Interpolation in surfaces

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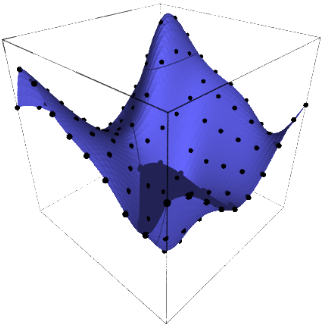
Introducction



A **surface** is a set of points such that (locally) is homeomorphic to an open subset of a plane.

The **interpolation** consist of, given some points that belong to an unknown surface, to obtain an equation such that the surface defined by the equation passes throught all the original points.

Introducction

A surface S in $\mathbb{R}$ is (locally) the graph of a function

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad f(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y)),$$

that is, for some (open) set U,

$$S = \{(f_1(x,y), f_2(x,y), f_3(x,y))\colon x, y \in U\}.$$

There are two tangent directions in each point (x_0, y_0) of the surface,

$$\frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial f}{\partial y}(x_0, y_0),$$

and one **normal direction** (that must be different from zero)

$$N(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \times \frac{\partial f}{\partial y}(x_0, y_0),$$

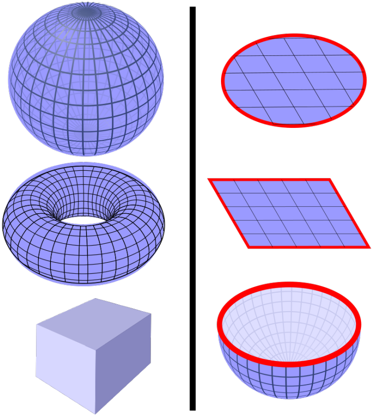
Introducction

A surface is defined as the imagen of several functions as above, such that they define the same set in the intersections.

The normal to the surface induces an **orientation**: two tangen vectors are positively oriented if their cross product "points to the same direction" as the normal.

A surface is **orientable** if we may define it as the imagen of several functions that induce the same orientation in the intersections.

Introducction



A surface is **closed** if “it has no borders”.

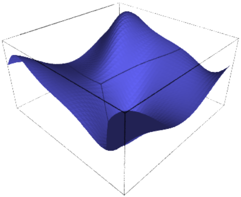
A surface is **compact** if it is closed and bounded (contained in a sphere).

Introducction

A special surface is the graph of a function of two variables $f(x, y)$, that is

$$S = \{(x, y, f(x, y) \colon x, y \in U\},$$

where U is some region of $\mathbb{R}^2$.



Introducción

In this section we will consider the interpolation of a function of two variables.

$$(x, y) \to f(x, y).$$

We assume we have a list of points where we have measured/evaluated the function.

Accordingly to how the points are distributed, we shall consider two cases:

- Regular grid.
- Scattered points.

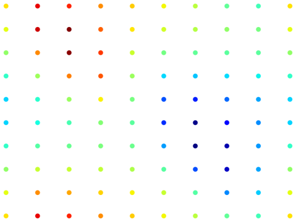
Introduction

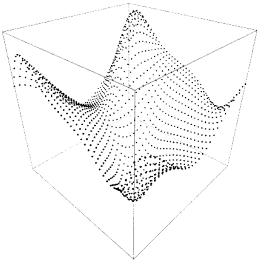
We assume we have some $h > 0$ and vectors:

$$pX = (x_0, x_1, \ldots, x_n), \quad pY = (y_0, y_1, \ldots, y_n)$$

Moreover, we assume we know the set

$$Z = \{f(x,y) \colon x \in pX,\ y \in pY\}.$$





Bilinear interpolation

Assume (x, y) is a point in the rectangle of the grid defined by

$$R = \{(x, y)\colon x_i \le x \le x_{i+1},\ y_j \le y \le y_{j+1}\}.$$

We define the interpolation function $L(x, y)$ with the following procedure:

1. We do a linear interpolation of (x_i, y_j), (x_{i+1}, y_j) to obtain $L(x, y_j)$.
2. We do a linear interpolation of (x_i, y_{j+1}), (x_{i+1}, y_{j+1}) to obtain $L(x, y_{j+1})$.
3. We do a linear interpolation of $L(x, y_j)$ and $L(x, y_{j+1})$ to obtain $L(x, y)$.

