0\to\mathcal{V}\to\mathcal{B}^*\overset{\pi}{\to}\mathcal{A}^*\to 0$$

where $\mathcal{V}^2=0$. If we consider the inclusion $G\subset\mathcal{A}^*$, then $\pi^{-1}(G)$ is an extension of groups of G by $V\overset{L}{\simeq}\pi^{-1}(1)$, where $L(v):=1+v$.

Inversely, let be an extension of groups $1\to N\to G'\overset{\pi}{\to}G\to 1$ where $N=1+V$. Let is consider the inclusion $\mathcal{V}\to\mathcal{K}G'^{\cdot}$, $v\mapsto(1+v)-1$, where $(1+v)\in N$. Let us denote $v=(1+v)-1\in\mathcal{K}G'^{\cdot}$. Let $\mathbb{I}=(\lambda g'\cdot v-\pi(g')(\lambda v))_{g'\in G',\lambda\in\mathcal{K},v\in\mathcal{V}}$ be the bilateral ideal functor of $\mathcal{K}G'^{\cdot}$. Then it holds that the kernel of the natural epimorphism $\mathcal{K}G'^{\cdot}/\mathbb{I} \to \mathcal{K}G^{\cdot}$ is $\mathcal{V}$. Taking closure, if we denote $\mathcal{B}^* = \overline{\mathcal{K}G'^{\cdot}/\mathbb{I}}$, we have an extension of algebras

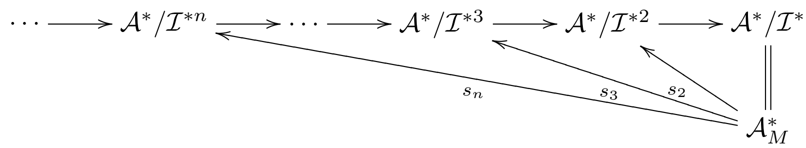
$$0 \to \mathcal{V} \to \mathcal{B}^* \to \mathcal{A}^* \to 0$$

Both assignments are mutually inverse.

Now let us generalize the Principal Theorem of Wedderburn-Malcev to algebra schemes.

Theorem 8.8. *Let K be an algebraically closed field, let $\mathcal{A}^*$ be a $\mathcal{K}$-algebra scheme, let $\mathcal{A}^*_M$ be its maximal semisimple quotient scheme and let $\mathcal{I}^*$ be the radical of $\mathcal{A}^*$. The morphism $\mathcal{A}^* \to \mathcal{A}^*_M$ has a section of algebra functors, which is the only one up to conjugations by elements of $1+\mathcal{I}^*$.*

Proof. $\mathcal{A}^*_M$ is a semisimple algebra scheme, then it is a product of algebras of matrices. Therefore, $\mathcal{A}^*_M \bar{\otimes} \mathcal{A}^*_M$ is a product of algebras of matrices, then it is semisimple. By Lemma 7.5 every extension of algebra schemes of $\mathcal{A}^*_M$ by any $\mathcal{A}^*_M \bar{\otimes} \mathcal{A}^*_M$-module scheme is trivial. $\mathcal{A}^*/\mathcal{I}^{*2}$ is an extension of algebra schemes of $\mathcal{A}^*_M$ by $\mathcal{I}^*/\mathcal{I}^{*2}$, therefore, the epimorphism $\pi_2 : \mathcal{A}^*/\mathcal{I}^{*2} \to \mathcal{A}^*_M$ has a section s_2. Let $\pi : \mathcal{A}^*/\mathcal{I}^{*3} \to \mathcal{A}^*/\mathcal{I}^{*2}$ be the natural epimorphism and let $\mathcal{B}^* = \pi^{-1}(s_2(\mathcal{A}^*_M)) \subset \mathcal{A}^*/\mathcal{I}^{*3}$. $\mathcal{B}^*$ is an algebra scheme extension of $\mathcal{A}^*_M$ by $\mathcal{I}^{*2}/\mathcal{I}^{*3}$, therefore, there exists a section $s' : \mathcal{A}^*_M \to \mathcal{B}^*$. As $\mathcal{B}^* \subset \mathcal{A}^*/\mathcal{I}^{*3}$, we have a morphism $s_3 : \mathcal{A}^*_M \to \mathcal{A}^*/\mathcal{I}^{*3}$. Acting this way, we finally obtain a commutative diagram of arrows



that defines the section $\mathcal{A}^*_M \to \mathcal{A}^*$ we looked for, since $\mathcal{A}^* = \lim\limits_{\overleftarrow{n}} \mathcal{A}^*/\mathcal{I}^{*n}$ by Theorem 6.17.

