A DIRECT PROOF OF THE THEOREM ON FORMAL FUNCTIONS

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Abstract. We give a direct and elementary proof of the theorem on formal functions by studying the behaviour of the Godement resolution of a sheaf of modules under completion.

Introduction

Let $\pi\colon X \to \operatorname{Spec} A$ be a proper scheme over a ring A. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module and $Y \subset \operatorname{Spec} A$ a closed subscheme. Let us denote by $^\wedge$ the completion along Y (respectively, along $\pi^{-1}(Y)$). The theorem on formal functions states that

$$H^i(X,\mathcal{M})^\wedge = H^i(X,\hat{\mathcal{M}})$$

Two important corollaries of this theorem are Stein's factorization theorem and Zariski's Main Theorem ([H] III, 11.4, 11.5).

Hartshorne [H] gives a proof of the theorem on formal functions for projective schemes (over a ring). Grothendieck [G] proves it for proper schemes. He first gives sufficient conditions for the commutation of the cohomology of complexes of A-modules with inverse limits (0, 13.2.3 [G]); secondly, he gives a general theorem on the commutation of the cohomology of sheaves with inverse limits (0, 13.3.1 [G]); finally, he laboriously checks that the theorem on formal functions is under the hypothesis of this general one (4.1.5 [G]).

In this paper we give the "obvious direct proof" of the theorem on formal functions. Very briefly, we prove that the completion of the Godement resolution of a coherent sheaf is a flasque resolution of the completion of the coherent sheaf and that taking sections in the Godement complex commutes with completion.

1. Theorem on formal functions

Definition 1. Let X be a scheme, $\mathfrak{p} \subset \mathcal{O}_X$ a sheaf of ideals and $\mathcal{M}$ an $\mathcal{O}_X$-module. The $\mathfrak{p}$-adic completion of $\mathcal{M}$, denoted by $\widehat{\mathcal{M}}$, is

$$\widehat{\mathcal{M}} := \lim_{\substack{\leftarrow \\ n}} \mathcal{M}/\mathfrak{p}^n\mathcal{M}$$

If $U = \operatorname{Spec} A$ is an affine open subset and $I = \mathfrak{p}(U)$, one has a natural morphism

$$\Gamma(U,\mathcal{M}) \otimes_A A/I^n \to \Gamma(U,\mathcal{M}/\mathfrak{p}^n\mathcal{M})$$

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and then a morphism

$$\Gamma(U,\mathcal{M})^\wedge \to \Gamma(U,\widehat{\mathcal{M}})$$

where $\Gamma(U,\mathcal{M})^\frown$ is the I-adic completion of $\Gamma(U,\mathcal{M})$.

Definition 2. We say that $\mathcal{M}$ is affinely $\mathfrak{p}$-acyclic if for any affine open subset U and any natural number n, the sheaves $\mathcal{M}$ and $\mathcal{M}/\mathfrak{p}^n\mathcal{M}$ are acyclic on U and the morphism $\Gamma(U,\mathcal{M})\otimes_A A/I^n \to \Gamma(U,\mathcal{M}/\mathfrak{p}^n\mathcal{M})$ is an isomorphism. In particular, $\Gamma(U,\mathcal{M})^\wedge \to \Gamma(U,\widehat{\mathcal{M}})$ is an isomorphism.

Every quasi-coherent module is affinely $\mathfrak{p}$-acyclic.

Notations: For any sheaf F, let us denote

$$0\to F\to C^0F\to C^1F\to\cdots\to C^nF\to\cdots$$

its Godement resolution. We shall denote $C^{\cdot}F = \underset{i\geq 0}{\oplus} C^iF$ and $F_i = \operatorname{Ker}(C^iF \to C^{i+1}F)$. One has that $C^0F_i = C^iF$.

