A CHARACTERIZATION OF LINEARLY SEMISIMPLE GROUPS

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Abstract. Let $G = \operatorname{Spec} A$ be an affine K-group scheme and $\tilde{A} = \{w \in A^*\colon \dim_K A^* \cdot w \cdot A^* < \infty\}$. Let $\langle -, - \rangle : A^* \times \tilde{A} \to K$, $\langle w, \tilde{w} \rangle := tr(w\tilde{w})$, be the trace form. We prove that G is linearly reductive if and only if the trace form is non-degenerate on A^*.

1. Introduction

Let K be a field and let $\bar{K}$ be its algebraic closure. A finite K-algebra (associative with unit) B is separable (i.e., $B \otimes_K \bar{K} = \prod_i M_{n_i}(\bar{K})$) and $g.c.d.\{n_i, \operatorname{char} K\} = 1$, for every i, if and only if the trace form of B is non-degenerate.

Let G be a finite group of order n and let KG be the group ring of G over K. G is linearly semisimple if and only if KG is a separable K-algebra. By Maschke's theorem G is linearly semisimple if and only if $g.c.d.\{n, \operatorname{char} K\} = 1$. It holds that G is linearly semisimple if and only if the trace form of KG is non-degenerate. This theorem has been generalized to finite-dimensional Hopf algebras and Frobenius algebras (see [LS] and [M]).

Now let $G = \operatorname{Spec} A$ be an affine K-group scheme. Let $\tilde{A} = \{w \in A^*\colon \dim_K A^* \cdot w < \infty \text{ and } \dim_K w \cdot A^* < \infty\} \subseteq A^*$, which is a bilateral ideal of A^*. Let $tr\colon \tilde{A} \to K$ be defined by $tr(\tilde{w})$ = trace of the endomorphism of $\tilde{A}$, $\tilde{w'} \mapsto \tilde{w}\tilde{w}'$, and $\langle -, - \rangle : A^* \times \tilde{A} \to K$, $\langle w, \tilde{w} \rangle := tr(w\tilde{w})$, the associated trace form.

In this paper we prove that G is linearly semisimple if and only if the trace form is non-degenerate on A^*, that is, $\varphi\colon A^* \to \tilde{A}^*$, $\varphi(w) := \langle w, - \rangle$ is injective. We also prove that G is linearly semisimple if and only if the trace form is non-degenerate on $\tilde{A}$, that is, $\phi\colon \tilde{A} \to \tilde{A}^*$, $\phi(\tilde{w}) := \langle -, \tilde{w} \rangle$ is injective, and $\tilde{A}$ is dense in A^*.

$G = \operatorname{Spec} A$ is linearly semisimple if and only if there exists $w_G \in {A^*}^G$ such that $w_G(1) = 1$ (see [Sw]). Let $*\colon A \to A$, $a \mapsto a^*$ be the morphism induced on A by the morphism $G \to G$, $g \mapsto g^{-1}$ and let $F\colon A \to A^*$, $F(a) := w_G(a^* \cdot -)$ be the "Fourier transform". In this situation we prove that $\phi\colon \tilde{A} \to \tilde{A}^*$ maps bijectively onto A and that the inverse morphism $\phi^{-1}\colon A \to \tilde{A}$ coincides with the Fourier Transform.

2. Preliminary results

If G is a smooth algebraic group over an algebraically closed field one may only consider the rational points of G in order to resolve many questions in the theory of linear representations of G. In general, for any K-group scheme, one may regard the functor of points of the group scheme and its linear representations and in this way K-groups schemes and their linear representations may be handled as mere sets, as it is well known.

Let R be a commutative ring (associative with unit). All functors considered in this paper are covariant functors over the category of commutative R-algebras (with unit). Given an R-module M, the functor $\mathcal{M}$ defined by $\mathcal{M}(S) := M \otimes_R S$ is called a quasi-coherent $\mathcal{R}$-module. The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R)$ establish an equivalence between the category of R-modules and the category of quasi-coherent $\mathcal{R}$-modules ([A1, 1.12]). In particular, $\mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') = \mathrm{Hom}_R(M, M')$.