Bilinear interpolation

Alternatively, we may consider

$$L(x, y) = a_0 + a_1 x + a_2 y + a_3 xy,$$

and obtain a_0, a_1, a_2, a_3 from the (linear) equations

$$L(x_i, y_j) = f(x_i, y_j), \quad L(x_{i+1}, y_j) = f(x_{i+1}, y_j)$$

$$L(x_i, y_{j+1}) = f(x_i, y_{j+1}), \quad L(x_{i+1}, y_{j+1}) = f(x_{i+1}, y_{j+1})$$

Bicubic spline interpolation

Assume (x, y) is a point in the rectangle R. We define the bicubic interpolation $B(x, y)$ as

$$B(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij} x^i y^j,$$

where $\{a_{ij}\}$ are obtained solving the (linear) system:

$$B(x,y) = f(x,y), \quad x \in \{x_i, x_{i+1}\},\ y \in \{y_j, y_{j+1}\}$$

$$\frac{\partial B}{\partial x}(x,y) = \frac{\partial f}{\partial x}(x,y), \quad x \in \{x_i, x_{i+1}\},\ y \in \{y_j, y_{j+1}\}$$

$$\frac{\partial B}{\partial y}(x,y) = \frac{\partial f}{\partial y}(x,y), \quad x \in \{x_i, x_{i+1}\},\ y \in \{y_j, y_{j+1}\}$$

$$\frac{\partial B}{\partial xy}(x,y) = \frac{\partial f}{\partial xy}(x,y), \quad x \in \{x_i, x_{i+1}\},\ y \in \{y_j, y_{j+1}\}$$

Bicubic spline interpolation

The values of the derivatives of f at the points are estimated from the values of the neighbors:

$$\frac{\partial f}{\partial x}(x_i, y_j) = \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{x_{i+1} - x_{i-1}}.$$

$$\frac{\partial f}{\partial y}(x_i, y_j) = \frac{f(x_i, y_{j+1}) - f(x_i, y_{j-1})}{y_{i+1} - y_{i-1}}.$$

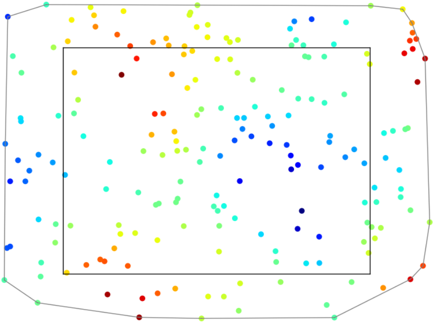
And analogously, the value of the cross derivative:

$$\frac{\partial f}{\partial xy}(x_i, y_j) = \frac{\frac{\partial f}{\partial x}(x_i, y_{j+1}) - \frac{\partial f}{\partial x}(x_i, y_{j-1})}{y_{i+1} - y_{i-1}}.$$

Introduction

Assume we have a list of points $P = \{(x_0, y_0), \ldots, (x_n, y_n)\}$ and we know the set

$$Z = \{f(x, y) \colon (x, y) \in P\}.$$



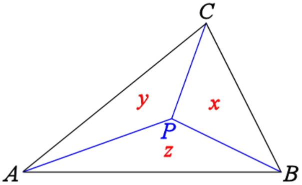
Linear interpolation

First we triangularize P with Delaunay triangulation.

Assume that P is a point of the triangle A, B, C. Then we define the linear interpolation as

$$I(P) = xf(A) + yf(B) + zf(C),$$

where, x, y, z are the coordinates of P in a barycentric system of coordinates.