Lemma 3. *Let X be a scheme, $\mathfrak{p}$ a coherent ideal and $\mathcal{M}$ an $\mathcal{O}_X$-module. Denote $I=\Gamma(X,\mathfrak{p})$ and assume that $\mathfrak{p}$ is generated by a finite number of global sections (this holds for example when X is affine). For any open subset $V\subseteq X$ one has*

$$\Gamma(V,C^0(\mathfrak{p}\mathcal{M})) = I\cdot\Gamma(V,C^0\mathcal{M})$$

In particular, the natural morphism $\mathfrak{p}C^0M\to C^0(\mathfrak{p}\mathcal{M})$ is an isomorphism.

Proof. If J is a finitely generated ideal of a ring A and M_i is a collection of A-modules, then $J\cdot\prod M_i = \prod(J\cdot M_i)$. Now, by hypothesis $\mathfrak{p}$ is generated by a finite number of global sections $f_1,\dots,f_r$. Let $J=(f_1,\dots,f_r)$. Then

$$\Gamma(V,C^0(\mathfrak{p}\mathcal{M})) = \prod_{x\in V}\mathfrak{p}_x\cdot\mathcal{M}_x = \prod_{x\in V} J\cdot\mathcal{M}_x = J\cdot\prod_{x\in V}\mathcal{M}_x = J\cdot\Gamma(V,C^0\mathcal{M})$$

Since $I\cdot\prod_{x\in V}\mathcal{M}_x$ is contained in $\Gamma(V,C^0(\mathfrak{p}\mathcal{M}))$ one concludes. In particular, if V is affine, then $\Gamma(V,C^0(\mathfrak{p}\mathcal{M})) = I_V\cdot\Gamma(V,C^0\mathcal{M})$, with $I_V=\Gamma(V,\mathfrak{p})$. It follows that $\mathfrak{p}C^0M\to C^0(\mathfrak{p}\mathcal{M})$ is an isomorphism. □

Proposition 4. *Let X be a scheme and let $\mathfrak{p}$ be a coherent ideal. For any $\mathcal{O}_X$-module $\mathcal{M}$ one has:*

1. *$\mathfrak{p}C^i\mathcal{M} = C^i(\mathfrak{p}\mathcal{M})$ and $(C^i\mathcal{M})/\mathfrak{p}(C^i\mathcal{M}) = C^i(\mathcal{M}/\mathfrak{p}\mathcal{M})$, for any i.*
2. *$C^0\mathcal{M}$ is affinely $\mathfrak{p}$-acyclic.*
3. *$\widehat{C^0\mathcal{M}}$ is flasque. Moreover, if $\mathfrak{p}$ is generated by a finite number of global sections, then*

$$\Gamma(X,\widehat{C^0\mathcal{M}}) = \Gamma(X,C^0\mathcal{M})^\wedge$$

Proof. 1. We may assume that X is affine. Hence $\mathfrak{p}C^0\mathcal{M} = C^0(\mathfrak{p}\mathcal{M})$ by the previous lemma and $(C^0\mathcal{M})/\mathfrak{p}C^0\mathcal{M} = C^0\mathcal{M}/C^0(\mathfrak{p}\mathcal{M}) = C^0(\mathcal{M}/\mathfrak{p}\mathcal{M})$. From the exact sequence

$$\mathcal{M}/\mathfrak{p}\mathcal{M}\to C^0\mathcal{M}/\mathfrak{p}C^0\mathcal{M}\to\mathcal{M}_1/\mathfrak{p}\mathcal{M}_1\to 0$$

and the isomorphism $C^0\mathcal{M}/\mathfrak{p}C^0\mathcal{M} = C^0(\mathcal{M}/\mathfrak{p}\mathcal{M})$ it follows that $\mathcal{M}_1/\mathfrak{p}\mathcal{M}_1 = (\mathcal{M}/\mathfrak{p}\mathcal{M})_1$ and $\mathfrak{p}\mathcal{M}_1 = (\mathfrak{p}\mathcal{M})_1$. Consequently $\mathfrak{p}C^1\mathcal{M} = \mathfrak{p}C^0(\mathcal{M}_1) = C^0(\mathfrak{p}\mathcal{M}_1) = C^0((\mathfrak{p}\mathcal{M})_1) = C^1(\mathfrak{p}\mathcal{M})$, and analogously $C^1\mathcal{M}/\mathfrak{p}C^1\mathcal{M} = C^1(\mathcal{M}/\mathfrak{p}\mathcal{M})$. Repeating this argument one concludes 1.