If $\mathbb{M}$ and $\mathbb{N}$ are functors of $\mathcal{R}$-modules, we will denote by $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ the functor of $\mathcal{R}$-modules

$$\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})(S) := \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{N}_{|S}),$$

where $\mathbb{M}_{|S}$ is the functor $\mathbb{M}$ restricted to the category of commutative S-algebras. We denote $\mathbb{M}^* = \mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ and given an R-module M, then

$$\mathcal{M}^*(S) = \mathrm{Hom}_S(M \otimes_R S, S) = \mathrm{Hom}_R(M, S).$$

We will say that $\mathcal{M}^*$ is an $\mathcal{R}$-module scheme. By [A1, 1.10], it holds that

$$\mathcal{M}^{**} = \mathcal{M}.$$

Remark 2.1. *Every morphism $M^* \to M'^*$ considered in this paper will be a functorial morphism, that is, we will actually have a morphism $\mathcal{M}^* \to \mathcal{M}'^*$. Observe that the dual morphism is a morphism $\mathcal{M}' \to \mathcal{M}$, which is induced by a morphism $M' \to M$. Then $M^* \to M'^*$ is the dual morphism of $M' \to M$.*

Let $G = \mathrm{Spec}\, A$ be an affine R-group and let $G^{\cdot}$ be the functor of points of G, that is, $G^{\cdot}(S) = \mathrm{Hom}_{R-alg}(A, S)$. $\mathcal{A}^*$ is an R-algebra scheme, that is, besides from being an R-module scheme it is a functor of algebras ($\mathcal{A}^*(S)$ is a S-algebra such that $S \subseteq Z(\mathcal{A}^*(S))$). One has a natural morphism of functors of monoids $G^{\cdot} \to \mathcal{A}^*$, because $G^{\cdot}(S) = \mathrm{Hom}_{R-alg}(A, S) \subset \mathrm{Hom}_R(A, S) = \mathcal{A}^*(S)$. If $\mathbb{M}^*$ is a functor of $\mathcal{R}$-modules (resp. $\mathcal{R}$-algebras), any morphism of functors of sets (resp. monoids) $G^{\cdot} \to \mathbb{M}^*$ factorizes uniquely through a morphism of functors of modules (resp. algebras) $\mathcal{A}^* \to \mathbb{M}^*$ ([A1, 5.3]). Therefore, the category of linear (and rational) representations of G is equal to the category of quasi-coherent $\mathcal{A}^*$-modules ([A1, 5.5]).

Remark 2.2. *For abbreviation, we sometimes use $g \in G$ or $m \in \mathbb{M}$ to denote $g \in G^{\cdot}(S)$ or $m \in \mathbb{M}(S)$ respectively. Given $m \in \mathbb{M}(S)$ and a morphism of R-algebras $S \to S'$, we still denote by m its image by the morphism $\mathbb{M}(S) \to \mathbb{M}(S')$.*

Definition 2.3. *Let $R = K$ be a field. We will say that an affine K-group scheme G is linearly semisimple if it is linearly reductive, that is, if every linear representation of G is completely reducible.*

G is linearly semisimple if and only if $\mathcal{A}^*$ is semisimple, i.e., $A^* = \prod_i A_i^*$, where A_i^* are simple (and finite) K-algebras ([A1, 6.8]). If K is an algebraically closed field, then $A_i^* = \mathrm{End}_K V_i$, where V_i are the simple G-modules (up to isomorphism), by Wedderburn's theorem.

Let $V_0 = K$ be the trivial representation of G. The unit of $A_0^* = \mathrm{End}_K V_0$, $w_G := 1_0 \in A_0^* \subset A^*$ is said to be the "invariant integral on G". The invariant integral on G is characterized by being G-invariant and normalized, that is, $w_G(1) = 1$ ([A2, 2.8]). Let $\mathbb{N} = \mathbb{M}^*$ be a functor of G-modules (equivalently, a functor of $\mathcal{A}^*$-modules). The morphism $w_G \cdot : \mathbb{N} \to \mathbb{N}, n \mapsto w_G \cdot n$ is the unique projection of G-modules of $\mathbb{N}$ onto $\mathbb{N}^G$, and $\mathbb{N}^G = w_G \cdot \mathbb{N}$ (see [A2, 2.3, 3.3]).