Let s_1, s_2 be two sections of algebra schemes of the epimorphism $\mathcal{A}^* \to \mathcal{A}^*_M$. The induced morphisms $\bar{s}_1, \bar{s}_2 : \mathcal{A}^*_M \to \mathcal{A}^*/\mathcal{I}^{*^2}$ differ on an element of $\mathrm{Der}_{\mathcal{K}}(\mathcal{A}^*_M, \bar{\mathcal{I}}^*) = \mathrm{Hom}_{\mathcal{A}^*_M \otimes_{\mathcal{K}} \mathcal{A}^{*\circ}_M}(\bar{\Delta}_{\mathcal{A}^*_M}, \bar{\mathcal{I}}^*)$, where $\bar{\mathcal{I}}^* = \mathcal{I}/\mathcal{I}^{*2}$. Moreover, the natural morphism

$$\bar{\mathcal{I}}^*(K) = \mathrm{Hom}_{\mathcal{A}^*_M \otimes_{\mathcal{K}} \mathcal{A}^{*\circ}_M}(\mathcal{A}^*_M \otimes \mathcal{A}^{*\circ}_M, \bar{\mathcal{I}}^*) \to \mathrm{Hom}_{\mathcal{A}^*_M \otimes_{\mathcal{K}} \mathcal{A}^{*\circ}_M}(\bar{\Delta}_{\mathcal{A}^*_M}, \bar{\mathcal{I}}^*)$$

is surjective because $\mathrm{Ext}^1_{\mathcal{A}^*_M \bar{\otimes} \mathcal{A}^{*\circ}_M}(\mathcal{A}^*_M, \bar{\mathcal{I}}^*) = 0$ by Theorem 7.10. In conclusion, there exists an $i_1 \in \mathcal{I}^*(K)$ such that $\bar{s}_2(m) = (1+i_1) \cdot \bar{s}_1(m) \cdot (1+i_1)^{-1}$. Let s'_2 be the composite of s_1 with the automorphism of $\mathcal{A}^*$ which is to conjugate by $1+i_1$. The induced morphisms $\bar{s}_2$, $\bar{s}'_2 : \mathcal{A}^*_M \to \mathcal{A}^*/\mathcal{I}^{*3}$ differs on an element of $\mathrm{Der}_{\mathcal{K}}(\mathcal{A}^*_M, \bar{\mathcal{I}}^{*2}) = \mathrm{Hom}_{\mathcal{A}^*_M \otimes_{\mathcal{K}} \mathcal{M}^{*\circ}}(\bar{\Delta}_{\mathcal{A}^*_M}, \bar{\mathcal{I}}^{*2})$, where $\bar{\mathcal{I}}^{*2} = \mathcal{I}^{*2}/\mathcal{I}^{*3}$. But the natural morphism $\bar{\mathcal{I}}^{*2}(K) = \mathrm{Hom}_{\mathcal{A}^*_M \otimes_{\mathcal{K}} \mathcal{A}^{*\circ}_M}(\mathcal{A}^*_M \otimes \mathcal{A}^{*\circ}_M, \bar{\mathcal{I}}^{*2}) \to \mathrm{Hom}_{\mathcal{A}^*_M \otimes_{\mathcal{K}} \mathcal{M}^{*\circ}}(\bar{\Delta}_{\mathcal{A}^*_M}, \bar{\mathcal{I}}^{*2})$ is surjective. Therefore there exists an $i_2 \in \mathcal{I}^{*2}(K)$ such that $\bar{s}_2$ is the composite of $\bar{s}'_2$ and the automorphism of conjugation by $1+i_2$. Then, modulo $\mathcal{I}^{*3}$, s_2 is equal to the composite of s_1 and the conjugation by $1+i_1+i_2$. Arguing this way we obtain that s_2 is equal to the composite of s_1 and the conjugation by an element of $1+\mathcal{I}^*$. □

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