2. Denote $\mathcal{N} = C^0\mathcal{M}$. By (1), $\mathcal{N}/\mathfrak{p}^n\mathcal{N}$ is acyclic on any open subset. From the long exact sequence of cohomology associated to $0 \to \mathfrak{p}^n\mathcal{N} \to \mathcal{N} \to \mathcal{N}/\mathfrak{p}^n\mathcal{N} \to 0$ and the acyclicity of $\mathfrak{p}^n\mathcal{N}$ (by (1)) one obtains that

$$\Gamma(U, \mathcal{N}/\mathfrak{p}^n\mathcal{N}) = \Gamma(U, \mathcal{N})/\Gamma(U, \mathfrak{p}^n\mathcal{N}).$$

Moreover, if U is affine $\Gamma(U, \mathfrak{p}^n\mathcal{N}) = \mathfrak{p}^n(U)\Gamma(U, \mathcal{N})$, by Lemma 3. We have concluded.

3. Let us prove that $\mathcal{N} = \widehat{C^0\mathcal{M}}$ is flasque. It suffices to prove that its restriction to any affine open subset is flasque, so we may assume that X is affine. Let us denote $I = \mathfrak{p}(X)$. For any open subset V, one has as in the proof of (2)

$$\Gamma(V, \widehat{\mathcal{N}}) = \lim_{\substack{\leftarrow \\ n}} \Gamma(V, \mathcal{N}/\mathfrak{p}^n\mathcal{N}) = \lim_{\substack{\leftarrow \\ n}} \Gamma(V, \mathcal{N})/\Gamma(V, \mathfrak{p}^n\mathcal{N})$$

and by Lemma 3, $\Gamma(V, \mathfrak{p}^n\mathcal{N}) = I^n\Gamma(V, \mathcal{N})$. In conclusion, $\Gamma(V, \widehat{\mathcal{N}}) = \Gamma(V, \mathcal{N})^{\frown}$. One concludes that $\widehat{\mathcal{N}}$ is flasque because $\mathcal{N}$ is flasque and the I-adic completion preserves surjections. The same arguments prove the second part of the satement. □

Proposition 5. *If $\mathcal{M}$ is affinely $\mathfrak{p}$-acyclic, then $\widehat{C^\cdot\mathcal{M}}$ is a flasque resolution of $\widehat{\mathcal{M}}$.*

Proof. We already know that $\widehat{C^\cdot\mathcal{M}}$ is flasque. Let us prove now that $\mathcal{M}_1$ is affinely $\mathfrak{p}$-acyclic. From the exact sequence

$$0 \to \mathcal{M}/\mathfrak{p}^n\mathcal{M} \to C^0(\mathcal{M}/\mathfrak{p}^n\mathcal{M}) \to \mathcal{M}_1/\mathfrak{p}^n\mathcal{M}_1 \to 0$$

one has that $\mathcal{M}_1/\mathfrak{p}^n\mathcal{M}_1$ is acyclic on any affine open subset. Moreover, taking sections on an affine open subset $U = \operatorname{Spec} A$, one obtains the exact sequence (let us denote $I = \mathfrak{p}(U)$)

$$0 \to \Gamma(U, \mathcal{M}) \otimes_A A/I^n \to \Gamma(U, C^0\mathcal{M}) \otimes_A A/I^n \to \Gamma(U, \mathcal{M}_1/\mathfrak{p}^n\mathcal{M}_1) \to 0$$

and then $\Gamma(U, \mathcal{M}_1) \otimes_A A/I^n = \Gamma(U, \mathcal{M}_1/\mathfrak{p}^n\mathcal{M}_1)$, i. e. $\mathcal{M}_1$ is affinely $\mathfrak{p}$-acyclic.