3. Trace form

Let us assume that $R = K$ is a field.

Let $\mathcal{A}^*$ be a $\mathcal{K}$-algebra scheme and let $\tilde{A} = \{w \in A^* \colon \dim_K A^* \cdot w < \infty, \dim_K w \cdot A^* < \infty\}$. It holds that $\tilde{A}$ is a bilateral ideal of A^*.

Remark 3.1. *Functorially, the $\mathcal{K}$-quasi-coherent module associated to $\tilde{A}$, $\tilde{\mathcal{A}}$, is the maximal $\mathcal{K}$-quasi-coherent bilateral ideal $\mathcal{I} \subset \mathcal{A}^*$: Given $w \in \tilde{A}$, consider the morphism of $\mathcal{A}^*$-modules $f \colon \mathcal{A}^* \to \mathcal{A}^*$, $f(w') := w' \cdot w$. Let $\mathcal{B}^* = \operatorname{Im} f = \mathcal{A}^*/\ker f$. Observe that $B^* = A^* \cdot w$, then $\dim_K B^* < \infty$ and $\mathcal{B}^*$ is a $\mathcal{K}$-quasi-coherent module. Since $B^* \subset \tilde{A}$, then $\mathcal{B}^* \subset \tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ is a left $\mathcal{A}^*$-submodule of $\mathcal{A}^*$. Likewise, $\tilde{\mathcal{A}}$ is a right $\mathcal{A}^*$-submodule of $\mathcal{A}^*$. Hence, $\tilde{\mathcal{A}} \subseteq \mathcal{I}$. If $\mathcal{M}$ is a $\mathcal{K}$-quasi-coherent left $\mathcal{A}^*$-module (resp. right $\mathcal{A}^*$-module) and $m \in M$, then $\dim_K A^* \cdot m < \infty$ (resp. $\dim_K m \cdot A^* < \infty$) by* [A1, 4.7]. *Now it follows easily that $\mathcal{I} = \tilde{\mathcal{A}}$.*

Definition 3.2. *Let $\mathcal{A}^*$ be a $\mathcal{K}$-algebra scheme. Let $tr \colon \tilde{\mathcal{A}} \to \mathcal{K}$ be defined by $tr(\tilde{w}) :=$ trace of the endomorphism of $\tilde{A}$, $\tilde{w}' \mapsto \tilde{w}\tilde{w}'$. We will say that*

$$\langle -,- \rangle : \mathcal{A}^* \times \tilde{\mathcal{A}} \to \mathcal{K},\ \langle w, \tilde{w} \rangle := tr(w\tilde{w})$$

is the trace form of $\mathcal{A}^*$.

We have the associated "polarities" to the trace form

$$\varphi \colon \mathcal{A}^* \to \tilde{\mathcal{A}}^*,\ \varphi(w) := \langle w, - \rangle\,, \qquad \phi \colon \tilde{\mathcal{A}} \to \mathcal{A}^{**} = \mathcal{A},\ \phi(\tilde{w}) := \langle -, \tilde{w} \rangle\,.$$

The dual morphism $\phi^* \colon \mathcal{A}^* \to \tilde{\mathcal{A}}^*$ of ϕ is equal to φ:

$$(\phi^*(w))(\tilde{w}) = w(\phi(\tilde{w})) = (\phi(\tilde{w}))(w) = \langle w, \tilde{w} \rangle = \varphi(w)(\tilde{w})$$

Likewise, the dual morphism of φ is ϕ.