Radial basis interpolation

A **radial basis function** ϕ is a function that deppends only of the distance, that is, $\phi(r) = \phi(\|r\|)$ for any $r \in \mathbb{R}^n$. Let

$x_0, \ldots, x_n \in \mathbb{R}^n$ the points where we know the value of our function f. Fixed a radial basis function, ϕ, a **radial basis**

interpolation is defined at any point $x \in \mathbb{R}^n$ by

$$I(x) = \sum_{i=0}^{n} w_i \phi(x - x_i),$$

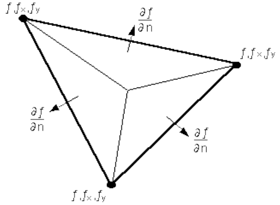
where $\{w_i\}$ are obtained imposing $I(x_i) = f(x_i)$, $i = 0, \ldots, n$

Clough-Tocher interpolation

Clough-Tocher interpolation is an extension of bilinear interpolation to scattered data.

We start obtaining the Delaunay triangulation of P.

We divide each triangle in three using the barycenter.
In each of the triangles we interpolate by a cubic polynomial.
We impose conditions of the derivatives and on the continuity and differentiability in the border of the triangles.



Definition

An **oriented triangle** is a list of three points. We say two oriented triangles (A_1, A_2, A_3), (B_1, B_2, B_3) are **equivalent** if one of the following conditions holds:

1. $A_1 = B_1$, $A_2 = B_2$, $A_3 = B_3$.
2. $A_1 = B_2$, $A_2 = B_3$, $A_3 = B_1$.
3. $A_1 = B_3$, $A_2 = B_1$, $A_3 = B_2$.

The **edges** of the triangle (A_1, A_2, A_3) are the tuples (A_1, A_2), (A_2, A_3), (A_3, A_1).

We say that the edges (A_1, A_2) and (A_2, A_1) have **opposite** orientation.

A triangle (A_1, A_2, A_3) induces an orientation in each of the vertices.

Definition

A triangulated closed oriented surface is a set of triangles such that the union of all triangles defines a closed oriented surface.

It has the following properties:

- All the faces with a common vertex induce the same orientation in that vertex.
- Each edge is included in exactly two faces. Moreover, it has opposite orientation in each one (closed surface).

Loop

Loop subdivision scheme smooths a surface defined by a triangulation.

Let $e = (v_1, v_2)$ be an edge of the triangulation.

Let $\{u_1, u_2\}$ the vertices of the faces containing the edge e and different from v_1, v_2.

Define a new vertex v_e as

$$v_e = \frac{3}{8}(v_1 + v_2) + \frac{1}{8}(u_1 + u_2)$$

Loop

Now, let v be a vertex of the triangulation. Let $v_1, \ldots, v_n$ the vertices sharing an edge with v.

Define a new vertex v' as

$$v' = (1 - n\alpha)v + \alpha \sum_{i=1}^{n} v_i,$$

where $\alpha = 3/16$ if $n = 3$, and if $n > 3$,

$$\alpha = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2\right)$$

Loop

For each triangle (A, B, C) of the triangulation let a, b, c are the edges.

Replace the triangle (A, B, C) by the following triangles:

$$(A', v_c, v_b), (B', v_a, v_c), (C', v_b, v_a), (v_a, v_b, v_c),$$

where v_a, v_b, v_c are the new vertices defined from the edges a, b, c as above, and A', B', C' are the new vertices defined from the vertices A, B, C.

Butterfly

Butterfly subdivision scheme interpolates a surface defined by a triangulation.

Let $e = (v_1, v_2)$ be an edge of the triangulation.

Let $\{u_1, u_2\}$ the vertices of the faces containing the edge e and different from v_1, v_2.

Let $\{u_{11}, u_{12}, u_{21}, u_{22}\}$ the vertices of the faces containing the edges $(v_1, u_1), (v_1, u_2), (v_2, u_1), (v_2, u_2)$ and different from v_1, v_2, u_1, u_2.

Define a new vertex:

$$v_e = \frac{8}{16}(v_1 + v_2) + \frac{2}{16}(u_1 + u_2) - (u_{11} + u_{12} + u_{21} + u_{22})$$

Butterfly

For each triangle (A, B, C) of the triangulation let a, b, c are the edges.

Replace the triangle (A, B, C) by the following triangles:

$$(A, v_c, v_b), (B, v_a, v_c), (C, v_b, v_a), (v_a, v_b, v_c),$$

where v_a, v_b, v_c are the new vertices defined from the edges a, b, c as above