Now, taking inverse limit in the above exact sequence (and taking into account that the I-adic completion preserves surjections) one obtains the exact sequence

$$0 \to \Gamma(U, \widehat{\mathcal{M}}) \to \Gamma(U, \widehat{C^0\mathcal{M}}) \to \Gamma(U, \widehat{\mathcal{M}_1}) \to 0$$

Therefore the sequence $0 \to \widehat{\mathcal{M}} \to \widehat{C^0\mathcal{M}} \to \widehat{\mathcal{M}_1} \to 0$ is exact. Conclusion follows easily. □

Remark 6. In the proof of the preceding proposition it has been proved that if $\mathcal{M}$ is affinely $\mathfrak{p}$-acyclic, then $\widehat{\mathcal{M}}$ is acyclic on any affine subset.

Lemma 7. *Let A be a noetherian ring and $I \subseteq A$ an ideal. If $0 \to M' \to M \to N \to 0$ is an exact sequence of A-modules and N is finitely generated, then the I-adic completion $0 \to \widehat{M'} \to \widehat{M} \to \widehat{N} \to 0$ is exact.*

Proof. Let $L \subseteq M$ be a finite submodule surjecting on N and $L' = L \cap M'$ which is also finite because A is noetherian. The exact sequences

$$0 \to L \to M \to M/L \to 0, \quad 0 \to L' \to M' \to M'/L' \to 0, \quad 0 \to L' \to L \to N \to 0$$

remain exact after I-adic completion, because L and L' are finite (this is a consequence of Artin-Rees lemma (10.10 [A])). Since $M/L \simeq M'/L'$ one concludes. □

Theorem 8 (on formal functions). *Let $f\colon X \to Y$ be a proper morphism of locally noetherian schemes, $\mathfrak{p}$ a coherent sheaf of ideals on Y and $\mathfrak{p}\mathcal{O}_X$ the ideal induced in X. For any coherent module $\mathcal{M}$ on X, the natural morphisms (where completions are made by $\mathfrak{p}$ and $\mathfrak{p}\mathcal{O}_X$ respectively)*

$$\widehat{R^i f_* \mathcal{M}} \to R^i f_*(\widehat{\mathcal{M}})$$

are isomorphisms. If $Y = \operatorname{Spec} A$, then

$$H^i(X,\mathcal{M})^\wedge = H^i(X,\widehat{\mathcal{M}})$$

Proof. The question is local on Y, so we may assume that $Y = \operatorname{Spec} A$ is affine. It suffices to show that $H^i(X,\mathcal{M})^\frown = H^i(X,\widehat{\mathcal{M}})$. It is clear that $\mathfrak{p}\mathcal{O}_X$ is generated by its global sections. As usual, we denote $I = \Gamma(X,\mathfrak{p}\mathcal{O}_X)$.

Let $C^\cdot\mathcal{M}$ be the Godement resolution of $\mathcal{M}$. Then $\widehat{C^\cdot M}$ is a flasque resolution of $\widehat{\mathcal{M}}$ (by Proposition 5) and $\Gamma(X,\widehat{C^\cdot\mathcal{M}}) = \Gamma(X,C^\cdot\mathcal{M})^\frown$ (by Proposition 4, (3)). Then we have to prove that the natural map

$$H^i(X,\mathcal{M})^\wedge = [H^i\Gamma(X,C^\cdot\mathcal{M})]^\wedge \to H^i(\Gamma(X,C^\cdot\mathcal{M})^\wedge) = H^i(\Gamma(X,\widehat{C^\cdot\mathcal{M}})) = H^i(X,\widehat{\mathcal{M}})$$

is an isomorphism. Let us denote by d_i the differential of the complex $\Gamma(X,C^\cdot\mathcal{M})$ on degree i. Completing the exact sequences

$$0 \to \operatorname{Ker} d_i \to \Gamma(X,C^i\mathcal{M}) \to \operatorname{Im} d_i \to 0$$