Definition 3.3. *Let $G = \operatorname{Spec} A$ an affine K-group scheme. We will say that the trace form of $\mathcal{A}^*$ is the trace form of G.*

Let C be a K-algebra ($K \subseteq Z(C)$). If M is a left C-module (resp. a right C-module), then $M^* = \operatorname{Hom}_K(M, K)$ is a right C-module (resp. a left C-module): $(w \cdot c)(m) := w(c \cdot m)$ (resp. $(c \cdot w)(m) := w(m \cdot c)$) for all $w \in M^*$, $m \in M$ and $c \in C$.

The morphism $\varphi \colon \mathcal{A}^* \to \tilde{\mathcal{A}}^*$ is a morphism of left and right $\mathcal{A}^*$-modules:

$$\begin{aligned} \varphi(w_1 w_2)(\tilde{w}) &= tr(w_1 w_2 \tilde{w}) = tr(w_2 \tilde{w} w_1) = \varphi(w_2)(\tilde{w} w_1) = (w_1 \varphi(w_2))(\tilde{w}) \\ \varphi(w_1 w_2)(\tilde{w}) &= tr(w_1 w_2 \tilde{w}) = \varphi(w_1)(w_2 \tilde{w}) = (\varphi(w_1) w_2)(\tilde{w}) \end{aligned}$$

for all $w_1, w_2 \in \mathcal{A}^*$ and $\tilde{w} \in \tilde{\mathcal{A}}$. Likewise, ϕ is a morphism of left and right $\mathcal{A}^*$-modules.

If $A^* = \prod_i A_i^*$, where A_i^* are finite K-algebras, then $\tilde{A} = \oplus_i A_i^*$. In this case we have the morphism $\phi \colon \tilde{A} = \oplus A_i^* \to \oplus_i A_i = A$. Given $1_i := (0, \ldots, \overset{i}{1}, \ldots, 0) \in \tilde{A}$, then $\phi(1_i) = tr_{A_i^*}$ (where $tr_{A_i^*} \in A_i$ is the trace, tr, on A_i^*). Obviously, $\tilde{A}$ is an A^*-module generated by $\{1_i\}_i$, and

$$\langle 1_i, 1_j \rangle = tr(1_i \cdot 1_j) = \begin{cases} \dim_K A_i^* & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Example 3.4. *Let* $G_m = \operatorname{Spec}\mathbb{C}[x,1/x]$ *be the multiplicative group. It holds that* $\mathbb{C}[x,1/x]^* = \prod_{\mathbb{Z}}\mathbb{C}$, $\widetilde{\mathbb{C}[x,1/x]} = \oplus_{\mathbb{Z}}\mathbb{C}$ *and*

$$\phi\colon \widetilde{\mathbb{C}[x,1/x]} = \bigoplus_{\mathbb{Z}}\mathbb{C} \to \bigoplus_{n\in\mathbb{Z}}\mathbb{C}\cdot x^n = \mathbb{C}[x,1/x], \quad \phi((a_n)_{n\in\mathbb{Z}}) = \sum_n a_n x^n.$$

Let $G_a = \operatorname{Spec}\mathbb{C}[x]$ *be the additive group. It holds that* $\mathbb{C}[x]^* = \mathbb{C}[[z]]$ *(*$z^n(x^m) = \delta_{nm}\cdot n!$*) and* $\widetilde{\mathbb{C}[[z]]} = 0$.

Let V be a linear representation of an affine K-group scheme $G = \operatorname{Spec}A$. The associated character $\chi_V \in A$ is defined by $\chi_V(g) =$ trace of the linear endomorphism $V \to V$, $v \mapsto g\cdot v$, for every $g \in G$ and $v \in V$.

Let $G = \operatorname{Spec}A$ be a linearly semisimple affine K-group scheme. One has $A^* = \prod_i A_i^*$, where A_i^* are finite simple algebras. Assume for simplicity that K is an algebraically closed field, then $A_i^* = \operatorname{End}_K(V_i)$. Observe that $tr_{A_i^*} = n_i\cdot\chi_{V_i}$, where $n_i = \dim_K V_i$.

Let $V_0 = K$ be the trivial representation of G and let $w_G := 1_0 \in A^*$ be the "invariant integral on G".

Given $a \in A$, then $w_G\cdot a \in K = A^G$. Hence $w_G\cdot a = (w_G\cdot a)(1) = a(w_G) = w_G(a)$.

One has $w_G\cdot A_j = 1_0\cdot A_j = 0$, if $j \neq 0$ and $w_G\cdot a_0 = a_0$ for all $a_0 \in A_0$. Hence, $w_G\cdot\chi_{V_j} = 0$ if V_j is not the trivial representation, and $w_G\cdot\chi_{V_0} = \chi_{V_0} = 1$. Moreover, since $\chi_{V\oplus V'} = \chi_V + \chi_{V'}$, one has

$$w_G(\chi_V) = w_G\cdot\chi_V = \dim_K V^G. \tag{1}$$

4. Characterization of linearly semisimple groups

Let $G = \operatorname{Spec}A$ be an affine K-group scheme.

Let $* : A \to A$, $a \mapsto a^*$ be the morphism induced by the morphism $G \to G$, $g \mapsto g^{-1}$. In the following proof, if V_i is a linear representation of G, we will consider V_i^* as a left G-module by $(g*w)(v) = w(g^{-1}\cdot v)$. One has that $\chi_{V_i}^* = \chi_{V_i^*}$, because the trace of $g^{-1} \in G$ operating on V_i is equal to the trace of g operating on V_i^* (which operates by the transposed inverse of g).

Theorem 4.1. *Let* $G = \operatorname{Spec}A$ *be a linearly semisimple affine* K*-group scheme and let* $w_G \in A^*$ *be its invariant integral. Let*

$$F\colon A \to A^*,\ F(a) := w_G(a^*\cdot -)$$

be the "Fourier Transform" (where $w_G(a^*\cdot -)(a') := w_G(a^*\cdot a')$*). Then* $F(A) = \tilde{A}$ *and* F *is equal to the inverse morphism of the polarity* ϕ *associated to the trace form of* G*.*

Proof. First let us prove that $F : \mathcal{A} \to \mathcal{A}^*$, $F(a) := w_G(a^*\cdot -)$ is a morphism of left G-modules. For every point $g \in G$,

$$\begin{aligned} w_G((g\cdot a)^*\cdot -) &\overset{*}{=} w_G((a^*\cdot g^{-1})\cdot -) \overset{**}{=} w_G(((a^*\cdot g^{-1})\cdot -)\cdot g) = w_G(a^*\cdot(-\cdot g)) \\ &= g\cdot(w_G(a^*\cdot -)) \end{aligned}$$

where $\overset{*}{=}$ is due to $(g\cdot a)^*(g') = (g\cdot a)(g'^{-1}) = a(g'^{-1}\cdot g) = a((g^{-1}\cdot g')^{-1}) = (a^*\cdot g^{-1})(g')$, and $\overset{**}{=}$ is due to $g\cdot w_G = w_G$. Likewise, F is a morphism of right G-modules. Then F is a morphism of left and right $\mathcal{A}^*$-modules and $F(A) \subseteq \tilde{A}$.

Assume K is an algebraically closed field. Then $A^* = \prod_i A_i^*$, where $A_i^* = \operatorname{End}_K(V_i) = \mathrm{M}_{n_i}(K)$. Observe that $M_{n_i}(K) \simeq M_{n_i}(K)^*$ as left and right $M_{n_i}(K)$-modules. Then $\tilde{A} = \oplus_i M_{n_i}(K)^*$ and $A = \oplus_i M_{n_i}(K)$ are isomorphous as left and right A^*-modules. Let us see that $F \circ \phi = z\cdot$, where $z = (z_i) \in \prod_i K = Z(A^*)$:

$$F \circ \phi \in \operatorname{Hom}_{A^* \otimes A^*}(\tilde{A}, \tilde{A}) = \operatorname{Hom}_{A^* \otimes A^*}(A, A) \subseteq \operatorname{Hom}_{A^* \otimes A^*}(A^*, A^*) = Z(A^*).$$