we obtain the exact sequences

$$0 \to \widehat{\operatorname{Ker} d_i} \to \Gamma(X,\widehat{C^i\mathcal{M}}) \to \widehat{\operatorname{Im} d_i} \to 0$$

because, as we shall see below, the I-adic topology of $\Gamma(X,C^i\mathcal{M})$ induces in $\operatorname{Ker} d_i$ the I-adic topology. Hence

$$H^i(X,\mathcal{M})^\wedge = (\operatorname{Ker} d_i/\operatorname{Im} d_{i-1})^\wedge \overset{\text{Lemma 7}}{=\!=\!=} \widehat{\operatorname{Ker} d_i}/\widehat{\operatorname{Im} d_{i-1}} = H^i(X,\widehat{\mathcal{M}})$$

Let $\mathcal{M}_i$ be the kernel of $C^i\mathcal{M} \to C^{i+1}\mathcal{M}$ (recall that $C^i\mathcal{M} = C^0\mathcal{M}_i$). Let us prove that the I-adic topology of $\Gamma(X,C^i\mathcal{M})$ induces the I-adic topology on $\operatorname{Ker} d_i = \Gamma(X,\mathcal{M}_i)$. Intersecting the equality $I^n\Gamma(X,C^0\mathcal{M}_i) = \Gamma(X,C^0(\mathfrak{p}^n\mathcal{M}_i))$ with $\Gamma(X,\mathcal{M}_i)$, one obtains that the induced topology on $\Gamma(X,\mathcal{M}_i)$ is given by the filtration $\{\Gamma(X,\mathfrak{p}^n\mathcal{M}_i)\}$. Hence it suffices to show that this filtration is I-stable. Since $\mathfrak{p}^n\mathcal{M}_i = (\mathfrak{p}^n\mathcal{M})_i$ (see the proof of 4 1.), it is enough to prove that the filtration $\{\Gamma(X,(\mathfrak{p}^n\mathcal{M})_i)\}$ is I-stable; this is equivalent to show that $\oplus_{n=0}^\infty\Gamma(X,(\mathfrak{p}^n\mathcal{M})_i)$ is a D_IA-module generated by a finite number of homogeneous components, where $D_IA = \underset{n=0}{\overset{\infty}{\oplus}} I^n$. By the exact sequence

$$\underset{n=0}{\overset{\infty}{\oplus}}\Gamma(X,C^{i-1}(\mathfrak{p}^n\mathcal{M})) \to \underset{n=0}{\overset{\infty}{\oplus}}\Gamma(X,(\mathfrak{p}^n\mathcal{M})_i) \to \underset{n=0}{\overset{\infty}{\oplus}}H^i(X,\mathfrak{p}^n\mathcal{M}) \to 0$$

it suffices to see the statement for the first and the third members. For the first one is obvious because $\Gamma(X,C^{i-1}(\mathfrak{p}^n\mathcal{M})) = I^n\Gamma(X,C^{i-1}\mathcal{M})$. For the third one, it suffices to see that it is a finite D_IA-module. Let $X' = X \times_A D_IA$, $\pi\colon X' \to X$ the natural projection and $\mathcal{M}' = \underset{n=0}{\overset{\infty}{\oplus}}\mathfrak{p}^n\mathcal{M}$ the obvious $\mathcal{O}_{X'}$-module. Since $H^i(X',\mathcal{M}')$ is a finite D_IA-module, one concludes from the equalities $H^i(X',\mathcal{M}') = H^i(X,\pi_*\mathcal{M}') = \underset{n=0}{\overset{\infty}{\oplus}}H^i(X,\mathfrak{p}^n\mathcal{M})$, because $\pi_*\mathcal{M}' = \underset{n=0}{\overset{\infty}{\oplus}}\mathfrak{p}^n\mathcal{M}$. □

Remark 9. Reading carefully the above proof, it is not difficult to see that one has already showed that $H^i(X,\mathcal{M})^\wedge = \lim\limits_{\substack{\leftarrow \\ n}} H^i(X,\mathcal{M}/\mathfrak{p}^n\mathcal{M})$.

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