As $F(tr_{A_i^*}) = F(\phi(1_i)) = z \cdot 1_i = z_i \cdot 1_i$, then $F(\chi_{V_i}) = (z_i/n_i) \cdot 1_i$. Observe that

$$F(\chi_{V_i})(\chi_{V_j}) = w_G(\chi_{V_i^* \otimes V_j}) = w_G \cdot \chi_{V_i^* \otimes V_j} \overset{\text{Eq.1}}{=} \dim_K \operatorname{Hom}_G(V_i, V_j) = \delta_{ij}.$$

Hence, $1 = F(\chi_{V_i})(\chi_{V_i}) = ((z_i/n_i) \cdot 1_i)(\chi_{V_i}) = (z_i/n_i) \cdot n_i = z_i$ and $F \circ \phi = \mathrm{Id}$. Then ϕ is an injective morphism. Since $\phi(A_i^*) \subseteq A_i$, then $\phi(A_i^*) = A_i$ and $\phi\colon \tilde{A} = \oplus_i A_i^* \to \oplus_i A_i = A$ is an isomorphism. As a consequence, $\phi \circ F = \mathrm{Id}$. □

Theorem 4.2. *Let $G = \operatorname{Spec} A$ be an affine K-group scheme. G is linearly semisimple if and only if the polarity $\phi\colon \tilde{A} \to A$ associated to the trace form of G is an isomorphism.*

Proof. Let us assume that $\phi\colon \tilde{A} \simeq A$ is a left and right A^*-module isomorphism. Observe that $1 \in A$ is left and right G-invariant. Let $w_G = \phi^{-1}(1)$, then $g \cdot w_G = w_G = g(1) \cdot w_G$ and $w_G \cdot g = w_G = g(1) \cdot w_G$. Given $w \in A^*$, $w \cdot w_G = w(1) w_G = w_G \cdot w$. Then it is easy to check that $tr(w_G) = w_G(1)$ and in general $\langle w', w_G \rangle = w'(1) w_G(1)$. As $\langle -, w_G \rangle = \phi(w_G) \neq 0$, therefore $w_G(1) \neq 0$. Then, $w' = w_G / w_G(1)$ is a normalized invariant integral on G and G is linearly semisimple.

Reciprocally, if G is linearly semisimple, by the previous theorem ϕ is an isomorphism, whose inverse morphism is the Fourier transform. □

Definition 4.3. *We will say that the trace form of $G = \operatorname{Spec} A$ is non-degenerate on A^* if the associated polarity, $\varphi\colon A^* \to \tilde{A}^*$, $\varphi(w) := \langle w, - \rangle$, is an injective morphism.*

Theorem 4.4. *Let $G = \operatorname{Spec} A$ be an affine K-group scheme. G is linearly semisimple if and only if its trace form is non-degenerate on A^*.*

Proof. If $\varphi\colon A^* \to \tilde{A}^*$ is injective, then dually $\phi\colon \tilde{A} \to A$ is surjective. Let us denote by $i\colon \tilde{A} \to A^*$ the natural inclusion. From the commutative diagram

$$\begin{array}{ccc} \tilde{A} & \overset{\phi}{\twoheadrightarrow} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i^*} \\ A^* & \overset{\varphi}{\hookrightarrow} & \tilde{A}^* \end{array}$$

it follows that $\phi\colon \tilde{A} \to A$ is an isomorphism and, by Theorem 4.2, G is linearly semisimple.

Reciprocally, if G is linearly semisimple, by the previous theorem $\phi\colon \tilde{A} \to A$ is an isomorphism, then dually $\varphi\colon A^* \to \tilde{A}^*$ is an isomorphism. □

Given a vector space V, on V^* we will consider the topology whose closed sets are $\{V'^0 = (V/V')^*\}_{V' \subset V}$. Given a vector subspace $W \subseteq V^*$ we denote $W^0 = \{v \in V : v(W) = 0\}$. The closure of W in V^* coincides with W^{00}. (Functorially, the closure of $\mathcal{W}$ in $\mathcal{V}^*$ is the minimum $\mathcal{K}$-scheme of submodules of $\mathcal{V}^*$ which contains $\mathcal{W}$). The closure of $\tilde{A}$ in A^* is a bilateral ideal.

Definition 4.5. *We will say that the trace form of $G = \operatorname{Spec} A$ is non-degenerate on $\tilde{A}$ if its associated polarity $\phi\colon \tilde{A} \to \tilde{A}^*$, $\phi(\tilde{w}) = \langle -, \tilde{w}\rangle$ is an injective morphism.*

Theorem 4.6. *Let $G = \operatorname{Spec} A$ be an affine K-group scheme. G is a linearly semisimple group if and only if $\tilde{A}$ is dense in A^* and its trace form is non-degenerate on $\tilde{A}$.*

Proof. Let us assume that the trace form of $G = \operatorname{Spec} A$ is non-degenerate on $\tilde{A}$. That is, the morphism $\phi\colon \tilde{A} \to A$ is injective. Since $\tilde{A}$ is dense in A^*, $\ker\varphi \cap \tilde{A} = \ker\phi = 0$. Let $w \in \ker\varphi$, then $\tilde{A}\cdot w \subset \ker\varphi \cap \tilde{A} = 0$. $A^*\cdot w = 0$ since $\tilde{A}\cdot w = 0$ is dense in $A^*\cdot w$. In conclusion, $w = 0$ and φ is injective and G is linearly semisimple, by Theorem 4.4.

Inversely, if G is linearly semisimple then $A^* = \prod_i A_i^*$, where A_i^* is a finite dimensional simple algebra, for all i, $\tilde{A} = \oplus_i A_i^*$ (which is dense in A^*) and $\oplus_i A_i^*$ is isomorphic to $\oplus_i A_i$ through ϕ.

□

5. Pontrjagin's duality

Now our aim is to write Theorem 4.6 in terms of the classical Pontrjagin's duality, which establishes the duality between compact commutative groups and discrete commutative groups.

Recall the closure of $\tilde{A}$ in A^*, $\bar{\tilde{A}}$, is a bilateral ideal. $C^* = A^*/\bar{\tilde{A}}$ is an inverse limit of finite K-algebras, by [A1, 4.12]. Hence, $\tilde{A}$ is dense in A^* if and only if for every morphism of algebra schemes $\pi\colon \mathcal{A}^* \to \mathcal{B}^*$, where $\dim_K B^* < \infty$, it holds that $\pi(\tilde{A}) = \pi(A^*)$. Since B^* injects into $\operatorname{End}_K(B^*)$, we obtain the following proposition.

Proposition 5.1. *Let $G = \operatorname{Spec} A$ be an affine K-group scheme. $\tilde{A}$ is dense in A^* if and only if for every linear representation, $G^{\cdot} \to \mathbb{E}nd_{\mathcal{K}}(\mathcal{V})$, where $\dim_K V < \infty$, the image of the induced morphism $A^* \to \operatorname{End}_K(V)$ coincides with the image of $\tilde{A}$.*

Let us assume that $\tilde{A}$ is dense in A^*. We want to determine the maximal bilateral ideals of $\tilde{A}$.

Let V be a G-module. For every $v \in V$, $A^*\cdot v \subset V$ is finite dimensional by [A1, 4.6]. Then, $v \in \tilde{A}\cdot v = A^*\cdot v$. In particular, for every $\tilde{w} \in \tilde{A}$ it holds that $\tilde{w} \in \tilde{A}\cdot\tilde{w}$.

Let $I \subseteq \tilde{A}$ be an ideal and $w \in A^*$, then $w\cdot I \subseteq I$: $w\cdot I \subseteq \tilde{A}\cdot(w\cdot I) \subseteq \tilde{A}\cdot I \subseteq I$. Then, every bilateral ideal $I \subset \tilde{A}$ is a left and right A^*-submodule, then $\mathcal{I}$ is a left and right $\mathcal{A}^*$-module, by [A1, 4.6]. If $I \subsetneq \tilde{A}$ is a maximal bilateral ideal, then $\dim_K \tilde{A}/I < \infty$, because if $\mathcal{M}$ is a left and right quasi-coherent $\mathcal{A}^*$-module and $m \in M$, then the left and right quasi-coherent $\mathcal{A}^*$-module generated by m is finite dimensional, by [A1, 4.7].

Let $I \subset \tilde{A}$ be a bilateral ideal of finite codimension and let be $\tilde{w} \in \tilde{A}$, by Proposition 5.1, such that $\tilde{w}\cdot\colon \tilde{A}/I \to \tilde{A}/I$ is the identity morphism. Then, $\bar{\tilde{w}}\cdot\bar{v} = \bar{v}$, for every $\bar{v} \in \tilde{A}/I$, that is, $\bar{\tilde{w}}$ is a left unit of $\tilde{A}/I$. Likewise, there exists a right unit, $\tilde{w}'$ in $\tilde{A}/I$, then there exists a unit $1 = \tilde{w}'\cdot\bar{\tilde{w}} \in \tilde{A}/I$. The morphism $\pi\colon \mathcal{A}^* \to \tilde{\mathcal{A}}/\mathcal{I}$, $\pi(w) = w\cdot 1$ is a surjective morphism of $\mathcal{K}$-algebra schemes. If $\mathcal{J}^* = \ker\pi$, then $A^*/J^* = \tilde{A}/I$.

If $\mathcal{B}^*$ is an algebra scheme, then $\mathcal{B}^* = \lim\limits_{\overleftarrow{i}} \mathcal{B}_i^*$ is an inverse limit of algebra schemes $\mathcal{B}_i^*$, where $\dim_K B_i < \infty$, by [A1, 4.12]. Then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{K}-alg}(\tilde{\mathcal{A}}, \mathcal{B}^*) &= \lim_{\overleftarrow{i}} \operatorname{Hom}_{\mathcal{K}-alg}(\tilde{\mathcal{A}}, \mathcal{B}_i^*) = \lim_{\overleftarrow{i}} \operatorname{Hom}_{\mathcal{K}-alg}(\mathcal{A}^*, \mathcal{B}_i^*) \\ &= \operatorname{Hom}_{\mathcal{K}-alg}(\mathcal{A}^*, \mathcal{B}^*).\end{aligned}$$

Let $G = \operatorname{Spec} A$ be an affine K-group scheme and let $\hat{G}$ be the "dual group", that is, the set of irreducible representations of G, up to isomorphism. If $\tilde{A}$ is dense in A^*, then

$$\begin{aligned}\operatorname{Spec} \tilde{A} &:= \{\text{Maximal bilateral ideals } I \subsetneq \tilde{A}\} \\ &= \{\text{Maximal bilateral ideal schemes } \mathcal{J}^* \subset \mathcal{A}^*\} \overset{[A1,\ 6.11\text{-}12]}{=} \hat{G}.\end{aligned}$$

Definition 5.2. *We will say that $\hat{G}$ is discrete if $\tilde{A}$ is dense in A^* and $\tilde{A}$ is separable (that is, $\tilde{A}$ is a direct sum of algebras of matrices by base change to the algebraic closure of K).*

Now we can rewrite Theorem 4.6 as follows.

Theorem 5.3. *Let $G = \operatorname{Spec} A$ be an affine K-group scheme. G is linearly semisimple if and only if $\hat{G}$ is discrete.*

Proof. If G is linearly semisimple, then $A^* = \prod_i A_i^*$, where A_i^* is a finite simple algebra for all i. By base change G is linearly semisimple, then A_i^* is a direct sum of algebras of matrices by base change to the algebraic closure of K. Since $\tilde{A} = \oplus_i A_i^*$, then $\tilde{A}$ is dense in A^* and it is separable.

If $\tilde{A}$ is separable then its trace form is non-degenerate. Hence, if $\tilde{A}$ is separable and dense in A^*, by Theorem 4.6, G is linearly semisimple. □

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